1.

(a) i: value function: $V(k_t)=\lambda+\xi\log k_t$

policy function: $k_{t+1}=\pi(k_t)=\gamma A k_t^{\alpha}$

Bellman Equation:

$$V(x)=\sup_{y\in G(x)}\{u(x,y)+\beta V(y)\}.$$

$\Rightarrow$ $V(x)=\sup\{\ln(Ax^{\alpha}-y)+\beta V(y)\}$

$V_y(x)=\dfrac{-1}{Ax^{\alpha}-y}+\beta V_y(y)=0$ 则 $\dfrac{1}{Ax^{\alpha}-y}=\beta V_y(y)$ ①

$V_x(x)=\dfrac{A\alpha x^{\alpha-1}}{Ax^{\alpha}-y}$ 又 $y=\pi(x)$

可化为 $V_x(x)=\dfrac{A\alpha x^{\alpha-1}}{Ax^{\alpha}-\pi(x)}$，设 $\pi(x)=\gamma A x^{\alpha}$

可化为 $V_x(x)=\dfrac{A\alpha x^{\alpha-1}}{Ax^{\alpha}-\gamma A(\gamma Ax^{\alpha})^{\alpha}}$. ②

联立①②可得:

$$\frac{1}{Ax^{\alpha}-\pi(x)}=\frac{\beta A\alpha\,\pi(x)^{\alpha-1}}{A[\pi(x)]^{\alpha}-\pi[\pi(x)]}$$ ，将$\pi(x)$代入.

$$\frac{1}{Ax^{\alpha}-\gamma Ax^{\alpha}}=\frac{\beta\cancel{A}\alpha(\gamma A\cancel{x^{\alpha}})^{\alpha-1}}{\cancel{A}(\gamma A\cancel{x^{\alpha}})^{\alpha}-\cancel{\gamma A}(\gamma A\cancel{x^{\alpha}})^{\alpha}}$$

$\gamma=\alpha\beta$

$$V(x) = \lambda + \xi \ln x = \sup \left\{ \ln(Ax^{\alpha} - y) + \beta(\lambda + \xi \ln y) \right\}$$

对x求导

$$\frac{\xi}{x} = \frac{A\alpha x^{\alpha-1}}{Ax^{\alpha} - y}$$

$$\frac{\xi}{x} = \frac{A\alpha x^{\alpha-1}}{Ax^{\alpha} - \gamma A x^{\alpha}}, \text{且 } \gamma = \alpha\beta$$

$$\Rightarrow \xi = \frac{\alpha}{1-\alpha\beta}$$

$$V(x) = \lambda + \xi \ln x = \ln(Ax^{\alpha} - y) + \beta(\lambda + \xi \ln y)$$

$$\lambda + \frac{\alpha}{1-\alpha\beta} \ln x = \ln(Ax^{\alpha} - \alpha\beta A x^{\alpha}) + \beta\left(\lambda + \frac{\alpha}{1-\alpha\beta} \cdot \ln \alpha\beta A x^{\alpha}\right)$$

$$\lambda + \frac{\alpha}{1-\alpha\beta} \ln x = \ln A(1-\alpha\beta) + \alpha \ln x + \beta\lambda + \frac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta A + \frac{\alpha^2\beta}{1-\alpha\beta} \ln x$$

$$(1-\beta)\lambda + \cancel{\frac{\alpha}{1-\alpha\beta}} \ln x = \ln A(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta A + \cancel{\frac{\alpha}{1-\alpha\beta}} \ln x.$$

$$\lambda = \frac{\ln A(1-\alpha\beta)}{1-\beta} + \frac{\alpha\beta \ln(\alpha\beta A)}{(1-\alpha\beta)(1-\beta)}$$

2. $\max_{[C(t),a(t)]_{t=0}^{1}} \int_0^1 e^{-\rho t} u(C(t))\,dt$

s.t. $\dot{a}(t) = r a(t) + w - C(t)$, $a(0)=a_0$ $a(1)=0$

(a). $C(t) = \dot{a}(t) - r a(t) + w$

$$L = \max \int_0^1 e^{-\rho t} u(r a(t) - \dot{a}(t) + w)\,dt$$

$$\frac{\partial L}{\partial f} - \frac{d(\frac{\partial L}{\partial f'})}{dt} = 0 \quad , \quad f = a(t)$$

$$\Rightarrow e^{-\rho t} u'(r a(t) - \dot{a}(t) + w)\cdot r - \frac{-d(e^{-\rho t} u'(r a(t) - \dot{a}(t) + w)}{dt} = 0$$

$$e^{-\rho t} u'(r a(t) - \dot{a}(t) + w)\cdot r = -\frac{d(e^{-\rho t} u'(r a(t) - \dot{a}(t) + w)}{dt}$$

$$= -[-\rho e^{-\rho t} u'(r a(t) - \dot{a}(t) + w) + e^{-\rho t} u''(r a(t) - \dot{a}(t) + w) \cdot [r \dot{a}(t) - \ddot{a}(t)]$$

$$\Rightarrow u'(r a(t) - \dot{a}(t) + w)\cdot r = \rho u'(r a(t) - \dot{a}(t) + w) - u''(r a(t) - \dot{a}(t) + w)[r\dot{a}(t) - \ddot{a}(t)]$$

$$(\rho - r) u'(r a(t) - \dot{a}(t) + w) = u''(r a(t) - \dot{a}(t) + w)[r\dot{a}(t) - \ddot{a}(t)]$$

(b) $C(t) = r a(t) + w - \dot{a}(t) \Rightarrow \dot{C}(t) = r\dot{a}(t) - \ddot{a}(t)$

则(a)结果可化为:

$$(\rho - r) u'(C(t)) = u''(C(t))\,\dot{C}(t)$$

$$\Rightarrow \frac{u''(C(t))\dot{C}(t)}{u'(C(t))} = \rho - r$$

(c). 由
$$\begin{cases} \dot{\lambda}(t) = -[f_x(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_x(t, \hat{x}(t), \hat{y}(t))] \\ \lambda(t_1) = 0 \\ f_y(t, \hat{x}(t), \hat{y}(t)) + \lambda(t) g_y(t, \hat{x}(t), \hat{y}(t)) = 0 \quad \text{for all } t \in [0, t_1] \end{cases}$$

对应到本题中:

$$\begin{cases} e^{-\rho t} u'(c(t)) = \lambda(t) \\ \dot{\lambda}(t) = -\lambda(t) r \end{cases}$$

$$\Rightarrow -\rho \cancel{e^{-\rho t}} u'(c(t)) + \cancel{e^{-\rho t}} u''(c(t))\dot{c}(t) = \cancel{e^{-\rho t}} u'(c(t))$$

$$\Rightarrow \frac{u''(c(t))\dot{c}(t)}{u'(c(t))} = \rho - r$$

(d) 当 $u(c) = \log c$ 时.

$$\frac{u''(c(t))\dot{c}(t)}{u'(c(t))} = \frac{-\frac{1}{c(t)^2}\cdot \dot{c}(t)}{\frac{1}{c(t)}} = -\frac{\dot{c}(t)}{c(t)} = \rho - r$$

$$\Rightarrow \frac{\dot{c}(t)}{c(t)} = r-\rho \quad \Rightarrow \ln c(t) - \ln c(0) = (r-\rho)t \quad ③$$

又 $c(t) = ra(t) + w - \dot{a}(t)$ $\quad a(0) = a_0 \quad a(1) = 0$

$$\ln c(t) = \ln(ra(t) + w - \dot{a}(t))$$

$$\Rightarrow \ln c(0) = \ln(ra(0) + w - \dot{a}(0)) = \ln(ra_0 + w - \dot{a}(0))$$

由③式

$$\ln c(t) = (r-\rho)t + \ln(ra_0 + w - \dot{a}(0))$$

$$c(t) = e^{(r-\rho)t}(ra_0 + w - \dot{a}(0))$$

当 $u(c) = [\theta - e^{-\beta c(t)}]$ 时,

$$u'(c) = -\beta(1-e^{-\beta c(t)}) = \beta e^{-\beta c(t)}$$

$$u''(c) = -\beta^2 e^{-\beta c(t)}$$

$$\frac{u''(c)\dot{c}(t)}{u'(c)} = \frac{-\beta^2 e^{-\beta c(t)}\dot{c}(t)}{\beta e^{-\beta c(t)}} = -\beta \dot{c}(t) = \rho - r$$

$$\dot{c}(t) = \frac{r-\rho}{\beta}$$

$$c(t) = \frac{r-\rho}{\beta}t + m \quad ,m为常数$$

$$又 c(0) = r a_0 + w - \dot{a}(0)$$

$$\Rightarrow m = r a_0 + w - \dot{a}(0)$$

$$\Rightarrow c(t) = \frac{r-\rho}{\beta}t + r a_0 + w - \dot{a}(0)$$