9. Given the following set of data

$$\left\{ f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1 \right\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

Let $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $H_3'(x) = 3a_3x^2 + 2a_2x + a_1$

$$\begin{aligned} H_3(-1) &= -a_3 + a_2 - a_1 + a_0 = 1 \\ H_3'(-1) &= 3a_3 - 2a_2 + a_1 = 1 \\ H_3'(1) &= 3a_3 + 2a_2 + a_1 = 2 \\ H_3(2) &= 8a_3 + 4a_2 + 2a_1 + a_0 = 1 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}}_{= B} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$\det(B) = 0 \Rightarrow$ The Hermite-Birkhoff interpolating polynomial $H_3(x)$ does not exist because the coefficient matrix is singular.

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \qquad (8.75)$$

called the *Padé approximation*. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + \ldots.$$

[*Solution*: $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

We know that $\cos(x) - r(x) = O(x^8)$
Equivalently,

$$(1+b_2x^2)r(x) = (1+b_2x^2)\cos(x) + O(x^8)$$

Then

$$\begin{aligned} (1+b_2x^2)\cos x &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) + b_2x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + O(x^8) \\ &= 1 + \left(b_2 - \frac{1}{2}\right)x^2 + \left(\frac{1}{24} - \frac{b_2}{2}\right)x^4 + \left(-\frac{1}{720} + \frac{b_2}{24}\right)x^6 + O(x^8) \end{aligned}$$

and

$$(1+b_2x^2)r(x) = a_0 + a_2x^2 + a_4x^4$$

Ater comparing the coefficients, we can know that

$$\begin{cases} a_0 = 1 \\ a_2 = \left(b_2 - \frac{1}{2}\right) = -\frac{7}{15} \\ a_4 = \left(\frac{1}{24} - \frac{b_2}{2}\right) = \frac{1}{40} \\ \left(-\frac{1}{720} + \frac{b_2}{24}\right) = 0 \Rightarrow b_2 = \frac{1}{30} \end{cases}$$