6.25

$$\langle Px, Py \rangle = (Px)^t Py = x^t P^t P y = x^t y = \langle x, y \rangle. \quad ■$$

6.26

$$(AB)^t AB = B^t A^t A B = B^t B = I$$

$\therefore (AB)^{-1} = (AB)^t \rightsquigarrow$ orthogonal mtx.

$$(BA)^t (BA) = A^t B^t B A = A^t A = I$$

$\therefore (BA)^{-1} = (BA)^t \rightsquigarrow$ orthogonal mtx.

■

6.33

$$P_\sigma = \begin{bmatrix} e_{\sigma(1)}^t \\ \vdots \\ e_{\sigma(n)}^t \end{bmatrix}$$

$$\rightsquigarrow P_\sigma P_\sigma^t = \begin{bmatrix} e_{\sigma(1)}^t \\ \vdots \\ e_{\sigma(n)}^t \end{bmatrix} \cdot \begin{bmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{bmatrix}$$

$$= \left[\langle e_{\sigma(i)}, e_{\sigma(j)} \rangle \right]_{ij} = I.$$

$\therefore P_\sigma$: orthogonal mtx

■

6.34. $\exists\, u \in M_{m,1}(\mathbb{R}),\ v \in M_{n,1}(\mathbb{R})$.

(a). $(u \otimes v)_{ij} = u_i v_j \rightsquigarrow u \otimes v = [\, u v_1 \ \cdots \ u v_n \,]$

$= u \cdot [\, v_1 \cdots v_n \,] = u v^t \in M_{m,n}(\mathbb{R})$

(b) $(u \otimes v) \cdot v = u v^t \cdot v = u \cdot \|v\|_2^2.$

■

6.36 SVD : $\forall A \in M_{n \times n}(\mathbb{R})$, $\exists$ orthogonal mtx U, V and a diag matrix Σ, s.t. $A = U \Sigma V^t$.

(a) $A^tA = V\Sigma U^t \cdot U \Sigma V^t = V\Sigma \cdot \Sigma V^t = V\Sigma^2 V^t$.

(b) $\because$ A^tA is a sym mtx, A is diagonalizable.

$A^tA = V\Sigma^2V^t \rightsquigarrow \underset{\text{egual}}{\Sigma^2} = U^t(A^tA)V$

$\therefore$ egual of Σ^2 is square of diag. entries in Σ ■

(c), (d) : coding prob.

6.40 : coding prob.

7.2.

(a) $A = \begin{bmatrix} 1 & 9 \\ -1 & 5 \end{bmatrix}$

$\|A\|_1 = \max_j \sum_i |a_{ij}| = 14$

$A^tA = \begin{bmatrix} 2 & 4 \\ 4 & 106 \end{bmatrix}$

$\|A\|_2 = \max_i |\sqrt{s_i}| = 10.3031$

$\|A\|_\infty = \max_i \sum_j |a_{ij}| = 10$

$\|A\|_F = \sqrt{82+26} = 6\sqrt{3}$

(b) $A = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 4 & 1 \end{bmatrix}$

$\|A\|_1 = \max_j \sum_i |a_{ij}| = 9$

$A^tA = \begin{bmatrix} 4 & 10 & 6 \\ 10 & 41 & 19 \\ 6 & 19 & 10 \end{bmatrix}$

$\|A\|_2 = \max_i |\sqrt{s_i}| = 7.2652$

$\|A\|_\infty = \max_i \sum_j |a_{ij}| = 10$

$\|A\|_F = \sqrt{38+17} = \sqrt{55}$

7.5. $\|A\|_1 = \max_{1 \le k \le n} \sum |a_{ik}|$

(proof)

$$\|A\|_1 = \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{x \neq 0} \frac{\sum_i |\sum_j a_{ij} x_j|}{\sum_j |x_j|}$$

$$\le \max_{x \neq 0} \frac{\sum_j \sum_i |a_{ij}| \cdot |x_j|}{\sum_j |x_j|} = \max_{x \neq 0} \frac{\sum_j |x_j| \left(\sum_i |a_{ij}| \right)}{\sum_j |x_j|}$$

$$\le \max_{\substack{x \neq 0 \\ k}} \frac{\left(\sum_i |a_{ik}| \right) \cdot \sum_j |x_j|}{\sum_j |x_j|} = \max_{\substack{x \neq 0 \\ k}} \sum_i |a_{ik}|$$ (upper bound)

$x = e_k \rightarrow \frac{\|Ax\|_1}{\|x\|_1} = \sum_i |a_{ik}|$.

($\exists$ x for equality)

7 $\exists$ x for upper bound. upperbound P maximum. ∎

7.18

$$\|Ax\|_2^2 = \sum_i \left(\sum_j a_{ij} x_j\right)^2 \leq \sum_i \left(\sum_j a_{ij}^2\right)\left(\sum_j x_j^2\right) = \|A\|_F^2 \cdot \|x\|_2^2$$

C.S. inequality

$\therefore$ Euclidean norm is compatible with Frobenius Norm. $\blacksquare$

7.19.

(a) Using the well known fact: $\|A\|_2 = \max_i \sigma_i$ (σ_i : singular values of matrix)

$$\|A\|_{\infty,\text{schatt}} := \lim_{p\to\infty} \left(\sum_{i=1}^n \sigma_i^p\right)^{\frac{1}{p}} = \lim_{p\to\infty} \left(\sum_{i=1}^n \left(\frac{\sigma_i}{\sigma}\right)^p\right)^{\frac{1}{p}} \cdot \sigma$$

(σ : $\max_i (\sigma_i)$)

$$= \lim_{p\to\infty} \Big(\underbrace{m_\sigma}_{\sigma} \cdot 1 + \sum_{\substack{i=1 \\ \sigma_i \neq \sigma}}^n \left(\frac{\sigma_i}{\sigma}\right)^p \Big)^{\frac{1}{p}} \cdot \sigma = \sigma$$

multiplicity factor of σ

(b) $\|A\|_{2,\text{schatt}}^2 = \sum_{i=1}^n \sigma_i^2 = \operatorname{tr}(A^tA) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} a_{ij} = \|A\|_F^2$ $\blacksquare$

7.20.

(a) A: symmetric $\Rightarrow$ $\|A\|_2 = \rho(A)$

(proof) singular value is equal of A^tA. $\exists$ A being sym. matrix, diagonalizable. $\Rightarrow$ $(P\Lambda P^t)(P\Lambda P^t) = \underline{P\Lambda^2P^t}$ ($\Lambda = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$)

$$\therefore \|A\|_2 = \max_i \sqrt{s_i} = \max_i |\lambda_i| = \rho(A).$$ $\blacksquare$

(b) $\|A^{-1}\|_2$ so eigenval of $(A^{-1})^t A^{-1}$ → singular value of A^{-1}.

singular value of A is spectrum of A^tA.

so ∵ $A^tA = P\Lambda P^t$ is diagonalizable, $(A^tA)^{-1} = P\Lambda^{-1}P^t$

Also, A^tA is positive-semidefinite matrix, all diagonal

values of Λ are non-negative. (∵ $x^tA^tAx = (Ax)^tAx \geq 0$)

∵ eigval of A^tA is same as eigval of AA^t.

$$(A^tA)^{-1} = (A^{-1})(A^t)^{-1} = (A^{-1})(A^t)^{-1} \longleftrightarrow (A^t)^{-1}A^{-1}$$

same eigval.

$$\therefore \|A\|_2 = \max_i \sqrt{\lambda_i} = \sigma_{max}. \quad (\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n))$$

$$\|A^{-1}\|_2 = \max_i \frac{1}{\sqrt{\lambda_i}} = \frac{1}{\min_i \sqrt{\lambda_i}} = \frac{1}{\sigma_{min}}$$

∵ A is nonsingular, $\nexists\, i$ s.t. $\lambda_i = 0$.

■

7.24. Coding problem

7.31.

(a) $\det(\lambda I - A) = (\lambda - 0.6)^4(\lambda + 0.7) = 0$

so $\lambda = 0.6,\ -0.7$.

(b)
$$\begin{pmatrix} 0.6 & 1 & 6 & -1 & 5 \\ 0 & 0.6 & 1 & 1 & 0 \\ 0 & 0 & 0.6 & 1 & 3 \\ 0 & 0 & 0 & 0.6 & 1 \\ 0 & 0 & 0 & 0 & -0.7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0.6 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$\therefore$ $x_5 = 0$ $x_4 = 0$ $x_3 = 0$ $x_2 = 0$. $\rightsquigarrow$ $[1, 0, 0, 0, 0]^t$.

$$\begin{pmatrix} 0.6 & 1 & 6 & -1 & 5 \\ 0 & 0.6 & 1 & 1 & 0 \\ 0 & 0 & 0.6 & 1 & 3 \\ 0 & 0 & 0 & 0.6 & 1 \\ 0 & 0 & 0 & 0 & -0.7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = -0.7 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$\Downarrow$

$$(A + 0.7I)x = \begin{pmatrix} 1.3 & 1 & 6 & -1 & 5 \\ 0 & 1.3 & 1 & 1 & 0 \\ 0 & 0 & 1.3 & 1 & 3 \\ 0 & 0 & 0 & 1.3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = 0$$

$\hookrightarrow$ $rk(A + 0.7I) = 4$

$\rightarrow$ $ker(A + 0.7I) = 5 - 4 = 1$.

$\therefore$ $\nexists$ basis for eigenvectors. ($\because$ dim of eigenspace is only 2)

(c) 0.7.

(d) ~ (g) : code.

(h)

i) $\because$ A being symmetric, A is diagonalizable. ($P^t A P = \Lambda$).

$\rightsquigarrow$ $A = P \Lambda P^t$. If $\rho(A) > 1$, $\exists$ diagonal comp. of Λ^n s.t. as $n \to \infty$ $\Rightarrow$ such element $\to \infty$.

$\therefore$ some elements in A tends to get larger.

$\because$ sym mtx generated by rand() fn. probability that all spectrums of A is smaller than 1 conv. to 0.

(∵ Fixing orthogonal mtx P, chance that picking a real num from $\mathbb{R}$, with absolute value is smaller than 1 is virtually zero)

∴ At (g), powers of sym matrix always tends to diverge.

ii) ∃ non-symmetric case, using Schur's triangulization,

∃ orthogonal mtx P s.t. $A = PTP^t$ (T: upper triangular mtx).

Then, we can separate T into, $T = \Lambda + T'$, where

Λ: diagonal mtx, T': strictly upper triangular mtx.

↓

$A^N = PT^NP^t = P(\Lambda + T')^N P^t$. ∵ T' is nilpotent,

∃ $k \le n$ ($T' \in M_{n \times n}(\mathbb{R})$) s.t. $(T')^k = 0$.

so $A^N = P\left(\sum_{i=0}^{n} \binom{N}{i} \Lambda^{N-i} T'^{i} \right) P^t$ ... (*)

If ∃ spectrum of A, greater than 1, some components in (*) will diverge as $N \to \infty$

If all spectrum of A is smaller than 1, (*) goes to zero, as $\lim_{n \to \infty} \frac{(k)^i}{r^k} = 0$ ($\forall i \in \mathbb{N}$).

(e). (f), 로부터 확인 가능.

■

7.33

proposal: $\max_{1 \le i,j \le n} |a_{ij}^{(n)}| \le \|A\|$.

(proof)

i) Frobenius Norm

$$\max_{1 \le i,j \le n} |a_{ij}^{(n)}| \le \left(\sum_{i,j} a_{ij}^2 \right)^{\frac{1}{2}} = \|A\|_F.$$

ii) Induced Norm

Assump) $\underbrace{a_{\bar{i}\bar{j}}}_{} = \max_{1 \le i,j \le n} |a_{ij}^{(n)}|$

↳ Maximum absolute value occurs at $(\bar{i}, \bar{j})$.

$$\max_{x \ne 0} \frac{\|Ax\|}{\|x\|} \ge \frac{\|Ae_{\bar{j}}\|}{\|e_{\bar{j}}\|} = \| [a_{1\bar{j}} \ a_{2\bar{j}} \ \cdots \ a_{n\bar{j}}] \|$$

$$\ge |a_{\bar{i}\bar{j}}| = \max_{1 \le i,j \le n} |a_{ij}^{(n)}|$$

■

8.15.

(a) $\left[\dfrac{\sum_{i=1}^{20} |x_i|}{|\sum_{i=1}^{20} x_i|} \right] \cdot 19 \text{ eps}.$

(b) $(1.06) \cdot (19)$ eps.

8.17

$b=10,\ p=4.$

$p = 0.5672 \times 10^2,\ q = 0.1678 \times 10^2,\ r = 0.9815 \times 10^2.$

$p \otimes (q \oplus r) = 0.5672 \times 10^2 \otimes (\underbrace{0.1678 \times 10^2 \oplus 0.9815 \times 10^2}_{\underline{1.1493}_4})$

$= 0.5672 \times 10^2 \otimes 0.1149 \times 10^3$

$= \underline{0.6517 \times 10^4}$. (1)

$p \otimes q + p \otimes r = 0.5672 \times 10^2 \otimes 0.1678 \times 10^2$

$\oplus\ 0.5672 \times 10^2 \otimes 0.9815 \times 10^2$

$= 0.9518 \times 10^3 + 0.5567 \times 10^4 = \underline{0.6519 \times 10^4}$ (2)

(1) $\neq$ (2). ∎

8.18. $b=10,\ p=7$

$x = 0.1526359 \times 10^4,\ y = 0.5258231 \times 10^2,\ z = 0.3000333 \times 10^{-3}$

$(x \oplus y) \oplus z = 0.1578939 \times 10^4 \oplus 0.3000333 \times 10^{-3}$

$= 0.1578939 \times 10^4$ (1)

$x \oplus (y \oplus z) = 0.1526359 \times 10^4 \oplus 0.5258261 \times 10^2$.

$= (0.1526359 + 0.0052583) \times 10^4 = 0.1578940 \times 10^4$ (2)

(1) $\neq$ (2) ∎

B.32

(a), (b), (c) → coding problem

(d) The term $\frac{A^n}{n!}$ gets very large before it conv. to 0.

⇒ inaccuracy with large values.

(shown at code files

B.35.

(a) $$\frac{-10000+\sqrt{10000^2+4(0.001)^2}}{0.0002} = 9.094947017729282 \times 10^{-9}$$

(b) code

(c) sympy library also shows some error!!

■