1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

Using $u(x) = \int_0^1 G(x,s)f(s)ds$ and $f(x) = \frac{1}{x}$

$$\Rightarrow u(x) = (1-x)\int_0^x 1\, ds + x\int_x^1 \left(\frac{1}{s}-1\right) ds$$

$$(1-x)\int_0^x 1\, ds = (1-x)x \quad \text{and} \quad x\int_x^1 \left(\frac{1}{s}-1\right) ds = x\left([\ln s - s]_{s=x}^1\right) = x(-1-\ln x + x)$$

Then $u(x) = (1-x)x + x(-1-\ln x + x) = -x\ln x$

Discussion:

On the interval $(0,1)$, the function $u(x) = -x\ln x$ is twice continuously differentiable $\Rightarrow u \in C^2(0,1)$

We know that as $x \to 0^+$,

$u(x) \longrightarrow 0$, $u'(x) = -(\ln x + 1) \longrightarrow \infty$, $u''(x) = -\frac{1}{x} \longrightarrow -\infty$

Therefore, $u(0)$ is not well-defined.

4. Cerify the summation by parts formula

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1}-v_h)w_{j+1},$$

and show that, for $v_h \in V_h^0$,

$$(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2.$$

Claim: $\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j = W_nV_n - W_0V_0 - \sum_{j=0}^{n-1}(V_{j+1}-V_j)W_{j+1}$

<pf>: $\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j = \sum_{j=0}^{n-1} W_{j+1}V_j - \sum_{j=0}^{n-1} W_jV_j$

We can shift the index,

$$\sum_{j=0}^{n-1} W_{j+1}V_j = \sum_{j=1}^{n} W_jV_{j-1} = W_nV_{n-1} + \sum_{j=1}^{n-1} W_jV_{j-1}$$

Hence,

$$\begin{aligned}
\sum_{j=0}^{n-1}(W_{j+1}-W_j)V_j &= W_nV_{n-1} - W_0V_0 + \sum_{j=1}^{n-1} W_jV_{j-1} - \sum_{j=1}^{n-1} W_jV_j \\
&= W_nV_{n-1} - W_0V_0 - \sum_{j=1}^{n-1}(V_j - V_{j-1})W_j \\
&= W_nV_n - W_nV_n + W_nV_{n-1} - W_0V_0 - \sum_{j=1}^{n-1}(V_j - V_{j-1})W_j \\
&= W_nV_n - W_0V_0 - (V_n - V_{n-1})W_n - \sum_{j=1}^{n-1}(V_j - V_{j-1})W_j \\
&= W_nV_n - W_0V_0 - \sum_{j=1}^{n}(V_j - V_{j-1})W_j \\
&= W_nV_n - W_0V_0 - \sum_{j=0}^{n-1}(V_{j+1} - V_j)W_{j+1}
\end{aligned}$$

Let the grid be $x_j = j \cdot h$ with $h = \frac{1}{n}$
Define the discrete inner product

$$(u_h, v_h)_h := h \sum_{j=1}^{n-1} u_j v_j .$$

and the discrete Laplacian

$$(L_h v_h)_j := -\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, \quad j = 1, \dots, n-1$$

with $v_0 = v_n = 0$ for $v_h \in V_h^0$
Set the forward difference $g_j := \frac{v_{j+1} - v_j}{h}$ for $j = 0, \dots, n-1$
Then

$$(L_h v_h)_j = \frac{g_j - g_{j-1}}{h}$$

Therefore,

$$(L_h v_h, v_h)_h = h \sum_{j=1}^{n-1} \frac{g_j - g_{j-1}}{h} \cdot v_j = \sum_{j=1}^{n-1} (g_j - g_{j-1}) v_j$$

Apply the above with $w_j := g_{j-1}$ $(w_{j+1} - w_j = g_j - g_{j-1})$ and using $v_0 = v_n = 0$

$$\sum_{j=1}^{n-1} (g_j - g_{j-1}) v_j = -g_1 v_1 - \sum_{j=1}^{n-1} (v_{j+1} - v_j) g_j = -g_0 v_1 - \sum_{j=1}^{n-1} h g_j^2$$

Hence,

$$(L_h v_h, v_h)_h = -g_0 v_1 - \sum_{j=1}^{n-1} h g_j^2 = -\frac{v_1}{h} \times v_1 + h \sum_{j=1}^{n-1} \left(\frac{v_{j+1} - v_j}{h}\right)^2 = -\frac{1}{h} v_1^2 + \frac{1}{h} \sum_{j=1}^{n-1} (v_{j+1} - v_j)^2$$
$$= \frac{1}{h} \sum_{j=0}^{n-1} (v_{j+1} - v_j)^2$$

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).
[*Solution*: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \dots, k-1$ and $l = k+1, \dots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

We want to show that $G^k = hG(\cdot, x_k)$ satisfies the discrete system

$$L_h G^k = e^k,$$

where e^k is the unit vector with 1 at position k and 0 elsewhere.

We know that

$$G(x,s) = \begin{cases} x(1-s), & 0 \le x \le s, \\ s(1-x), & s \le x \le 1. \end{cases}$$

For a fixed $s = x_k$, then $G(x, x_k)$ is a piecewise linear function of x.
Let the discrete Laplacian be

$$(L_h v)_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, \quad j = 1, \dots, n-1$$

with $v_0 = v_n = 0$
We evaluate $L_h G(\cdot, x_k)$ at discrete points $x_j = j h$
Since $G(x, x_k)$ is linear on every subinterval $[x_{l-1}, x_l]$ $\forall l \neq k$,
then the second difference at any such node is zero:

$$(L_h G)(x_l) = 0, \quad \text{for } l = 1, 2, \dots, k-1, k+1, \dots, n-1$$

This means $L_h G$ is nonzero only at the node x_k

At x_k and $x_k = k \cdot h$

$$G(x_{k+1}, x_k) = x_k(1 - x_{k+1}), \quad G(x_k, x_k) = x_k(1 - x_k), \quad G(x_{k-1}, x_k) = x_{k-1}(1 - x_k)$$

$$= hk - h^2 k(k+1) \qquad = hk(1 - hk) \qquad = hk - h - h^2 k(k-1)$$

Then

$$(L_h G)(x_k) = -\frac{G(x_{k+1}, x_k) - 2G(x_k, x_k) + G(x_{k-1}, x_k)}{h^2}$$

$$= -\frac{-h^3}{h^2} = h$$

Thus we have

$$(L_h G)(x_j) = \begin{cases} 0 & , j \neq k \\ h & , j = k \end{cases}$$

That is,

$$L_h G = h e^k$$

Therefore, the discrete Green's function G^k defined by $L_h G^k = e^k$ satisfies $G^k = h G(\cdot, x_k)$

$\Rightarrow G^k(x_j) = h G(x_j, x_k)$