

Happy number

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A **happy number** is a number defined by the following process: Starting with any positive integer, replace the number by the sum of the squares of its digits, and repeat the process until the number either equals 1 (where it will stay), or it loops endlessly in a cycle that does not include 1. Those numbers for which this process ends in 1 are happy numbers, while those that do not end in 1 are unhappy numbers (or sad numbers).^[1]

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Overview

More formally, given a number $n = n_0$, define a sequence n_1, n_2, \dots where n_{i+1} is the sum of the squares of the digits of n_i . Then n is happy if and only if there exists i such that $n_i = 1$.

If a number is happy, then all members of its sequence are happy; if a number is unhappy, all members of the sequence are unhappy.

For example, 19 is happy, as the associated sequence is:

$$\begin{aligned} 1^2 + 9^2 &= 82 \\ 8^2 + 2^2 &= 68 \\ 6^2 + 8^2 &= 100 \\ 1^2 + 0^2 + 0^2 &= 1. \end{aligned}$$

The 143 happy numbers up to 1,000 are:

1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100, 103, 109, 129, 130, 133, 139, 167, 176, 188, 190, 192, 193, 203, 208, 219, 226, 230, 236, 239, 262, 263, 280, 291, 293, 301, 302, 310, 313, 319, 320, 326, 329, 331, 338, 356, 362, 365, 367, 368, 376, 379, 383, 386, 391, 392, 397, 404, 409, 440, 446, 464, 469, 478, 487, 490, 496, 536, 556, 563, 565, 566, 608, 617, 622, 623, 632, 635, 637, 638, 644, 649, 653, 655, 656, 665, 671, 673, 680, 683, 694, 700, 709, 716, 736, 739, 748, 761, 763, 784, 790, 793, 802, 806, 818, 820, 833, 836, 847, 860, 863, 874, 881, 888, 899, 901, 904, 907, 910, 912, 913, 921, 923, 931, 932, 937, 940, 946, 964, 970, 973, 989, 998, 1000 (sequence A007770 in the OEIS).

The happiness of a number is unaffected by rearranging the digits, and by inserting or removing any number of zeros anywhere in the number.

Happy prime

A happy prime is a number that is both happy and prime. The happy primes below 500 are

7, 13, 19, 23, 31, 79, 97, 103, 109, 139, 167, 193, 239, 263, 293, 313, 331, 367, 379, 383, 397, 409, 487 (sequence A035497 in the OEIS).

All numbers, and therefore all primes, of the form $10^n + 3$ or $10^n + 9$ for n greater than 0 are happy. This does not mean that these are the only happy primes, as evidenced by the sequence above. To see this, note that

- All such numbers will have at least two digits;
- The first digit will always be 1 due to the 10^n
- The last digit will always be either 3 or 9.
- Any other digits will always be 0 (and therefore will not contribute to the sum of squares of the digits).
 - The sequence for numbers ending in 3 is: $1^2 + 3^2 = 10 \rightarrow 1^2 = 1$.
 - The sequence for numbers ending in 9 is: $1^2 + 9^2 = 82 \rightarrow 8^2 + 2^2 = 64 + 4 = 68 \rightarrow 6^2 + 8^2 = 36 + 64 = 100 \rightarrow 1$.

The palindromic prime $10^{150006} + 7426247 \times 10^{75\,000} + 1$ is also a happy prime with 150,007 digits because the many 0's do not contribute to the sum of squared digits, and

$1^2 + 7^2 + 4^2 + 2^2 + 6^2 + 2^2 + 4^2 + 7^2 + 1^2 = 176$, which is a happy number. Paul Jobling discovered the prime in 2005.^[6]

As of 2010, the largest known happy prime is $2^{42643801} - 1$ (Mersenne prime). Its decimal expansion has 12,837,064 digits.^[7]

Happy numbers in other bases

The definition of happy numbers depends on the decimal (i.e., base 10) representation of the numbers. The definition can be extended to other bases.

To represent numbers in other bases, we may use a subscript to the right to indicate the base. For instance, **100**₂ represents the number 4, and

$$123_5 = 1 \cdot 5^2 + 2 \cdot 5 + 3 = 38.$$

Then, it is easy to see that there are happy numbers in every base. For instance, the numbers

$$1_b, 10_b, 100_b, 1000_b, \dots$$

are all happy, for any base b .

By a similar argument to the one above for decimal happy numbers, unhappy numbers in base b lead to cycles of numbers less than **1000** _{b} . If $n < \mathbf{1000}_b$, then the sum of the squares of the base- b digits of n is less than or equal to

$$3(b-1)^2,$$

which can be shown to be less than b^3 , for $b \geq 5$. This shows that once the sequence reaches a number less than **1000** _{b} , it stays below **1000** _{b} , and hence must cycle or reach 1.

In base 2, all numbers are happy. All binary numbers larger than 1000_2 decay into a value equal to or less than 1000_2 , and all such values are happy: The following four sequences contain all numbers less than 1000_2 :

$111_2 \rightarrow 11_2 \rightarrow 10_2 \rightarrow 1$
 $110_2 \rightarrow 10_2 \rightarrow 1$
 $101_2 \rightarrow 10_2 \rightarrow 1$
 $100_2 \rightarrow 1$.

Since all sequences end in 1, we conclude that all numbers are happy in base 2. This makes base 2 a *happy base*.

The only known happy bases are 2 and 4. There are no others less than 500,000,000.^[8]

Base 3 is also a special case in that the happiness (or sadness) of a number is an indication also of being odd (or Even). Specifically, because $3 - 1 = 2$, the sum of every digit of a base 3 number will indicate divisibility by 2 IFF the sum of digits ends in 0 or 2. This is the general application of the test for 9-divisibility in base 10. In balanced ternary, the digits are 1, -1 and 0. The square of both 1 and -1 are 1, and $1 + 1$ is 2, which is the only balanced ternary cycle. For every pair of digits 1 or -1, their sum is 0 and the sum of their squares is 2 and if there are an even number of 1, -1 sets, the number divisible by 2 and sad and if odd, it is happy. In this case, the result always end in a one-digit cycle of 0, 1 or 2, repeated infinitely. In unbalanced ternary, the digits square to 1 and 4, and in this case there are 5 loops: 0, 1, $2 \rightarrow 4 \rightarrow 2$, 5 and 8. While all even numbers are sad because they end in the 0, 2 or 8 cycle, some odd numbers are also sad because they end in 5 or 1, and are thus occasionally sad.

In base 12, there is no other happy number between 10 (decimal 12) and 100 (decimal 144), there are 3 fixed points: 1, 25, 75, and 4 cycles:

$5 \rightarrow 21 \rightarrow 5$ (length 2)
 $8 \rightarrow 54 \rightarrow 35 \rightarrow 26 \rightarrow 88 \rightarrow 68 \rightarrow 118 \rightarrow 56 \rightarrow 51 \rightarrow 22 \rightarrow 8$ (length 10)
 $18 \rightarrow 55 \rightarrow 42 \rightarrow 18$ (length 3)
 $68 \rightarrow 84 \rightarrow 68$ (length 2)

The numbers 25 (decimal) and 75 (decimal 125) are Armstrong numbers in base 12. As of A161949, there are no 2-digit Armstrong numbers in base 10.

In hexadecimal, there is only 1 fixed point: 1, and 1 cycle:

D, A9, B5, 92, 55, 32, D (length 7)

The status in base 16 is similar to base 10.

Cubing the digits rather than squaring

A variation to the happy numbers problem is to find the sum of the cubes of the digits rather than the sum of the squares of the digits. For example, working in base 10, 1579 is happy, since:

$1^3 + 5^3 + 7^3 + 9^3 = 1 + 125 + 343 + 729 = 1198$
 $1^3 + 1^3 + 9^3 + 8^3 = 1 + 1 + 729 + 512 = 1243$
 $1^3 + 2^3 + 4^3 + 3^3 = 1 + 8 + 64 + 27 = 100$
 $1^3 + 0^3 + 0^3 = 1$

In the same way that when summing the squares of the digits (and working in base 10) each number above $243 (= 3 \cdot 81)$ produces a number that is strictly smaller, when summing the cubes of the digits each number above $2916 (= 4 \cdot 729)$ produces a number that is strictly smaller.

By conducting an exhaustive search of [1,2916] one finds that for summing the cubes of digits base 10 there are happy numbers and eight different types of unhappy number:

those that eventually reach **153**, **370**, **371**, or **407**, which perpetually produce themselves.

those that eventually reach the loops:

133 → **55** → **250** → **133** → **55** → **250**...

217 → **352** → **160** → **217** → **352** → **160**...,

as well as those that alternate between **1459** and **919** or between **136** and **244**.

All multiples of three end in 153. This fact can be proved by the exhaustive search up and noting that a number is a multiple of three if and only if the sum of digits is a multiple of three if and only if the sum of its cubed digits are a multiple of three. By similar reasoning, all happy numbers when summing up odd powers (e.g. cubes, 5th powers, 7th powers, etc.) of their digits must have a remainder of 1 when dividing by 3.

All numbers that are congruent to 2 (mod 3) end in either 371 or 407.

The only positive whole numbers that are the sum of the cubes of their digits are 1, 153, 370, 371 and 407 (sequence A046197 in the OEIS).

Higher powers

For higher powers, the density of happy numbers declines.

Taking the sum of the fourth powers of the digits, one can find that most numbers in the range 1-100 end in the loop:

13139, 6725, 4338, 4514, 1138, 4179, 9219, 13139, 6725, 4338, 4514, 1138, 4179, 9219, etc.

as well as those that alternate between 2178 and 6514 and those that end in **1634**, **8208**, or **9474**, which perpetually produce themselves.

Origin

The origin of happy numbers is not clear. Happy numbers were brought to the attention of Reg Allenby (a British author and senior lecturer in pure mathematics at Leeds University) by his daughter, who had learned of them at school. However, they "may have originated in Russia" (Guy 2004:§E34).

Popular culture

In the 2007 *Doctor Who* episode "42", a sequence of happy primes (313, 331, 367, 379) is used as a code for unlocking a sealed door on a spaceship about to collide with a star. When the Doctor learns that nobody on the spaceship besides himself has heard of happy numbers, he asks, "Don't they teach recreational mathematics anymore?"

The contestants in the 2012 University Challenge final were asked to identify a sequence of numbers as happy primes in a picture round.

Programming example

The examples below apply the 'happy' process described in the definition of happy given at the top of this article, repeatedly; after each time, they check for both halt conditions: reaching 1, and repeating a number. Everything else is book-keeping (for example, the Python example precomputes the squares of all 10 digits).

A simple test in Python to check if a number is happy:^[9]

```
def square(x):
    return int(x) * int(x)

def happy(number):
    return sum(map(square, list(str(number))))

def is_happy(number):
    seen_numbers = set()
    while number > 1 and (number not in seen_numbers):
        seen_numbers.add(number)
        number = happy(number)
    return number == 1
```

When the algorithm ends in a cycle of repeating numbers, this cycle always includes the number 4, so it is not even necessary to store previous numbers in the sequence:

```
def is_happy(number):
    if number == 1:
        return True
    elif number == 4:
        return False
    else:
        return is_happy(happy(number))
```

See also

- Fortunate number
- Harshad number
- Lucky number

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Categories: Base-dependent integer sequences | Recreational mathematics

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