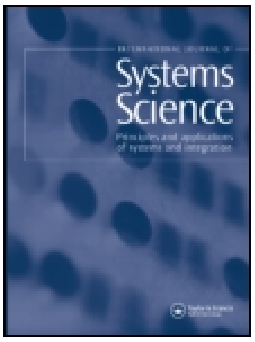

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Distributed robust finite-time nonlinear consensus protocols for multi-agent systems

Zongyu Zuo[a,b,c,*] and Lin Tie[c]

[a]*The Seventh Research Division, Beijing University of Aeronautics and Astronautics, Beijing, China;* [b]*Science and Technology on Aircraft Control Laboratory, Beijing University of Aeronautics and Astronautics, Beijing, China;* [c]*School of Automation Science and Electrical Engineering, Beijing University of Aeronautics and Astronautics, Beijing, China*

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This paper investigates the robust finite-time consensus problem of multi-agent systems in networks with undirected topology. Global nonlinear consensus protocols augmented with a variable structure are constructed with the aid of Lyapunov functions for each single-integrator agent dynamics in the presence of external disturbances. In particular, it is shown that the finite settling time of the proposed general framework for robust consensus design is upper bounded for any initial condition. This makes it possible for network consensus problems to design and estimate the convergence time offline for a multi-agent team with a given undirected information flow. Finally, simulation results are presented to demonstrate the performance and effectiveness of our finite-time protocols.

Keywords: distributed control; finite-time stability; multi-agents; network consensus; undirected graph; variable structure

1. Introduction

Network consensus problem, due to a broad application of multi-agent systems including formation control (Abdessameud & Tayebi, 2011), flocking (Toner & Tu, 1998), and attitude alignment (Ren, 2010), has emerged as a challenging topic in control area and has been intensively studied in the literature (Olfati & Murray, 2004; Ren, Beard, & Atkins, 2007; Yang, Wang, & Shi, 2013). In some applications involving multi-agent systems, groups of agents are required to agree upon certain quantities of interest, which is called *consensus* or *agreement problem*. To achieve agreement, distributed neighbour-based feedback laws (Jadbabaie, Lin, & Morse, 2003) (called *protocols*) are usually used to interconnect the agents in networks.

In the analysis of consensus problems, an important performance index for a proposed consensus protocol is convergence rate (Wang & Xiao, 2010). Olfati-Saber and Murray (2004) proposed a linear consensus protocol and demonstrated that the *algebraic connectivity* of a interaction graph, i.e., the second smallest eigenvalue of the graph Laplacian, qualified the convergence rate, which motivated some researchers to seek proper interaction topology with larger algebraic connectivity (Kim & Mesbahi, 2006). On the other hand, *finite-time consensus* problem has been promoted to achieve high-speed convergence. The benchmark work due to Bhat and Bernstein (2000) related the regularity properties of the Lyapunov functions to the settling time function in finite-time stability analysis for autonomous systems. Xiao, Wang, and Jia (2008), Wang and Xiao (2010), and Xiao, Wang, and Chen (2011) expanded the finite-time control idea (Bhat & Bernstein, 2000) to multi-agent systems with single-integrator dynamics. The works due to Wang and Hong (2008) and Zhao, Duan, Wen, and Zhang (2013) presented the finite-time consensus and tracking control, respectively, for second-order multi-agent systems via using homogeneity with dilation. Cao, Ren, and Meng (2010) focused on decentralised finite-time formation tracking of multi-autonomous vehicles with the introduction of decentralised finite-time sliding mode estimators, and they proposed a variable structure-based protocol to achieve finite-time consensus tracking without velocity measurement for first-order agent networks. Meng, Jia, and Du (2013) incorporated iterative learning control (ILC) actions into the terminal state/output in the finite-time consensus problem.

Unfortunately, the explicit expressions for the bound of the finite settling time depend upon initial states of the multi-agent networks in the aforementioned consensus protocols, which prohibit their practical applications since the knowledge of initial conditions is unavailable in advance. In addition, information variables of the agents in a multi-agent network team may not be measured precisely because they are often corrupted by various disturbances and noises. Robust consensus protocols with the desired H_∞ or H_2 performance have been investigated in numerous papers, such as Lin, Jia, and Li (2008), Li, Duan, and Chen (2011), and Liu and Jia (2012), to name a few. However, these robust consensus protocols may not realise

*Corresponding author. Email: zzybobby@buaa.edu.cn

the finite-time convergence. Motivated by this and taking into account the external disturbances, we present a general framework for constructing a new class of robust nonlinear finite-time consensus protocols for first-order multi-agent networks with undirected information flow in this paper. The main contributions of the proposed results are as follows: (1) the consensus design framework generalises the existing finite-time protocols in literature; (2) the explicit estimation of finite settling time of the proposed protocols is independent of initial states but merely relies on design parameters, algebraic connectivity, and group order; (3) a variable structure is incorporated into the framework to achieve the robustness to the external disturbance in each agent's dynamics. Thus, the convergence time can be assigned *a priori* for a fixed undirected interaction graph and a given group of agents in the presence of external disturbances.

This paper is organised as follows. Sections 2 and 3 present the preliminaries on graph theory notions and finite-time consensus definitions, respectively. Section 4 presents the main results of this paper, and Section 5 gives the stability analysis. In Section 6, an illustrative example is discussed. Finally, the paper is ended by concluding remarks in Section 7.

2. Graph theory notions

A weighted graph $\mathcal{G}(A) = \{\mathcal{V}, \mathcal{E}, A\}$ consists of a node set $\mathcal{V}(\mathcal{G}) = \{v_1, v_2, \ldots, v_N\}$, an edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacent matrix $A = [a_{ij}] \in \mathbb{R}^{N\times N}$. An edge (v_i, v_j) on $\mathcal{G}(A)$ denotes that the state of node v_i is available to node v_j, but not necessarily vice versa. In contrast, $(v_i, v_j) \in \mathcal{E}$ implies $(v_j, v_i) \in \mathcal{E}$ in a weighted *undirected graph*, i.e., the nodes v_i and v_j can sense each other. If $(v_i, v_j) \in \mathcal{E}$, then node v_i is called a *neighbour* of node v_j, or v_i and v_j are said to be *adjacent*. The set of all neighbours of v_j is denoted by $\mathcal{N}_j = \{i : (v_i, v_j) \in \mathcal{E}\}$. The weighted adjacent matrix A of a directed graph is defined such that $a_{ij} > 0$ for $(v_j, v_i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. For a weighted undirected graph A is defined similarly except that $a_{ij} = a_{ji}$, $\forall i \neq j$. A *path* on $\mathcal{G}$ from v_i to v_j is a sequence of distinct vertices $\{v_i, \ldots, v_j\}$ such that consecutive vertices are adjacent. An undirected graph is called *connected* if, for any two nodes, there exists a path.

3. Problem formulation

The information consensus problem involves finding a dynamic feedback law that enables a group of agents in a network to reach agreement upon certain qualities of interest with undirected or directed information flow (Wang & Hong, 2008). This paper will investigate finite-time consensus protocols based on undirected information flow for multi-agent systems.

Consider a group of N continuous-time agents with dynamics in the form of

$$\dot{x}_i(t) = u_i(t) + d_i(t), \quad i \in \mathcal{I}_N = \{1, 2, \ldots, N\} \tag{1}$$

where $x_i \in \mathbb{R}^M$ denotes the information state of agent $i \in \mathcal{I}_N$, $u_i \in \mathbb{R}^M$ the control input, called *protocol*, and $d_i \in \mathbb{R}^M$ the external disturbance and $M \in \mathbb{Z}_+$ is shared by all agents. Without loss of generality in the consensus design, we take $M = 1$ for simplicity in the sequel. The results to be developed later are valid for the vector state case by introducing Kronecker product, and also feasible for a class of feedback linearisable systems $\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i + d_i(t)$ with $g_i(x_i)$ invertible for all x_i.

Assumption 1: *The external disturbances are all uniformly bounded by a positive constant d, i.e.,* $|d_i(t)| \leq d, (i \in \mathcal{I}_N)$.

With a given protocol $u_i, i \in \mathcal{I}_N$, the closed-loop system in Equation (1) is said to reach or achieve *consensus*, if, for $\forall x_i(0)$ and $\forall i, j \in \mathcal{I}_N$, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$. It is said to achieve *finite-time consensus* if, for $\forall x_i(0)$ and $\forall i, j \in \mathcal{I}_N$, there is a *settling time* $T \in [0, \infty)$ such that

$$\begin{cases} \lim_{t\to T} |x_i(t) - x_j(t)| \to 0 \\ x_i(t) = x_j(t), \forall t \geq T \end{cases} \tag{2}$$

If the final agreement state satisfies

$$\lim_{t\to\infty} x_i(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(0), \forall i \in \mathcal{I}_N \tag{3}$$

then the system is said to achieve *average consensus*.

Stack the states of N agents as a vector $x = [x_1, x_2, \ldots, x_N]^T$, and let $1_N = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$ and $\text{span}(1_N) = \{\xi \in \mathbb{R}^N : \xi = \bar{x}1_N, \bar{x} \in \mathbb{R}\}$.

Lemma 3.1: *Let* $L(A) = [l_{ij}] \in \mathbb{R}^{N\times N}$ *denote the graph Laplacian of* $\mathcal{G}$*, which is defined by*

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{N} a_{ik} & i = j \\ -a_{ij} & i \neq j \end{cases}$$

$L(A)$ has the following properties:

(1) 0 *is an eigenvalue of* $L(A)$ *and* 1_N *is the associated eigenvector.*
(2) $x^T L(A)x = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_j - x_i)^2$, *and the semi-positive definiteness of* $L(A)$ *implies that all eigenvalues of* $L(A)$ *are real and non-negative.*
(3) *If* $\mathcal{G}(A)$ *is connected, the second smallest eigenvalue of* $L(A)$*, which is denoted by* $\lambda_2(L_A)$ *and called the algebraic connectivity of* $\mathcal{G}(A)$*, is larger than zero.*

(4) *The algebraic connectivity of* $\mathcal{G}(A)$ *is equal to* $\min_{x\neq 0, 1_N^T x=0}\frac{x^T L(A)x}{x^T x}$, *and therefore, if* $1_N^T x = 0$, *then* $x^T L(A)x \geq \lambda_2(L_A)x^T x$.

Since the finite-time consensus is closely related to the finite-time stability, the following lemma due to Bhat and Bernstein (2000)) is needed to establish one of our main results.

Lemma 3.2: *Consider the following system of differential equations:*

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{4}$$

where $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^N$, $f(x) : \mathbb{R}^N \to \mathbb{R}^N$ *is continuous on* $\mathbb{R}^N$, *and* $f(0) = 0$. *Suppose there exists a continuous positive definite function* $V(x) : \mathbb{R}^N \to \mathbb{R}$ *such that there exist real numbers* $c > 0$ *and* $\alpha \in (0, 1)$ *such that*

$$\dot{V}(x) + c(V(x))^\alpha \leq 0, \quad x \in \mathbb{R}^N \backslash \{0\}$$

Then, the origin is a globally finite-time stable equilibrium of Equation (4) *and the settling time is*

$$T(x_0) \leq \frac{1}{c(1-\alpha)} V(x_0)^{(1-\alpha)}$$

To streamline the technical proofs of the main results, another two lemmas are needed.

Lemma 3.3: *Let* $\xi_1, \xi_2, \ldots, \xi_N \geq 0$ *and* $0 < p \leq 1$. *Then,*

$$\sum_{i=1}^{N} \xi_i^p \geq \left(\sum_{i=1}^{N} \xi_i\right)^p \tag{5}$$

Proof: Inequality (5) holds trivially for $\xi_1 = \xi_2 = \cdots = \xi_N = 0$, and otherwise using the fact $x^p \geq x$ for $\forall x \in (0, 1]$ and $0 < p \leq 1$, we have

$$\frac{\sum_{i=1}^N \xi_i^p}{(\sum_{i=1}^N \xi_i)^p} = \sum_{i=1}^{N}\left(\frac{\xi_i}{\sum_{i=1}^N \xi_i}\right)^p \geq \sum_{i=1}^{N} \frac{\xi_i}{\sum_{i=1}^N \xi_i} = 1$$

which proves Inequality (5). □

Lemma 3.4: *Let* $\xi_1, \xi_2, \ldots, \xi_N \geq 0$ *and* $p > 1$. *Then,*

$$\sum_{i=1}^{N} \xi_i^p \geq N^{1-p}\left(\sum_{i=1}^{N} \xi_i\right)^p \tag{6}$$

Proof: Based on *Hölder inequality*

$$\sum_{i=1}^{N} \xi_i \eta_i \leq \left(\sum_{i=1}^{N} \xi_i^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{N} \eta_j^q\right)^{\frac{1}{q}}$$

for $\forall \xi_i, \eta_i > 0$, $i \in \mathcal{I}_N$ and $\forall p, q \in (1, \infty)$ satisfying $1/p + 1/q = 1$, Equation (6) is a straightforward result by letting $[\eta_1, \eta_2, \ldots, \eta_N]^T = 1_N$. □

4. Main results

In this section, we shall develop a general framework of the finite-time consensus protocols for a group of multi-agent systems in Equation (1). Before that and to clarify the design idea, we first present our motivation and a retrofit lemma is given.

Lemma 3.2 gives sufficient conditions for the finite-time convergence. However, the upper-bound estimate of the settling time T depends on the system initial states, which restricts the practical applications. And larger initial disagreement leads to larger settling time. Motivated by this analysis, if the settling time has *a priori* estimate that relies only on design parameters, then the finite settling time is bounded for any initial state.

Lemma 4.1: *Consider a scalar system*

$$\dot{y} = -\alpha y^{\frac{m}{n}} - \beta y^{\frac{p}{q}}, \quad y(0) = y_0 \tag{7}$$

where m, n, p, q are all positive ***odd*** *integers satisfying m > n and p < q,* $\alpha > 0$ *and* $\beta < 0$. *Then, the equilibrium of Equation* (7) *is globally finite-time stable and the settling time is bounded by*

$$T < \frac{1}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p} \tag{8}$$

Proof: Let $V(y) = y^2 \geq 0$. Differentiating $V(y)$ along system (7) yields

$$\begin{aligned}\dot{V}(y) &= 2y\left(-\alpha y^{\frac{m}{n}} - \beta y^{\frac{p}{q}}\right) \\ &= -2\alpha(y^2)^{\frac{m+n}{2n}} - 2\beta(y^2)^{\frac{p+q}{2q}} \\ &= -2\left(\alpha V^{\frac{m+n}{2n}-\frac{p+q}{2q}} + \beta\right)V^{\frac{p+q}{2q}}\end{aligned} \tag{9}$$

The fact $\alpha V^{(m+n)/2n-(p+q)/2q} > 0$ implies that $\dot{V}(y) \leq -2\beta V^{(p+q)/2q}$. In view of $0 < \frac{p+q}{2q} < 1$, the system in Equation (7) is globally finite-time stable due to Lemma 3.2. Further, since $V(y) = 0$ in Equation (9) is a trivial case, assuming $V(y) \neq 0$, we have

$$\begin{aligned}&\frac{1}{V^{\frac{p+q}{2q}}}\frac{\mathrm{d}V}{\mathrm{d}t} = -2\left(\alpha V^{\frac{m+n}{2n}-\frac{p+q}{q}} + \beta\right) \\ &\Rightarrow \frac{q}{q-p}\frac{\mathrm{d}V^{\frac{q-p}{2q}}}{\mathrm{d}t} = -\left(\alpha V^{\frac{m+n}{2n}-\frac{p+q}{q}} + \beta\right)\end{aligned} \tag{10}$$

Let $z = V^{(q-p)/2q}$. Equation (10) can be written as

$$\frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z = -\frac{q-p}{q}\mathrm{d}t$$

where $\varepsilon \triangleq \frac{q(m-n)}{n(q-p)}$. Let $\varphi(z) = \int_0^z \frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z$. Integrating both sides of the preceding equation yields

$$\varphi(z(t)) = \varphi(z(0)) - \frac{q-p}{q}t$$

Since the function $\varphi(z)$ is monotonically increasing, $\varphi(z) = 0$ if and only if $z = 0$, which implies $V = 0$. Thus, we have

$$\lim_{t\to T(y_0)} V = 0$$

where $T(y_0)$ denotes the settling time function given by

$$T(y_0) = \frac{q}{q-p}\varphi(z(0)) = \frac{q}{q-p}\varphi(y^{(q-p)/q}(0))$$

Toward this end, it can be verified that $T(y_0)$ is bounded by

$$\begin{aligned}\lim_{y_0\to\infty} T(y_0) &= \lim_{z_0\to\infty}\frac{q}{q-p}\varphi(z(0)) = \frac{q}{q-p}\varphi(\infty)\\ &= \frac{q}{q-p}\left(\int_0^1 \frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z + \int_1^\infty \frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z\right)\\ &< \frac{q}{q-p}\left(\int_0^1 \frac{1}{\beta}\mathrm{d}z + \int_1^\infty \frac{1}{\alpha z^{1+\varepsilon}}\mathrm{d}z\right)\\ &= \frac{q}{q-p}\left(\frac{1}{\beta}+\frac{1}{\alpha\varepsilon}\right) = \frac{1}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}\end{aligned}$$

Using the fact that $V(y(t)) = 0$ implies $y(t) = 0$ completes the proof. □

Remark 1: The settling time function is upper bounded by *a priori* value that is not dependent on the system initial state y_0 but only on the design parameters, i.e., m, n, p, q, α, and β. This implies that the convergence time can be offline designed or estimated.

Remark 2: It is worth noting from Equation (7) that if $|y| \gg 1$, then $|y|^{m/n} \gg |y| \gg |y|^{p/q}$ implies that the first nonlinear term in Equation (7) dominates the convergence speed, while if $|y| \ll 1$, $|y|^{m/n} \ll |y| \ll |y|^{p/q}$ implies that the second nonlinear term in Equation (7) is the dominant one.

Remark 3: It is worthwhile to note that estimation (8) is much less conservative for larger $\varepsilon \gg 1$. Besides, for $0 < \varepsilon \le 1$, a less conservative upper-bound estimation for the settling time, instead of Inequality (8), can be obtained as

$$\begin{aligned}&\lim_{y_0\to\infty} T(y_0)\\ &= \frac{q}{q-p}\left(\int_0^1 \frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z + \int_1^\infty \frac{1}{\alpha z^{1+\varepsilon}+\beta}\mathrm{d}z\right)\\ &< \frac{q}{q-p}\left(\int_0^1 \frac{1}{\alpha z^2+\beta}\mathrm{d}z + \int_1^\infty \frac{1}{\alpha z^{1+\varepsilon}}\mathrm{d}z\right)\\ &= \frac{q}{q-p}\left(\frac{1}{\sqrt{\alpha\beta}}\tan^{-1}\sqrt{\frac{\alpha}{\beta}} + \frac{1}{\alpha\varepsilon}\right)\end{aligned} \tag{11}$$

With the above preparation, we are now in a position to present two new robust nonlinear protocols that solve the finite-time consensus problem in the presence of external disturbances and the settling time are bounded for any initial states:

$$\begin{aligned}u_i &= \alpha\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{m}{n}} + \beta\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{p}{q}}\\ &\quad + \gamma\,\mathrm{sgn}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)\end{aligned} \tag{12}$$

$$\begin{aligned}u_i &= \alpha\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{\frac{m}{n}} + \beta\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)^{\frac{p}{q}}\\ &\quad + \gamma\,\mathrm{sgn}\left(\sum_{j\in\mathcal{N}_j}(x_j - x_i)\right)\end{aligned} \tag{13}$$

where $\gamma \ge d$ and sgn(·) denotes the signum function.

Remark 4: If set $m = n = p = q = 1$ and $\gamma = 0$, the two protocols are degenerated into the typical linear consensus protocol (Ren & Beard, 2005), which achieves asymptotical agreement. Setting $\alpha = \gamma = 0$ yields two protocols in Equations (12) and (13) equivalent to the ones obtained in Xiao et al. (2008)) and Wang and Xiao (2010)), respectively. Letting $m = n = 1$ and $\beta = 1$ in Equation (12) obtains the protocol proposed by Cao et al. (2010)). $\alpha = \beta = 0$ in Equation (12) yields the discontinuous protocols studied by Cortés (2006)). It follows from Remark 2 intuitively that protocols (12) and (13) have a faster convergence rate than the existing ones mentioned above.

5. Stability analysis

In this section, we present the finite-time consensus proofs for the multi-agent system (1).

Theorem 5.1: *Suppose that the undirected graph $\mathcal{G}$ with A of system* (1) *is connected. Then, the distributed consensus protocol* (12) *achieves the global finite-time consensus.*

Proof: Consider the following semi-positive definite function:

$$V\left(x(t)\right) = \frac{1}{2}x^T L_A x = \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(x_j(t) - x_i(t)\right)^2 \quad (14)$$

Since $\mathcal{G}(A)$ is connected, zero is a simple eigenvalue of L_A which implies that $V(x(t)) = 0$ if and only if $x(t) \in \text{span}\{1_N\}$ (Ren et al., 2007). The symmetry of A gives that

$$\frac{\partial V(x)}{\partial x_i} = -\sum_{j=1}^{N} a_{ij}(x_j - x_i)$$

Thus, the derivative of $V(x)$ versus time is

$$\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{N}\frac{\partial V(x)}{\partial x_i}\dot{x}_i = -\alpha\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{m+n}{n}} \\
&\quad - \beta\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^{\frac{p+q}{q}} \\
&\quad - \gamma\sum_{i=1}^{N}\left|\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right| \\
&\quad - \sum_{i=1}^{N} d_i\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right) \\
&\le -\alpha\sum_{i=1}^{N}\left(\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{m+n}{2n}} \\
&\quad - \beta\sum_{i=1}^{N}\left(\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{p+q}{2q}} \\
&\quad - (\gamma - d)\sum_{i=1}^{N}\left|\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right| \\
&\le -\alpha N^{\frac{n-m}{2n}}\left(\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{m+n}{2n}} \\
&\quad - \beta\left(\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2\right)^{\frac{p+q}{2q}}
\end{aligned} \quad (15)$$

where Lemmas 3.3 and 3.4 are inserted in view of $(m + n)/2n \in (1, \infty)$ and $(p+q)/2q \in (0, 1)$, respectively. The semi-positive property of L_A ensures that there exists a semi-positive matrix $Q \in R^{N*N}$ such that $L_A = Q^T Q$. By Lemma 3.1, for $V(x) \neq 0$, we have

$$\begin{aligned}
\frac{\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_j} a_{ij}(x_j - x_i)\right)^2}{V(x)} &= \frac{x^T L_A^T L_A x}{\frac{1}{2}x^T L_A x} \\
&= \frac{2x^T Q^T Q Q^T Q x}{x^T Q^T Q x} >= 2\lambda_2(QQ^T) = 2\lambda_2(L_A)
\end{aligned} \quad (16)$$

where $\lambda_2(L_A) > 0$ ensured by Lemma 3.1. With Equation (16) inserted into Equation (15), we have

$$\begin{aligned}
\dot{V} &\le -\alpha N^{\frac{n-m}{2n}}\left(2\lambda_2(L_A)V\right)^{\frac{m+n}{2n}} - \beta\left(2\lambda_2(L_A)V\right)^{\frac{p+q}{2q}} \\
&= -\left[\alpha N^{\frac{n-m}{2n}}\left(2\lambda_2(L_A)V\right)^{\frac{m+n}{2n}-\frac{p+q}{2q}} + \beta\right]\left(2\lambda_2(L_A)V\right)^{\frac{p+q}{2q}}
\end{aligned} \quad (17)$$

If $V \neq 0$, then let $y = \sqrt{2\lambda_2(L_A)V}$ be the solution of the following differential equation:

$$\dot{y}(t) = -\alpha N^{\frac{n-m}{2n}}\lambda_2(L_A)y(t)^{\frac{m}{n}} - \beta\lambda_2(L_A)y(t)^{\frac{p}{q}} \quad (18)$$

where $\dot{y} = \lambda_2(L_A)\dot{V}/\sqrt{2\lambda_2(L_A)V}$ is used. By Lemma 4.1 and *comparison principle* of differential equations (Khalil, 2005), we obtain

$$\lim_{t\to T} V(x) = 0$$

where the settling time is, from Inequality (8), bounded by

$$T < \frac{1}{\lambda_2(L_A)}\left(\frac{N^{\frac{m-n}{2n}}}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}\right)$$

Thus,

$$\lim_{t\to T}|x_j(t) - x_i(t)| = 0, \quad \forall i, j \in \mathcal{I}_n$$

□

Remark 5: The upper bound of the settling time T is only related to the design parameters of protocol (12), the algebraic connectivity of $\mathcal{G}(A)$, and the order N of the multi-agent system.

Remark 6: Invoking Remark 3, we can obtain a less conservative bound for T:

$$T < \frac{q}{(q-p)\lambda_2(L_A)}\left(\frac{N^{\frac{m-n}{4n}}}{\sqrt{\alpha\beta}}\tan^{-1}\sqrt{\frac{\alpha N^{\frac{n-m}{2n}}}{\beta}} + \frac{1}{\alpha\varepsilon}\right) \quad (19)$$

where $0 < \varepsilon = \frac{q(m-n)}{n(q-p)} \le 1$.

Theorem 5.2: *Suppose that the undirected graph $\mathcal{G}$ with A of system* (1) *is connected. Then, the distributed consensus protocol* (13) *achieves the global finite-time average consensus.*

Proof: Since $a_{ij} = a_{ji}$ for $\forall i, j \in \mathcal{I}_N$ and $(\cdot)^{m/n}$, $(\cdot)^{p/q}$ and signum function sgn($\cdot$) are all odd functions with respect to ($\cdot$), it can be verified that $\sum_{i=1}^{N} \dot{x}_i(t) = 0$. Let $x^* = (1/N)\sum_{i=1}^{N} x_i(t)$ and $\delta_i(t) = x_i(t) - x^*$. It follows from $\sum_{i=1}^{N} \dot{x}_i(t) = 0$ that x^* is time invariant and, therefore, $\dot{\delta}_i(t) = \dot{x}_i(t)$. Then, the group disagreement vector (Olfati and Murray, 2004) can be written as $\delta(t) = [\delta_1(t), \delta_2(t), \ldots, \delta_N(t)]^T$. Consider the following candidate Lyapunov function:

$$V(\delta(t)) = \frac{1}{2}\sum_{i=1}^{N} \delta_i^2(t)$$

Noting from the definition of x^* that $\sum_{j=1}^{N}(x_i - x_j) = N(x_i - x^*)$, the protocol (13) can be rewritten as

$$u_i = \alpha \sum_{j\in\mathcal{N}_j} a_{ij}(\delta_j - \delta_i)^{\frac{m}{n}} + \beta \sum_{j\in\mathcal{N}_j} a_{ij}(\delta_j - \delta_i)^{\frac{p}{q}} - \gamma\,\mathrm{sgn}(\delta_i) \tag{20}$$

Differentiating $V(\delta(t))$ versus time yields

$$\begin{aligned}
\dot{V}(\delta(t)) &= \sum_{i=1}^{N} \delta_i \dot{\delta} = \alpha \sum_{i=1}^{n} \delta_i \sum_{j=1}^{N} a_{ij}(\delta_j - \delta_i)^{\frac{m}{n}} \\
&\quad + \beta \sum_{i=1}^{N} \delta_i \sum_{j=1}^{N} a_{ij}(\delta_j - \delta_i)^{\frac{p}{q}} \\
&\quad - \gamma \sum_{i=1}^{N} |\delta_i| + \sum_{i=1}^{N} \delta_i d_i \\
&\le \frac{1}{2}\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\delta_i - \delta_j)(\delta_j - \delta_i)^{\frac{m}{n}} \\
&\quad + \frac{1}{2}\beta \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\delta_i - \delta_j)(\delta_j - \delta_i)^{\frac{p}{q}} \\
&\quad - (\gamma - d)\sum_{i=1}^{N} |\delta_i| \\
&\le -\frac{1}{2}\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\delta_j - \delta_i)^{\frac{m+n}{n}} \\
&\quad - \frac{1}{2}\beta \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\delta_j - \delta_i)^{\frac{p+q}{q}} \\
&= -\frac{1}{2}\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} \left[a_{ij}^{\frac{2n}{m+n}}(\delta_j - \delta_i)^2\right]^{\frac{m+n}{2n}} \\
&\quad - \frac{1}{2}\beta \sum_{i=1}^{N}\sum_{j=1}^{N} \left[a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2\right]^{\frac{p+q}{2q}} \\
&\le -\frac{1}{2}\alpha N^{\frac{n-m}{n}} \left[\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2n}{m+n}}(\delta_j - \delta_i)^2\right]^{\frac{m+n}{2n}} \\
&\quad - \frac{1}{2}\beta \left[\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^{\frac{2q}{p+q}}(\delta_j - \delta_i)^2\right]^{\frac{p+q}{2q}}
\end{aligned} \tag{21}$$

where Lemmas 3.3 and 3.4 are inserted. Let $G_B = \sum_{i,j=1}^{N} b_{ij}(\delta_j - \delta_i)^2$ and $G_C = \sum_{i,j=1}^{N} c_{ij}(\delta_j - \delta_i)^2$ with $b_{ij} \triangleq a_{ij}^{2n/(m+n)}$ and $c_{ij} \triangleq a_{ij}^{2q/(p+q)}$, for $\forall i, j \in \mathcal{I}_N$, respectively. Let $B = [b_{ij}]_{N\times N}$ and $C = [c_{ij}]_{N\times N}$. Then, we have $G_B(\delta) = 2\delta^T L_B \delta$ and $G_C(\delta) = 2\delta^T L_C \delta$, where L_B and L_C are, respectively, the graph Laplacians of $\mathcal{G}(B)$ and $\mathcal{G}(C)$. It follows from the definitions of b_{ij} and c_{ij} that both $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are connected if $\mathcal{G}(A)$ is connected. Since $1_N^T\delta = 0$, Lemma 3.1 gives that

$$\begin{cases} G_B \ge 2\lambda_2(L_B)\delta^T\delta \\ G_C \ge 2\lambda_2(L_C)\delta^T\delta \end{cases} \tag{22}$$

where $\lambda_2(L_B) > 0$ and $\lambda_2(L_C) > 0$. Noting that $G_B/V \ge 4\lambda_2(L_B)$ and $G_C/V \ge 4\lambda_2(L_C)$ if $V(\delta) \ne 0$, we have

$$\begin{aligned}
\dot{V}(\delta) &\le -\frac{1}{2}\alpha N^{\frac{n-m}{n}} \left[\frac{G_B(\delta)}{V}V\right]^{\frac{m+n}{2n}} - \frac{1}{2}\beta \left[\frac{G_C(\delta)}{V}V\right]^{\frac{p+q}{2q}} \\
&= -\frac{1}{2}\alpha N^{\frac{n-m}{n}} [4\lambda_2(L_B)V]^{\frac{m+n}{2n}} - \frac{1}{2}\beta [4\lambda_2(L_C)V]^{\frac{p+q}{2q}} \\
&\le -\frac{1}{2}\alpha N^{\frac{n-m}{n}} (4\underline{\lambda}V)^{\frac{m+n}{2n}} - \frac{1}{2}\beta (4\underline{\lambda}V)^{\frac{p+q}{2q}} \\
&= -\frac{1}{2}\left[\alpha N^{\frac{n-m}{n}} (4\underline{\lambda}V)^{\frac{m+n}{2n} - \frac{p+q}{2q}} + \beta\right](4\underline{\lambda}V)^{\frac{p+q}{2q}}
\end{aligned} \tag{23}$$

where $\underline{\lambda} = \min\{\lambda_2(L_B), \lambda_2(L_C)\} > 0$. Let $y = \sqrt{4\underline{\lambda}V}$ if $V \ne 0$, and let

$$\dot{y}(t) = -\alpha N^{\frac{n-m}{n}} \underline{\lambda} y(t)^{\frac{m}{n}} - \beta\underline{\lambda} y(t)^{\frac{p}{q}} \tag{24}$$

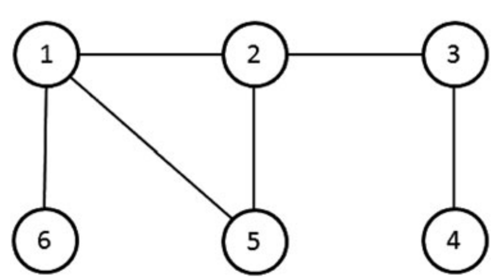
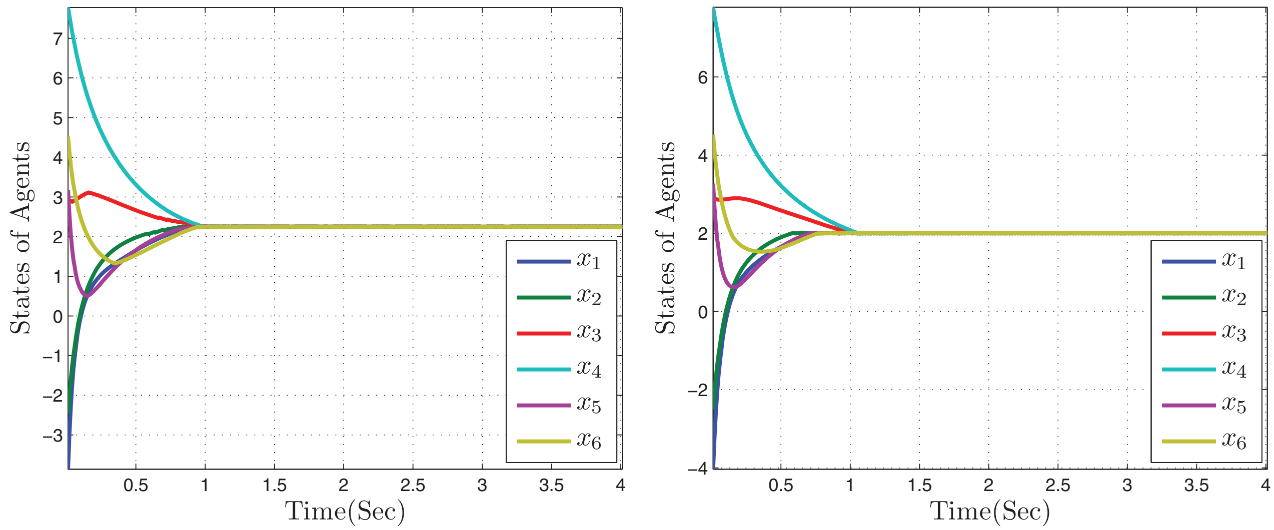
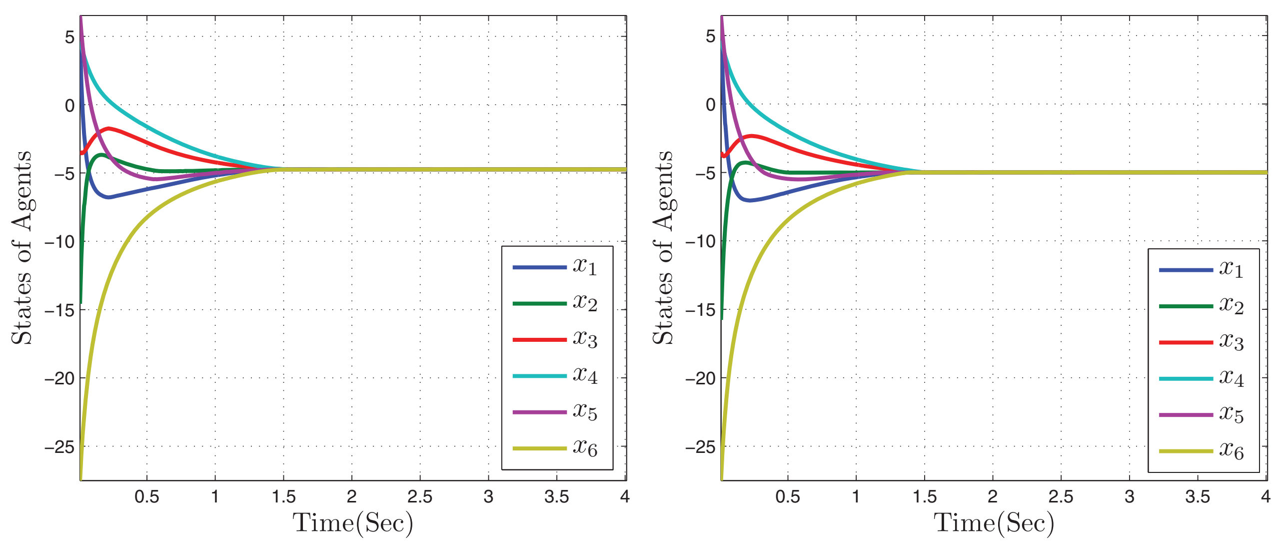


Figure 1. A connected graph.



(a) Protocol (12) (b) Protocol (13)

Figure 2. State trajectories of six agents under scenario (a).



(a) Protocol (12) (b) Protocol (13)

Figure 3. State trajectories of six agents under scenario (b).

where $\dot{y} = 2\underline{\lambda}\dot{V}/\sqrt{4\underline{\lambda}V}$ is used. Similarly, by Lemma 4.1 and *comparison principle*, we conclude that

$$\lim_{t\to T} \delta(t) = 0$$

where the settling time is, from Inequality (8), bounded by

$$T < \frac{1}{\underline{\lambda}}\left(\frac{N^{\frac{m-n}{n}}}{\alpha}\frac{n}{m-n} + \frac{1}{\beta}\frac{q}{q-p}\right)$$

Since $\delta(t) \to 0$ implies $x_i(t) \to x^*$ for $\forall i \in \mathcal{I}_n$, the protocol solves the global finite-time average consensus problem. □

Remark 7: Similarly, Remark 3 presents a less conservative bound for T:

$$T < \frac{q}{(q-p)\underline{\lambda}}\left(\frac{N^{\frac{m-n}{2n}}}{\sqrt{\alpha\beta}}\tan^{-1}\sqrt{\frac{\alpha N^{\frac{n-m}{n}}}{\beta}} + \frac{1}{\alpha\varepsilon}\right) \tag{25}$$

where $0 < \varepsilon = \frac{q(m-n)}{n(q-p)} \le 1$.

Remark 8: It can be found that the settling time is upper bounded by the design parameters, the group order N, and the *smaller* algebraic connectivity of $\mathcal{G}(B)$ and $\mathcal{G}(C)$. As Xiao and Wang (2008)) pointed out, different from

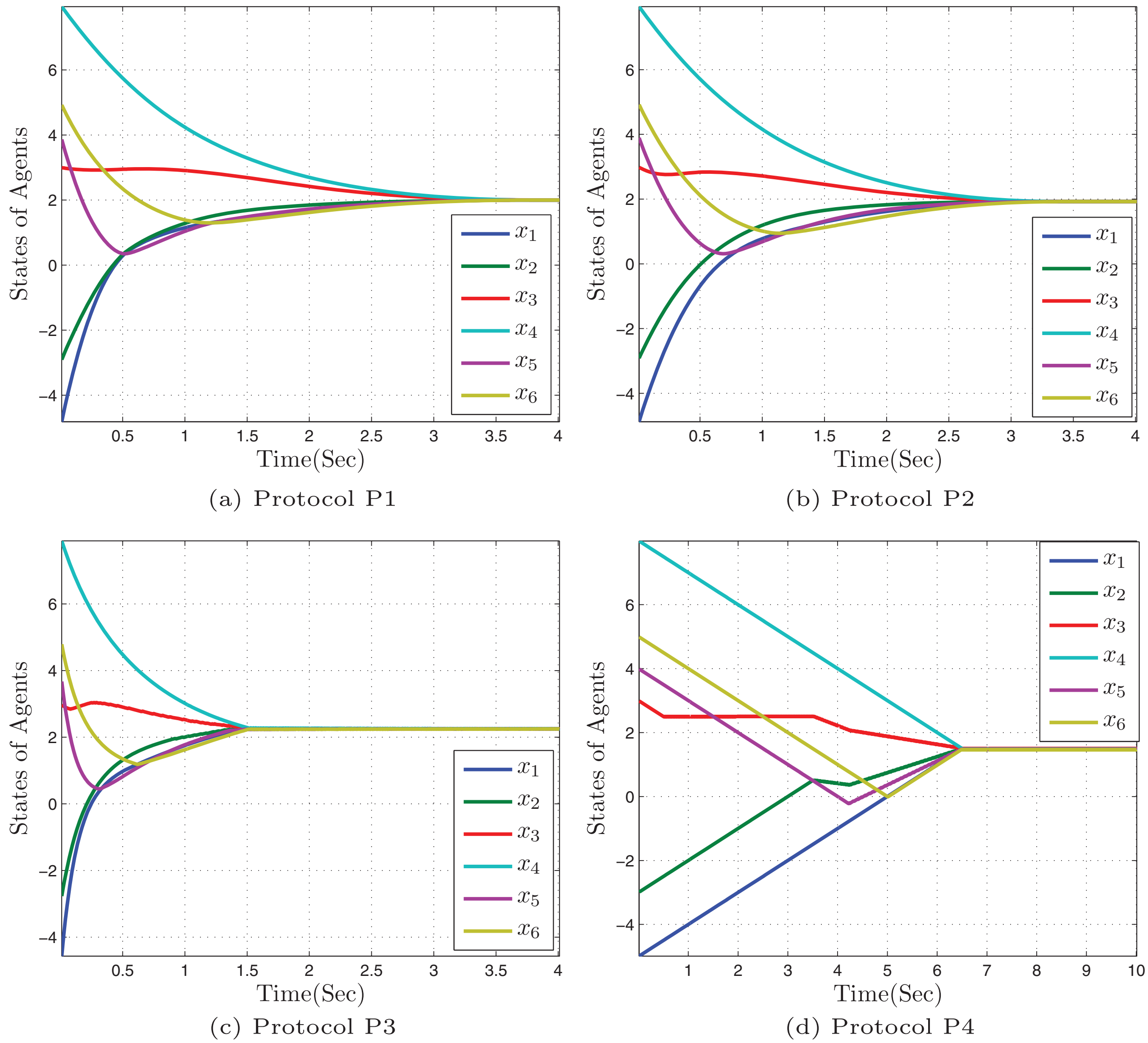


Figure 4. State trajectories of six agents under scenario (a).

Theorem 5.1, the Lyapunov function used in Theorem 5.2 does not depend on the network topology which makes it possible to analyse the stability of system in networks with switching topology.

Remark 9: It is worth mentioning that the selection of design parameter should reach a compromise between convergence rate and energy consumption. This means that given a desired settling time, large group disagreement requires large energy to reach consensus.

6. Simulation example

Here we take a six-agent system ($N = 6$) in a network with an undirected topology (see Figure 1) as an example. In the simulation studies, non-zero weights $a_{ij} = 2$ and design parameters $\alpha = \beta = 2, m = 9, n = 7, p = 3,$ and $q = 5$ are set for each protocol. The external disturbances injected into each agent dynamics are $d_1 = d_3 = d_5 = \sin(0.1t)$ and $d_2 = d_4 = d_6 = \cos(0.2t)$. The algebraic connectivity of $\mathcal{G}(A)$ is 0.8262, while the algebraic connectivity corresponding to $\mathcal{G}(B)$ and $\mathcal{G}(C)$ are 0.7709 and 0.9010, respectively. The estimated upper bounds of the settling time for protocols (12) and (13) are 4.2490 s and 5.4091 s, respectively. Since $\varepsilon = 5/7 < 1$, less conservative bounds for T can be calculated as 3.3589 s and 3.6499 s, respectively, using Equations (19) and (25). Consider two initial scenarios: (a) $x_a(0) = [-5, -3, 3, 8, 4, 5]^T$ and (b) $x_b(0) = [10, -20, -3, 5, 2, -24]^T$.

The numerical results given in Figures 2 and 3 show that the settling time of the two consensus protocols under different initial conditions are about 1 s and 1.5 s, respectively, which demonstrate the correctness of the estimated upper bounds in Theorems 5.1 and 5.2. From the numerical results, the steady-state values of the closed-loop system in Equation (1) via protocol (13) are, respectively, $x_a^* = \frac{1}{6}\sum_1^6 x_{ai}(0) = 2$ in Figure 2(b) and $x_b^* = \frac{1}{6}\sum_1^6 x_{bi}(0) = -5$ in Figure 3(b), which demonstrates that unlike the protocol (12), the protocol (13) achieves average consensus.

For comparison, we consider four existing protocols in literature: (P1) $u_i = \beta\sum_{j\in\mathcal{N}_i} a_{ij}\mathrm{sgn}(x_j - x_i)|x_j - x_i|^{p/q}$ (Wang & Xiao, 2010); (P2) $u_i = \beta\mathrm{sgn}[\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)]|\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)|^{p/q}$ (Xiao et al., 2008); (P3) $u_i = \alpha\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i) + \gamma\mathrm{sgn}[\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)]$ (Cao et al., 2010); and (P4) $u_i = \gamma\mathrm{sgn}[\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)]$ (Cortés , 2006), where the design parameters α, β, γ, p, and q are not re-tuned. It can be seen from Figure 4 that the settling time of the closed systems under these four protocols are longer than 1 s, which demonstrates as claimed in Remark 4 that our protocols proposed in this paper have faster convergence rate (see Figure 1). It is worth noting that the estimation of the finite settling time obtained for (P1)–(P4) in the aforementioned papers depends on initial states of the multi-agent networks, while the estimation for Equations (12) and (13) does not.

7. Conclusion

In this paper, a general framework for robust nonlinear finite-time consensus protocols was developed for multi-agent systems with first-order dynamics in the presence of external disturbances. With the proposed consensus protocols, the bound for finite settling time of each protocol is independent of initial conditions of the agents in a given group. Thus, the consensus reaching time can be assigned in advance via appropriately choosing design parameters for a given undirected communication topology and arbitrary initial states. However, much remains to be done and the extension to be addressed includes the cases with time-variant network topologies, communication time delays, and higher order agent dynamics.

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Notes on contributors

Zongyu Zuo received his BS degree in automatic control from Central South University, Hunan, China, in 2005, and PhD degree in control theory and applications from Beihang University (formerly Beijing University of Aeronautics and Astronautics), Beijing, China, in 2011. He is currently an assistant professor at the School of Automation Science and Electrical Engineering, Beihang University. His research interests are in the fields of nonlinear system control, control of unmanned aerial vehicles, and coordination of multi-agent system.

Lin Tie received his BS and PhD degrees from Beihang University (Beijing University of Aeronautics and Astronautics), Beijing, China, in 2005 and 2011, respectively. He worked as a postdoctoral fellow at Beihang University from 2011 to 2013. Currently, he is an assistant professor at the School of Automation Science and Electrical Engineering, Beihang University. His research interests include nonlinear control systems, discrete-time systems, and matrix analysis.

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