3D Geometric Transformations

Spring 2016

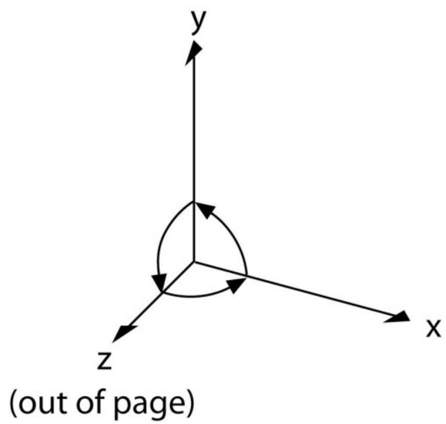
Seungyong Lee
POSTECH

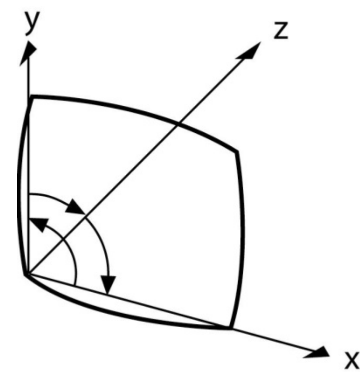
Overview

- 3D geometric transformations
  - similar to 2D geometric transformations
  - 3D rotations
- Chapter summary
  - 3D coordinate systems
  - basic and general transformations
  - general rotations and quaternions
- Related materials
  - Angel: Chapter 3
  - H&B: Chapter 11

3D Coordinate Systems

- Right-handed/left-handed coordinate system





$$M_{R \leftarrow L} = M_{L \leftarrow R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Coordinate Systems (2)

- Homogeneous coordinate system
  - from Cartesian to homogeneous coordinates
    - $(x, y, z) \rightarrow (x, y, z, 1)$
  - equivalent homogeneous coordinates
    - $(x, y, z, w) = (\alpha x, \alpha y, \alpha z, \alpha w), \alpha \neq 0$
  - from homogeneous to Cartesian coordinates
    - $(x, y, z, w) \rightarrow (x/w, y/w, z/w)$
  - $w = 0$?
    - point at infinity
    - can specify a 3D direction

Basic Transformations

- Translation
  - $T(d_x, d_y, d_z)\,[x \ \ y \ \ z \ \ 1]^T = [x + d_x \ \ y + d_y \ \ z + d_z \ \ 1]^T$

$$T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scaling
  - uniform/nonuniform scaling

$$T(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Transformations (2)

- Rotation
  - rotations about x, y, z axes

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Transformations (3)

- Reflections

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Shear transformations

$$SH_{xy}(sh_x, sh_y) = \begin{bmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Basic Transformations (4)

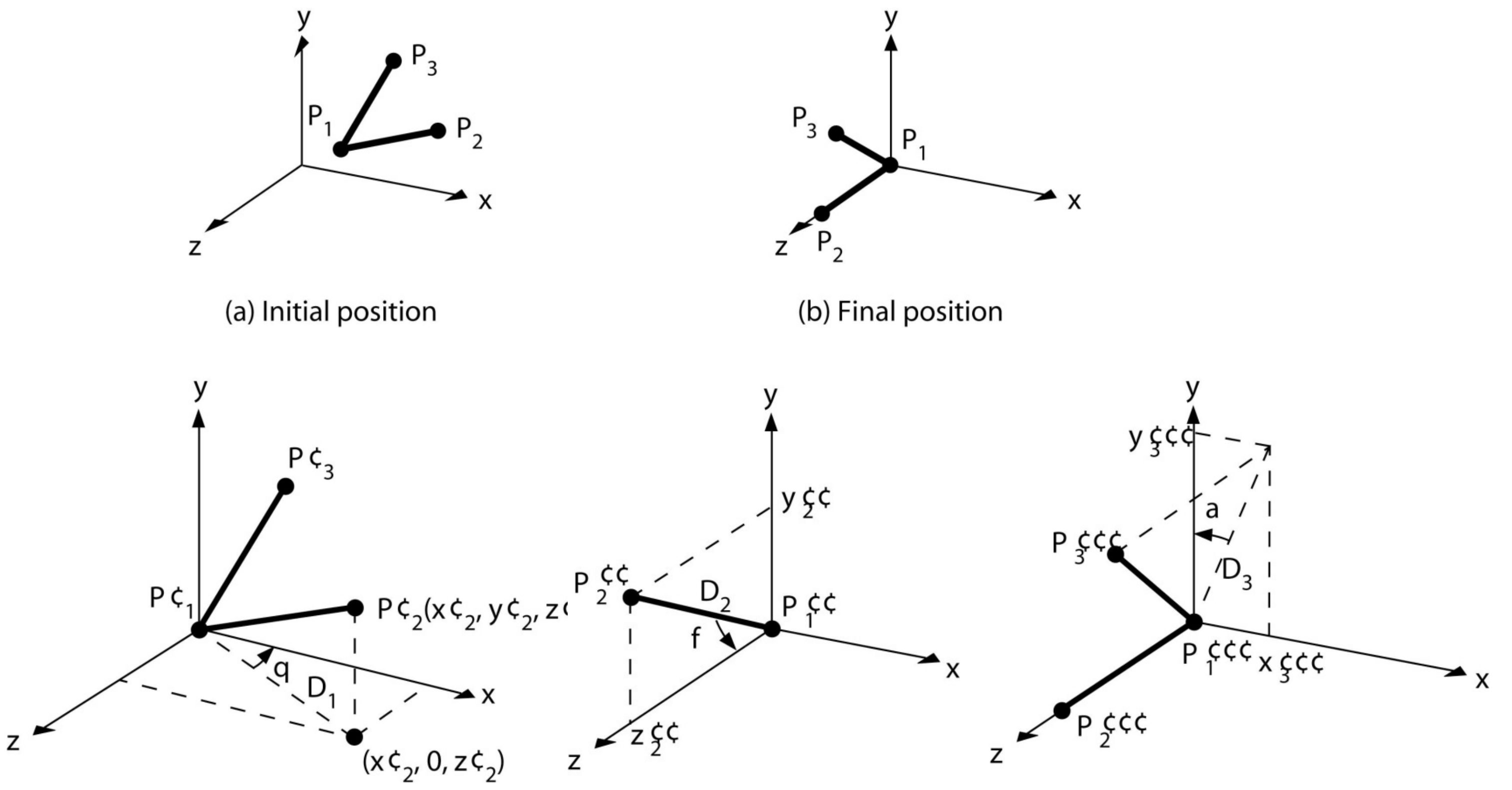
- Inverse transformations
  - translation
    - $T(d_x, d_y, d_z)^{-1} = T(-d_x, -d_y, -d_z)$
  - scaling
    - $S(s_x, s_y, s_z)^{-1} = S(1/s_x, 1/s_y, 1/s_z)$
  - rotation
    - $R_x(\theta)^{-1} = R_x(-\theta)$, $R_y(\theta)^{-1} = R_y(-\theta)$, $R_z(\theta)^{-1} = R_z(-\theta)$

General Transformations

- Composition of basic (primitive) transformations
  - concatenation of 4×4 matrix multiplications
  - finally represented by a single 4×4 matrix
- (e.g.) scaling about an arbitrary point
  - $T(x_1, y_1, z_1)\ S(s_x, s_y, s_z)\ T(-x_1, -y_1, -z_1)$

General Transformations (2)

- General transformation example
  - $R \cdot T = R_z(\alpha)R_x(\phi)R_y(\theta - 90)T(-x_1, -y_1, -z_1)$



(a) Initial position

(b) Final position

General Transformations (3)

- Transformation matrix
  - translation, scaling, rotation, shear

$$M = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

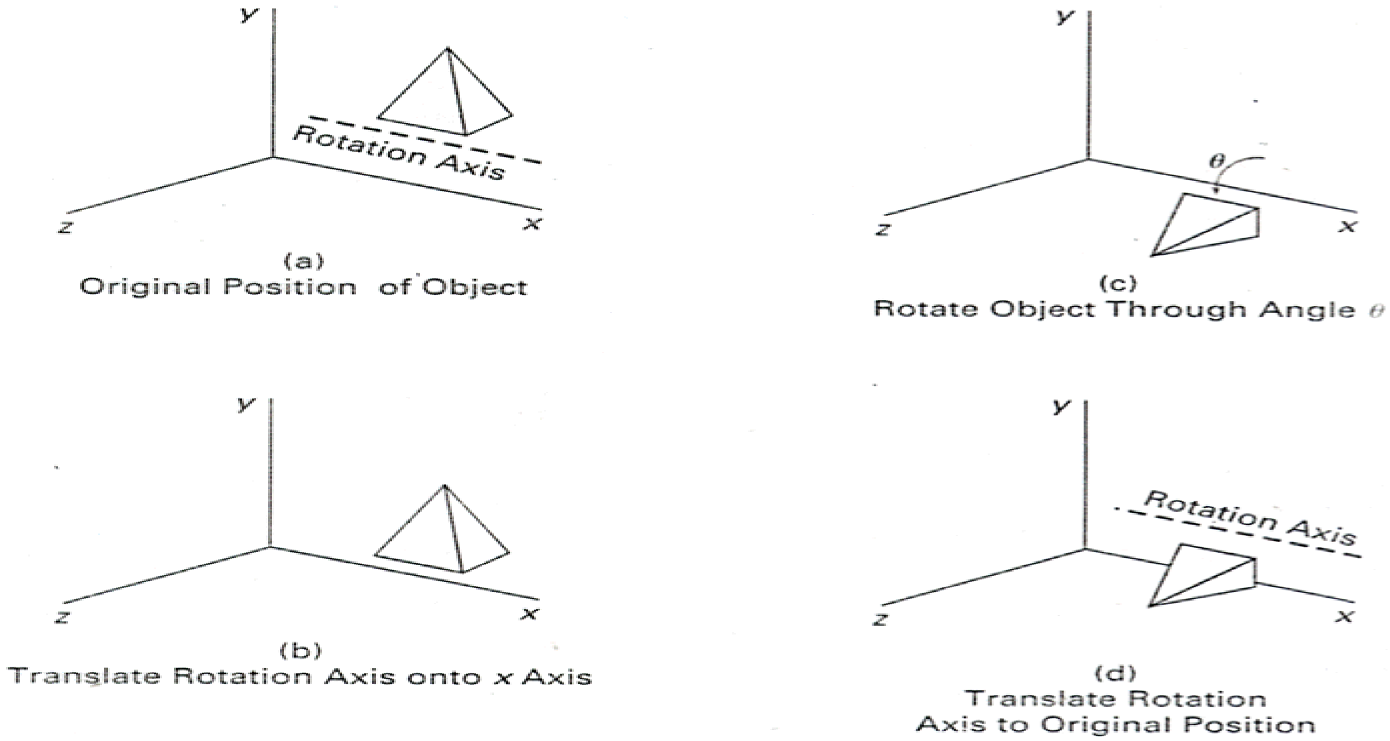
- Rigid-body transformation
  - the upper left 3 x 3 submatrix is orthonormal
  - orthonormal matrix: $B^{-1} = B^T$
- Affine transformation
- Perspective transformation?

General Transformations (4)

- Transformation process
  - the order is important in a composite transformation
  - efficient computation
    - $[x' \; y' \; z']^T = M\,[x \; y \; z]^T = R\,[x \; y \; z]^T + T$
- OpenGL transformation process
  - same as the 2D case

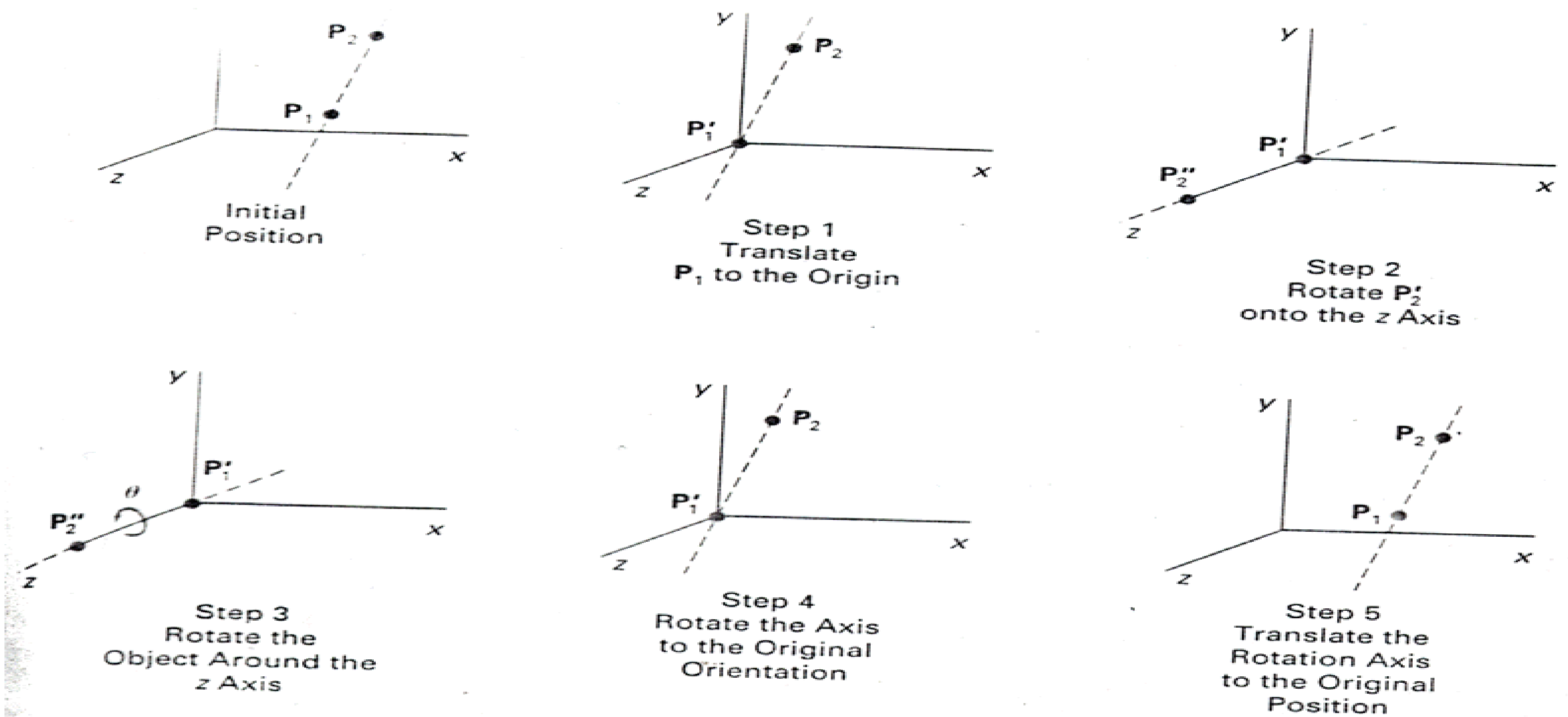
General rotations

- Rotation about an axis parallel to a coordinate axis
  - $T^{-1} R_x(\theta) T$



General rotations (2)

- Rotation about an arbitrary axis
  - $R(\theta) = T^{-1} R_x^{-1}(\alpha) R_y^{-1}(\beta) R_z(\theta) R_y(\beta) R_x(\alpha) T$



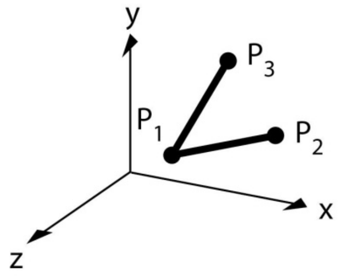
General rotations (3)

- General rotational matrix
  - results from a sequence of rotations
  - upper left 3 x 3 submatrix is orthonormal
  - $UU^T = U^TU = I$
  - 2D example?

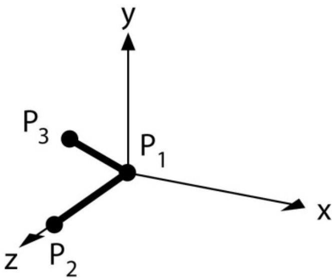
$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow R\begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, R\begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, R\begin{bmatrix} r_{31} \\ r_{32} \\ r_{33} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

General rotations (4)

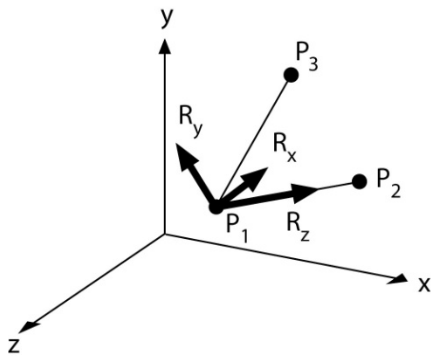
- General transformation example revisited
  - $R \cdot T = R_z(\alpha) R_x(\phi) R_y(\theta - 90) T(-x_1, -y_1, -z_1)$



(a) Initial position



(b) Final position



$$R_z = \begin{bmatrix} r_{31} & r_{32} & r_{33} \end{bmatrix}^T = \frac{P_1 P_2}{|P_1 P_2|}$$

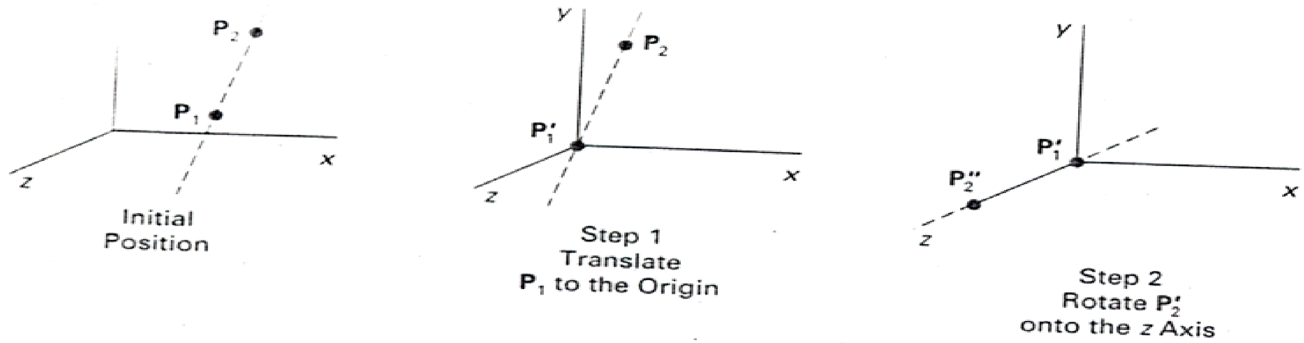
$$R_x = \begin{bmatrix} r_{11} & r_{12} & r_{13} \end{bmatrix}^T = \frac{P_1 P_3 \times P_1 P_2}{|P_1 P_3 \times P_1 P_2|}$$

$$R_y = \begin{bmatrix} r_{21} & r_{22} & r_{23} \end{bmatrix}^T = R_z \times R_x$$

$$R = \begin{bmatrix} R_x & R_y & R_z \end{bmatrix}^T$$

General rotations (5)

- Rotation about an arbitrary axis revisited
  - $R(\theta) = T^{-1} R_x^{-1}(\alpha) R_y^{-1}(\beta) R_z(\theta) R_y(\beta) R_x(\alpha) T$



$$u_z = u, u_y = \frac{u \times \vec{x}}{|u \times \vec{x}|}, u_x = u_y \times u_z$$

$$R = R_y(\beta) R_x(\alpha) = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$$

$$R(\theta) = T^{-1} R^{-1} R_z(\theta) RT$$

Rotations with Quaternions

- What is a quaternion?
  - consists of one real part and three imaginary parts
  - an extension of complex number to 4D
    - q = (s, a, b, c) = s + ia + jb + kc
- Quaternion arithmetic
  - $i^2 = j^2 = k^2 = -1$
  - $ij = -ji = k, \; jk = -kj = i, \; ki = -ik = j$
  - $q_1 + q_2 = (s_1 + s_2) + i(a_1 + a_2) + j(b_1 + b_2) + k(c_1 + c_2)$

Rotations with Quaternions (2)

- Vector representation
  - $q = (s, \vec{v})$
  - $q_1 + q_2 = (s_1 + s_2, \vec{v}_1 + \vec{v}_2)$
  - $q_1 q_2 = (s_1 s_2 - \vec{v}_1 \cdot \vec{v}_2, s_1 \vec{v}_2 + s_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$
- Length of a quaternion
  - $|q|^2 = s^2 + v \cdot v$
  - unit quaternion: $|q| = 1$
- Conjugate and inverse of a quaternion
  - conjugate: $\overline{q} = (s, -v)$
  - inverse: $qq^{-1} = q^{-1}q = (1, \vec{0})$, $q^{-1} = \dfrac{1}{|q|^2}(s, -v)$
  - for unit quaternion q, $q^{-1} = \overline{q}$

Rotations with Quaternions (3)

- Rotations with quaternions

  u: a unit vector passing through the origin

  θ : the specified rotation angle about axis u, $0 \le \theta < 2\pi$

  $$q = (s, v), \text{ where } s = \cos\frac{\theta}{2}, v = u\sin\frac{\theta}{2}$$

  $$P = (0, p), P' = qPq^{-1} = (0, p')$$

  $$p' = s^2 p + v(p \cdot v) + 2s(v \times p) + v \times (v \times p)$$

- Successive rotations
  - can be handled by quaternion multiplication

  $$p' = q_1 p q_1^{-1},\ p'' = q_2 p' q_2^{-1}$$

  $$p'' = q_2\left(q_1 p q_1^{-1}\right)q_2^{-1} = \left(q_2 q_1\right)p\left(q_1^{-1} q_2^{-1}\right) = \left(q_2 q_1\right)p\left(q_2 q_1\right)^{-1}$$

Rotations with Quaternions (4)

- Quaternion and rotational matrix
  - q = (s, v) = (s, a, b, c) corresponds to

$$R = \begin{bmatrix} 1-2b^2-2c^2 & 2ab-2sc & 2ac+2sb \\ 2ab+2sc & 1-2a^2-2c^2 & 2bc-2sa \\ 2ac-2sb & 2bc+2sa & 1-2a^2-2b^2 \end{bmatrix}$$

    - ex) rotation about z axis by θ

$$s=\cos\frac{\theta}{2}, v=(0,0,1)\sin\frac{\theta}{2}$$

$$1-2\sin^2\frac{\theta}{2}=\cos\theta, 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}=\sin\theta$$

$$R=R_z(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - a rotation matrix can be converted to a quaternion

Rotations with Quaternions (5)

- Other properties

$q = -q$?

$$-q = \left(-\cos\frac{\theta}{2}, -u\sin\frac{\theta}{2}\right) = \left(\cos\left(\frac{360^\circ - \theta}{2}\right), -u\sin\left(\frac{360^\circ - \theta}{2}\right)\right)$$

$q_1 q_2 \neq q_2 q_1$

Summary

- Homogeneous coordinates
- Basic transformations
- General transformations
  - composite transformation by a 4×4 matrix
  - same transformation process as the 2D case
- Rotations
  - general rotation by a sequence of simple rotations
  - coordinate frame transformation by computing an orthogonal submatrix
  - rotations with quaternions

Summary (2)

- Rotation matrix computation
  - general rotation
    - quaternion → rotation matrix
    - glRotate(angle, x, y, z)

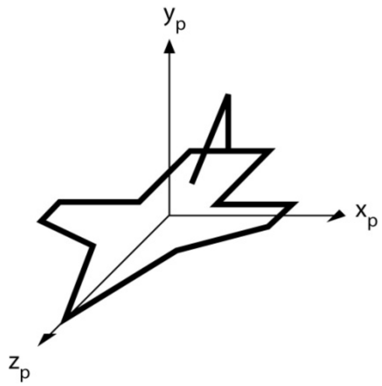
$$R = \begin{bmatrix} xx(1-c)+c & xy(1-c)-zs & xz(1-c)+ys \\ yx(1-c)+zs & yy(1-c)+c & yz(1-c)-xs \\ xz(1-c)-ys & yz(1-c)+xs & zz(1-c)+c \end{bmatrix} \qquad \begin{aligned} c &= \cos(angle) \\ s &= \sin(angle) \\ \|(x,y,z)\| &= 1 \end{aligned}$$

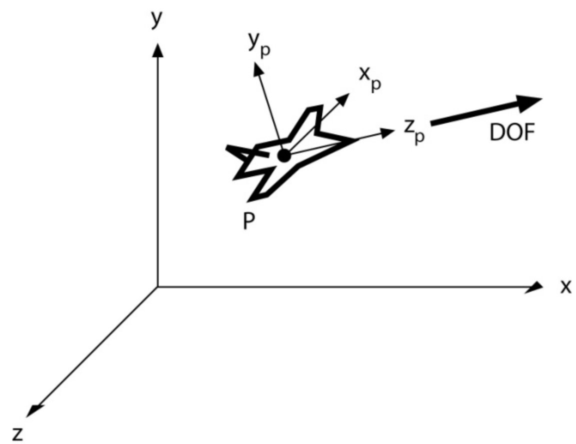
  - rotation of a frame into the principal axes
    - orthogonal submatrix
  - rotation of one frame into another one
    - two orthogonal submatrices → rotation matrix

Supplementary Slides

General rotations

- Another example





$$R = \begin{bmatrix} & & & 0 \\ \left|\vec{y} \times DOF\right| & \left|DOF \times (\vec{y} \times DOF)\right| & \left|DOF\right| & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$