7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

For $\mathrm{Re}(z) > 0$,

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, dt$$

Integrate by parts with $u = t^z$ and $dv = e^{-t}$.
Then $du = z t^{z-1} dt$ and $v = -e^{-t}$.
Hence,

$$\Gamma(z+1) = \cancel{\left[-e^{-t} t^z\right]_0^\infty} + z \int_0^\infty e^{-t} t^{z-1} dt$$

$$= z \int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z)$$

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1 - \frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

Let y be the exact solution, $f(x,y) = y'(x)$.
Using Taylor :

$$y_{n+1} - y_n = h u'(x_n) + \frac{h^2}{2} u''(x_n) + \frac{h^3}{6} u'''(x_n) + O(h^4)$$

and

$$f(x_{n+1}, y_{n+1}) = y'(x_{n+1}) = y'(x_n) + h u''(x_n) + \frac{h^2}{2} u'''(x_n) + O(h^3)$$

Hence, the truncation error

$$T_{n+1} = (y_{n+1} - y_n) - h\left[(1 - \tfrac{\alpha}{2}) u' + \tfrac{\alpha}{2}(u' + h u'' + \tfrac{h^2}{2} u''')\right]$$

$$= \frac{h^2}{2}(1-\alpha) u'' + \frac{h^3}{12}(2 - 3\alpha) u''' + O(h^4)$$

- For any α, $T_{n+1} = O(h^2) \Rightarrow$ the method is consistent
- For $\alpha \neq 1$, $T_{n+1} = O(h^2) \Rightarrow$ Order 1
- For $\alpha = 1$, $T_{n+1} = O(h^3) \Rightarrow$ Order 2 (CN)

Take $\alpha = 1$ for $y' = 10y$, $y(0) = 1$

Then

$$U_{n+1} = U_n + \frac{h}{2}(f_n + f_{n+1}) = U_n + \frac{h}{2}(-10U_n - 10U_{n+1})$$

Solve for U_{n+1}:

$$(1+5h)U_{n+1} = (1-5h)U_n \Rightarrow U_{n+1} = \left(\frac{1-5h}{1+5h}\right)U_n$$

With $U_0 = 1$, $U_n = \left(\frac{1-5h}{1+5h}\right)^n$

Absolute stability:

For the question $y' = \lambda y$, CN has amplification

$$R(z) = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}, \quad z = \lambda h$$

Its stability region is $\{z : Re(z) < 0\}$

Here $\lambda = -10 \Rightarrow z = -10h$ with $h > 0$,

$$|R(-10h)| = \left|\frac{1-5h}{1+5h}\right| < 1 \quad \text{for all } h > 0.$$

Hence, the method is absolutely stable for every stepsize $h > 0$ for this problem.