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A new class of finite-time nonlinear consensus protocols for multi-agent systems

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This paper is devoted to investigating the finite-time consensus problem for a multi-agent system in networks with undirected topology. A new class of global continuous time-invariant consensus protocols is constructed for each single-integrator agent dynamics with the aid of Lyapunov functions. In particular, it is shown that the settling time of the proposed new class of finite-time consensus protocols is upper bounded for arbitrary initial conditions. This makes it possible for network consensus problems that the convergence time is designed and estimated offline for a given undirected information flow and a group volume of agents. Finally, a numerical simulation example is presented as a proof of concept.

Keywords: distributed control; finite-time stability; multi-agent systems; network consensus; undirected graph

1. Introduction

Network consensus problem, due to a broad application of multi-agent systems including formation control (Abdessameud & Tayebi, 2011; Bullo, Cortés, & Martínez, 2009), flocking (Toner & Tu, 1998) and attitude alignment (Ren, 2010), has emerged as a challenging topic in control area and has been intensively studied in the literature (Olfati & Murray, 2004; Ren, Beard, & Atkins, 2007). In some applications involving multi-agent systems, groups of agents are required to agree upon certain quantities of interest, which is called *consensus* or *agreement problem*. To achieve agreement, distributed neighbour-based feedback laws (called *protocols*) are usually used to interconnect the agents in networks.

In the analysis of consensus problems, an important performance index for a proposed consensus protocol is convergence rate (Wang & Xiao, 2010). Olfati and Murray (2004) proposed a linear consensus protocol and demonstrated that *algebraic connectivity* of the interaction graph, i.e. the second smallest eigenvalue of the graph Laplacian, qualifies the convergence rate, which motivated some researchers to seek proper interaction topology with larger algebraic connectivity (Kim & Mesbahi, 2006). On the other hand, *finite-time consensus* problem has been promoted to achieve high-speed convergence. The benchmark work by Bhat and Bernstein (2000) related the regularity properties of the Lyapunov functions and the settling-time function in finite-time stability analysis for autonomous systems. Xiao and Wang (2007), Wang and Xiao (2010) and Xiao, Wang, and Chen (2011) expanded the finite-time control idea (Bhat & Bernstein, 2000) to multi-agent systems with single-integrator dynamics. The works by Wang and Hong (2008) and Zhao, Duan, Wen, and Zhang (2013) presented finite-time consensus and tracking control, respectively, for second-order multi-agent systems via using homogeneity with dilation. Cao, Ren, and Meng (2010) focused on decentralised finite-time formation tracking of multi-autonomous vehicles with the introduction of decentralised finite-time sliding mode estimators. Meng and Jia (2011, 2012) incorporated iterative learning control (ILC) actions into output feedbacks in the finite-time consensus tracking problem, while Liu and Geng (2013) introduced optimal control idea into the finite-time formation control design.

Unfortunately, the convergence time estimates of existing finite-time consensus protocols are closely dependent upon initial states of the agents in networks, which limit the practical applications, since the knowledge of initial conditions is not available in advance. Motivated by this, we propose a new class of nonlinear finite-time consensus protocols for multi-agent systems with single-integrator dynamics in networks with undirected information flow. The upper bound of the settling time of the proposed protocols no longer depends on initial states but merely on the design parameters, the algebraic connectivity and the group volume. Therefore, the convergence time can be designed or estimated as a priori for a given undirected interaction graph and a group volume of agents. The explicit expression of settling-time function obtained in this paper shows that the systems will take more time to reach agreement

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for larger group volume of agents and/or smaller algebraic connectivity.

This paper is organised as follows: Sections 2 and 3 present the preliminaries on graph theory notions and finite-time consensus definitions, respectively. Section 4 presents the protocol design, and Section 5 gives the stability analysis. In Section 6, an illustrative example is discussed. Finally, the paper is ended by concluding remarks in Section 7.

2. Graph theory notions

A weighted graph $\mathcal{G}(A) = \{\mathcal{V}, \mathcal{E}, A\}$ consists of a node set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_n\}$, an edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$ and an adjacent matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. An edge (v_i, v_j) on $\mathcal{G}(A)$ denotes the state of node v_i available to node v_j, but not necessarily vice versa. In contrast, $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ in a weighted *undirected graph*, i.e. the nodes v_i and v_j can sense each other. If $(v_i, v_j) \in \mathcal{E}$, then node v_i is called a *neighbour* of node v_j, or v_i and v_j is said to be *adjacent*. The set of all neighbours of v_j is denoted by $\mathcal{N}_j = \{v_i : (v_i, v_j) \in \mathcal{E}\}$. The weighted adjacent matrix A of a directed graph is defined such that $a_{ij} > 0$ for $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For a weighted undirected graph, A is defined similarly except that $a_{ij} = a_{ji}$, $\forall i \neq j$. A *path* on $\mathcal{G}$ from v_i to v_j is a sequence of distinct vertices $\{v_i, \ldots, v_j\}$ such that consecutive vertices are adjacent. An undirected graph is called *connected* if, for any two nodes, there exists a path.

3. Problem formulation

The information consensus problem involves finding a dynamic feedback law that enables a group of agents in a network to reach agreement upon certain qualities of interest with undirected or directed information flow (Wang & Hong, 2008). This paper will investigate finite-time consensus protocols based on undirected information flow for multi-agent systems.

Consider a group of n continuous-time agents with dynamics in the form of

$$\dot{x}_i(t) = u_i(t),\ i \in \mathcal{I}_n = \{1, 2, \ldots, n\}, \tag{1}$$

where $x_i \in \mathbb{R}^N$ denotes the information state of agent $i \in \mathcal{I}_n$, $u_i \in \mathbb{R}^N$ is the state feedback called *protocol* and $N \in \mathbb{Z}_+$ is shared by all agents. Without loss of generality in the consensus design, $N = 1$ is assumed for simplicity in the sequel. The results to be developed later are valid for the vector state case by introducing Kronecker product.

With a given protocol u_i, $i \in \mathcal{I}_n$, the closed-loop system in Equation (1) is said to achieve *asymptotic consensus*, if, for $\forall x_i(0)$ and $\forall i, j \in \mathcal{I}_n$, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$. It is said to achieve *finite-time consensus* if, for $\forall x_i(0)$ and $\forall i, j \in \mathcal{I}_n$, there exists a *settling time* $T \in [0, \infty)$ such that

$$\begin{cases} \lim_{t\to T} |x_i(t) - x_j(t)| \to 0 \\ x_i(t) = x_j(t),\ \forall t \geq T. \end{cases} \tag{2}$$

If the final agreement state satisfies

$$\lim_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{k=1}^{n} x_k(0), \quad \forall i \in \mathcal{I}_n, \tag{3}$$

then the system is said to achieve *average consensus*.

Stack the states of n agents as a vector $x = [x_1, x_2, \ldots, x_n]^T$, and let $1_n = [1, 1, \ldots, 1]^T \in \mathbb{R}^n$ and $\mathrm{span}(1_n) = \{\xi \in \mathbb{R}^n : \xi = r1_n, r \in \mathbb{R}\}$.

Lemma 3.1 (Olfati & Murray, 2004)**:** *Let $L_A = [l_{ij}] \in \mathbb{R}^{n\times n}$ denote the graph Laplacian of $\mathcal{G}$, which is defined by*

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik} & i = j \\ -a_{ij} & i \neq j. \end{cases}$$

$L(A)$ has the following properties:

1. *0 is an eigenvalue of $L(A)$ and 1_n is the associated eigenvector;*
2. *$x^T L_A x = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_j - x_i)^2$ and the semi-positive definiteness of L_A implies that all eigenvalues of L_A are real and non-negative;*
3. *if $\mathcal{G}(A)$ is connected, the second smallest eigenvalue of L_A, which is denoted by $\lambda_2(L_A)$ and called the algebraic connectivity of $\mathcal{G}(A)$ is greater than zero;*
4. *the algebraic connectivity of $\mathcal{G}(A)$ is equal to $\min_{x\neq 0, 1_n^T x=0} \frac{x^T L_A x}{x^T x}$, and therefore, if $1_n^T x = 0$, $x^T L_A x \geq \lambda_2(L_A) x^T x$.*

Since finite-time consensus is closely related to finite-time stability, the following lemma by Bhat and Bernstein (2000) is needed to establish one of our main results.

Lemma 3.2: *Consider the system of differential equations,*

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0, \tag{4}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}^n$ and $f(0) = 0$. Suppose there exists a continuous positive definite function $V(x) : \mathbb{R}^n \to \mathbb{R}$ such that there exist real numbers $c > 0$ and $\alpha \in (0, 1)$, such that

$$\dot{V}(x) + c(V(x))^{\alpha} \leq 0,\ x \in \mathbb{R}^n \backslash \{0\}.$$

Then, the origin is a globally finite-time stable equilibrium of Equation (4) and the settling time is

$$T(x_0) \leq \frac{1}{c(1-\alpha)} V(x_0)^{(1-\alpha)}.$$

To streamline the technical proofs of the main results, another two lemmas are needed.

Lemma 3.3: *Let $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $0 < p \leq 1$. Then,*

$$\sum_{i=1}^{n} \xi_i^p \geq \left(\sum_{i=1}^{n} \xi_i\right)^p. \tag{5}$$

Proof: Using the fact $x^p \geq x$ for $\forall x, p \in (0, 1]$, we have

$$\frac{\sum_{i=1}^{n} \xi_i^p}{(\sum_{i=1}^{n} \xi_i)^p} = \sum_{i=1}^{n} \left(\frac{\xi_i}{\sum_{i=1}^{n} \xi_i}\right)^p \geq \sum_{i=1}^{n} \frac{\xi_i}{\sum_{i=1}^{n} \xi_i} = 1,$$

which is equivalent to Equation (5). □

Lemma 3.4: *Let $\xi_1, \xi_2, \ldots, \xi_n \geq 0$ and $p > 1$. Then,*

$$\sum_{i=1}^{n} \xi_i^p \geq n^{1-p} \left(\sum_{i=1}^{n} \xi_i\right)^p. \tag{6}$$

Proof: Based on the *Hölder inequality,*

$$\sum_{i=1}^{n} \xi_i \eta_i \leq \left(\sum_{i=1}^{n} \xi_i^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} \eta_j^q\right)^{\frac{1}{q}}$$

for $\forall \xi_i, \eta_i > 0$, $i \in \mathcal{I}_n$ and $\forall p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$, Equation (6) is a straightforward result by letting $[\eta_1, \eta_2, \ldots, \eta_n]^T = 1_n^T$. □

4. Protocol design

In this section, we shall develop a class of new nonlinear finite-time continuous protocols for a group of multi-agent systems in Equation (1). Before that and to make the idea clear, we first present our motivation and give a retrofit lemma.

Lemma 3.2 gives the sufficient conditions for finite-time convergence. However, the upper-bound estimate of the settling time T depends on the system initial states, which restrict the practical applications. Motivated by this analysis, if the settling time has an a-priori estimate that depends merely on design parameters, then the settling time can be estimated for any initial state.

Lemma 4.1: *Consider a scalar system*

$$\dot{y} = -\alpha y^{2-\frac{p}{q}} - \beta y^{\frac{p}{q}},\ y(0) = y_0, \tag{7}$$

where $\alpha, \beta > 0$, p, q are both positive odd integers satisfying $p < q$. Then, the equilibrium of Equation (7) is finite-time stable and the settling time is bounded by

$$T \leq \frac{q\pi}{2\sqrt{\alpha\beta}(q-p)}. \tag{8}$$

Proof: Let $V(y) = y^2 \geq 0$. Differentiating $V(y)$ along system (7) yields

$$\begin{aligned} \dot{V}(y) &= 2y\left(-\alpha y^{2-\frac{p}{q}} - \beta y^{\frac{p}{q}}\right) \\ &= -2\alpha(y^2)^{\frac{3q-p}{2q}} - 2\beta(y^2)^{\frac{p+q}{2q}} \\ &= -2\left(\alpha V^{\frac{q-p}{q}} + \beta\right) V^{\frac{p+q}{2q}}. \end{aligned} \tag{9}$$

The fact $\alpha V^{(q-p)/q} > 0$ implies $\dot{V}(y) \leq -2\beta V^{(p+q)/2q}$. In view of $0 < (p+q)/(2q) < 1$, the system in Equation (7) is finite-time stable due to Lemma 3.2. Further, since $V(y) = 0$ in Equation (9) is a trivial case, assuming $V(y) \neq 0$, we have

$$\begin{aligned} &\frac{1}{V^{\frac{p+q}{2q}}}\frac{\mathrm{d}V}{\mathrm{d}t} = -2\left(\alpha V^{\frac{q-p}{q}} + \beta\right) \\ &\Rightarrow \frac{q}{q-p}\frac{\mathrm{d}V^{\frac{q-p}{2q}}}{\mathrm{d}t} = -\left(\alpha V^{\frac{q-p}{q}} + \beta\right). \end{aligned} \tag{10}$$

Let $z = V^{(q-p)/2q}$. Equation (10) can be written as

$$\frac{1}{\alpha z^2 + \beta}\mathrm{d}z = -\frac{q-p}{q}\mathrm{d}t,$$

integrating both sides of which yields

$$\begin{aligned} &\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\frac{\alpha}{\beta}}z(t)\right) \\ &= \frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\frac{\alpha}{\beta}}z(0)\right) - \frac{q-p}{q}t. \end{aligned}$$

Since the function $\tan^{-1}(\cdot)$ is monotonically increasing, $\tan^{-1}(z) = 0$ if and only if $z = 0$ which implies $V = 0$. Thus, we have

$$\lim_{t \to T(y_0)} V = 0,$$

where $T(y_0)$ denotes the settling-time function given by

$$\begin{aligned} T(y_0) &= \frac{q}{q-p}\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\frac{\alpha}{\beta}}z(0)\right) \\ &= \frac{q}{q-p}\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\frac{\alpha}{\beta}}y_0^{\frac{q-p}{q}}\right). \end{aligned}$$

To this end, it can be verified that $T(y_0)$ is bounded by

$$\begin{aligned} \lim_{y_0 \to \infty} T(y_0) &= \lim_{y_0 \to \infty} \frac{q}{q-p}\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\left(\sqrt{\frac{\alpha}{\beta}}y_0^{\frac{q-p}{q}}\right) \\ &= \frac{q\pi}{2\sqrt{\alpha\beta}(q-p)}. \end{aligned}$$

Using the fact $V(y(t)) = 0$ implies $y(t) = 0$, which completes the proof. □

Remark 1: The settling-time function is upper bounded by a value that is not dependent on the system initial state y_0 but only on the design parameters, i.e. p, q, α and β. This promises that the convergence time can be designed or estimated.

Remark 2: It is worth noting that if $\lfloor y\rceil \gg 1$, then $\lfloor y\rceil^{2-p/q} \gg \lfloor y\rceil \gg \lfloor y\rceil^{p/q}$ implies that the first nonlinear term in Equation (7) dominates the convergence speed, while if $\lfloor y\rceil \ll 1$, $\lfloor y\rceil^{2-p/q} \ll \lfloor y\rceil \ll \lfloor y\rceil^{p/q}$ implies the second nonlinear term in Equation (7) is the dominant one.

With the above preparation, we are now in a position to present two new nonlinear continuous protocols that solve the finite-time consensus problem and the settling time is bounded for any initial state,

$$u_i = \alpha\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{2-\frac{p}{q}} + \beta\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{p}{q}}, \tag{11}$$

$$u_i = \alpha \sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{2-\frac{p}{q}} + \beta \sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{\frac{p}{q}}. \tag{12}$$

It is worth mentioning in advance that the protocol (12) solves global finite-time *average* consensus problem, while the protocol (11) only achieves global finite-time consensus, which will be clear soon in the next section.

Remark 3: Provided that the interaction topology is time invariant, protocols (11) and (12) are both continuous functions with respect to state variables. If we set $p = q = 1$, the two protocols are degenerated into the typical linear consensus protocol presented by Ren and Beard (2005) which achieves asymptotical agreement. It follows from Remark **2** intuitively that protocols (11) and (12) have a faster convergence rate than the linear one (Ren & Beard, 2005) and the combined linear–nonlinear one (Cao et al., 2010).

5. Stability analysis

In this section, we present the finite-time consensus proofs for multi-agent system (1).

Theorem 5.1: *Suppose that the undirected graph $\mathcal{G}$ with A of system* (1) *is connected. Then, the distributed consensus protocol* (11) *achieves the global finite-time stability.*

Proof: Consider the following semi-positive definite function:

$$V(x(t)) = \frac{1}{2}x^T L_A x = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(x_j(t) - x_i(t)\right)^2.$$

Since $\mathcal{G}(A)$ is connected, zero is a simple eigenvalue of L_A which implies that $V(x(t)) = 0$ if and only if $x(t) \in \text{span}\{1_n\}$ (Ren et al., 2007). The symmetry of A results that

$$\frac{\partial V(x)}{\partial x_i} = -\sum_{j=1}^{n} a_{ij}(x_j - x_i).$$

Thus, the derivative of $V(x)$ versus time is

$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n}\frac{\partial V(x)}{\partial x_i}\dot{x}_i = -\alpha\sum_{i=1}^{n}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{3q-p}{q}} \\
&\quad - \beta\sum_{i=1}^{n}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{p+q}{q}} \\
&= -\alpha\sum_{i=1}^{n}\left(\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{3q-p}{2q}} \\
&\quad - \beta\sum_{i=1}^{n}\left(\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{p+q}{2q}} \\
&\leq -\alpha n^{\frac{p-q}{2q}}\left(\sum_{i=1}^{n}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{3q-p}{2q}} \\
&\quad - \beta\left(\sum_{i=1}^{n}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{p+q}{2q}},
\end{aligned} \tag{13}$$

where Lemmas 3.3 and 3.4 are inserted in view of $(3q - p)/2q \in (1, \infty)$ and $(p + q)/2q \in (0, 1]$. The semi-positive property of L_A ensures that there exists $Q \in \mathbb{R}^{n\times n}$ such that $L_A = Q^T Q$. By Lemma 3.1, for $V(x) \neq 0$, we have

$$\begin{aligned}
\frac{\sum_{i=1}^{n}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2}{V(x)} &= \frac{x^T L_A^T L_A x}{\frac{1}{2}x^T L_A x} \\
= \frac{2x^T Q^T Q Q^T Q x}{x^T Q^T Q x} &= \frac{2x^T Q^T L_A^T Q x}{x^T Q^T Q x} \geq 2\lambda_2(L_A),
\end{aligned} \tag{14}$$

where $\lambda_2(L_A) > 0$ by Lemma 3.1. With Equation (14) being inserted into Equation (13), we have

$$\begin{aligned}
\dot{V} &\leq -\alpha n^{\frac{p-q}{2q}}\left(2\lambda_2(L_A)V\right)^{\frac{3q-p}{2q}} - \beta\left(2\lambda_2(L_A)V\right)^{\frac{p+q}{2q}} \\
&= -\left[\alpha n^{\frac{p-q}{2q}}\left(2\lambda_2(L_A)V\right)^{\frac{q-p}{q}} + \beta\right]\left(2\lambda_2(L_A)V\right)^{\frac{p+q}{2q}}.
\end{aligned} \tag{15}$$

If $V \neq 0$, then let $y = \sqrt{2\lambda_2(L_A)V}$ be the solution of the following differential equation:

$$\dot{y}(t) = -\alpha n^{\frac{p-q}{2q}} \lambda_2(L_A) y(t)^{\frac{2q-p}{q}} - \beta\lambda_2(L_A) y(t)^{\frac{p}{q}}, \quad (16)$$

where $\dot{y} = \lambda_2(L_A)\dot{V}/\sqrt{2\lambda_2(L_A)V}$ is used. By Lemma 4.1 and Comparison Principle of differential equations (Khalil, 2005), we obtain

$$\lim_{t\to T} V(x) = 0,$$

where the settling time is given by

$$\begin{aligned} T &= \frac{qn^{\frac{q-p}{4q}}}{\sqrt{\alpha\beta}\lambda_2(L_A)(q-p)} \tan^{-1}\left(n^{\frac{p-q}{4q}}\sqrt{\frac{\alpha}{\beta}}V(x_0)\right) \\ &\le \frac{\pi q n^{\frac{q-p}{4q}}}{2\sqrt{\alpha\beta}\lambda_2(L_A)(q-p)}. \end{aligned} \quad (17)$$

Thus,

$$\lim_{t\to T} |x_j(t) - x_i(t)| = 0, \forall i, j \in \mathcal{I}_n.$$

□

Remark 4: The upper bound of the settling time T is only related to the design parameters of protocol (11), the algebraic connectivity of $\mathcal{G}(A)$ and the volume n of the multi-agent system.

Theorem 5.2: *Suppose that the undirected graph $\mathcal{G}$ with A of system* (1) *is connected. Then, the distributed consensus protocol* (12) *achieves the global finite-time average consensus.*

Proof: Since $a_{ij} = a_{ji}$ for $\forall i, j \in \mathcal{I}_n$ and $(\cdot)^{2-p/q}$ and $(\cdot)^{p/q}$ are both continuous odd functions with respect to $(\cdot)$, invoking Equation (12) we have

$$\begin{aligned} \sum_{i=1}^{n} \dot{x}_i(t) &= \sum_{i=1}^{n} u_i(t) = \alpha \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{2-\frac{p}{q}} \\ &\quad + \beta \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{\frac{p}{q}} = 0. \end{aligned}$$

Let $x^* = (1/n)\sum_{i=1}^{n} x_i(t)$ and $\delta_i(t) = x_i(t) - x^*$. It follows from $\sum_{i=1}^{n} \dot{x}_i(t) = 0$ that x^* is time invariant and, therefore, $\dot{\delta}_i(t) = \dot{x}_i(t)$. Then, the group disagreement vector (Olfati & Murray, 2004) can be written as $\delta(t) = [\delta_1(t), \delta_2(t), \ldots, \delta_n(t)]^T$. Consider the following candidate Lyapunov function:

$$V(\delta(t)) = \frac{1}{2}\sum_{i=1}^{n} \delta_i^2(t).$$

Differentiating $V(\delta(t))$ versus time yields

$$\begin{aligned} \dot{V}(\delta(t)) &= \sum_{i=1}^{n} \delta_i\dot{\delta}_i = \alpha\sum_{i=1}^{n}\delta_i\sum_{j=1}^{n} a_{ij}(\delta_j - \delta_i)^{2-\frac{p}{q}} \\ &\quad + \beta\sum_{i=1}^{n}\delta_i\sum_{j=1}^{n} a_{ij}(\delta_j - \delta_i)^{\frac{p}{q}} \\ &= \frac{1}{2}\alpha\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\delta_i - \delta_j)(\delta_j - \delta_i)^{2-\frac{p}{q}} \\ &\quad + \frac{1}{2}\beta\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\delta_i - \delta_j)(\delta_j - \delta_i)^{\frac{p}{q}} \\ &= -\frac{1}{2}\alpha\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\delta_j - \delta_i)^{\frac{3q-p}{q}} \\ &\quad - \frac{1}{2}\beta\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\delta_j - \delta_i)^{\frac{p+q}{q}} \\ &= -\frac{1}{2}\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}^{\frac{2q}{3q-p}}(\delta_j - \delta_i)^2\right]^{\frac{3q-p}{2q}} \\ &\quad - \frac{1}{2}\beta\sum_{i=1}^{n}\sum_{j=1}^{n}\left[a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2\right]^{\frac{p+q}{2q}} \\ &\le -\frac{1}{2}\alpha n^{\frac{p-q}{q}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{\frac{2q}{3q-p}}(\delta_j - \delta_i)^2\right]^{\frac{3q-p}{2q}} \\ &\quad - \frac{1}{2}\beta\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2\right]^{\frac{p+q}{2q}}, \end{aligned} \quad (18)$$

where Lemmas 3.3 and 3.4 are inserted. Let $G_A = \sum_{i,j=1}^{n} b_{ij}(\delta_j - \delta_i)^2$ and $G_B = \sum_{i,j=1}^{n} c_{ij}(\delta_j - \delta_i)^2$ with $b_{ij} \triangleq a_{ij}^{2q/(3q-p)}$ and $c_{ij} \triangleq a_{ij}^{2q/(p+q)}$, for $\forall i, j \in \mathcal{I}_n$, respectively. Let $B = [b_{ij}]_{n\times n}$ and $C = [c_{ij}]_{n\times n}$. Then, we have $G_B(\delta) = 2\delta^T L_B \delta$ and $G_C(\delta) = 2\delta^T L_C \delta$, where L_B and L_C are, respectively, the graph Laplacians of $\mathcal{G}(B)$ and $\mathcal{G}(C)$. It follows from the definitions of b_{ij} and c_{ij} that both $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are connected if $\mathcal{G}(A)$ is connected. Since $1_n^T\delta = 0$, Lemma 3.1 gives

$$\begin{cases} G_B \ge 2\lambda_2(L_B)\delta^T\delta \\ G_C \ge 2\lambda_2(L_C)\delta^T\delta, \end{cases} \quad (19)$$

where $\lambda_2(L_B) > 0$ and $\lambda_2(L_C) > 0$. Noting that $G_B/V \ge 4\lambda_2(L_B)$ and $G_C/V \ge 4\lambda_2(L_C)$ if $V(\delta) \neq 0$, we have

$$\begin{aligned} \dot{V}(\delta) &\le -\frac{1}{2}\alpha n^{\frac{p-q}{q}}\left[\frac{G_B(\delta)}{V}V\right]^{\frac{3q-p}{2q}} - \frac{1}{2}\beta\left[\frac{G_C(\delta)}{V}V\right]^{\frac{p+q}{2q}} \\ &= -\frac{1}{2}\alpha n^{\frac{p-q}{q}}\left[4\lambda_2(L_B)V\right]^{\frac{3q-p}{2q}} - \frac{1}{2}\beta\left[4\lambda_2(L_C)V\right]^{\frac{p+q}{2q}} \end{aligned}$$

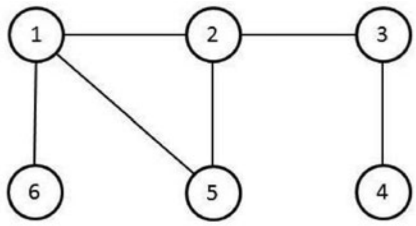

Figure 1. A connected graph.

$$\leq -\frac{1}{2}\alpha n^{\frac{p-q}{q}}(4\underline{\lambda}V)^{\frac{3q-p}{2q}} - \frac{1}{2}\beta(4\underline{\lambda}V)^{\frac{p+q}{2q}}$$
$$= -\frac{1}{2}\left[\alpha n^{\frac{p-q}{q}}(4\underline{\lambda}V)^{\frac{q-p}{q}} + \beta\right](4\underline{\lambda}V)^{\frac{p+q}{2q}}, \tag{20}$$

where $\underline{\lambda} = \min\{\lambda_2(L_B), \lambda_2(L_C)\} > 0$. Let $y = \sqrt{4\underline{\lambda}V}$ if $V \neq 0$ and let

$$\dot{y}(t) = -\alpha n^{\frac{p-q}{q}}\underline{\lambda}y(t)^{\frac{2q-p}{q}} - \beta\underline{\lambda}y(t)^{\frac{p}{q}}, \tag{21}$$

where $\dot{y} = 2\underline{\lambda}\dot{V}/\sqrt{4\underline{\lambda}V}$ is used. Similarly, by *Lemma* 4.1 and Comparison Principle we conclude

$$\lim_{t\to T}\delta(t) = 0,$$

where the settling time is given by

$$T = \frac{qn^{\frac{q-p}{2q}}}{\underline{\lambda}\sqrt{\alpha\beta}(q-p)}\tan^{-1}\left(n^{\frac{p-q}{2q}}\sqrt{\frac{\alpha}{\beta}}V(\delta_0)\right)$$
$$\leq \frac{\pi qn^{\frac{q-p}{2q}}}{2\underline{\lambda}\sqrt{\alpha\beta}(q-p)}. \tag{22}$$

Since $\delta(t) \to 0$ implies $x_i(t) \to x^*$ for $\forall i \in \mathcal{I}_n$, the protocol solves the global finite-time average consensus problem.

Remark 5: It can be found that the settling time is upper bounded by the design parameters, the volume n of a given agent team and the *minimum* algebraic connectivity of $\mathcal{G}(B)$ and $\mathcal{G}(C)$. As Xiao and Wang (2007) pointed out, different from Theorem 5.1, the Lyapunov function used in Theorem 5.2 does not depend on the network topology which makes it possible for stability analysis of system in networks with switching topology.

Remark 6: It is noted that the input constraint should be taken into account and a compromise on convergence rate is inevitable for large initial group disagreement, although arbitrarily fast convergence can be assigned by parameter selection.

6. Simulation example

Here we take a six-agent system ($n = 6$) in a network with an undirected topology (Figure 1) as an example. In the simulations non-zero $a_{ij} = 2$, $\alpha = \beta = 2$, $p = 7$ and $q = 9$ for each protocol. The algebraic connectivity of $\mathcal{G}(A)$ is 0.8262, while the ones corresponding to $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are 0.7709 and 0.9010, respectively. The estimated upper bounds of the settling time for protocols (11) and (12) are 4.7231 and 5.5917 s by Equations (17) and (22), respectively. Consider two initial conditions: (1) $x(0) = [-5, -2, 4, 6, 4, 5]^T$ and (2) $x(0) = [10, -20, -3, 9, 4, -30]^T$.

The numerical results given in Figures 2 and 3 show that the settling time of the two consensus protocols under different initial conditions are both about 3 s, which demonstrates the correctness of the estimated bounds in Theorems 5.1 and 5.2. For each initial condition, the average consensus

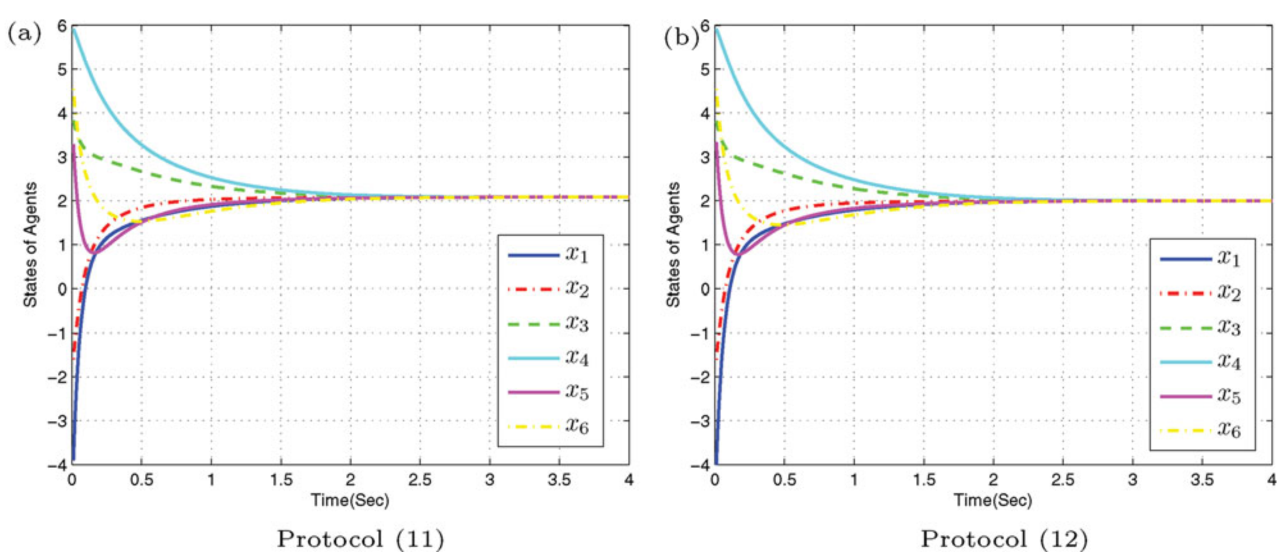

Figure 2. State trajectories of 6 agents under case (i). (a) Protocol (11). (b) Protocol (12).

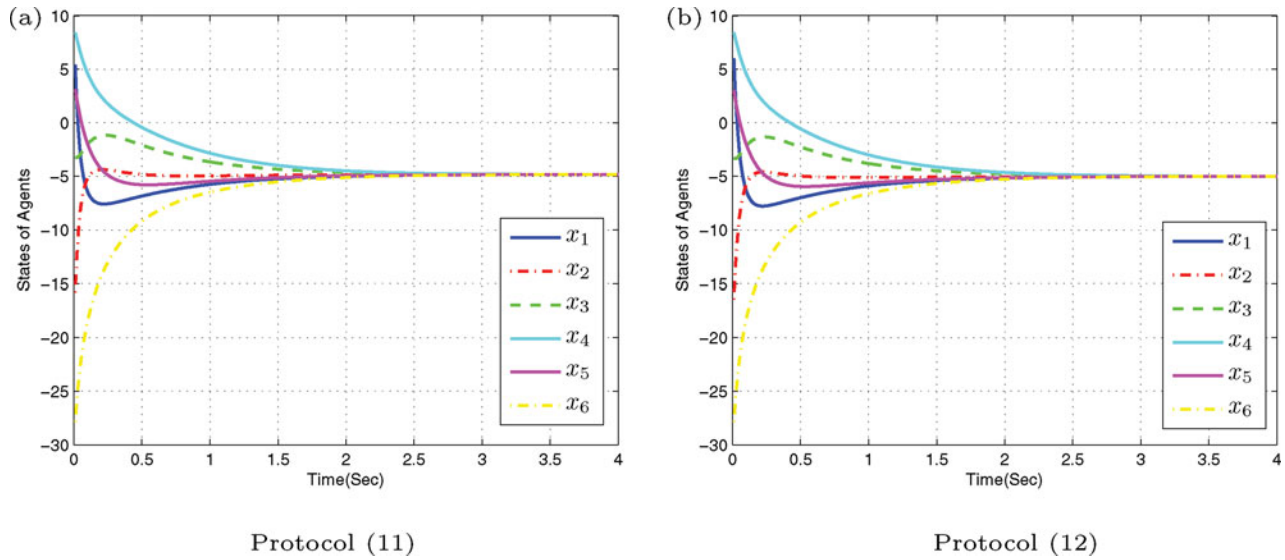

Figure 3. State trajectories of 6 agents under case (ii). (a) Protocol (11). (b) Protocol (12).

values can be computed via $x^* = \frac{1}{6}\sum_{i=1}^{6} x(0)$ as 2 and -5, respectively, which are coincident with those in Figures 2(b) and 3(b).

7. Conclusion

In this paper, a new class of nonlinear finite-time consensus algorithms was developed for a multi-agent system with first-order agent dynamics. With the proposed consensus algorithm, the upper bound of settling time for each protocol is closely related only to the design parameters, the algebraic connectivity of a given undirected interaction graph and the agent numbers of a given team. Thus, the consensus reaching time can be estimated a priori offline via appropriately selecting design parameters for arbitrary initial states. However, much remains to be done and the extension to be addressed includes the cases with time-variant network topologies and communication time delays.

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