12.3.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \quad (\because \text{odd fn})$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx = 0 \quad (\because \text{odd fn})$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} -\sin kx\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin kx\,dx\right]$$

$$= -\frac{2}{k\pi}\left(\cos \pi k - \cos\frac{\pi}{2}k\right) = \begin{cases} \frac{2}{\pi k} & (k:\text{odd}) \\ 0 & (k \equiv 0 \pmod 4) \\ -\frac{4}{\pi k} & (k \equiv 2 \pmod 4) \end{cases}$$

$$\therefore f(x) = a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$= \sum_{k=1}^{\infty}\left(\frac{2}{(2k-1)\pi}\sin((2k-1)x) - \frac{2}{(2k-1)\pi}\sin(2\cdot(2k-1)x)\right)$$

■

12.4.

(a) $\cos(n+1)\theta = \cos n\theta \cos\theta - \sin n\theta \cdot \sin\theta$

$= \cos n\theta \cos\theta - \sin^2\theta \cdot \cos(n-1)\theta - \sin\theta\cos\theta\sin(n-1)\theta$

$= \cos n\theta \cos\theta - \sin^2\theta \cdot \cos(n-1)\theta - \cos\theta(-\cos n\theta + \cos\theta\cos(n-1)\theta)$

$= 2\cos n\theta \cos\theta - \cos(n-1)\theta$

(b)

i) $n=0$ 일 때 $\cos 0\theta = 1$

$n=1$ 일 때 $\cos\theta$

ii) T_n: Chebyshev polynomial 일 때 $\exists\, T_n \;\forall n \in \mathbb{Z}_{\geq 0}$

$\exists\, n=0, n=1$. trivial

$T_{n+2}(x) = 2x\,T_{n+1}(x) - T_n(x)$

If true for $n+1, n$ ($n \geq 1$),

$$T_{n+2} = 2x \cdot T_{n+1} - T_n \in \mathbb{Z}[n+2].$$

($\mathbb{Z}[n]$: set of poly of order n, with integer coeff) "

iii), iv) $T_n(x) = \cos(n\cos^{-1}x)$ $\quad$ ($x = \cos\theta$)

WTS $\max\limits_{-1\leq x\leq 1} T_n(x) = 1$.

$\because$ $0 \leq \cos^{-1}x \leq \pi$ $\because$ $-1 \leq x \leq 1$,

$\exists$ $x' \in (-1, 1)$ s.t. $\cos^{-1}x' = \dfrac{2\pi}{n}$

$$\rightarrow T_n(x') = \cos\frac{2\pi}{n}\cdot n = 1$$

$\because$ $T_n(x) = \cos(n\cos^{-1}x) \leq 1$. $\quad \max\limits_{-1\leq x\leq 1} T_n(x) = 1$

v) $T_n(x) = \cos(n\cos^{-1}x)$

$\hookrightarrow$ $\deg(T_n) = n$. $\quad |\{x : T_n(x) = 0,\ -1 \leq x \leq 1\}| \leq n$

Setting $x_i = \cos\left(\dfrac{2i-1}{2n}\pi\right)$. $\quad$ ($i = 1, \cdots, n$)

$\because$ $\cos x$ is decreasing on $(0, \pi)$ $\quad x_i \neq x_j$ if $i \neq j$.

Also, $T_n(x_j) = \cos\left(\dfrac{n(2i-1)}{2n}\pi\right) = 0$

$\therefore$ $\exists$ n root of $T_n(x) = 0$ in $[-1, 1]$ with

$$x_i = \frac{2i-1}{2n}\pi \quad (i = 1, \cdots, n)$$

vi) : Proved in i).

$$\underbrace{\cos(n+1)\theta}_{T_{n+1}} = \underbrace{2\cos n\theta}_{2T_n}\ \underbrace{\cos\theta}_{x} - \underbrace{\cos(n-1)\theta}_{T_{n-1}}. \quad //$$

vii)

$T_0 = 0$

$T_1 = x$

$T_2 = 2x^2 - 1$

$T_3 = 4x^3 - 3x$

$T_4 = 8x^4 - 8x^2 + 1$

$T_5 = 16x^5 - 20x^3 + 5x$

(c)

i) $$\int_0^{\pi} \cos m\theta \cos n\theta = \int_0^{\pi} \frac{1}{2}\left(\cos(m+n)\theta + \cos(m-n)\theta \right) d\theta$$

$$= \begin{cases} 0 & (m \neq n) \\ \frac{1}{2}\pi & (m = n,\ m, n \neq 0) \\ \pi & (m = n = 0) \end{cases}$$

ii) $$\int_{-1}^{1} T_m(x)\, T_n(x) \cdot \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos m\theta \cdot \cos n\theta \cdot \frac{-\sin\theta}{\sin\theta}\, d\theta$$

$$\therefore \text{ If } m \neq n, \quad \int_0^{1} T_m \cdot T_n \cdot \frac{1}{\sqrt{1-x^2}}\, dx = 0 \qquad \blacksquare$$

12.5.

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p \cdot \frac{y_{i+1} - y_{i-1}}{2h} + q\, y_i = r(x_i).$$

$$\therefore \left(1 + \frac{ph}{2}\right) y_{i+1} + (h^2 q - 2)\, y_i + \left(1 - \frac{ph}{2}\right) y_{i-1} = h^2 r(x_i).$$

$$(i = 2, \cdots, n)$$

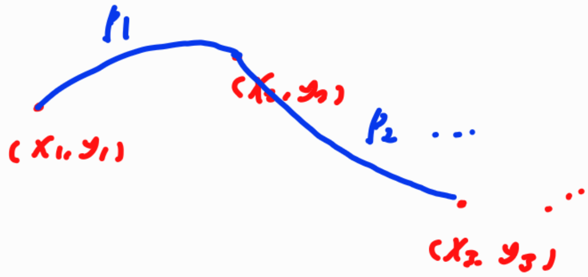
$$\therefore \begin{pmatrix} h^2q-2 & 1+\frac{ph}{2} & & & \\ 1-\frac{ph}{2} & h^2q-2 & 1+\frac{ph}{2} & & \\ & 1-\frac{ph}{2} & h^2q-2 & 1+\frac{ph}{2} & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} (\frac{ph}{2}-1)YA + h^2r(x_2) \\ h^2r(x_3) \\ \vdots \\ h^2r(x_{n-1}) \\ -(\frac{ph}{2}+1)YB + h^2r(x_n) \end{pmatrix}$$

12.6.

$p_i(x_i) = y_i \ (i=1,\dots,n)$ n

$p_i(x_{i+1}) = y_{i+1} \ (i=1,\dots,n)$ n

$p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}) \ (i=1,\dots,n-1)$ n-1



∵ ∃ $3n$ variables. we need 1 more.

→ $p_1''(x_1) = 0$

12.11, 12.14 → code problem

13.1.

$\forall x \in \mathbb{R}^n$, $x^tAx > 0$. Then, A is nonsingular. (∵ if not.

$\dim\ker A \geq 1 \leadsto \exists x \neq 0$ s.t. $Ax = 0$ (*))

∵ A^{-1} is nonsingular, $\forall y \in \mathbb{R}^t$. $\exists!\, x$ s.t. $Ax = y$.

$\therefore y^t \cdot A^{-1} \cdot y = x^tA^tA^{-1}Ax = x^tA^tx = x^tAx > 0$ ■

13.6. $k(A) = \|A^{-1}\| \cdot \|A\|$

$k(R) = \|R^{-1}\| \cdot \|R\|$.

A: pd → its eigval is non-negative, and real num.

∵ A is diagonalizable, $\|A\|_2 = \max_i |\lambda_i|$ (λ_i: eigval of A)

$\|A^{-1}\|_2$: pd ~~ $\|A^{-1}\|_2 = \max_i \frac{1}{|\lambda_i|} = \frac{1}{\min_i |\lambda_i|}$

$\exists\ A = R^t \cdot R$, $\quad \|R\|_2 = \max_i \sqrt{\lambda_i}$

$\|R^{-1}\|_2 = \max_i \sqrt{\delta_i}$. with δ_i : eigval of $(R^{-1})^t R^{-1}$.

trivially, $(R^{-1})^t R^{-1}$ is p.d. Also, eigval of RR^t = eigval of R^tR

$\therefore\ \|R^{-1}\|_2 = \max_i \frac{1}{\sqrt{\lambda_i}} = \frac{1}{\min_i \sqrt{\lambda_i}}$

$\therefore\ k(A) = k(R)^2$. ■

13.14. $A, B > 0$

~~ $x^t(A+B)x = x^tAx + x^tBx > 0$, $(\forall x \in \mathbb{R}^n)$

$\therefore\ A+B > 0$ ■

13.15. If not, $\dim\ker A \geq 1$ ~~ $\exists x \neq 0$ s.t. $Ax = 0$

$\rightarrow\ x^tAx = x^t \cdot 0 = 0$ $(*)$.

$\therefore$ A is non-singular. ■

13.19, 13.22 : code problem

14.5.

$A^tA = (QR)^t(QR) = R^tQ^tQR = R^tR$. ■

14.9.

$e_1 = \left(\frac{1}{\sqrt{a^2+c^2}}(a, c)\right)^t$. $\quad u_2 = \begin{bmatrix} b \\ d \end{bmatrix} - \langle e_1, \begin{bmatrix} b \\ d \end{bmatrix} \rangle \cdot e_1$

$$= \begin{bmatrix} b \\ d \end{bmatrix} - \left[\frac{ab+cd}{\sqrt{a^2+c^2}} \cdot \frac{1}{\sqrt{a^2+c^2}} \cdot (a, c)\right]^t$$

$$= \begin{bmatrix} b \\ d \end{bmatrix} - \begin{bmatrix} a \\ c \end{bmatrix} \cdot \frac{ab+cd}{a^2+c^2} = \begin{bmatrix} \dfrac{\cancel{a^2b}+bc^2-\cancel{a^2b}-acd}{a^2+c^2} \\ \dfrac{a^2d+\cancel{c^2d}-abc-\cancel{c^2d}}{a^2+c^2} \end{bmatrix}$$

$$= \begin{bmatrix} (bc-ad) \cdot \dfrac{c}{a^2+c^2} \\ (ad-bc) \cdot \dfrac{a}{a^2+c^2} \end{bmatrix} \cdots (*)$$

$$\rightarrow e_2 = \frac{u_2}{\|u_2\|} = \left[\frac{1}{\sqrt{a^2+c^2}} (-c, a) \right]^t$$

$$\therefore A = \begin{pmatrix} \dfrac{a}{\sqrt{a^2+c^2}} & \dfrac{-c}{\sqrt{a^2+c^2}} \\ \dfrac{c}{\sqrt{a^2+c^2}} & \dfrac{a}{\sqrt{a^2+c^2}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{a^2+c^2} & \dfrac{ab+cd}{\sqrt{a^2+c^2}} \\ 0 & \dfrac{-bc+ad}{\sqrt{a^2+c^2}} \end{pmatrix}$$

(b) From (a), $(*)$ goes to 0 when $ad = bc$

$\Updownarrow$

$\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}$ linearly dependent

$\therefore$ Q is not orthogonal ... ■

14.9. $\exists A \in M_{m,n}(\mathbb{R})$, $m \geq n$, with A being a full rank.

(a) A^tA : pd ($\because$ $\langle Ax, Ax \rangle = x^tA^tAx$ and $\exists$ column of A is linearly indpt, $\langle Ax, Ax \rangle = 0 \leftrightarrow Ax = 0 \leftrightarrow x = 0$).

Then using Cholesky decomp $\exists$UT R, with $r_{ii} > 0$

s.t. $A^tA = R^tR$.

(b) $\exists r_{ii} > 0$, $(\forall i)$, R: UT $\rightarrow \det R = \prod_{i=1}^{n} r_{ii} > 0$.

$\therefore$ R is nonsingular. $\rightarrow$ $Q := AR^{-1}$ is well-defined.

(c) $Q^t Q = (R^{-1})^t A^t A R^{-1} = (R^{-1})^t \cdot R^t \cdot R \cdot R^{-1} = I \cdot I = I$.

$\therefore$ Q is orthogonal ■

14.10. Reduced decomposition is unique

(a) $A^t A = (\hat{Q}\hat{R})^t \hat{Q}\hat{R} = \hat{R}^t \cdot (\hat{Q}^t \cdot \hat{Q}) \hat{R} = \hat{R}^t \cdot \hat{R}$.

(b) $\because$ Cholesky decomposition is unique, $\hat{R} = R$.

$\therefore$ $A = \hat{Q} R = QR$. $\because$ R has positive diagonal entries,

R is invertible so $\hat{Q} = Q$. ■

14.16. code

(b) → From the code, we can see that $Q \cdot R = A$, but $Q^t \cdot Q \neq I$.

(c) → Yes!!! ($\because$ $Q^t \cdot Q = I$. from the code)
$Q \cdot R = A$

14.18 code

(a) Q is orthogonal → $Qy = b$ so $y = Q^t \cdot b$.

(b) R is UT so use back substitution.