Ch9

1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

$E_0(f) = \frac{h_0^3}{3} f''(\xi)$, $h_0 = \frac{b-a}{2}$ and $E_1(f) = -\frac{h_1^3}{12} \cdot f''(\xi)$, $h_1 = b-a$

Let $h = b-a$, then $h_1 = h$ and $h_0 = \frac{h}{2}$

$|E_0(f)| = \frac{h^3}{24} \cdot f''(\xi)$ and $|E_1(f)| = \frac{h^3}{12} \cdot f''(\xi)$

Therefore, $|E_1(f)| \simeq 2 \cdot |E_0(f)|$

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

(a.) When $f(x) = 1$, $I_2(1) = \frac{2}{3}[2 \cdot (1) - 1 + 2 \cdot (1)] = 2$ and $\int_{-1}^{1} 1\, dx = 2$

When $f(x) = x$, $I_2(x) = \frac{2}{3}[2 \cdot (-\frac{1}{2}) + 2 \cdot (\frac{1}{2})] = 0$ and $\int_{-1}^{1} x\, dx = 0$

When $f(x) = x^2$, $I_2(x^2) = \frac{2}{3}[2 \cdot (\frac{1}{4}) + 2 \cdot (\frac{1}{4})] = \frac{2}{3}$ and $\int_{-1}^{1} x^2 dx = \frac{2}{3}$

When $f(x) = x^3$, $I_2(x^3) = \frac{2}{3}[2 \cdot (-\frac{1}{8}) + 2 \cdot (\frac{1}{8})] = 0$ and $\int_{-1}^{1} x^3 dx = 0$

When $f(x) = x^4$, $\underline{I_2(x^4)} = \frac{2}{3}[2 \cdot (\frac{1}{16}) + 2 \cdot (\frac{1}{16})] = \underline{\frac{1}{6}}$ and $\underline{\int_{-1}^{1} x^4 dx = \frac{2}{5}}$

Therefore, the degree of exactness is $\boxed{r=3}$

By the general error theory of Lagrange-type quadrature, if the rule integrates exactly all polynomials up to degree r, then the error involves the derivative of order $n+1$:

$$E(f) = C \cdot f^{(r+1)}(\xi), \quad \xi \in (-1, 1) \quad \text{for some nonzero constant } C$$

Here $r=3$, so the error depends on $f^{(4)}(\xi)$. Moreover, since the integration is over an interval of length $h = b-a = 2$, one more factor of h arises from the integration of the interpolation error polynomial. Thus $E(f) = O(h^{r+2}) = O(h^5)$
Therefore, the order of infinitesimal is $\boxed{p=5}$

(b.) When $f(x) = 1$, $I_4(1) = \frac{1}{4}[1 + 3 \cdot (1) + 3 \cdot (1) + 1] = 2$ and $\int_{-1}^{1} 1\, dx = 2$

When $f(x) = x$, $I_4(x) = \frac{1}{4}[-1 + 3 \cdot (-\frac{1}{3}) + 3 \cdot (\frac{1}{3}) + 1] = 0$ and $\int_{-1}^{1} x\, dx = 0$

When $f(x) = x^2$, $I_4(x^2) = \frac{1}{4}[1 + 3 \cdot (\frac{1}{9}) + 3 \cdot (\frac{1}{9}) + 1] = \frac{2}{3}$ and $\int_{-1}^{1} x^2 dx = \frac{2}{3}$

When $f(x) = x^3$, $I_4(x^3) = \frac{1}{4}[1 + 3 \cdot (\frac{-1}{27}) + 3 \cdot (\frac{1}{27}) + 1] = 0$ and $\int_{-1}^{1} x^3 dx = 0$

When $f(x) = x^4$, $\underline{I_4(x^4)} = \frac{1}{4}[1 + 3 \cdot (\frac{1}{81}) + 3 \cdot (\frac{1}{81}) + 1] = \underline{\frac{14}{27}}$ and $\underline{\int_{-1}^{1} x^4 dx = \frac{2}{5}}$

Therefore, the degree of exactness is $\boxed{r=3}$

By (a.), the order of infinitesimal is $\boxed{p=5}$

5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

When $f(x)=1$, $I_w(1) = \int_0^1 x^{\frac{1}{2}} dx = \frac{2}{3}$ and $Q(1) = a \cdot 1 = a$

$I_w(1) = Q(1) \Rightarrow a = \frac{2}{3}$

When $f(x)=x$, $I_w(x) = \int_0^1 x^{\frac{3}{2}} dx = \frac{2}{5}$ and $Q(x) = \frac{2}{3} \cdot x_1$

$I_w(x) = Q(x) \Rightarrow \frac{2}{5} = \frac{2}{3}x_1 \Rightarrow x_1 = \frac{3}{5}$

When $f(x)=x^2$, $I_w(x^2) = \int_0^1 x^{\frac{5}{2}} dx = \frac{2}{7}$ and $Q(x^2) = \frac{2}{3} \cdot (\frac{3}{5})^2 = \frac{6}{25}$

Therefore, $a = \frac{2}{3}$, $x_1 = \frac{3}{5}$ and $r = 1$

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

$f(x)=1$, $Q(1) = \alpha_1 + \alpha_2$ and $I(1) = \int_0^1 1\, dx = 1 \Rightarrow \alpha_1 + \alpha_2 = 1$

$f(x)=x$, $Q(x) = \alpha_2 + \alpha_3$ and $I(x) = \int_0^1 x\, dx = \frac{1}{2} \Rightarrow \alpha_2 + \alpha_3 = \frac{1}{2}$

$f(x)=x^2$, $Q(x^2) = \alpha_2$ and $I(x^2) = \int_0^1 x^2 dx = \frac{1}{3} \Rightarrow \alpha_2 = \frac{1}{3}$

Therefore, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$ and $\alpha_3 = \frac{1}{6}$

$f(x)=x^3$, $Q(x^3) = \frac{1}{3}$ and $I(x^3) = \frac{1}{4}$

$\Rightarrow r = 2$