9.2.

(a) $m(n+n-1) = m(2n-1) \simeq 2mn$

(b) mp

(c) $2n+2n+1+n = 5n+1$

(d) $m(n-1) \simeq mn$

(e) $(m-1)\cdot n \simeq mn$

(f) $n-1 \simeq n$

9.5. $u \otimes v := u \cdot v$

```
function TENSORPROD(u,v)
    for i =1:m do
        for j =1:n do
            tensorprod(i,j)=u(i)*v(j)
        end for
    end for
    return TP
end function
```

9. 12

```
function BISOLVE(A,b)
    %n is the row(column) size of square matrix A
    x(n,1)=b(n,1)/A(n,n)
        for i = n-1:-1:1
            x(i,1)=(b(i,1)-A(i,i+1)*x(i+1))/A(i,i);
        end
    return x
end function
```

9.14

$$1+\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1}2+2\right) = 1+\sum_{i=2}^{n}(2(i-1)+2)$$

$$= 1+\sum_{i=2}^{n}2i = 1+n(n+1)-2 = n^2+n-1$$

$\blacksquare$

9.21, coding problem

9.25

(a) $k = \dfrac{n+1}{2}$.

(b) ~ (e) → coding problem

10.2.

$r = b - A\hat{x}$ be the residual. → $A\hat{x} = b - r$, and r is a small perturbation → $\|\hat{x} - x\|$ is really small. ■

10.6. $f(x) = \ln x$.

(a) $$c(x) = \frac{|x| \cdot |f'(x)|}{|f(x)|} = \frac{|x| \cdot \frac{1}{|x|}}{|\ln x|} = \frac{1}{|\ln x|}.$$

(b) Near $x=1$ → $\ln x \to 0$ → $c(x)$ goes very large

∴ ill-conditioned. ■

10.10.

(a) $$\frac{\dfrac{\|A(x+\delta x) - Ax\|}{\|Ax\|}}{\dfrac{\|\delta x\|}{\|x\|}} = \frac{\|x\|}{\|Ax\|} \cdot \left(\frac{\|A(\delta x)\|}{\|\delta x\|} \right)$$

$$\therefore \; C_{Ax}(x) = \frac{\|A\| \cdot \|x\|}{\|Ax\|}.$$

(b) $\dfrac{\|T\| \cdot \|x\|}{\|Tx\|}$ → $Tx = \begin{bmatrix} -9 \\ 3 \\ 20 \end{bmatrix}$ → $\dfrac{11 \cdot 2}{20} = 1.1$

(c)

$$\frac{\|A\|\cdot\|x\|}{\|Ax\|} = \|A\|\cdot\frac{\|x\|}{\|Ax\|} = \|A\|\cdot\frac{\|A^{-1}y\|}{\|y\|} \leq \|A\|\cdot\|A^{-1}\|.$$

$x = A^{-1}y$

■

(d) $A^{-1} = \begin{bmatrix} \frac{5}{8} & -\frac{1}{2} & \frac{3}{8} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{37}{24} & \frac{11}{6} & -\frac{19}{24} \end{bmatrix} \rightsquigarrow \|A^{-1}\|_\infty = \frac{100}{24} = \frac{25}{6}$ //

$\therefore\ 1.1 \leq 17\cdot\frac{25}{6} = \frac{425}{6} = 70.833\ ..$ ■

10.11

(a) $k_p(A) := \|A\|_p\cdot\|A^{-1}\|_p \geq \|A\cdot A^{-1}\|_p = \|I\|_p = 1$.

(b) $k_2(A) = \|A\|_2\cdot\|A^{-1}\|_2 = \|A^t\|_2\cdot\|(A^t)^{-1}\|_2 = k_2(A^t)$

$k_1(A) = \|A\|_1\cdot\|A^{-1}\|_1 = \|A^t\|_\infty\cdot\|(A^t)^{-1}\|_\infty = k_\infty(A^t)$

(c) $\|AB\|\cdot\|B^{-1}A^{-1}\| \leq \|A\|\cdot\|B\|\cdot\|B^{-1}\|\cdot\|A^{-1}\| = \|A\|\cdot\|A^{-1}\|\cdot\|B\|\cdot\|B^{-1}\|$

$= k(A)\cdot k(B)$

■

10.12

$A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{1-a^2}\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}$

$\lambda = \frac{1+a}{1-a^2}, \frac{1-a}{1-a^2} = \frac{1}{1-a}, \frac{1}{1+a}$

$(1-\lambda)^2 - a^2 = 0 \rightsquigarrow \lambda = 1+a, 1-a$

$\therefore\ c(A) = \left|\frac{1+a}{1-a}\right| (a>0),\ \left|\frac{1-a}{1+a}\right| (a \leq 0)$

$\therefore$ A is ill-conditioned when $a = \pm 1$.

When $a \to \infty$, $C(A) \to 1$. (not ill-conditioned)

$\blacksquare$

10.15, 10.28, 10.31 : coding problem.

11.1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 5 & 4 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

and $$E_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & -5 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and $$E_2 E_1 A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} = U$$

$$\therefore A = (E_2 E_1)^{-1} U = (E_1^{-1} E_2^{-1}) \cdot U.$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix}$$

$\blacksquare$

11.8.

Assuming $PA = LU$, $|l_{ij}| \leq 1$ when $j < i$.

$$\therefore \|L\|_F^2 \leq 1 + 2 + \cdots + n = \frac{1}{2} n(n+1).$$

$$\|L\|_\infty \leq 1 + \cdots + 1 = n$$

$$\|L\|_1 \leq 1 + \cdots + 1 = n$$

all $l_{ij}=1 \ \forall\, i \geq j$

$l_{ij}=0 \ \forall\, i<j$.

$\forall$ two norm,

$$\max_{\|x\|\neq 0} \frac{\|Lx\|_2^2}{\|x\|_2^2} \leq \max_{\|x\|\neq 0} \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{i} x_j\right)^2}{\sum_{i=1}^{n} x_i^2}$$

$$\leq \max_{\|x\|\neq 0} \frac{\sum_{i=1}^{n}\left(\sum_{j=1}^{n} x_j\right)^2}{\sum_{i=1}^{n} x_i^2} \leq \max_{\|x\|\neq 0} \frac{\sum_{i=1}^{n} (n)\left(\cancel{\sum_{i=1}^{n} x_i^2}\right)}{\cancel{\sum_{i=1}^{n} x_i^2}} = n^2$$

$\therefore \|L\|_2 \leq \sqrt{n^2} = n.$

11.20.

(a) i): $3 = |-2|+|1| \Rightarrow$ not strictly dom

ii) $1 > 0.3+0.2+0.4$

$2.5 > 1+0.15+0.9$

$9 > 1+1+0.9$ $\Rightarrow$ strictly dom.

$0.8 > 0.1+0.25+0.24$

iii) $2 > 0.5+0.5 \Rightarrow$ strictly dom.

(b) $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 5 & 2 \\ 1 & 1 & -4 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix}$

$EA = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 4 & \frac{5}{3} \\ 0 & 0 & -\frac{13}{3} \end{bmatrix}$, $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} = L$

$\therefore A = L \cdot U$. (Not requiring permutation mtx)

(c) set diagonally dominant mtx A.

(lemma) Diagonally dominant mtx is invertible.

$\because$ If not, $\exists$ $(k_1, \dots, k_n)^t \neq 0$ s.t.

$k_1[A]_1 + \dots + k_n[A]_n = 0$ $\Rightarrow$ def $M := \underset{i}{\operatorname{argmax}} |k_i|$.

Then, $k_1 a_{1M} + \dots + k_M a_{MM} + \dots + k_n a_{nM} = 0$.

$\Rightarrow$ $k_M a_{MM} = -\sum_{i \neq M} k_i a_{iM}$.

$$|k_M a_{MM}| = \left|\sum_{i \neq M} k_i a_{iM}\right| \leq \sum_{i \neq M} |k_i a_{iM}| \leq |k_M| \cdot \sum_{i \neq M} |a_{iM}|$$

$$< |k_M| \cdot |a_{MM}| \quad (*)$$

$\therefore$ Diagonally dom. matrix is invertible //

set A as diagonally dom. matrix. set A as A_n. and

def. A_{n-1}, E_{n-1} as $A_n = \left[\begin{array}{c|c} A_{n-1} & * \\ \hline E_{n-1} & * \end{array}\right]$, with $A_{n-1} \in M_{n-1,n-1}(\mathbb{R})$,

$E_{n-1} \in M_{1,n-1}(\mathbb{R})$. Then, A_{n-1} is diagonally dom. matrix which

means that A_{n-1} is non singular.

Then, E_{n-1} can be generated from row space of A_{n-1}. as

$$\sum_{j=1}^{n-1} l_{nj} [A_{n-1}]_j = E_{n-1}$$

$\Rightarrow$ setting $L_{n-1} = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline \Delta & 1 \end{array}\right]$. with $\Delta = (-l_{n1}\ -l_{n2}\ \cdots\ -l_{n,n-1})$,

$$U_{n-1} := L_{n-1}A = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline \Delta & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} A_{n-1} & * \\ \hline E_{n-1} & * \end{array}\right] = \left[\begin{array}{c|c} A_{n-1} & * \\ \hline 0 & ** \end{array}\right].$$

As A is not singular, $** \neq 0$.

" L_{n-1} : lower triangular ".

Then, $\exists\ A_{n-1}$ is diagonal dom, repeat the similar procedure.

def. A_{n-2}, E_{n-2} as $A_{n-1} = \left[\begin{array}{c|c} A_{n-2} & * \\ \hline E_{n-2} & * \end{array}\right]$, with $A_{n-2} \in M_{n-2,n-2}(\mathbb{R})$,

$E_{n-2} \in M_{1,n-2}(\mathbb{R})$. Then, A_{n-2} is diagonally dom. matrix which means that A_{n-2} is non singular.

Then, E_{n-2} can be generated from row space of A_{n-2}, as

$$\sum_{j=1}^{n-2} l_{n-1\,j}\,[A_{n-2}]_j = E_{n-2}$$

$\rightsquigarrow$ setting $L'_{n-2} = \left[\begin{array}{c|c} I_{n-2} & 0 \\ \hline \Delta & 1 \end{array}\right]$, with $\Delta = (-l_{n-1,1}, \cdots, -l_{n-1,n-2})$

Also, set $L_{n-2} := \left[\begin{array}{c|c} L'_{n-2} & 0 \\ \hline 0 & I_2 \end{array}\right]$. Then,

$$U_{n-2} := L_{n-2}U_{n-1} = \left[\begin{array}{c|c} L'_{n-2} & 0 \\ \hline \Delta & I_2 \end{array}\right] \cdot \left[\begin{array}{c|c} A_{n-1} & * \\ \hline 0 & * \end{array}\right] = \left[\begin{array}{c|c} L'_{n-2}A_{n-1} & * \\ \hline 0 & ** \end{array}\right]$$

with $L'_{n-2}A_{n-1} = \left[\begin{array}{c|c} A_{n-2} & * \\ \hline 0 & ** \end{array}\right]$.

As A_{n-1} is not singular, $** \neq 0$.

"L_{n-2} : lower triangular".

Repeating this step to $A_{n-2}, A_{n-3}, \cdots$, we can set

$$L_0 \cdot L_1 \cdot \cdots \cdot L_{n-1} A = U_0 \rightsquigarrow \text{upper triangular.}$$

Then, $A = (L_0 \cdots L_{n-1})^{-1} U_0 = LU$

$\therefore$ no permutation matrix required !!!

* L = lower triangular matrix

$\rightsquigarrow$ $L = ([L]^1, \cdots, [L]^n)$. Then, defining $L_1 := ([L]^1\ e_2 \cdots e_n)$

$L_2 = (e_1\ [L]^2, \cdots e_n)$, $\cdots$, $L_n = (e_1, \cdots, e_{n-1}, [L]^n)$,

$L = L_1 \cdot L_2 \cdot L_3 \cdot \cdots \cdot L_n$

setting $A = LU$. then $L^{-1}A = U$. $L^{-1} = L_n^{-1} \cdots L_2^{-1} \cdot L_1^{-1}$.

∴ $L_i^{-1} = (e_1, \ldots, -[A]^i, \ldots e_n)$,

$L_n^{-1} \cdots L_2^{-1} L_1^{-1} A = U$ is a formal Gauss ellimination with no row exchange and pivoting.

∴ If A can be expressed as LU, permutation is not required during Gauss ellimination. ■

11.21

(a) If A : invertible, diagonal elem of U is nonzero.

setting $D = \text{diag}(u_{11}, \ldots, u_{nn})$, $(U = (u_{ij}))$,

$A = LU = LD(D^{-1}U) = LDU'$ (U' : unit upper triangular..) ■

diagonal entries being zero

(b) A : sym, non singular

∴ A : invertible then $\exists A = LDU$. ∴ $A = A^t$,

$A = A^t = U^t D L^t$. then ∴ diagonal entries of U, L is 1.

$A = U^t D L^t$ is also LDU decomposition.

then $U^t \cdot (DL^t)$: LU decomp of A. ∴ $\exists!$ LU decomp of A,

$U^t = L$, $DU = DL^t$ then ∴ D is invertible,

$U^t = L$, $U = L^t$. ■

(c) No.

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \rightsquigarrow L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 4 \\ 0 & -15 \end{bmatrix}$$

11-22

(a) $D = \mathrm{diag}(d_1, \cdots, d_n)$, $B = \mathrm{diag}(\frac{1}{d_1}, \cdots, \frac{1}{d_n})$ $(d_i \neq 0 \ \forall i)$.

Then, $DB = BD = I$ ■

(b)

$$\left(I - \frac{2uu^t}{u^tu} \right)\left(I - \frac{2uu^t}{u^tu} \right)$$

$$= I - \frac{2uu^t}{u^tu} - \frac{2uu^t}{u^tu} + \frac{4uu^tuu^t}{(u^tu)^2}$$

$$= I - \frac{2uu^t}{u^tu} - \frac{2uu^t}{u^tu} + \frac{4uu^t}{(u^tu)} = I.$$ ■

(c) $L \cdot L^{-1} = ([L]_k \cdot [L^{-1}]^j)_{kj} = (\langle [L]_k^t, [L^{-1}]^j \rangle)_{kj}$

$[L]_k^t : e_k \quad (k \leq i)$

$$: [\, 0 \cdots \underbrace{\ell_{k,i}}_{i\text{th term}}, \cdots \underbrace{1}_{k\text{th term}} \; 0 \cdots 0 \,]^t \quad (k > i)$$

$[L^{-1}]^j = e_j \quad (j \neq i)$

$$= [\, 0, \cdots 0, \underbrace{1}_{i\text{th term}}, -\ell_{i+1,i}, \cdots, -\ell_{ni} \,]^t \quad (j = i)$$

If $k < i$, $j \neq i$: $\quad \langle [L]_k^t, [L^{-1}]^j \rangle = \delta_{kj}$.

If $k > i$, $j \neq i$: $\quad \langle [L]_k^t, [L^{-1}]^j \rangle = \delta_{kj}$.

If $j = i$, $k = i$: $\quad \langle [L]_k^t, [L^{-1}]^j \rangle = 1$

If $j = i$, $k < i$: $\quad \langle [L]_k^t, [L^{-1}]^j \rangle = 0$

If $j = i$, $k > i$: $\quad \langle [L]_k^t, [L^{-1}]^j \rangle = l_{k,i} - l_{k,i} = 0$.

$\therefore \langle [L]_k^t, [L^{-1}]^j \rangle = \delta_{kj}$, $\quad L \cdot L^{-1} = I$. $\blacksquare$

11.28. 11.38, 11.42 : coding problems