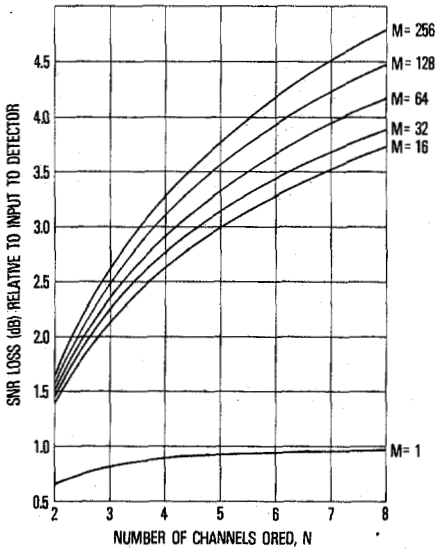

Fig. 8. SNR loss due to ORing and quantization for $P_D = 0.5$ and $P_F = 10^{-4}$.

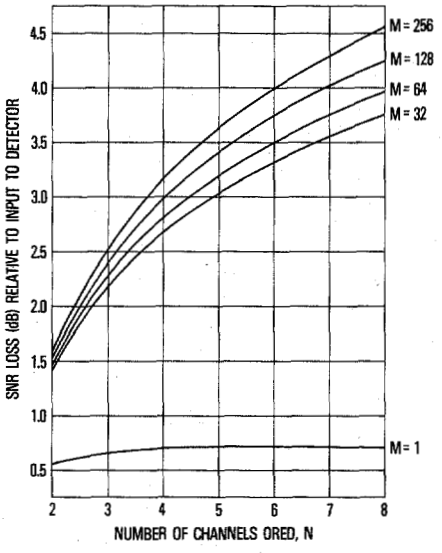

Fig. 9. SNR loss due to ORing and quantization for $P_D = 0.5$ and $P_F = 10^{-6}$.

$$P_F = \int_T^\infty p_0(z)\,dz = \Phi\left(\frac{\mu_{Z0} - T}{\sigma_{Z0}}\right) \tag{14}$$

where

P_D = detection probability

P_F = false alarm probability

T = threshold shown in Fig. 3

$p_0(z)$ and $p_1(z)$ = probability densities defined by (13).

Consider the case of $T = \mu_{Z1}$. For this case,

$$P_D = \Phi(0) = 0.5$$

$$P_F = \Phi\left(\frac{\mu_{Z0} - \mu_{Z1}}{\sigma_{Z0}}\right) = \Phi(-d_{0Z}). \tag{15}$$

Solving (15) for the output deflection coefficient d_{0Z} gives

$$d_{0Z} = -\Phi^{-1}(P_F) \quad \text{for } P_D = 0.5. \tag{16}$$

Therefore, the output deflection coefficient d_{0Z} defined previously by (11) is equivalent to (16) for a detection probability of 0.5. The output deflection coefficient for various false alarm probabilities [3] is listed in Table I.

Equations (8)–(12) were programmed on a digital computer. The value of the output deflection coefficient was set equal to $d_{0Z} = 2.33$, 3.09, 3.75, or 4.75. For each d_{0Z}, the required value of the input deflection coefficient was obtained for a specified number of channels N and the number of samples in the accumulator M. Equation (4) was then used to compute the ORing loss.

Results

The analytical results are presented in Figs. 6–9. The following conclusions can be drawn from an examination of Figs. 6–9:

1) The greater the number of channels ORed, the greater the ORing loss.

2) The lower the false alarm rate, the lower the ORing loss.

3) The more samples in the accumulator following the ORing device, the greater the ORing loss.

Conclusions

Exclusive ORing considerably decreases the amount of data that must be displayed. However, there will be a large signal-to-noise ratio loss if integration is done following ORing.

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Singular Value Decomposition and Improved Frequency Estimation Using Linear Prediction

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Abstract—**Linear-prediction-based (LP) methods for fitting multiple-sinusoid signal models to observed data, such as the forward-backward (FBLP) method of Nuttall [5] and Ulrych and Clayton [6], are very ill-conditioned. The locations of estimated spectral peaks can be greatly affected by a small amount of noise because of the appearance of outliers. LP estimation of frequencies can be greatly improved at low SNR by singular value decomposition (SVD) of the LP data matrix. The improved performance at low SNR is also better than that obtained by using the eigenvector corresponding to the minimum eigenvalue of the correlation matrix, as is done in Pisarenko's method and its variants.**

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I. Introduction

In previous related work, we presented results concerning the estimation of closely spaced frequencies of multiple sinusoids in noise using maximum likelihood (ML) and linear prediction (LP) methods [1], [2]. We have found that the threshold SNR, where the variance of the frequency estimation error departs significantly from the Cramer–Rao (CR) bound, is much lower for ML frequency estimation than for LP frequency estimation [1], [2]. This is a manifestation of the ill-conditioned nature of the LP frequency estimation [3].

In [2] we showed that the ill-conditioning of LP frequency estimation is not intrinsic. Good frequency estimates were obtained below the LP threshold SNR by extraction of the principal eigenvectors of the estimated correlation matrix which appears in the linear prediction normal equations.

Further, we showed in [3] and [4] that for purposes of estimating closely spaced frequencies, better LP coefficients can be obtained from the principal eigenvectors of the estimated correlation matrix than can be obtained by conventional LP calculations using the total matrix [5], [6].

In this correspondence, we wish to provide some new insights into our improvements to frequency estimation by linear prediction. We demonstrate in more detail the ill-conditioned nature of conventional LP methods, and we discuss the source of this ill conditioning. We also discuss the use of the singular value decomposition (SVD) of the data matrix. Henderson's work [7] is closely related to our work reported here.

II. Singular Value Decomposition and Linear Prediction

Let us assume that we want to approximate some complex valued data $x(k)$ by L sinusoids in complex valued form,

$$s(k) = \sum_{m=1}^{L} a_m z_m^k \qquad k = 0, 1, 2, \cdots N-1, \tag{1}$$

in which each of the complex numbers z_m has unit magnitude. And we assume that $x(k)$ consists of $s(k)$ plus broad-band complex valued noise $w(k)$:

$$x(k) = s(k) + w(k) \qquad k = 0, 1, 2 \cdots N-1. \tag{2}$$

In an LP method for estimating the z_m values or their angles, a coefficient vector $\boldsymbol{c}$ is sought which will approximately satisfy the linear prediction equations.

$$\begin{bmatrix} x(0) & x(1) & \cdots x(P-1) \\ x(1) & x(2) & \cdots x(P) \\ \vdots & \vdots & \vdots \\ x(N-P-1) & x(N-P) & \cdots x(N-1) \end{bmatrix} \begin{bmatrix} c(P) \\ c(P-1) \\ \vdots \\ c(1) \end{bmatrix} \approx \begin{bmatrix} x(P) \\ x(P+1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{3a}$$

or

$$A\,\boldsymbol{c} \simeq \boldsymbol{b}. \tag{3b}$$

The matrix A and vector $\boldsymbol{b}$ are slightly different if the data are used in both forward and backward directions [5], [6].

More specifically, of all the coefficient vectors which minimize the sum of squared magnitudes

$$\|\boldsymbol{b} - A\,\boldsymbol{c}\|^2 = \sum_{k=P}^{N-1} \left| x(k) - \sum_{n=1}^{P} c(n) \times (k-n) \right|^2 \tag{4}$$

we wish to find a vector $\boldsymbol{c}$ of minimum norm. The norm is defined by

$$\|c\|^2 = \boldsymbol{c}^*\boldsymbol{c} = \sum_{n=1}^{L} |c(n)|^2 \tag{5}$$

in which the asterisk stands for complex conjugate transpose.

The minimum norm solution to this minimum prediction error problem can be compactly written in terms of the pseudoinverse [8] A^I of the data matrix A. The solution is

$$\boldsymbol{c} = A^I \boldsymbol{b}. \tag{6}$$

We made use of this solution to obtain computationally simple improvements to earlier LP methods [9].

In the conventional LP methods [5], [6], [10]–[12], arrangements are made such that there is likely to be one vector $\boldsymbol{c}$ which minimizes the prediction error energy of (4). For this special case, we can use the following formula for the pseudoinverse [8]:

$$A^I = (A^*A)^{-1} A^*. \tag{7}$$

Or, equivalently, for this special case we can solve the normal equations

$$A^*A\,\boldsymbol{c} = \cdot\, A^*\boldsymbol{b} \tag{8}$$

in which A^*A is proportional to an estimated correlation matrix. As examples of such arrangements, Ulrych and Clayton [6] and Lang and McClellan [12] recommend that, for a fixed number of data samples N, the number of coefficients P should be approximately $N/3$ (or in between $N/3$ and $N/2$) for best frequency estimation.

How can singular value decomposition (SVD) be used to improve LP frequency estimation? To answer this question, we first present some well-known [8] SVD results. Then we apply them first to a noiseless case and then to the case of noisy data. More details are available in [3], [13], [14].

The SVD of a rectangular matrix A which has real or complex entries is given by the formula [8], [15]

$$\underset{(N-P \times P)}{A} = \underset{(N-P \times N-P)}{U} \underset{(N-P \times P)}{\sum} \underset{(P \times P)}{V^*}. \tag{8}$$

The dimensions of each matrix are written below the matrix. The square matrices U and V are unitary and Σ is a rectangular diagonal matrix of the same size as A with real nonnegative diagonal entries. These diagonal entries, called the singular values of A, are ordered in decreasing order with the largest in the upper left-hand corner. These singular values are the square roots of the eigenvalues of an estimated correlation matrix A^*A or AA^*. The asterisk is used to denote complex conjugate transpose. The columns of U and V are the eigenvectors of AA^* and A^*A, respectively.

The pseudoinverse A^I of (6) above is related to the SVD of A by the formula

$$A^I = V\,\Sigma^I\,U^* \tag{9}$$

in which Σ^I is obtained from Σ by replacing each positive diagonal entry by its reciprocal.

In the absence of noise, or at very high SNR, the solution of (6) is good. This is because the right-hand side vector $\boldsymbol{b}$ in (6) lies primarily along the L principal eigenvectors of A^*A. It can be shown that [14], if the number of prediction coefficients P satisfies the inequality

$$L \leqslant P \leqslant N - L, \tag{10}$$

then the signal-related zeros of the prediction error filter polynomial [16] derived from the prediction coefficient vector of (6) will be in their correct or approximately correct locations,

TABLE I
STANDARD DEVIATION OF FREQUENCY ESTIMATION ERROR (FOR $f_2 = 0.5$ Hz).
500 INDEPENDENT TRIALS WERE USED. σ_{CR} IS THE CRAMER-RAO BOUND.
PE: PRINCIPAL EIGENVECTORS METHOD. FBLP: FORWARD-BACKWARD
LINEAR PREDICTION METHOD. P = PREDICTION FILTER ORDER.
25 DATA SAMPLES WERE USED.

P	SNR=70dB, $\sigma_{CR}=.311\times10^{-5}$		SNR=30dB, $\sigma_{CR}=.311\times10^{-3}$		SNR=12dB, $\sigma_{CR}=.276\times10^{-2}$		SNR=7dB, $\sigma_{CR}=.491\times10^{-2}$	
	Standard Deviation		Standard Deviation		Standard Deviation		Standard Deviation	
	PE	FBLP	PE	FBLP	PE	FBLP	PE	FBLP
**2	$.529\times10^{-4}$	$.529\times10^{-4}$	$.629\times10^{-2}$	$.629\times10^{-2}$	*	*	*	*
4	$.110\times10^{-4}$	$.112\times10^{-4}$	$.191\times10^{-2}$	$.196\times10^{-2}$	*	*	*	*
6	$.676\times10^{-5}$	$.694\times10^{-5}$	$.680\times10^{-3}$	$.691\times10^{-3}$	*	*	*	*
8	$.510\times10^{-5}$	$.513\times10^{-5}$	$.514\times10^{-3}$	$.506\times10^{-3}$	$.473\times10^{-2}$	*	*	*
10	$.410\times10^{-5}$	$.448\times10^{-5}$	$.424\times10^{-3}$	*	$.345\times10^{-2}$	*	*	*
12	$.400\times10^{-5}$	$.426\times10^{-5}$	$.403\times10^{-3}$	*	$.313\times10^{-2}$	*	*	*
14	$.583\times10^{-5}$	*	$.394\times10^{-3}$	*	$.309\times10^{-2}$	*	$.484\times10^{-2}$	*
16	$.348\times10^{-5}$	*	$.355\times10^{-3}$	*	$.219\times10^{-2}$	*	$.433\times10^{-2}$	*
18	$.346\times10^{-5}$	*	$.345\times10^{-3}$	*	$.279\times10$	*	$.417\times10^{-2}$	*
20	$.332\times10^{-5}$	*	$.338\times10^{-3}$	*	$.264\times10^{-2}$	*	*	*
22	$.362\times10^{-5}$	$.347\times10^{-5}$	$.353\times10^{-3}$	*	$.275\times10^{-2}$	*	*	*
23	$.408\times10^{-5}$	$.400\times10^{-5}$	$.406\times10^{-3}$	*	$.443\times10^{-2}$	*	*	*
***24 $\frac{(N-L)}{2}$	$.514\times10^{-5}$	$.514\times10^{-5}$	$.517\times10^{-3}$	$.517\times10^{-3}$	$.670\times10^{-2}$	$.670\times10^{-2}$	*	*

* For these cases the frequency estimation error was large either due to a spurious zero of the prediction-error filter wandering too close to the unit circle (at large P) or due to merging of two zeros close to the signalfrequency locations.

** This corresponds to the Prony's method with the modification in which the data is used in both directions. PE and FBLP give identical results

*** This corresponds to the Kumaresan-Prony case. PE and FBLP (min. Norm Solution) give identical results.

and any extra zeros (if $P > L$) will be located inside the unit circle in the complex plane and hence separable from the signal zeros. We have assumed that the values of P and $N - P$ are both greater than L. When there is no noise so that $x(k) = s(k)$ in (2), the rank of A is L and there will be only L nonzero elements in the diagonal matrices Σ and Σ^I. The introduction of a small amount of noise tends to change the situation. The matrix A tends to take on its full possible rank [15]. The matrix Σ then has some (or likely all) of its formerly zero singular values become small nonzero values. These small diagonal values of Σ, introduced by noise, become large diagonal values of Σ^I in (9), and this is a cause for large perturbations in the prediction coefficient vector of (6).

As shown by the work of Wilkinson [17], the perturbations of the L originally nonzero values of Σ and the corresponding first L columns of U and V are relatively small. The main perturbation effects of a small amount of noise are due to the increase in rank of A over its rank in the noiseless case and the associated large perturbations in the directions of the rest of the columns of U and V. This major effect can be alleviated by replacing A in the case of noisy data by a lower rank approximation $\hat{A}$ prior to calculation of the vector $\boldsymbol{c}$ of prediction coefficients. In 1936, Eckart and Young developed the following procedure, based on the SVD for finding the best lower rank approximation to a given matrix [18].

Let the rank of A be r, and let $s(L)$ be the set of all $(N - P) \times P$ matrices of rank $L < r$. Then for all matrices B in $S(L)$

$$\|A - \hat{A}\| \leq \|A - B\| \tag{11}$$

where

$$\hat{A} = U \hat{\Sigma} V^* \tag{12}$$

and $\hat{\Sigma}$ is obtained from the matrix Σ of (8) by setting to zero all but its L largest singular values. The matrix norm of (11) is the Frobenius norm. That is,

$$\|A - B\|^2 = \text{trace}\,[(A - B)^*(A - B)]. \tag{13}$$

Hence, $\hat{A}$ is the best least squares approximation of lower rank L to the given data matrix.

Even with the availability of efficient algorithms for computing the SVD of A, the computation of $\hat{A}$ using (12) can be made more efficient. For example, all but the first L columns of U and V can be set equal to zero to form U_L and V_L. One is then led to the following additional forms of (12):

$$\hat{A} = U_L \hat{\Sigma} V_L^* = U_L U_L^* A = A V_L V_L^*. \tag{14}$$

Because the columns of U_L and V_L are principal eigenvectors (associated with the larger eigenvalues) of A^*A and AA^*, respectively, the relatively small size of one or the other of these latter matrices can lead one to compute $\hat{A}$ by: 1) computing the principal eigenvectors of AA^* or A^*A, whichever is easier; and 2) computing $\hat{A}$ from one of the last two expressions in (14).

We note from (14) that $U_L U_L^*$ and $V_L V_L^*$ are matrix filtering operations which operate directly on the data matrix A to form the signal estimate matrix $\hat{A}$. Because U_L and V_L are derived directly from the data (using only information about the approximate rank of the "signal-only" data matrix), the last two expressions in (14) provide a data-adaptive Wiener-filtering interpretation of the formation of $\hat{A}$ from A.

III. EXPERIMENTAL RESULTS

Here we present some experimental results to demonstrate the improvements which we discussed above. The data samples

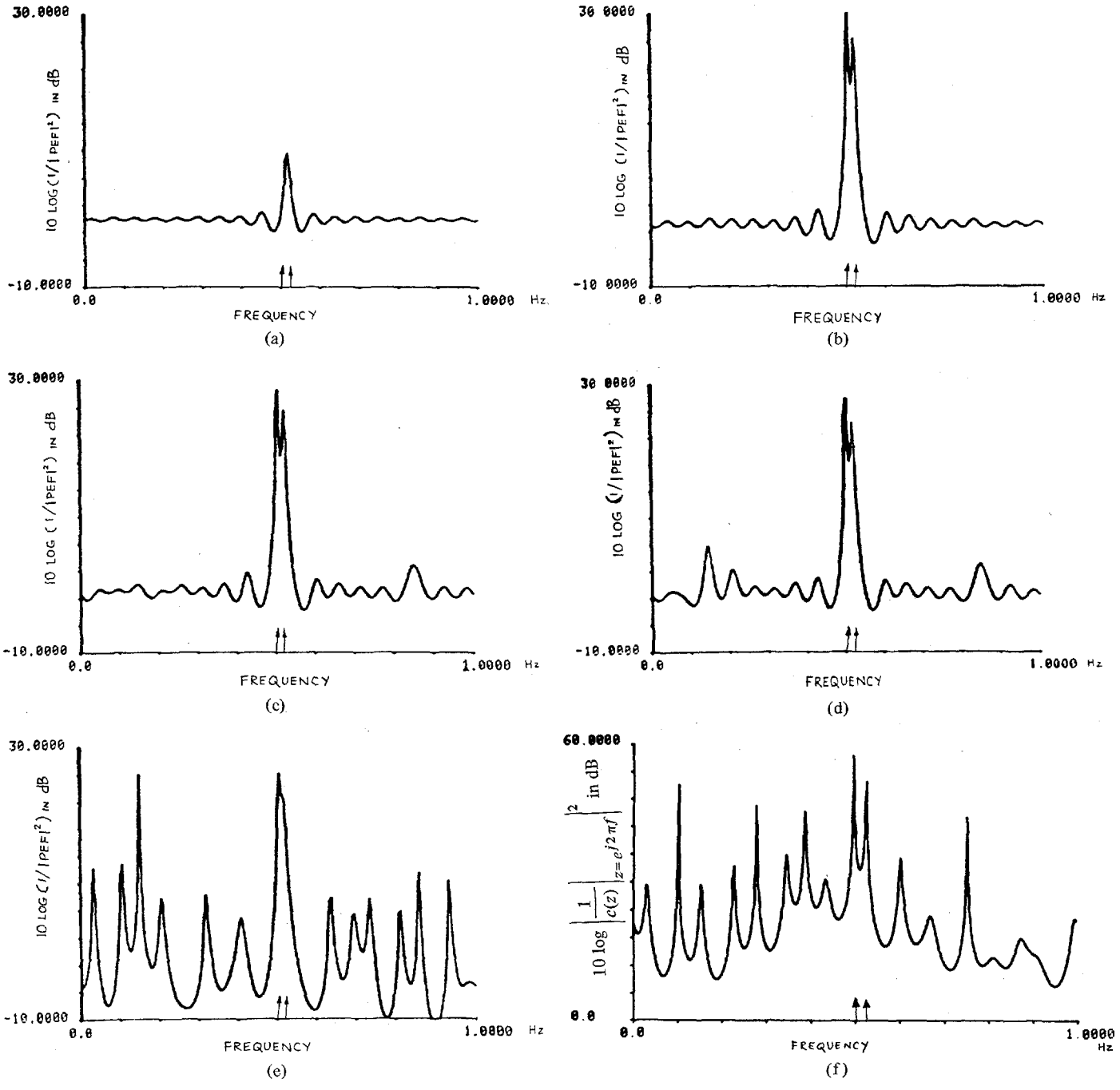

Fig. 1. Reciprocal of the prediction-error filter transfer function (PEF) using four different values L as working hypothesis. (a) $L = 1$, (b) $L = 2$, (c) $L = 3$, (d) $L = 4$. In (e) the minimum norm solution for the linear prediction equations was used. $P = 18$ and $N = 25$. (f) Reciprocal of the magnitude of a polynomial $c(z)$ (evaluated on the unit circle) constructed with the $L + 1$ elements of the eigenvector corresponding to the minimum eigenvalue of $A'^{*}A'$. A' is the augmented matrix $(-\boldsymbol{b}, A)$.

are generated using the formula

$$x(n) = e^{j(2\pi f_1 n + \phi_1)} + e^{j(2\pi f_2 n + \phi_2)} + w(n), \qquad n = 0, 1, \cdots N-1 \quad (15)$$

where $\phi_1 = \pi/4$, $\phi_2 = 0$, $f_1 = 0.52$ Hz, $f_2 = 0.5$ Hz, and $w(n)$ are independent complex Gaussian noise samples with variance $2\sigma^2$ for each real and imaginary part. SNR is defined as $10 \log \frac{1}{2}\sigma^2$. The sampling period is assumed to be 1 Hz. For each trial, a data block of 25 ($N = 25$) data samples is used. The frequency estimates are obtained by finding the two roots of the prediction error filter transfer function that are closest to the unit circle. Different values of P in the range of 2–24 are used. The angles of these roots are taken as the frequency estimates. The two methods used are the FBLP method discussed in [5], [6], [12] and the principal eigenvector (PE) method [13], [14] using SVD discussed above. The standard deviation of the frequency estimation error for f_2 is computed for 500 independent trials. The estimated standard deviations are tabulated in Table I for different SNR and P values. The corresponding CR bound values [19] are also given. The estimation bias was negligible in all cases except at 7 dB. The biases for the three values of $P = 14, 16, 18$ were about a third of the respective standard deviations. The main point in Table I is that the

FBLP method is primarily useful only at very high SNR values, whereas the PE method can be used at much lower SNR values. Also, by choosing the P value to be about $3N/4$ in the PE method which is not useful in the FBLP method, the PE method practically achieves the CR bound. The special case of $P = 2$ is a variant of Prony's method with the data being used in both the forward and backward directions.

The case of $P = 24$, called the Kumaresan–Prony (KP) case [3], [9], is simple to implement. It does not require SVD. It has superior performance at lower SNR values compared to the FBLP method at conventional values of P [6], [12]. For the two special cases of $P = 2$ and 24, the PE and the FBLP methods coincide if we find a prediction filter coefficient vector having minimum norm. This is because the rank of the data matrix A is only 2, and hence it has only two singular values.

In the second part of the experiment, we demonstrate the insensitivity of the frequency estimates to the assumed value of L if L is not known *a priori*. The same data set as in (15) is used. SNR is 10 dB. P is 18. For a particular data block of 25 data samples, the SVD of the $2(N - P) \times P$ data matrix showed two relatively large singular values, 4.83 and 3.8, and the rest were smaller than 0.95. Hence, it was easy to choose the value of L as 2. For more closely spaced sinusoidal signals, the magnitude of the second singular value would be smaller, making the choice of L more difficult.

We computed the prediction filter coefficients using SVD for various assumed values of L by setting the rest of the smaller singular values to zero (see [13]). For each assumed value of the number of signal components L, the corresponding spectral estimate, which is defined as the reciprocal of the magnitude squared of the transfer function of the prediction error filter, is plotted in Fig. 1(a)–(d). Fig. 1(e) shows the case of the minimum norm solution when none of the singular values is set to zero. For $L = 1$, the two sinusoids are not resolved. At least two singular values and corresponding eigenvectors in U and V are needed to resolve the two sinusoids. But the rest of the spectral peaks are quite damped. $L = 2$ corresponds to the ideal situation in which the working hypothesis of two signals is correct. For $L = 3$ and 4, the noise subspace perturbations start to introduce instabilities into the prediction coefficients, slightly effecting the extraneous spectral peaks. Occasionally, when the noise realization itself is close to a sinusoid of some frequency, one might see a large third peak for the cases when L is larger than 2. Fig. 1(e), corresponding to the minimum norm solution, shows large spurious peaks which exhibit the ill-conditioned nature of the problem. Fig. 1(f) shows the situation when the eigenvector corresponding to the minimum eigenvalue (in this case zero) is used as a polynomial and its reciprocal magnitude evaluated on the unit circle is plotted. This is considered a modification of the Pisarenko method [20], [21].

IV. Conclusions

Singular value decomposition (SVD) of the data matrix occurring in linear prediction equations is shown to be useful in finding the number of sinusoidal signals in the data, as well as in estimating their frequencies accurately. The resulting principal eigenvector (PE) method alleviates much of the ill-conditioned nature of the forward-backward linear prediction (FBLP) method by removing the noise subspace perturbation effects. Accurate parameter estimates are obtained when P is chosen as a large fraction of N. When P is chosen equal to $(N - L/2)$, the PE and FBLP methods (when the minimum norm prediction filter coefficients are found) are the same, and we called this case the Kumaresan–Prony (KP) case. This case does not require SVD calculations.

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A Linear Programming Approach to the Estimation of the Power Spectra of Harmonic Processes

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Abstract–**A new approach to the estimation of high resolution power spectra of harmonic processes is presented. This approach uses linear programming (LIP) to exploit the spiky character of the L_1 norm. Results for the LIP power spectral estimate are contrasted with those using various formulations of the maximum entropy algorithm.**

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