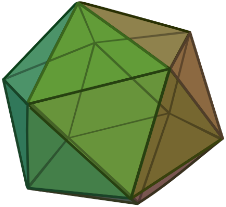
Least Annoying Problem About an Icosahedron

Aug 30, 2024



Where Is the Icosahedron?

Problem

Two strings of 0s and 1s are considered equivalent, if one can be obtained from the other by repeatedly removing or inserting (anywhere in the string) the following substrings: 00, 111, 0101010101. For example, 100110 and 010011 are equivalent, since we can do the following sequence of transformations:

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Given 4000 strings of total length at most $5 \cdot 10^4$, find out the groups of equivalent strings among them.

Simpler Problem

Problem

You are given a string s with 10^5 characters, each `a`, `b` or `c`. You must swap exactly two distinct letters to obtain a new string t. How many ways are there to do that in such a way that the string t is good? In this problem we define a good string as follows: empty string is good, if a string u is good then the strings aua, bub, cuc are good as well, and if two strings u and v are good, then their concatenation uv is also good.

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- Take A, B, C to be the matrix of reflections across three random lines
- Concretely, for a pair of numbers u, v, take

$$A = \frac{1}{u^2 + v^2} \begin{pmatrix} u^2 - v^2 & 2uv \\ 2uv & v^2 - u^2 \end{pmatrix}$$

and the same for B, C with different seeds.

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- How do we solve the problem given the way to efficiently tell if any substring is good?
- Divide & Conquer – need to count the number of flips making a string good where one element comes from the left half and the other comes from the right half
- For simplicity focus on flipping `a` with `b`. For every `a` in the left side compute the compressed representation of the left side with that `a` changed to `b`, and do the same on the right side. Then for every candidate on the left side count the number of times its inverse occurs on the right side (use matrices modulo prime p instead of floats).

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Group Theory Aside: Definition

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- A *group* $(G, \cdot, e)$ is a set with a binary operation $\cdot$ (usually called multiplication or addition depending on the context) and an identity e, such that
  1. Binary operation is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  2. Identity is reasonable: $a \cdot e = e \cdot a = a$
  3. Inverses exist: for any a exists b such that $a \cdot b = b \cdot a = e$.

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- Permutations under composition (associative, inverse permutations exist)
- Various symmetries in general (in some sense generalizes the previous two bullet points)

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- The underlying set consists of all words in letters $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$, where we identify words different by a series of

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- So we identified to each element of this group a matrix, and matrices we know how to work with, can compute and store in memory

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  1. $f(e_G) = e_H$
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- We defined a homomorphism from the weird word group W to the group of all invertible 2×2 matrices with mod p coefficients (or real numbers).
- The latter is much to understand, and a good enough proxy for the purposes of telling elements apart and doing computations. An element is not very likely to map to the identity matrix unless it is already the identity in W.

Still No Icosahedron?

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- Recall: two strings of 0s and 1s are considered equivalent, if one can be obtained from the other by repeatedly removing or inserting (anywhere in the string) the following substrings: 00, 111, 0101010101. For example, 100110 and 010011 are equivalent, since we can do the following sequence of transformations:

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- Any ideas?

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- The problem defines a word group again: relations are exactly given in the problem. If we rename the generators from $0, 1$ to a, b, it is $a^2 = e$, $b^3 = e$, $(ab)^5 = e$.

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- In fact, turns out this group is finite, and is isomorphic to the symmetries of the **icosahedron**.
- The same group is also isomorphic to the group of even permutations of length 5, usually denoted by A_5.
- But in contest we don't know any of this trivia, and how does it even help? We just want an AC!

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- Replace 0 with p, 1 with q, and compute the compositions for all the strings.
- Do this for a few pairs in case of collisions, since there are not that many permutations.