5. Prove the estimate (12.23). $\|\tau_h\|_h^2 \le 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2)$

[*Hint*: for each internal node x_j, $j = 1, \dots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2\, dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2\, dt\right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \dots, n-1$. On noting that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

$\left(\|v\|_h := h\sum_{j=1}^{n-1} v(x_j)^2, \quad x_j = jh, \; h = \frac{1}{n}\right)$

From (12.21), we have

$$\tau_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h)\right)$$

where

$$R_4(x_j+h) = \int_{x_j}^{x_j+h} \frac{(u'''(t) - u'''(x_j))}{2}(x_j+h-t)^2\, dt$$

$$R_4(x_j-h) = -\int_{x_j-h}^{x_j} \frac{(u'''(t) - u'''(x_j))}{2}(x_j-h-t)^2\, dt$$

Intgrating by parts, we have

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2 dt - \int_{x_j}^{x_j+h} u''(t)(x_j-h-t)^2 dt\right.$$

With $f = -u''$ and define

$$B_j = \frac{1}{h^2}\int_{x_j-h}^{x_j} f(t)(x_j-h-t)^2 dt$$

$$C_j = \frac{-1}{h^2}\int_{x_j}^{x_j+h} f(t)(x_j+h-t)^2 dt$$

Then

$$\tau_h(x_j) = f(x_j) + B_j + C_j$$

Using $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have

$$(\tau_h(x_j))^2 \le 3\left((f(x_j))^2 + B_j^2 + C_j^2\right)$$

Multiplying by h and summing over j,

$$\|\tau_h\|_h^2 = h\sum_{j=1}^{n-1}(\tau_h(x_j))^2 \le 3\|f\|_h^2 + 3h\sum_{j=1}^{n-1}(B_j^2 + C_j^2)$$

Then it remains to estimate $\sum(B_j^2 + C_j^2)$

By the Cauchy-Schwarz inequality for integral,

$$B_j^2 \le \frac{1}{h^4}\left(\int_{x_j-h}^{x_j} f(t)^2 dt\right)\left(\int_{x_j-h}^{x_j}(x_j-h-t)^4 dt\right)$$

Since

$$\int_{x_j-h}^{x_j}(x_j-h-t)^4 dt = \int_0^h s^4 ds = \frac{h^5}{5}$$

Then, we obtain

$$B_j^2 \le \frac{h}{5}\int_{x_j-h}^{x_j} f(t)^2 dt$$

Similarly,

$$C_j^2 \le \frac{h}{5}\int_{x_j}^{x_j+h} f(t)^2 dt$$

Hence,

$$h\sum_{j=1}^{n-1}(B_j^2 + C_j^2) \le \frac{h^2}{5}\sum_{j=1}^{n-1}\left(\int_{x_{j-1}}^{x_j} f^2 + \int_{x_j}^{x_{j+1}} f^2\right) \le \frac{2h^2}{5}\|f\|_{L^2(0,1)}^2 \le \|f\|_{L^2(0,1)}^2$$

Therefore,

$$\|\tau_h\|_h^2 \le 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right)$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2}x_j(1-x_j)$. $W_h = T_h g$, $W_h = \sum_{k=1}^{n-1} g(x_k) G^k$
[*Solution*: use the definition (12.25) with $g(x_k) = 1$, $k = 1, \dots, n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

from which, after straightforward computations, one gets the desired result.]

Since $g(x_k) = 1$ for all k, from (12.25) and problem 6 we have

$$T_h g(x_j) = \sum_{k=1}^{n-1} G^k(x_j) = \sum_{k=1}^{n-1} hG(x_j, x_k) = h\sum_{k=1}^{n-1} G(x_j, x_k) \quad \text{—— (1)}$$

From the form of Green's function, we know that
If $k \le j$, then $x_k \le x_j$ and $G(x_j, x_k) = x_k(1-x_j)$
If $k > j$, then $x_k > x_j$ and $G(x_j, x_k) = x_j(1-x_k)$
Thus, (1) becomes

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

Let $x_k = kh$, $h = \frac{1}{n}$, then

$$\sum_{k=1}^{j} x_k = \sum_{k=1}^{j} kh = h\,\frac{j(j+1)}{2}$$

and

$$\sum_{k=1}^{j}(1-x_k) = \sum_{k=j+1}^{n-1} 1 - h\sum_{k=j+1}^{n-1} k = (n-1-j) - h\left(\frac{(n-1)n}{2} - \frac{j(j+1)}{2}\right)$$

Furthermore, using $x_j = jh$, $h = \frac{1}{n}$

$$T_h g(x_j) = \frac{1}{n}\left[\left(1-\frac{j}{n}\right)\cdot\frac{1}{n}\cdot\frac{j(j+1)}{2} + \frac{j}{n}\cdot\left[(n-1-j) - \frac{1}{n}\left(\frac{(n-1)n}{2} - \frac{j(j+1)}{2}\right)\right]\right]$$

$$= \frac{1}{n^2}\left(1-\frac{j}{n}\right)\cdot\frac{j^2+j}{2} + \frac{j}{n^2}(n-1-j) - \frac{j}{n^3}\,\frac{n^2-n}{2} + \frac{j}{n^3}\,\frac{j^2+j}{2}$$

$$= \frac{j^2+j}{2n^2} - \cancel{\frac{j^3+j^2}{2n^3}} + \frac{2nj-2j-2j^2}{2n^2} - \frac{nj-j}{2n^2} + \cancel{\frac{j^3+j^2}{2n^3}}$$

$$= \frac{j^2+\cancel{j}-\cancel{2j}-2j^2+\cancel{j}}{2n^2} + \frac{2j-j}{2n} = \frac{j}{2n} - \frac{j^2}{2n^2}$$

and

$$\frac{1}{2}x_j(1-x_j) = \frac{1}{2}\cdot\frac{j}{n}\left(1-\frac{j}{n}\right) = \frac{j}{2n} - \frac{j^2}{2n^2}$$

Therefore,

$$T_h g(x_j) = \frac{1}{2}x_j(1-x_j), \quad j = 1, \dots, n-1$$

8. Prove Young's inequality (12.40).

$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \ \forall a, b \in \mathbb{R}, \forall \varepsilon > 0$

Let $a, b \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary.
Consider the square of any real number

$$\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}}\, b$$

Then we have

$$\left(\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}}\, b\right)^2 \ge 0$$

$$\Rightarrow \varepsilon a^2 - \cancel{2} \cdot \cancel{\sqrt{\varepsilon}}\, a \cdot \frac{1}{\cancel{2\sqrt{\varepsilon}}}\, b + \frac{1}{4\varepsilon} b^2 \ge 0$$

$$\Rightarrow \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \ge ab$$

Therefore, $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty}$ $\forall v_h \in V_h$.

Let $v_h \in V_h$. Recall the discrete norms

$$\|u_h\|_h^2 = h \sum_{j=1}^{n-1} u_h(x_j)^2, \quad \|u_h\|_{h,\infty} = \max_{1 \le j \le n-1} |u_h(x_j)|$$

For every j we have

$$|u_h(x_j)| \le \|v_h\|_{h,\infty} \Rightarrow u_h(x_j)^2 \le \|v_h\|_{h,\infty}^2$$

Therefore,

$$\|u_h\|_h^2 = h \sum_{j=1}^{n-1} u_h(x_j)^2 \le h \sum_{j=1}^{n-1} \|v_h\|_{h,\infty}^2 = h(n-1)\|u_h\|_{h,\infty}^2$$

Since the grid is uniform on $[0,1]$, $h = \frac{1}{n}$ and hence

$$h(n-1) = \frac{n-1}{n} < 1$$

Therefore, $\|v_h\|_h \le \|u_h\|_{h,\infty}$, $\forall v_h \in V_h$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.
[*Solution*: apply twice the second order centered finite difference operator L_h defined in (12.9).]

$Lu(x) = -u^{(iv)}(x)$. Let $u_h = \{u_j\}$ be the grid values $u_j \approx u(x_j)$, $x_j = jh$
Let $v_h = L_h u_h$, then

$$v_j = (L_h u_h)(x_j) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

Now apply L_h to v_h,

$$(L_h v_h)(x_j) = -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$$

Substitute the expression of v_{j+1}, v_j, v_{j-1}

$$V_{j+1} = -\frac{U_{j+2} - 2U_{j+1} + U_j}{h^2}$$

$$V_j = -\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}$$

$$V_{j-1} = -\frac{U_j - 2U_{j-1} + U_{j-2}}{h^2}$$

Hence,

$$(L_h V_h)(x_j) = -\frac{1}{h^2}\left(V_{j+1} - 2V_j + V_{j-1} \right)$$

$$= -\frac{1}{h^2}\left[-\frac{U_{j+2} - 2U_{j+1} + U_j}{h^2} + 2\,\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2} - \frac{U_j - 2U_{j-1} + U_{j-2}}{h^2} \right]$$

$$= \frac{1}{h^4}\left(U_{j+2} - 4U_{j+1} + 6U_j - 4U_{j-1} + U_{j-2} \right)$$