HW ch8 #5

Suppose the nodes are equally space. $x_j = -1 + j h$ for $j = 0, \dots, n$ with $h = \frac{2}{n}$ and n is even

Let $N = \frac{n}{2}$, so $x_N = -1 + N \cdot h = 0$ and $\{x_j\} = \{-N \cdot h, -(N-1)\cdot h, \dots, -h, 0, h, \dots, (N-1)\cdot h, N \cdot h\}$

Then $w_{n+1}(x) = \prod_{j=0}^{n} (x - x_j) = (x + N\cdot h)(x + (N-1)\cdot h) \cdot \ldots \cdot (x+h)\, x\, (x-h) \cdot \ldots \cdot (x - (N-1)h)(x - Nh)$ —— (1)

$\because x \in (x_{n-1}, x_n)$ $\therefore$ Take $x = r \cdot h$ with $N-1 < r < N$

From (1), $|w_{n+1}(x)| = \prod_{k=-N}^{N} |x - k \cdot h| = \prod_{k=-N}^{N} |r \cdot h - k \cdot h| = h^{n+1} \cdot \prod_{k=-N}^{N} |r - k|$

and $|(x - x_{n-1})(x - x_n)| = h^2 \cdot |(r - (N-1))(r - N)|$

Now we have $\dfrac{|w_{n+1}(x)|}{h^{n-1} \cdot |(x - x_{n-1})(x - x_n)|} = \prod_{k=-N}^{N-2} |r - k| = \prod_{j=-(N-2)}^{N} (r + j) = \overbrace{(r - (N-2)) \cdot (r - (N-3)) \cdot \ldots \cdot (r + N)}^{\text{共 } n-1 \text{ 項}}$ —— (2)

Note that $N-1 < r < N$, then $(r - (N-2)) \in (1, 2)$, $(r - (N-3)) \in (2, 3)$, ..., $(r + N) \in (n-1, n)$

Hence, $\prod_{j=-(N-2)}^{N} (r + j) \in ((n-1)!, n!)$ —— (3)

From (2), (3), we can know that

$$(n-1)! \cdot h^{n-1} \cdot |(x - x_{n-1})(x - x_n)| \leq |w_{n+1}(x)| \leq n! \cdot h^{n-1} \cdot |(x - x_n)(x - x_{n-1})|$$

HW ch8 #6

From #5, we have $w_{n+1}(x) = \prod_{k=-N}^{N} (x - k h)$ and $|w_{n+1}(x)|$ is an even function

Then $w_{n+1}(x + h) = \prod_{k=-N}^{N} (x + h - k \cdot h) = \prod_{k=-(N+1)}^{N-1} (x - k \cdot h)$

$$\Rightarrow \left| \frac{w_{n+1}(x+h)}{w_{n+1}(x)} \right| = \left| \frac{x + (N+1)\cdot h}{x - N \cdot h} \right| \quad \text{for } x \notin \{x_j\}$$

If $x \in (0, x_{n-1}) = (0, (N-1)h)$, then $N \cdot h - x > 0$ and $x + (N+1)\cdot h > N \cdot h - x$

$$\Rightarrow \left| \frac{w_{n+1}(x+h)}{w_{n+1}(x)} \right| = \frac{x + (N+1)\cdot h}{N \cdot h - x} > 1 \quad \text{for all } x \in (0, x_{n-1}) \text{ not at nodes}$$

So as x moves to the right by one mesh step, $|w_{n+1}|$ strictly increases on $(0, x_{n-1})$.

By continuity, $|w_{n+1}|$ on $[0, 1]$ attain its maximum over the last subinterval (x_{n-1}, x_n)

Finally, since $|w_{n+1}|$ is even, the same holds symmetric on the left, and overall the maximum on $[-1, 1]$ is reached for $x \in (x_{n-1}, x_n)$ and symmetrically on (x_0, x_1)

HW ch8 #8

Let $x_0 \in \mathbb{R}$ and fix $n \geq 0$

Define $L_{0;j}(x) := \frac{(x - x_0)^j}{j!} \in \mathbb{P}_n$ for $j = 0, \dots, n$

These are the one-node Hermite characteristic polynomials

$$\frac{d^p}{dx^p} L_{0;j}(x_0) = \begin{cases} 1, & p = j \\ 0, & p \neq j \end{cases}$$

Now, set $Hf(x) := \sum_{j=0}^{n} f^{(j)}(x_0) \cdot L_{0;j}(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$

Then, for each $k = 0, \dots, n$

$$(Hf)^{(k)}(x_0) = \sum_{j=0}^{n} f^{(j)}(x_0) \cdot L_{0;j}^{(k)}(x_0) = f^{(k)}(x_0)$$

so Hf satisfies the Hermite conditions at x_0

Therefore, the Hermite interpolating polynomial at the single node x_0 concides with the Taylor polynomial of order n at x_0.

HW 4.

The $n+1$ Chebyshev points of second kind are $x_j = \cos \frac{j \cdot \pi}{n}$, $j = 0, \dots, n$

For barycentric interpolation, the weights are $\lambda_j = \dfrac{1}{\prod_{k \neq j} (x_j - x_k)} = \dfrac{1}{w_{n+1}'(x_j)}$, where $w_{n+1}(x) = \prod_{k=0}^{n} (x - x_k)$

Since $\{x_j\}$ are $n+1$ Chebyshev 2nd kind nodes, then $w_{n+1}(x) = \frac{1}{2^{n-1}}(x^2-1)U_{n-1}(x)$, where $U_n(x)$ be the Chebyshev polynomial of 2nd kind

$\Rightarrow w_{n+1}'(x) = \frac{1}{2^{n-1}}\left[2x \cdot U_{n-1}(x) + (x^2-1)\,U_{n-1}'(x)\right]$

Let $x = \cos\theta$, $w_{n+1}(\cos\theta) = \frac{-1}{2^{n-1}} \sin^2\theta \, \frac{\sin(n\theta)}{\sin\theta} = \frac{-1}{2^{n-1}} \sin\theta \cdot \sin(n\theta)$

then $w_{n+1}'(x)\Big|_{x=\cos\theta} = \frac{d}{dx} w_{n+1}(\cos\theta) = \frac{d_\theta\, w_{n+1}(\cos\theta)}{-\sin\theta}$

Compute limit at $\theta = 0, \pi$,

$$w_{n+1}'(1) = \lim_{\theta\to 0} \frac{-2^{-(n-1)}\left(\cos\theta\sin(n\theta) + n\sin\theta\cos(n\theta)\right)}{-\sin\theta} = \frac{n}{2^{n-2}}$$

$$w_{n+1}'(-1) = \lim_{\theta\to \pi} \frac{-2^{-(n-1)}\left(\cos\theta\sin(n\theta) + n\sin\theta\cos(n\theta)\right)}{-\sin\theta} = \frac{(-1)^n \cdot n}{2^{n-2}}$$

Choose $c = \frac{n}{2^{n-1}}$ and do the rescaling by $w_j = c \cdot \lambda_j$

Then $w_0 = \frac{1}{2}$ and $w_n = \frac{(-1)^n}{2}$