# Design of Experiments for Quality

MAE 2019

MA4854 18S2

## Statistical Studies

- Statistics is about the collection, analysis, presentation and interpretation of data
- Most statistical studies are retrospective, *i.e.* with past data; some are observational with real time data, as in statistical process control charts
- Some studies need to be done but with no available data; hence data have to be generated – this is where techniques of Design of Experiments become relevant

# Reasons for DOE

- DOE is most useful when the subject of study is complex, new, or otherwise not fully understood
- DOE makes use of the most efficient (in terms of data collection effort) and most effective (in terms of providing information about the characteristics of the subject of study) approach to understanding the input-output relationship of a "black box" system
- DOE could reveal effects such as interactions of various orders among input factors that are otherwise not apparent – hence is a practical tool for discoveries especially in R&D and troubleshooting
- Understanding of the Mathematics of the theoretical foundation is not necessary for using DOE

# Terminology

- The independent variables (input) $x_1$, $x_2$, … are called *factors*
- Factors can be quantitative (*e.g*. time, temperature, rpm) or qualitative (*e.g*. male *vs* female, yes *vs* no)
- The dependent variable (output) $y$ is called *response*
- Factors are all controllable, and response is measurable
- Factors that are not controllable are generally known as *noise*
- The effects of all sources of noise are represented by one term $e$
- A replicate is response measured from an independent run, *i.e.* it is not a repeated measurement from the same response

## STATISTICAL EXPERIMENTS:

GENERATION

in addition to

COLLECTION,

ANALYSIS and

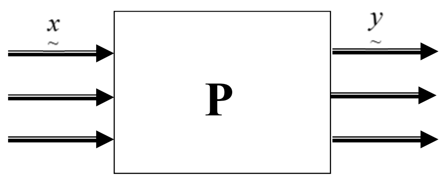
INTERPRETATION

of data

## ONE-FACTOR-AT-A-TIME

- Large number of observations
- Cannot detect interaction
- Effects of factors cannot be independently estimated
- Not possible to test the significance of individual effects
- Process optimization is difficult

## FRAMEWORK OF P-OPTIMIZATION




$$\underset{\sim}{y} = f\left(\underset{\sim}{x}\right)$$

| | | |
|---|---|---|
| Inputs | Outputs | |
| Independent variables | Dependent variables | *Systems* |
| Factors | Responses | *Mathematics* |
| Causes | Quality | *Statistics* |
| Parameters | Performance indices | *QA* |
| Key control characteristics | Key process characteristics | *Control* |
| | | *Engineering* |

## CODING EQUATIONS

$$X = \frac{\text{[ACTUAL]} - \text{[MID-POINT]}}{\text{[HALF RANGE]}}$$

HALF-RANGE = UNIT CHANGE IN $X$

# $2^k$ full factorial design

- Number of factors: $k$
- Factors could be quantitative (e.g. temperature, time) or qualitative (e.g. yes or no, machine A or machine B)
- Total number of experimental trials is equal to $2^k$ but this could be increased or decreased
- All factors are controllable
- All responses are measurable

In general, for $k$ variables, to write all possible combinations of two levels of the variables $x_1$, $x_2$, …, $x_k$ for a $2^k$ factorial design one can proceed by writing down columns as follows:

(1) For $x_1$ start with −1, +1, −1, +1, … and continue for all $2^k$ tests.

(2) For $x_2$ start with the first two tests as −1, the next two as +1, the next two as −1, etc., that is, −1, −1, +1, +1, −1, −1, … .

(3) For $x_3$ start with the first four tests as −1, the next four as +1, the next four as −1, etc., that is, −1, −1, −1, −1, +1, +1, +1, +1, −1, −1, −1, −1, … .

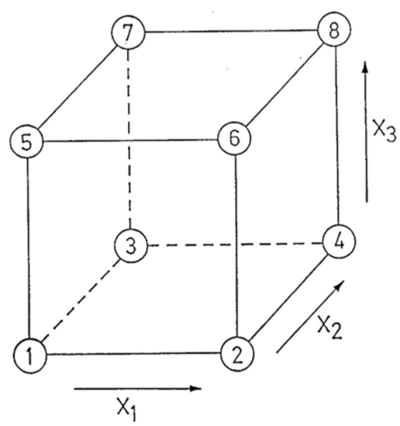
(4) Fox $x_4$ start with the first eight tests as −1, the next eight as +1, the next eight as −1, etc.

(5) Proceed in a similar way for $x_5$, $x_6$, …, etc. until finally reaching $x_k$.

(6) For $x_k$ start with the first $2^{k-1}$ tests as −1 and the next $2^{k-1}$ tests as +1.

## $2^3$ DESIGN

| $i$ | $x_1$ | $x_2$ | $x_3$ | $y_i$ |
|---|---|---|---|---|
| 1 | – | – | – | $y_1$ |
| 2 | + | – | – | $y_2$ |
| 3 | – | + | – | $y_3$ |
| 4 | + | + | – | $y_4$ |
| 5 | – | – | + | $y_5$ |
| 6 | + | – | + | $y_6$ |
| 7 | – | + | + | $y_7$ |
| 8 | + | + | + | $y_8$ |



## $2^3$

$$E_0 = \frac{1}{8}(+y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8)$$

$$E_1 = \frac{1}{4}(-y_1+y_2-y_3+y_4-y_5+y_6-y_7+y_8)$$

$$E_2 = \frac{1}{4}(-y_1-y_2+y_3+y_4-y_5-y_6+y_7+y_8)$$

$$E_3 = \frac{1}{4}(-y_1-y_2-y_3-y_4+y_5+y_6+y_7+y_8)$$

$$E_{12} = \frac{1}{4}(+y_1-y_2-y_3+y_4+y_5-y_6-y_7+y_8)$$

$$E_{13} = \frac{1}{4}(+y_1-y_2+y_3-y_4-y_5+y_6-y_7+y_8)$$

$$E_{23} = \frac{1}{4}(+y_1+y_2-y_3-y_4-y_5-y_6+y_7+y_8)$$

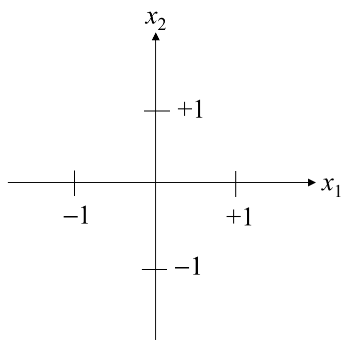
$$E_{123} = \frac{1}{4}(-y_1+y_2+y_3-y_4+y_5-y_6-y_7+y_8)$$

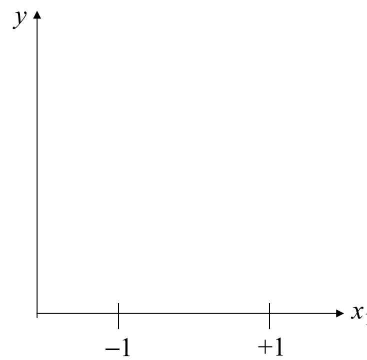
| | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | $y_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | – | – | – | + | + | + | – | $y_1$ |
| 2 | + | – | – | – | – | + | + | $y_2$ |
| 3 | – | + | – | – | + | – | + | $y_3$ |
| 4 | + | + | – | + | – | – | – | $y_4$ |
| 5 | – | – | + | + | – | – | + | $y_5$ |
| 6 | + | – | + | – | + | – | – | $y_6$ |
| 7 | – | + | + | – | – | + | – | $y_7$ |
| 8 | + | + | + | + | + | + | + | $y_8$ |

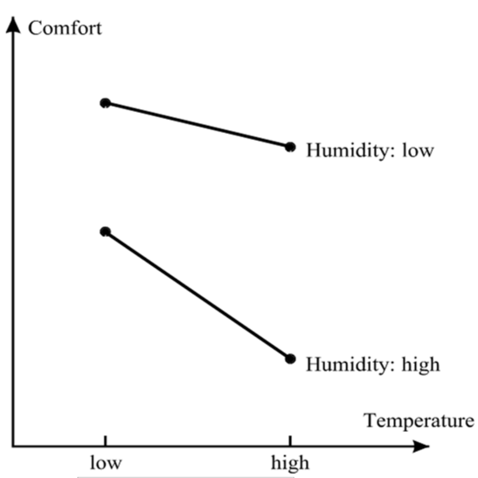
# Interactions

Diagonal rule and gradient rule

| $i$ | $x_1$ | $x_2$ | $y_i'$ | $y_i''$ |
|---|---|---|---|---|
| 1 | – | – | 21 | 12 |
| 2 | + | – | 23 | 15 |
| 3 | – | + | 26 | 13 |
| 4 | + | + | 28 | 19 |


$x_2$
+1
$x_1$
−1
+1
−1


$y$
$x_1$
−1
+1


Comfort
Humidity: low
Humidity: high
Temperature
low
high

## FACTORIAL EXPERIMENTS

## $2^2$

$$E_0 = \frac{1}{4}\left(+y_1 + y_2 + y_3 + y_4\right)$$

$$E_1 = \frac{1}{2}\left(-y_1 + y_2 - y_3 + y_4\right)$$

$$E_2 = \frac{1}{2}\left(-y_1 - y_2 + y_3 + y_4\right)$$

$$E_{12} = \frac{1}{2}\left(+y_1 - y_2 - y_3 + y_4\right)$$

## $2^3$

$$E_0 = \frac{1}{8}\left(+y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8\right)$$

$$E_1 = \frac{1}{4}\left(-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8\right)$$

$$E_2 = \frac{1}{4}\left(-y_1 - y_2 + y_3 + y_4 - y_5 - y_6 + y_7 + y_8\right)$$

$$E_3 = \frac{1}{4}\left(-y_1 - y_2 - y_3 - y_4 + y_5 + y_6 + y_7 + y_8\right)$$

$$E_{12} = \frac{1}{4}\left(+y_1 - y_2 - y_3 + y_4 + y_5 - y_6 - y_7 + y_8\right)$$

$$E_{13} = \frac{1}{4}\left(+y_1 - y_2 + y_3 - y_4 - y_5 + y_6 - y_7 + y_8\right)$$

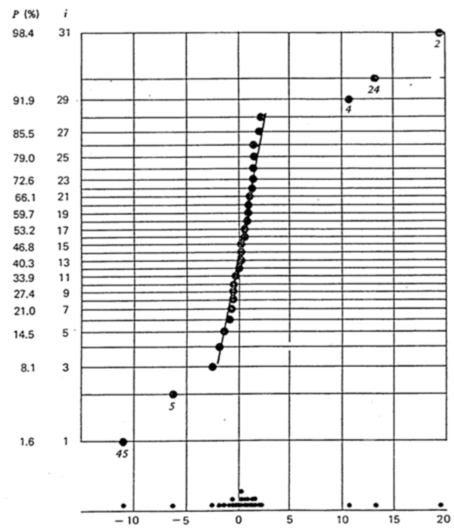
$$E_{23} = \frac{1}{4}\left(+y_1 + y_2 - y_3 - y_4 - y_5 - y_6 + y_7 + y_8\right)$$

$$E_{123} = \frac{1}{4}\left(-y_1 + y_2 + y_3 - y_4 + y_5 - y_6 - y_7 + y_8\right)$$

| $i$ | $x_1$ | $x_2$ | $x_1x_2$ | $y_i$ |
|---|---|---|---|---|
| 1 | – | – | + | $y_1$ |
| 2 | + | – | – | $y_2$ |
| 3 | – | + | – | $y_3$ |
| 4 | + | + | + | $y_4$ |

| | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | $y_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | – | – | – | + | + | + | – | $y_1$ |
| 2 | + | – | – | – | – | + | + | $y_2$ |
| 3 | – | + | – | – | + | – | + | $y_3$ |
| 4 | + | + | – | + | – | – | – | $y_4$ |
| 5 | – | – | + | + | – | – | + | $y_5$ |
| 6 | + | – | + | – | + | – | – | $y_6$ |
| 7 | – | + | + | – | – | + | – | $y_7$ |
| 8 | + | + | + | + | + | + | + | $y_8$ |

NORMAL PLOT OF EFFECTS




$P_i = 100(i - \frac{1}{2})/m$ for $i = 1, 2, \ldots, m$

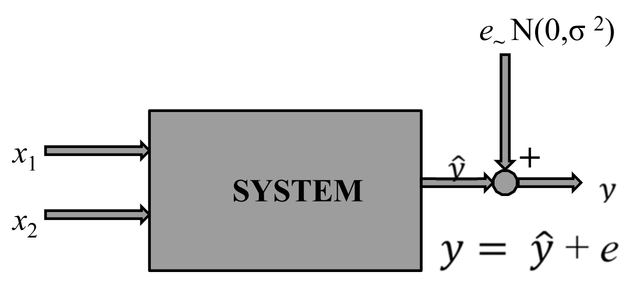
Experiment ‘A’

| $i$ | Temp | Press | FR | $y$% |
|---|---|---|---|---|
| 1 | – | – | – | 79.8 |
| 2 | + | – | – | 79.6 |
| 3 | – | + | – | 79.2 |
| 4 | + | + | – | 81.8 |
| 5 | – | – | + | 64.2 |
| 6 | + | – | + | 63.2 |
| 7 | – | + | + | 85.6 |
| 8 | + | + | + | 90.6 |

Analyze the experimental data, and assess the results by a (a) Pareto analysis (b) normal probability plot.

# Modeling *via* DOE

## Framework of DOE Modeling




Data components: $\hat{y} = f(x_1, x_2); e \sim N(0,\sigma^2)$

# PROCESS MODELLING

A MODEL is a mathematical representation of the essential input-output relationship of a process (or product).

It is useful because it

* DESCRIBES what has been observed;
* PREDICTS what is likely to happen;
* Is needed for CONTROL and OPTIMIZATION of the process (or product) behaviour.

## FACTORIALS EXPERIMENTS

ENGINEERING DATA

Factors: $x_1, x_2, x_3 \ldots$

Response: $y$

MATHEMATICAL MODELLING

$y$ : Observed response

$\hat{y}$ : Predicted response

$e$ : Error/residual

$$
\begin{aligned}
y &= \hat{y} + e \\
&= f(x_1, x_2, x_3, \ldots) + e
\end{aligned}
$$

STATISTICAL ANALYSIS

$e \sim N(0, \sigma^2)$

# MATHEMATICAL MODEL

$2^3$ Design

$$\hat{y} = E_0 + (E_1/2)x_1 + (E_2/2)x_2 + (E_3/2)x_3 + (E_{12}/2)x_1x_2 + (E_{13}/2)x_1x_3 + (E_{23}/2)x_2x_3 + (E_{123}/2)x_1x_2x_3$$

$$y = \hat{y} + e$$

$e \sim$ NID (o, $\sigma^2$)

Note: To drop insignificant terms
To add higher order terms
To check normality assumption

*Tools:* Pareto Analysis; Normal Probability Plot; Centerpoint Design; Histograms; Run Charts

## Replicated Factorial Design

| $i$ | $x_1$ | $x_2$ | $x_3$ | $y_{i1}$ | $y_{i2}$ | $\bar{y}_i$ | $s_i^2$ | d.f. |
|---|---|---|---|---|---|---|---|---|
| 1 | – | – | – | $y_{11}$ | $y_{12}$ | $\bar{y}_1$ | $s_1^2$ | 1 |
| 2 | + | – | – | $y_{21}$ | $y_{22}$ | $\bar{y}_2$ | $s_2^2$ | 1 |
| 3 | – | + | – | $y_{31}$ | $y_{32}$ | $\bar{y}_3$ | $s_3^2$ | 1 |
| 4 | + | + | – | $y_{41}$ | $y_{42}$ | $\bar{y}_4$ | $s_4^2$ | 1 |
| 5 | – | – | + | $y_{51}$ | $y_{52}$ | $\bar{y}_5$ | $s_5^2$ | 1 |
| 6 | + | – | + | $y_{61}$ | $y_{62}$ | $\bar{y}_6$ | $s_6^2$ | 1 |
| 7 | – | + | + | $y_{71}$ | $y_{72}$ | $\bar{y}_7$ | $s_7^2$ | 1 |
| 8 | + | + | + | $y_{81}$ | $y_{82}$ | $\bar{y}_8$ | $s_8^2$ | 1 |

Special case of $r = 2$:

$$s_i^2 = \frac{(y_{i1} - \bar{y}_i)^2 + (y_{i2} - \bar{y}_i)^2}{2-1} = \frac{(y_{i1} - y_{i2})^2}{2} = \frac{d_i^2}{2}$$

$s_p^2$ = average of $s_i^2$

## Experiment 'B'

| $i$ | Temp | Press | FR | $y_{i1}$ | $y_{i2}$ |
|---|---|---|---|---|---|
| 1 | – | – | – | 78.9 | 80.7 |
| 2 | + | – | – | 79.9 | 79.3 |
| 3 | – | + | – | 78.4 | 80.0 |
| 4 | + | + | – | 82.8 | 80.8 |
| 5 | – | – | + | 65.1 | 63.3 |
| 6 | + | – | + | 62.2 | 64.2 |
| 7 | – | + | + | 82.8 | 88.4 |
| 8 | + | + | + | 91.8 | 89.4 |

(a) Calculate the effects, and assess the significance of each magnitude.

(b) Examine the response consistency as a function of the factors and their interactions.

## $2^3 \times 2 + 4$: Design & Analysis

| $i$ | $x_1$ | $x_2$ | $x_3$ | $x_1x_2$ | $x_1x_3$ | $x_2x_3$ | $x_1x_2x_3$ | $y_{i1}$ | $y_{i2}$ | $y_{i3}$ | $y_{i4}$ | $\bar{y}_i$ | $s_i^2$ | d.f. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | – | – | – | + | + | + | – | $y_{11}$ | $y_{12}$ | | | $\bar{y}_1$ | $s_1^2$ | 1 |
| 2 | + | – | – | – | – | + | + | $y_{21}$ | $y_{22}$ | | | $\bar{y}_2$ | $s_2^2$ | 1 |
| 3 | – | + | – | – | + | – | + | $y_{31}$ | $y_{32}$ | | | $\bar{y}_3$ | $s_3^2$ | 1 |
| 4 | + | + | – | + | – | – | – | $y_{41}$ | $y_{42}$ | | | $\bar{y}_4$ | $s_4^2$ | 1 |
| 5 | – | – | + | + | – | – | + | $y_{51}$ | $y_{52}$ | | | $\bar{y}_5$ | $s_5^2$ | 1 |
| 6 | + | – | + | – | + | – | – | $y_{61}$ | $y_{62}$ | | | $\bar{y}_6$ | $s_6^2$ | 1 |
| 7 | – | + | + | – | – | + | – | $y_{71}$ | $y_{72}$ | | | $\bar{y}_7$ | $s_7^2$ | 1 |
| 8 | + | + | + | + | + | + | + | $y_{81}$ | $y_{82}$ | | | $\bar{y}_8$ | $s_8^2$ | 1 |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $y_{91}$ | $y_{92}$ | $y_{93}$ | $y_{94}$ | $\bar{y}_9$ | $s_9^2$ | 3 |
| | | Design matrix (orthogonal array) | | | Extension of design matrix for calculating the effects | | | | Raw data ----------→ | | | For $E_i$ Calculation | Pool to get $s_p^2$ | Pool to get $p$ |

$$s_p^2 = \frac{\Sigma \nu_i s_i^2}{\Sigma \nu_i}$$

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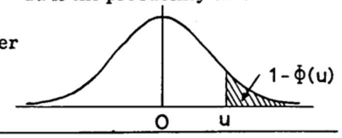
## AREAS IN TAIL OF THE NORMAL DISTRIBUTION

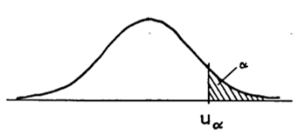
The function tabulated is $1 - \Phi(u)$ where $\Phi(u)$ is the cumulative distribution function of a standardised Normal variable u. Thus $1 - \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-u^2/2}\,du$ is the probability that a standardised Normal variable selected at random will be greater than a value of u $\left(= \frac{x-\mu}{\sigma}\right)$




| $\frac{(x-\mu)}{\sigma}$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |
| 0.1 | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| 0.2 | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| 0.3 | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| 0.4 | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| 0.5 | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| 0.6 | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| 0.7 | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| 0.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| 0.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| 1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| 1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| 1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| 1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| 1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |

## PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

The table gives the $100\alpha$ percentage points, $u_\alpha$, of a standardised Normal distribution where $\alpha = \frac{1}{\sqrt{2\pi}} \int_{u_\alpha}^\infty e^{-u^2/2}\,du$. Thus $u_\alpha$ is the value of a standardised Normal variate which has probability $\alpha$ of being exceeded.




| $\alpha$ | $u_\alpha$ | $\alpha$ | $u_\alpha$ | $\alpha$ | $u_\alpha$ | $\alpha$ | $u_\alpha$ | $\alpha$ | $u_\alpha$ | $\alpha$ | $u_\alpha$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| .50 | 0.0000 | .050 | 1.6449 | .030 | 1.8808 | .020 | 2.0537 | .010 | 2.3263 | .050 | 1.6449 |
| .45 | 0.1257 | .048 | 1.6646 | .029 | 1.8957 | .019 | 2.0749 | .009 | 2.3656 | .010 | 2.3263 |
| .40 | 0.2533 | .046 | 1.6849 | .028 | 1.9110 | .018 | 2.0969 | .008 | 2.4089 | .001 | 3.0902 |
| .35 | 0.3853 | .044 | 1.7060 | .027 | 1.9268 | .017 | 2.1201 | .007 | 2.4573 | .0001 | 3.7190 |
| .30 | 0.5244 | .042 | 1.7279 | .026 | 1.9431 | .016 | 2.1444 | .006 | 2.5121 | .00001 | 4.2649 |
| .25 | 0.6745 | .040 | 1.7507 | .025 | 1.9600 | .015 | 2.1701 | .005 | 2.5758 | .025 | 1.9600 |
| .20 | 0.8416 | .038 | 1.7744 | .024 | 1.9774 | .014 | 2.1973 | .004 | 2.6521 | .005 | 2.5758 |
| .15 | 1.0364 | .036 | 1.7991 | .023 | 1.9954 | .013 | 2.2262 | .003 | 2.7478 | .0005 | 3.2905 |
| .10 | 1.2816 | .034 | 1.8250 | .022 | 2.0141 | .012 | 2.2571 | .002 | 2.8782 | .00005 | 3.8906 |
| .05 | 1.6449 | .032 | 1.8522 | .021 | 2.0335 | .011 | 2.2904 | .001 | 3.0902 | .000005 | 4.4172 |

## PERCENTAGE POINTS OF THE t DISTRIBUTION

The table gives the value of $t_{\alpha;\nu}$ — the $100\alpha$ percentage point of the t distribution for $\nu$ degrees of freedom.

The values of t are obtained by solution of the equation:-

$$\alpha = \Gamma\{\tfrac{1}{2}(\nu+1)\}\,\{\Gamma(\tfrac{1}{2}\nu)\}^{-1}\,(\nu\pi)^{-1/2}\int_t^{\infty}(1 + x^2/\nu)^{-(\nu+1)/2}dx$$

Note. The tabulation is for one tail only i.e. for positive values of t. For $|t|$ the column headings for $\alpha$ must be doubled.


| $\alpha =$ | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 | 0.0005 |
|---|---|---|---|---|---|---|---|
| $\nu$ = 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.31 | 636.62 |
| 2 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.326 | 31.598 |
| 3 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.213 | 12.924 |
| 4 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 | 8.610 |
| 5 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 | 6.869 |
| 6 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 | 5.959 |
| 7 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 | 5.408 |
| 8 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 | 5.041 |
| 9 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 | 4.781 |
| 10 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 | 4.587 |
| 11 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 | 4.437 |
| 12 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 | 4.318 |
| 13 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 | 4.221 |
| 14 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 | 4.140 |
| 15 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 | 4.073 |

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