UCF ICPC Training Camp

Day II: Combinatorics

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Wednesday, March 22, 2022

Yongwhan Lim

Education

Part-time Jobs

Full-time Job

Workshops

Coach/Judge

https://www.yongwhan.io

Yongwhan Lim

- Currently:
    - a **Co-Founder** in a Stealth Mode Startup;
    - ICPC **Internship Director**;
    - Columbia ICPC **Head Coach**;
    - ICPC **Judge** for NAQ and Regionals;
    - **Lecturer** at MIT;
    - **Adjunct** (Associate in CS) at Columbia;

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Today's Format

9am ET - 10:20am ET	**Lecture**
10:30am ET - 12pm ET	**Lecture Exercises**
12pm ET - 12:45pm ET	**Lunch**
12:45pm ET - 3:45pm ET	**Practice Contest**
	UCF ICPC Training Camp Day 2
4pm ET - 5:20pm ET	**Review**

Request 1:1 Meeting, through Calendly

- Use calendly.com/yongwhan/one-on-one to request 1:1 meeting:
  - Mock Interview
  - Resume Critique
  - Career Planning
  - Practice Strategy
  - …
- Always inspired by driven students like yourself!
- Since I'd feel honored/thrilled to talk to you, do not feel shy to sign up!!

Lecture

Now, let's dive right into Combinatorics!

I. Catalan number;
II. Bell Number;
III. Euler Number;
IV. Bernoulli number;
V. Stirling numbers of the first kind; Stirling numbers of the second kind;

VI. Generating Function;
VII. Bernoulli polynomials; Bernoulli polynomials of the second kind;
VIII. Euler polynomial;
IX. Stirling polynomials;

I. Catalan Numbers: Motivating Examples

- Number of correct bracket sequence consisting of n opening and n closing brackets.
- The number of rooted full binary trees with n + 1 leaves (vertices are not numbered). A rooted binary tree is full if every vertex has either two children or no children.
- The number of ways to completely parenthesize n + 1 factors.
- The number of ways to connect the $2n$ points on a circle to form n disjoint chords.
- The number of non-isomorphic full binary trees with n internal nodes (i.e. nodes having at least one son).
- ...

I. Catalan Numbers (A000108)

- 1
- 1
- 2
- 5
- 14
- 42
- 132
- 429
- 1430
- ...

I. Catalan Numbers: Recursive Formula

$$C_0 = C_1 = 1$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}, n \geq 2$$

I. Catalan Numbers: Implementation

```
int catalan[MAX];
void init() {
    catalan[0] = catalan[1] = 1;
    for (int i=2; i<=n; i++) {
        catalan[i] = 0;
        for (int j=0; j < i; j++) {
            catalan[i] += (catalan[j] *
                                catalan[i-j-1]) % MOD;
            if (catalan[i] >= MOD) catalan[i] -= MOD;
        }
    }
}
```

I. Catalan Numbers: Analytical Formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

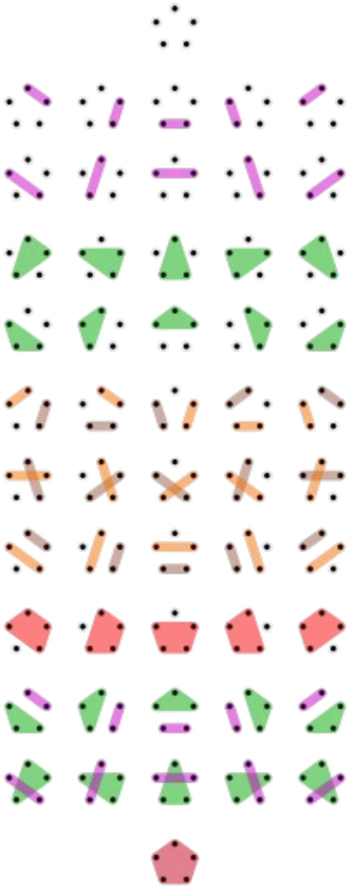
$$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}, n \geq 0$$

II. Bell Numbers

- Bell numbers count **the possible partitions of a set**.
- For example, when n=3 (e.g., {a,b,c}), we have:
    - {{a},{b},{c}};
    - {{a},{b,c}};
    - {{b},{a,c}};
    - {{c},{a,b}};
    - {{a,b,c}};

II. Bell Numbers ([A000110](A000110))

- 1
- 1
- 2
- 5
- 15
- 52
- 203
- 877
- 4140
- …



II. Bell Numbers: Recurrence & Explicit

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Binomial coefficient

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Stirling number of second kind

number of ways to partition a set of cardinality n into exactly k nonempty subsets

III. Euler Numbers

- Euler numbers are a sequence of integers defined by the Taylor series expansion:

$$\frac{1}{\cosh t} = \frac{2}{e^{t} + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n$$

III. Euler Numbers (A122045)

- 1
- 0
- -1
- 0
- 5
- 0
- -61
- 0
- 1385
- …

III. Euler Numbers: Explicit

$$E_n = 2^{2n-1}\sum_{\ell=1}^{n}\frac{(-1)^\ell S(n,\ell)}{\ell+1}\left(3\left(\frac{1}{4}\right)^{(\ell)} - \left(\frac{3}{4}\right)^{(\ell)}\right)$$

$S(n,\ell)$ **Stirling numbers of the second kind**

$x^{(\ell)} = (x)(x+1)\cdots(x+\ell-1)$ **Rising factorial**

IV. Bernoulli Numbers

- Taylor series expansions of the tangent and hyperbolic tangent functions;
- Faulhaber's formula for the sum of m-th powers of the first n positive integers;
- the Euler–Maclaurin formula;
- Certain values of the Riemann zeta function;

IV. Bernoulli Numbers: Numerator (A027641)

- 1
- -1
- 1
- 0
- -1
- 0
- 1
- 0
- -1
- ...

IV. Bernoulli Numbers: Denominator (A027642)

- 1
- 2
- 6
- 1
- 30
- 1
- 42
- 1
- 30
- …

IV. Bernoulli Numbers: Recurrence

$$B_m^- = \delta_{m,0} - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k^-}{m-k+1}$$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Kronecker delta

$B_n^+ = (-1)^n B_n^-$, or for integer n = 2 or greater

IV. Bernoulli Numbers: Explicit

$$B_m^- = \sum_{k=0}^{m} \sum_{v=0}^{k} (-1)^v \binom{k}{v} \frac{v^m}{k+1}$$

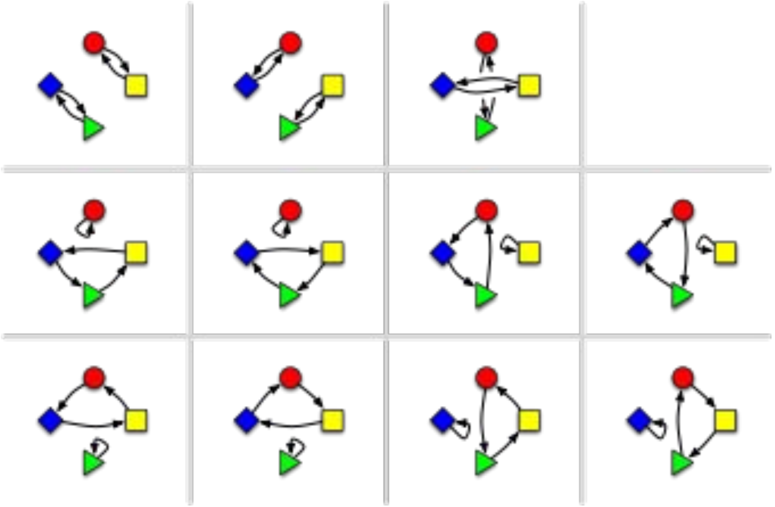
IV. Bernoulli Numbers: Riemann Zeta Function

$$B_n^{+} = -n\zeta(1-n) \qquad \text{for } n \geq 1$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

V-A. Stirling numbers of the first kind

- Count permutations according to their number of cycles (counting fixed points as cycles of length one)



$$\left[{4 \atop 2} \right] = 11$$

V-A. Stirling numbers of the first kind: Recurrence

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0$$

V-A. Stirling numbers of the first kind: Explicit

$$s(n, n-p) = \frac{1}{(n-p-1)!} \sum_{0 \le k_1, \ldots, k_p : \sum_1^p m k_m = p} (-1)^K \frac{(n+K-1)!}{k_1! k_2! \cdots k_p! \; 2!^{k_1} \, 3!^{k_2} \, \cdots (p+1)!^{k_p}}$$

V-B. Stirling numbers of the second kind

- the number of ways to partition a set of n objects into k non-empty subsets

V-B. Stirling numbers of the second kind: Recurrence

$$\left\{ {n+1 \atop k} \right\} = k \left\{ {n \atop k} \right\} + \left\{ {n \atop k-1} \right\} \quad \text{for } 0 < k < n$$

$$\left\{ {n \atop n} \right\} = 1 \quad \text{for } n \geq 0 \quad \text{and} \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0 \quad \text{for } n > 0.$$

V-B. Stirling numbers of the second kind: Explicit

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

VI. Generating Function

- a way of encoding an infinite sequence of numbers (a_n) by treating them as the **coefficients** of a formal **power series**.

VI. Ordinary Generating Function (OGF)

$$G(a_n;x)=\sum_{n=0}^{\infty}a_nx^n.$$

$$G(a_{m,n};x,y)=\sum_{m,n=0}^{\infty}a_{m,n}x^my^n$$

VI. Exponential Generating Function (EGF)

$$\mathrm{EG}(a_n; x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

VI. Generating Function: Example: Geometric Series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

1, 1, 1, 1, 1, ...

$$\sum_{n=0}^{\infty} (ax)^n = \frac{1}{1-ax}$$

1, a, a^2, a^3, a^4, a^5, ...

VII-A. Bernoulli Polynomials

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k$$

VII-B. Bernoulli Polynomials of the second kind

$$\frac{z(1+z)^x}{\ln(1+z)} = \sum_{n=0}^{\infty} z^n \psi_n(x), \qquad |z| < 1$$

VIII. Euler Polynomials

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

$$E_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{m-k}$$

IX. Stirling polynomials

$$\left(\frac{t}{1-e^{-t}}\right)^{x+1} = \sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!}$$

Further Topics

- Burnside's Lemma
- Placing Bishops on a Chessboard
- Balanced bracket sequences
- Counting labeled graphs
- Sums of powers
- Pólya enumeration theorem
- Probabilistic methods

Further Readings

- USACO Guide: https://usaco.guide/gold/combo?lang=cpp
- CP Algorithms: https://cp-algorithms.com/algebra/factorial-divisors.html

- *Concrete Mathematics*, Donald Knuth, et. al.
- *Generatingfunctionology*, Herbert Wilf
- *Combinatorics and Graph Theory*, John Harris, et. al.
- *Principles And Techniques In Combinatorics*, Chen Chuan-Chong, et. al.

A Terse Guide on ICPC Contest Strategies

- Please take a look at:
  - A Terse Guide on ICPC Contest Strategies for Columbia team.
  - In addition, we have Google Drive to Terse Guides, of course!

- These documents will be frequently expanded upon later.

Reminder! Discord Servers

- Join the following discord servers, if you have not already!!!

  [ICPC CodeForces Zealots] https://discord.gg/7bvMnMyF6G

Reminder! Practice makes PERFECT!

- Do as **many practice contests** as you can!
  - **Live Contests**
    - CodeForces: Division 1-4
    - AtCoder: Beginner; Regular; Grand;
    - LeetCode: Weekly/Biweekly
  - **ICPC North America Practice Contests** on:
    - **Sundays** from 1pm ET to 6pm ET
  - **Zealot Problem Sets**
    - **Everyday** (24 hours 7 days a week)!

Lecture Exercises

Lecture Exercise #1 (1800)

- https://codeforces.com/contest/300/problem/C

Solution

- https://codeforces.com/contest/300/submission/16421338

Lecture Exercise #2 (2500)

- https://codeforces.com/problemset/problem/407/C

Solution

- https://codeforces.com/contest/407/submission/6186385

Lecture Exercise #3 (2300)

- https://codeforces.com/contest/785/problem/D

Solution

- https://codeforces.com/contest/785/submission/37115054

Lecture Exercise #4 (1900)

- https://codeforces.com/problemset/problem/9/D

Solution

- https://codeforces.com/contest/9/submission/3014721

Lecture Exercise #5 (1600)

- https://codeforces.com/problemset/problem/888/D

Solution

- https://codeforces.com/contest/888/submission/33698050

Lecture Exercise #6 (2100)

- https://codeforces.com/problemset/problem/1606/E

Solution

- https://codeforces.com/contest/1606/submission/133732412

Lecture Exercise #7

- https://atcoder.jp/contests/dp/tasks/dp_y/

Solution

- https://atcoder.jp/contests/dp/submissions/3979598


LUNCH
BREAK


THANK YOU