1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\left\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\right\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

From Heun's method,

$$y_{n+1} = y_n + h\Phi(t_n, y_n, f_n; h), \text{ where } \Phi(t_n, y_n, f_n; h) = \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

Then

$$h\tau_{n+1} = y(t_{n+1}) - y(t_n) - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))]$$

We know that

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))ds$$

Then

$$h\tau_{n+1} = \underbrace{\int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]}_{E_1} + \underbrace{\frac{h}{2}[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]}_{E_2}$$

Estimate of E_1: trapezoidal integration error

Let $g(s) := f(s, y(s))$

From the trapezoidal rule error formula for g: $(g \in C^2)$

$$\int_{t_n}^{t_{n+1}} g(s)ds - \frac{h}{2}[g(t_n) + g(t_{n+1})] = -\frac{h^3}{12} g''(\xi_n) \quad \text{for some } \xi_n \in (t_n, t_{n+1})$$

Hence, $E_1 = O(h^3)$

Estimate of E_2: error from endpoint approximation

Since $f \in C^1$, f is locally Lipshitz, then $\exists$ a constant $L > 0$ s.t.

$$|f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + f(t_n, y_n))| \leq L\,|y_{n+1} - y_n - hf(t_n, y_n)|$$

Then

$$|E_2| \leq \frac{h}{2} \cdot L\,|y_{n+1} - y_n - hf(t_n, y_n)|$$

Using Taylor expansion in t_n and take $t = t_{n+1}$,

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi_n) \quad \text{for } \xi_n \in (t_n, t_{n+1})$$

$$= y(t_n) + hf(t_n, y_n) + \frac{h^2}{2}y''(\xi_n)$$

Then

$$|y(t_{n+1}) - y(t_n) - hf(t_n, y_n)| = \frac{h^2}{2}y''(\xi_n) \quad \text{for } \xi_n \in (t_n, t_{n+1})$$

Therefore,

$$|E_2| \le \frac{h^3}{4} L\, y''(\xi_n) \quad \text{for } \xi_n \in (t_n, t_{n+1})$$

Hence, $E_2 = O(h^3)$

From above, we have

$$h\,\tau_{n+1} = E_1 + E_2 = O(h^3) \quad \Rightarrow \tau_{n+1} = O(h^2)$$

Therefore, Heun's method has order 2 with respect to h.

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

From the CN method:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

We know that

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$$

Applying the trapezoidal rule with the error formula for $g(s) = f(s, y(s))$,
there exists $\xi_n \in (t_n, t_{n+1})$ s.t.

$$\int_{t_n}^{t_{n+1}} f(s, y(s))\,ds = \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

Then

$$y_{n+1} - y_n = \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

Dividing by h,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n))$$

Then

$$\tau_{n+1} = -\frac{h^2}{12} f''(\xi_n, y(\xi_n)) = O(h^2)$$

Therefore, CN method has order 2 with respect to h