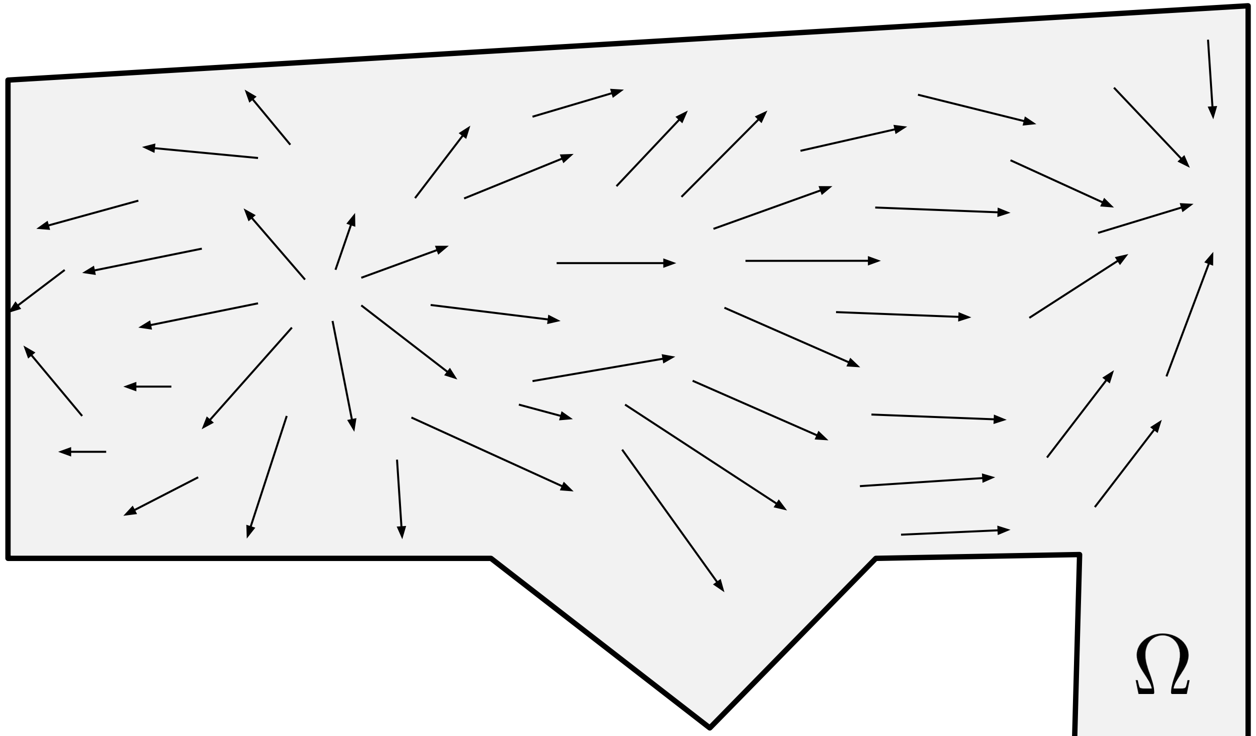
Geometry for Competitive Programming

Divergence

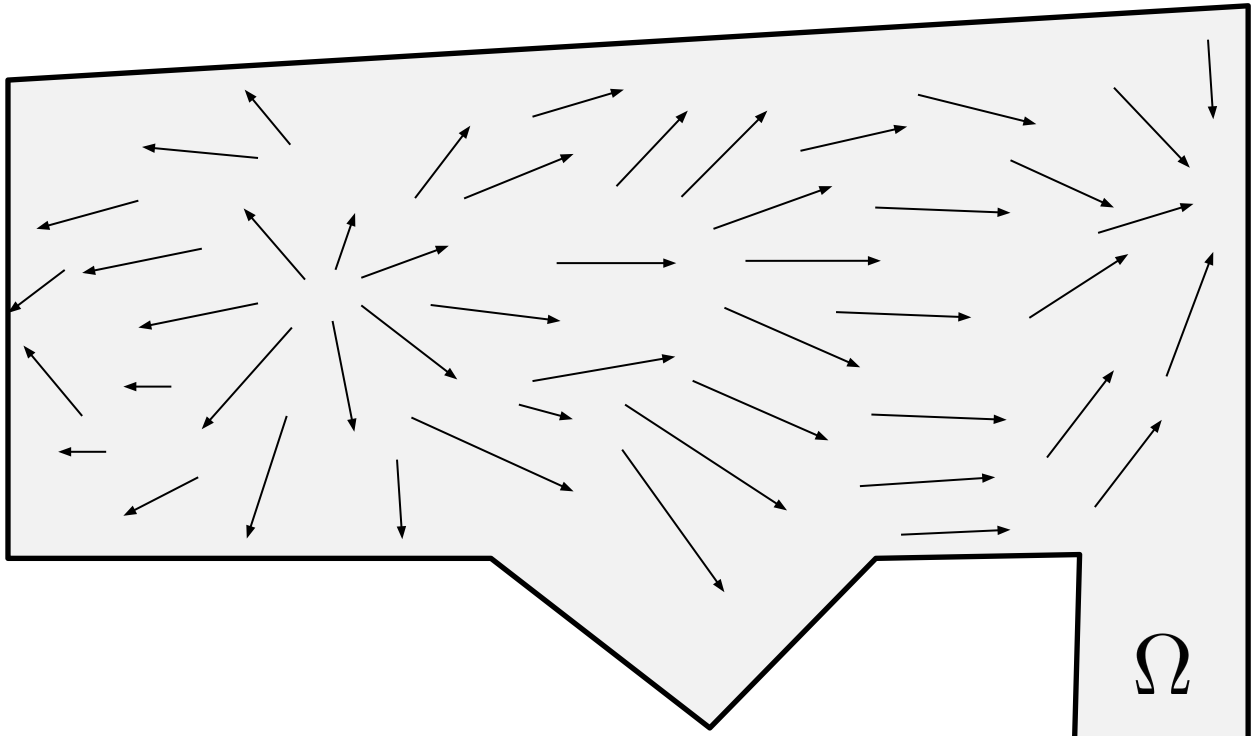
Vector field $\mathbf{v}(x, y) = (v_x(x, y), v_y(x, y))$ in a region Ω



E.g.: flow of corn in the unit states

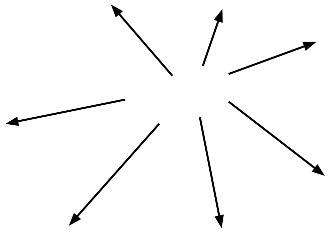
Divergence

Vector field $\mathbf{v}(x, y) = (v_x(x, y), v_y(x, y))$ in a region Ω



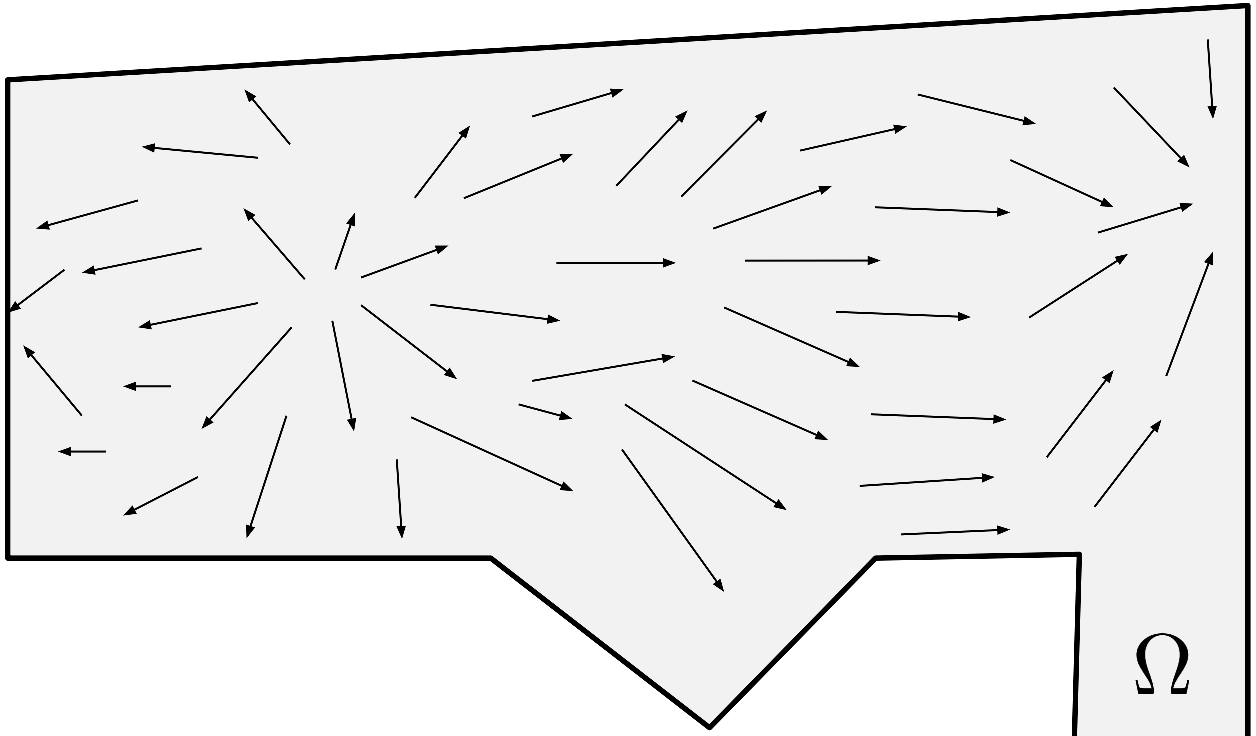
E.g.: flow of corn in the unit states

In some places there are **sources**:



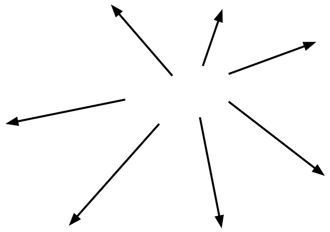
Divergence

Vector field $\mathbf{v}(x, y) = (v_x(x, y), v_y(x, y))$ in a region Ω

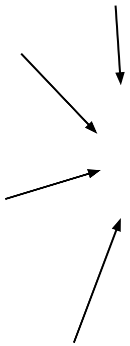


E.g.: flow of corn in the unit states

In some places there are **sources**:



Other places have **sinks**:

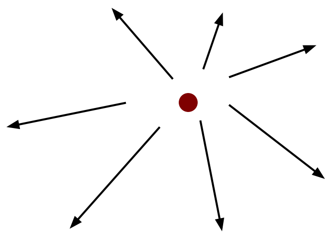


Divergence

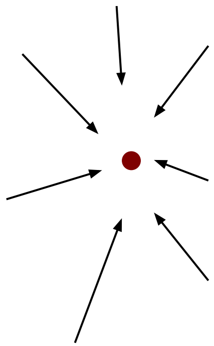
$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$$

Vector field $\mathbf{v}(x, y) = (v_x(x, y), v_y(x, y))$ in a region Ω

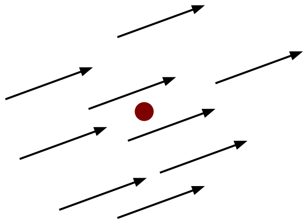
The **divergence** $\nabla \cdot \mathbf{v}$ measures how much a point is a source or sink



$\nabla \cdot \mathbf{v} > 0$



$\nabla \cdot \mathbf{v} < 0$



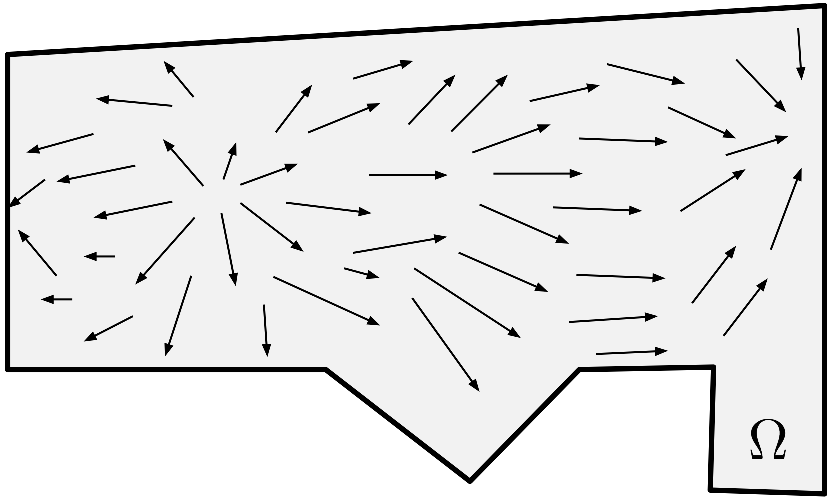
$\nabla \cdot \mathbf{v} = 0$

Divergence in Other Dimensions

Generalizes in the obvious way:

- 2D: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}$
- 3D: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$
- 1D: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} = v'$

Stokes's Theorem

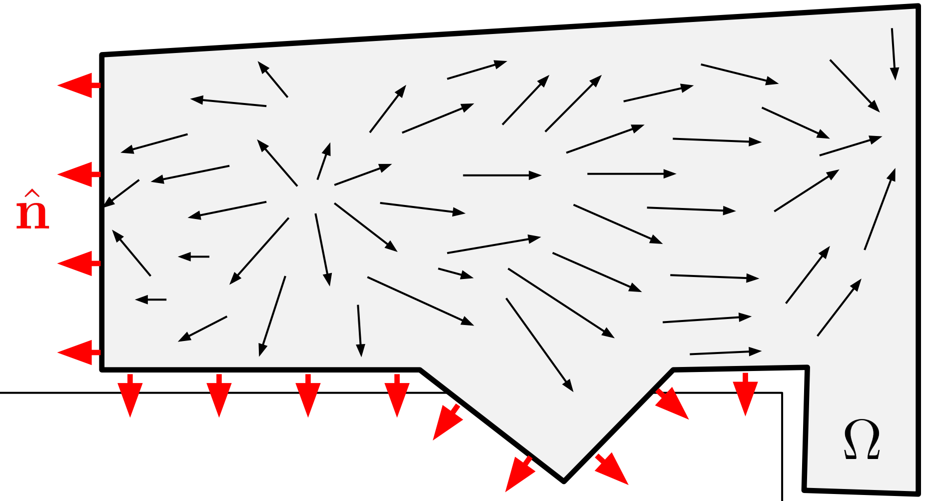


Let's say we want to measure the rate that corn is imported/exported from the United States

Approach #1: compute the divergence at every **interior** point and integrate it up:

$$\int_\Omega (\nabla \cdot \mathbf{v})(x, y)\, dA$$

Stokes's Theorem

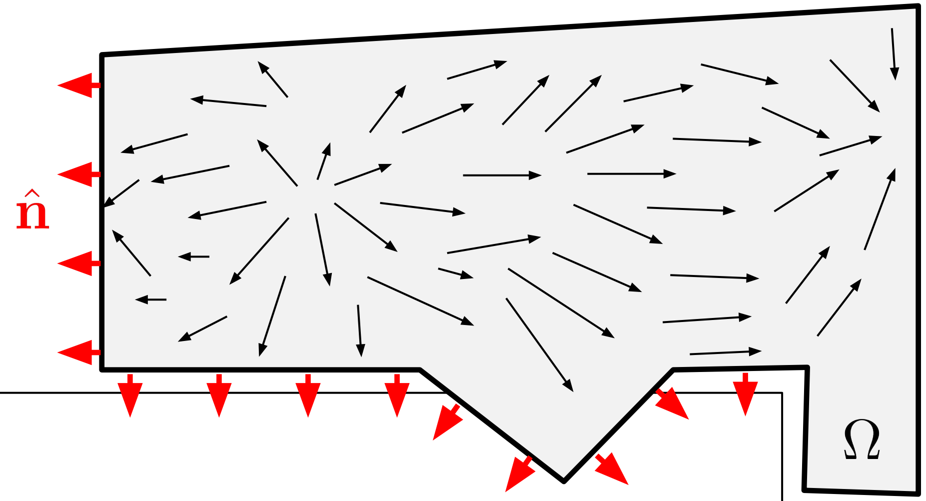


Let's say we want to measure the rate that corn is imported/exported from the United States

Approach #2: measure how much corn is entering or leaving through the border:

$$\int_{\partial\Omega} \mathbf{v}(x,y) \cdot \hat{\mathbf{n}}(x,y)\, ds$$

Stokes's Theorem



Let's say we want to measure the rate that corn is imported/exported from the United States

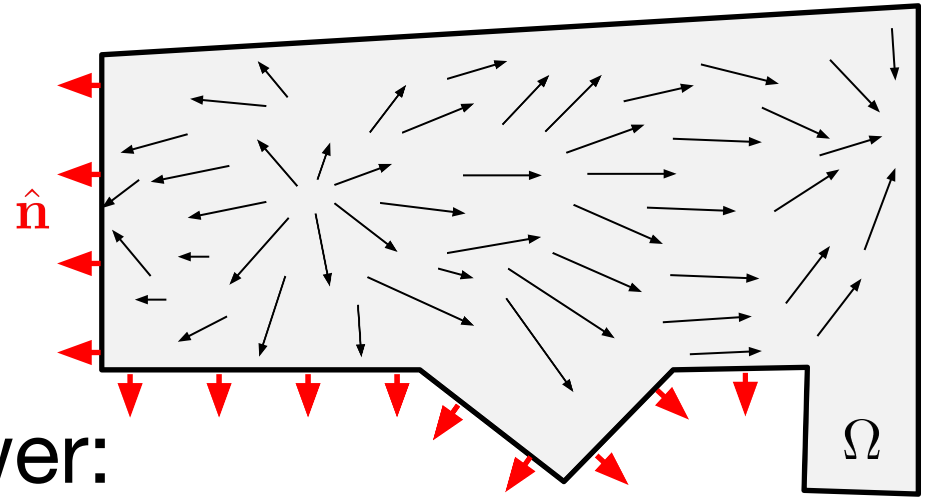
Approach #2: measure how much corn is entering or leaving through the border:

$$\int_{\partial\Omega} \mathbf{v}(x, y) \cdot \hat{\mathbf{n}}(x, y)\, ds$$

boundary of region

outward-pointing normal

Stokes's Theorem

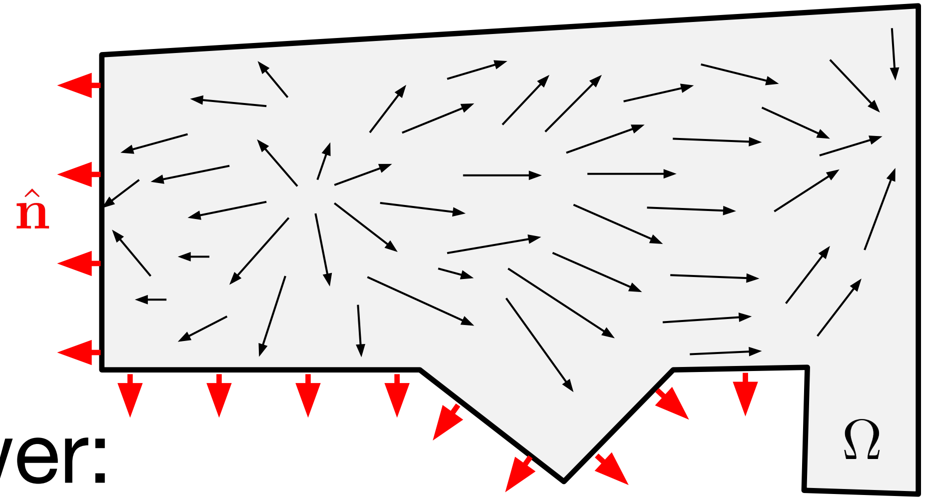


Both methods must give same answer:

[redacted]

this is the (divergence form of) Stokes's Theorem

Stokes's Theorem



Both methods must give same answer:

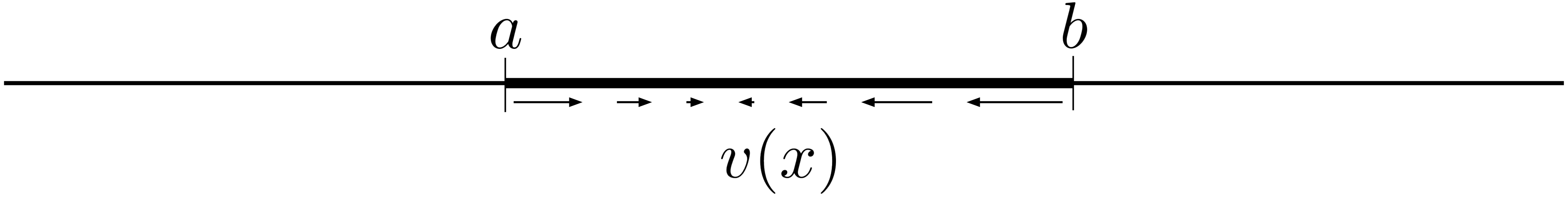
[redacted]

this is the (divergence form of) Stokes's Theorem

Turns annoying integrals on **interior** into much easier formulas on **boundary**

Stokes's Theorem: 1D Case

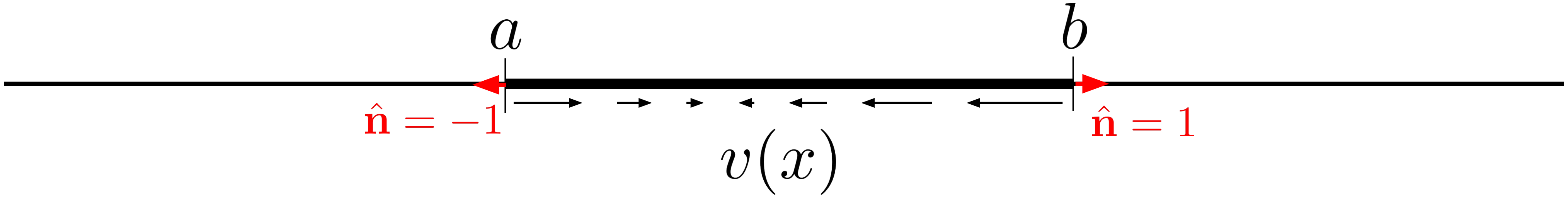
$$\Omega = [a, b]$$



[redacted]

Stokes's Theorem: 1D Case

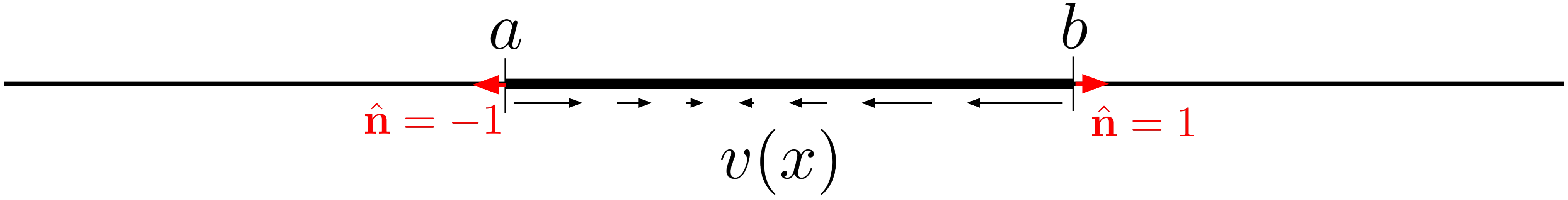
$$\Omega = [a, b]$$



[redacted]

Stokes's Theorem: 1D Case

$$\Omega = [a, b]$$

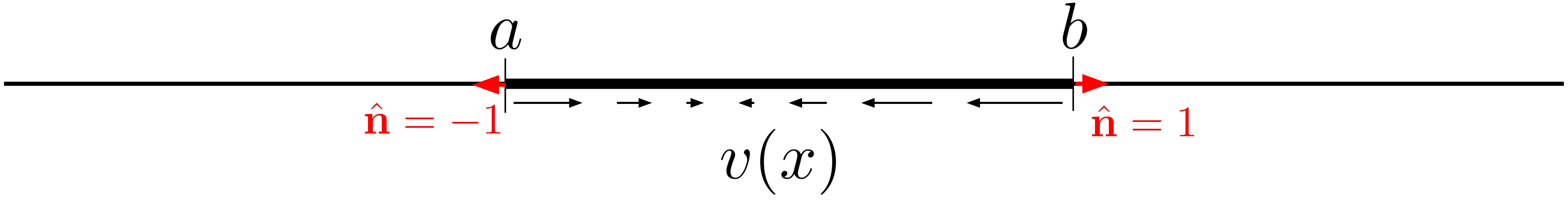


[redacted]

$$\int_a^b v'(x)\,dx$$

Stokes's Theorem: 1D Case

$$\Omega = [a, b]$$

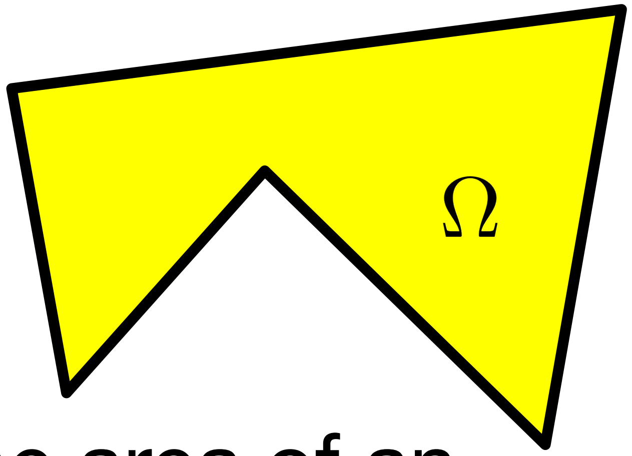


[redacted]

$$\int_a^b v'(x)\,dx$$

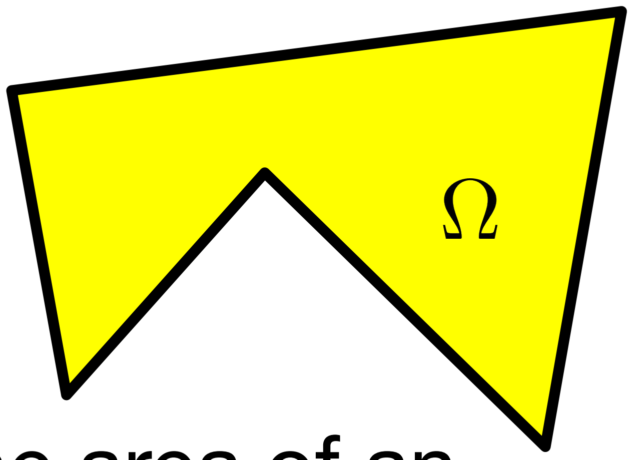
$$v(a) \cdot -1 + v(b) \cdot 1 = v(b) - v(a)$$

Application: Area of Polygon



Let's use Stokes's Theorem to compute the area of an arbitrary polygon

Application: Area of Polygon

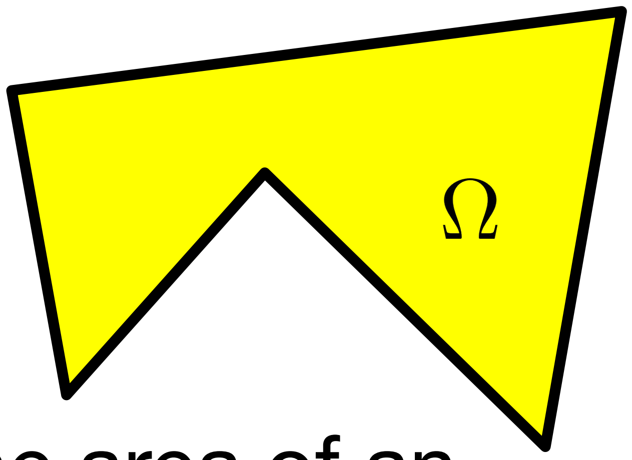


Let's use Stokes's Theorem to compute the area of an arbitrary polygon

Solution strategy:

1. Express area as integral over the **interior** of polygon
2. Write the integrand as divergence of a vector field
3. Use Stokes's Theorem to move integral to boundary
4. Simplify the boundary integral

Application: Area of Polygon



Let's use Stokes's Theorem to compute the area of an arbitrary polygon

$$A = \int_\Omega 1 \, dA$$

Solution strategy:

1. Express area as integral over the **interior** of polygon
2. Write the integrand as divergence of a vector field
3. Use Stokes's Theorem to move integral to boundary
4. Simplify the boundary integral

$$A = \int_\Omega 1 \, dA$$

Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x, y)$ and $v_y(x, y)$ satisfying $\nabla \cdot \mathbf{v} = 1$

$$A = \int_\Omega 1 \, dA$$

Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x, y)$ and $v_y(x, y)$ satisfying $\nabla \cdot \mathbf{v} = 1$

- one solution: $\mathbf{v}(x, y) = (x/2, y/2)$

$$A = \int_\Omega 1\, dA$$

Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x, y)$ and $v_y(x, y)$ satisfying $\nabla \cdot \mathbf{v} = 1$

- one solution: $\mathbf{v}(x, y) = (x/2, y/2)$
- another: $\mathbf{v}(x, y) = (x, 0)$

$$A = \int_{\Omega} 1 \, dA$$

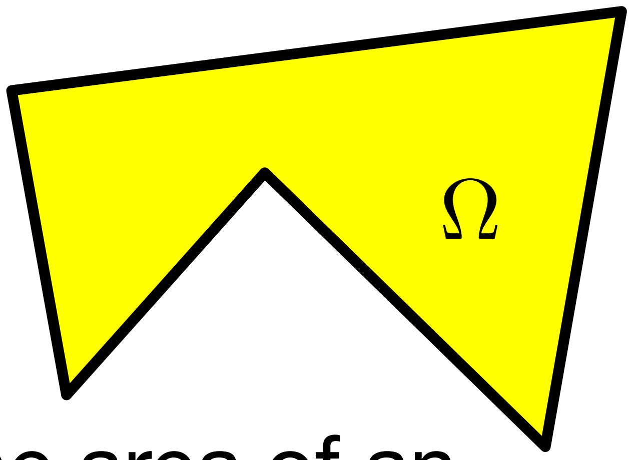
Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x, y)$ and $v_y(x, y)$ satisfying $\nabla \cdot \mathbf{v} = 1$

- one solution: $\mathbf{v}(x, y) = (x/2, y/2)$
- another: $\mathbf{v}(x, y) = (x, 0)$
- another: $\mathbf{v}(x, y) = \left(\sin \sqrt{e^{\tan y + 2}}, y\right)$

Application: Area of Polygon



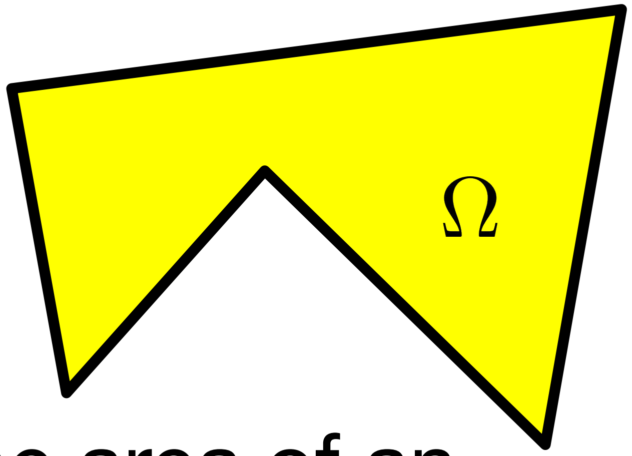
Let's use Stokes's Theorem to compute the area of an arbitrary polygon

$$A = \int_\Omega 1\, dA = \int_\Omega \nabla \cdot (x/2, y/2)\, dA$$

Solution strategy:

1. Express area as integral over the **interior** of polygon
2. <u>Write the integrand as divergence of a vector field</u>
3. Use Stokes's Theorem to move integral to boundary
4. Simplify the boundary integral

Application: Area of Polygon

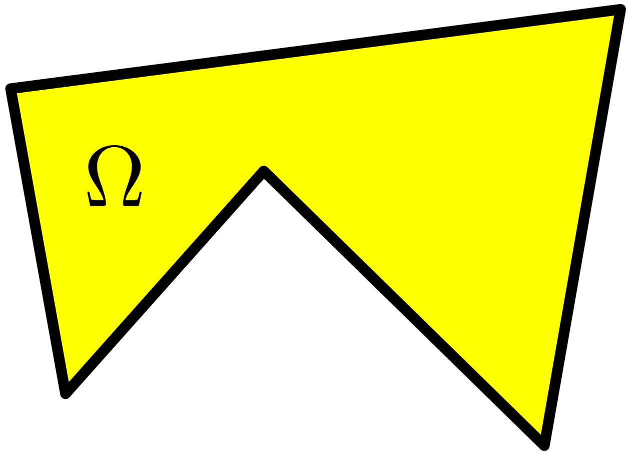


Let's use Stokes's Theorem to compute the area of an arbitrary polygon

Solution strate[redacted]

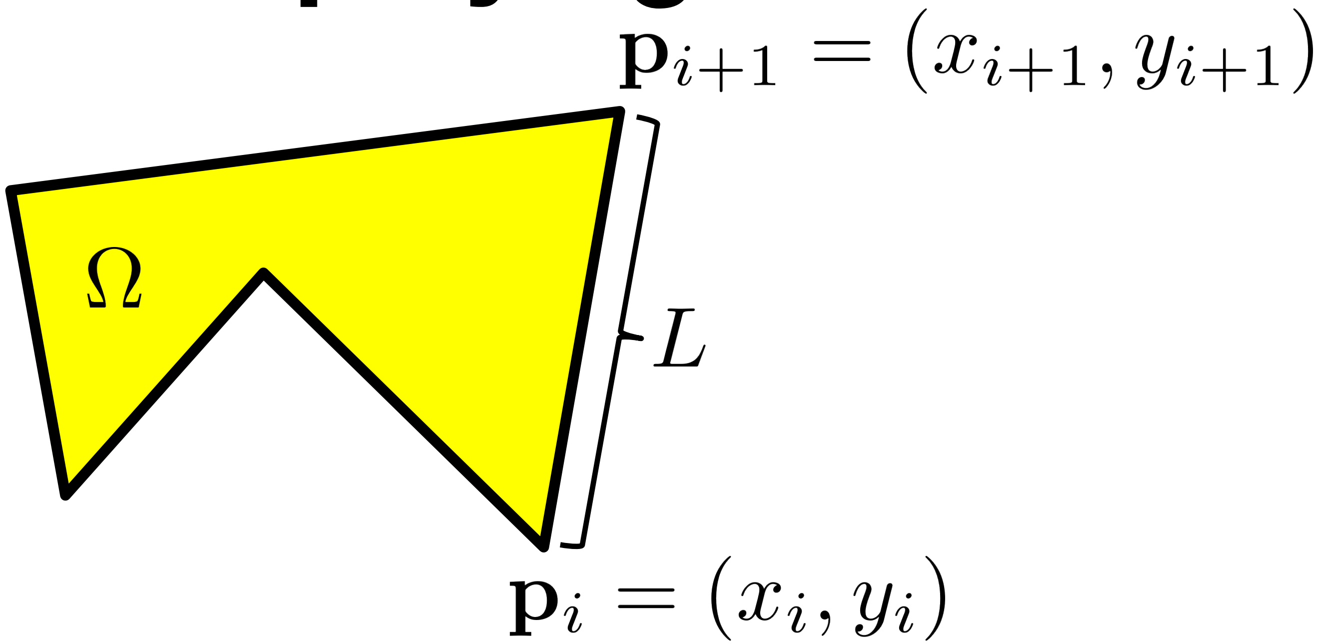
1. Express area as integral over the **interior** of polygon
2. Write the integrand as divergence of a vector field
3. <u>Use Stokes's Theorem to move integral to boundary</u>
4. Simplify the boundary integral

Simplifying the Boundary Integral



[redacted]

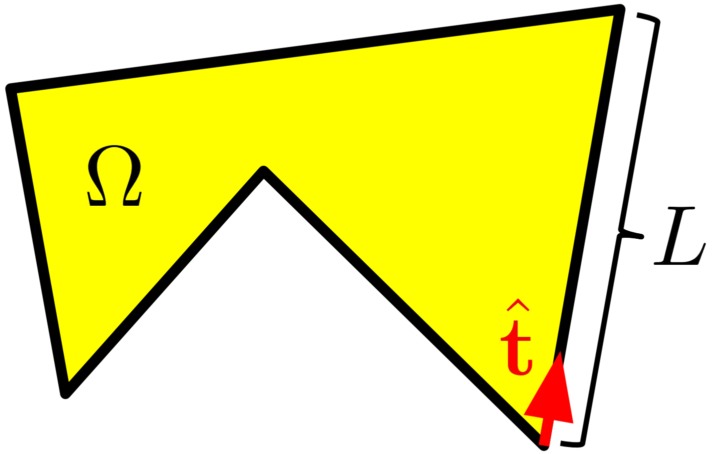
Simplifying the Boundary Integral



[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



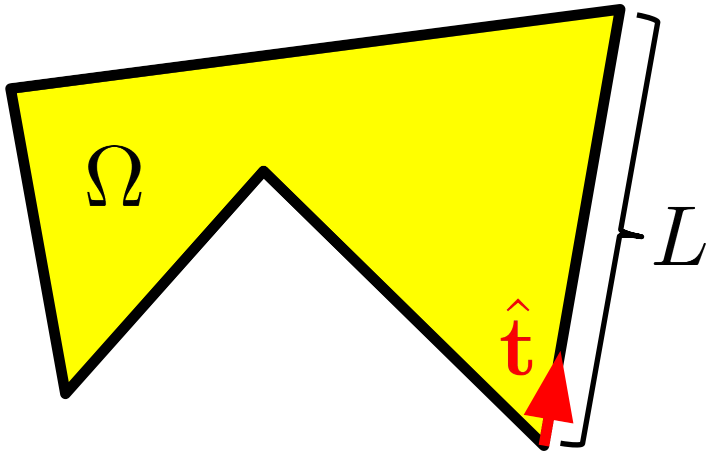
$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



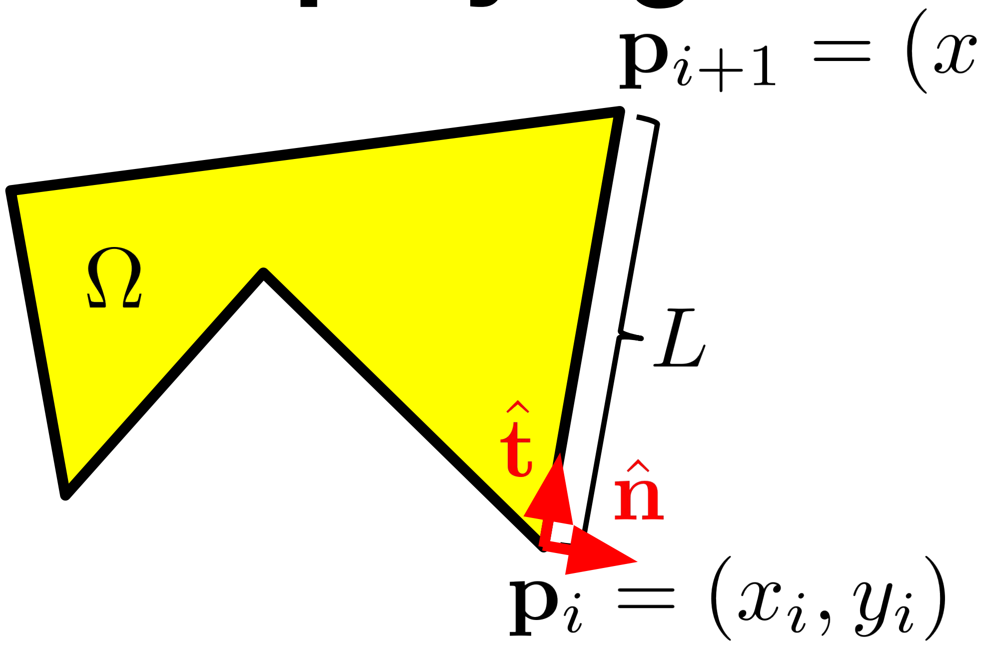
$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

$$\int_0^L (\mathbf{p}_i + s\hat{\mathbf{t}}) \cdot \hat{\mathbf{n}}\, ds$$

[redacted]

Simplifying the Boundary Integral



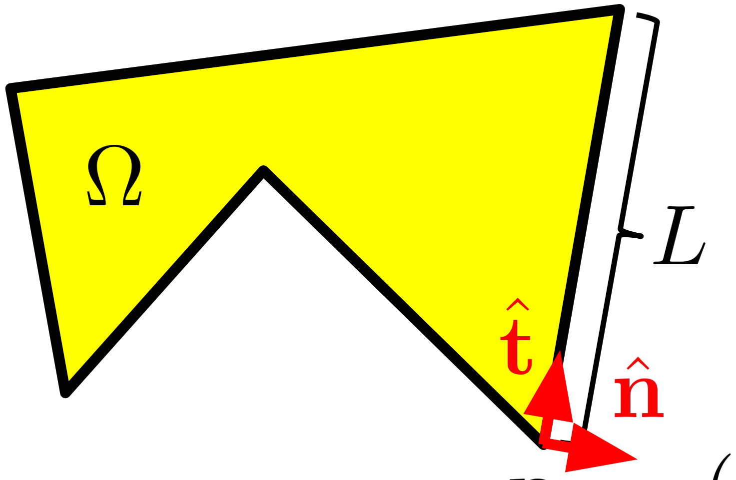
$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

$$\underbrace{\int_0^L (\mathbf{p}_i + s\hat{\mathbf{t}}) \cdot \hat{\mathbf{n}} \, ds}$$

[redacted]

Simplifying the Boundary Integral

$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$



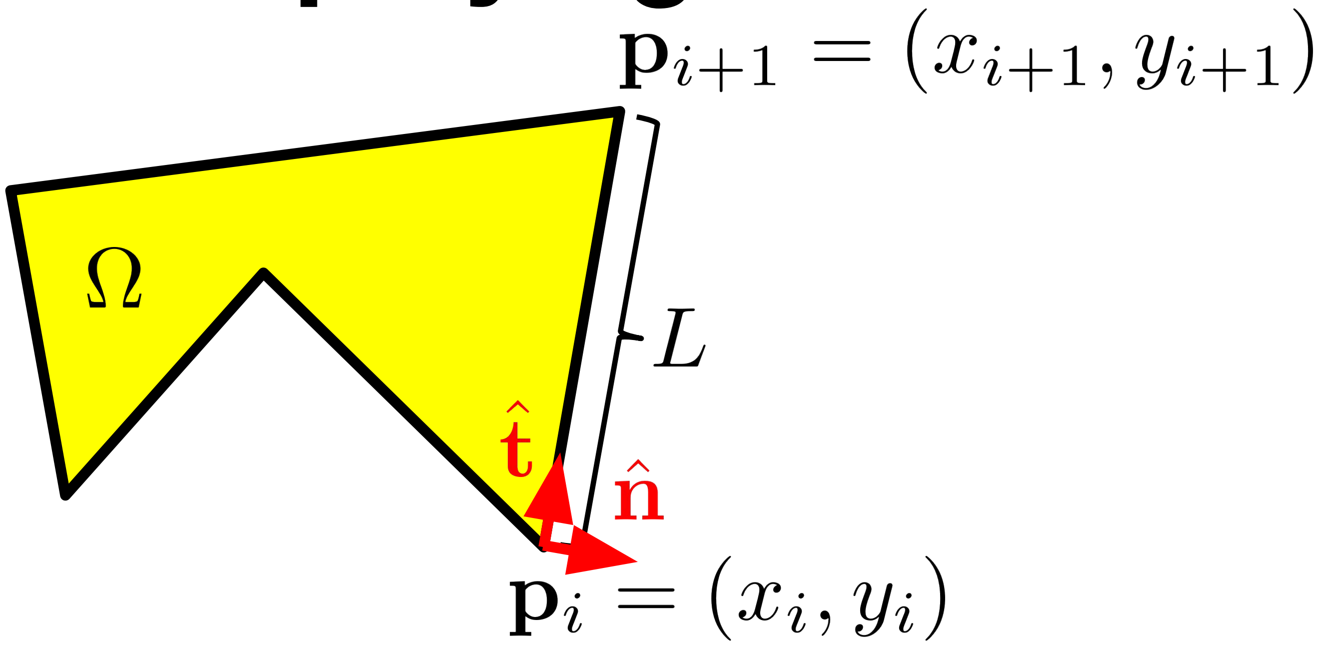
$\mathbf{p}_i = (x_i, y_i)$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

$$\underbrace{\int_0^L (\mathbf{p}_i + \cancel{s\hat{\mathbf{t}}}) \cdot \hat{\mathbf{n}}\, ds}$$

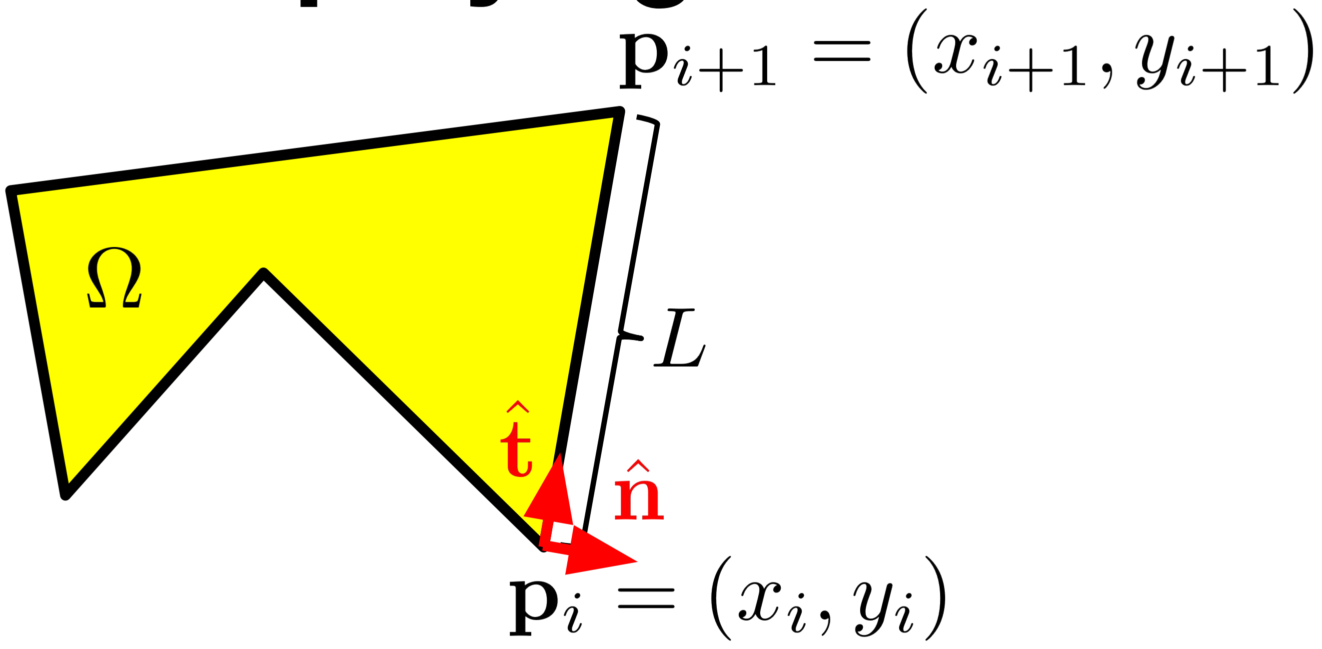
[redacted]

Simplifying the Boundary Integral



$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

Simplifying the Boundary Integral



$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

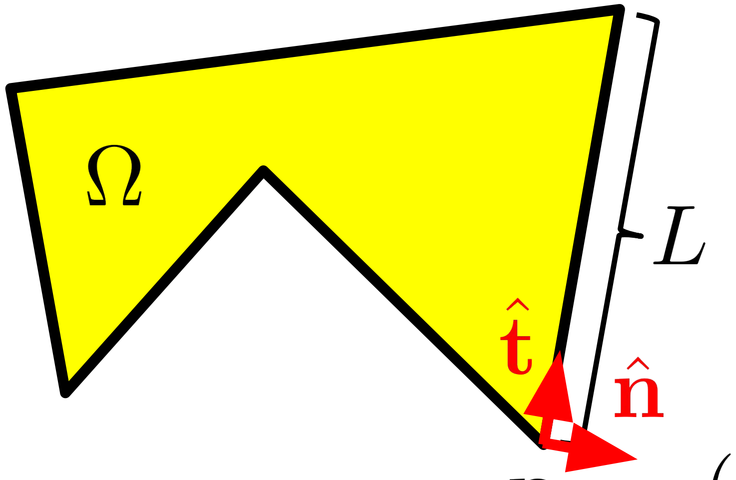
$$\hat{\mathbf{n}} = \hat{\mathbf{t}}^{\perp} = \frac{\mathbf{p}_{i+1}^{\perp} - \mathbf{p}_i^{\perp}}{L}$$

$L\mathbf{p}_i \cdot \hat{\mathbf{n}}$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

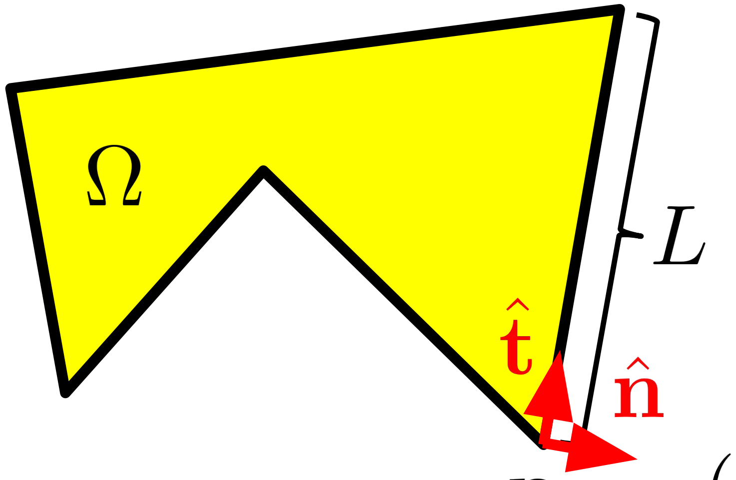
$$\hat{\mathbf{n}} = \hat{\mathbf{t}}^{\perp} = \frac{\mathbf{p}_{i+1}^{\perp} - \mathbf{p}_i^{\perp}}{L}$$

$$\mathbf{p}_i \cdot (\mathbf{p}_{i+1}^{\perp} - \mathbf{p}_i^{\perp})$$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

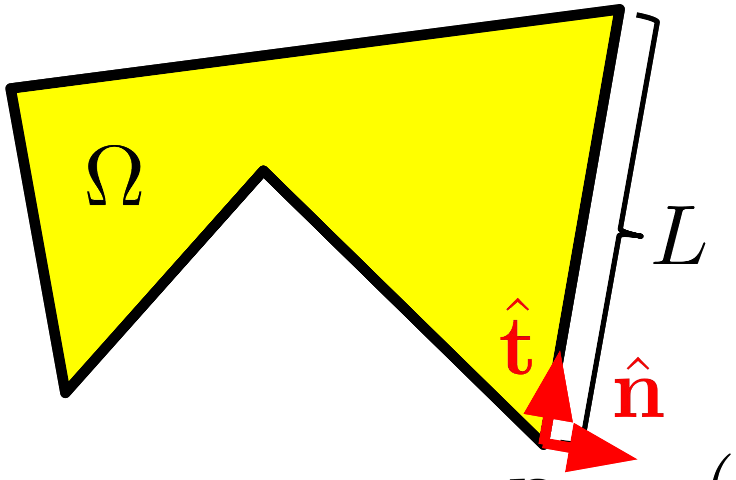
$$\hat{\mathbf{n}} = \hat{\mathbf{t}}^\perp = \frac{\mathbf{p}_{i+1}^\perp - \mathbf{p}_i^\perp}{L}$$

$$\mathbf{p}_i \cdot \mathbf{p}_{i+1}^\perp$$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

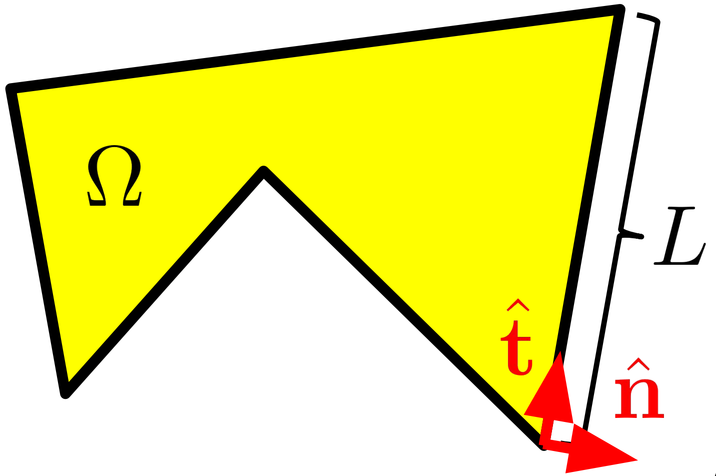
$$\hat{\mathbf{n}} = \hat{\mathbf{t}}^{\perp} = \frac{\mathbf{p}_{i+1}^{\perp} - \mathbf{p}_i^{\perp}}{L}$$

$$(x_i, y_i) \cdot (y_{i+1}, -x_{i+1})$$

[redacted]

Simplifying the Boundary Integral

$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$



$\mathbf{p}_i = (x_i, y_i)$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

$$\hat{\mathbf{n}} = \hat{\mathbf{t}}^\perp = \frac{\mathbf{p}_{i+1}^\perp - \mathbf{p}_i^\perp}{L}$$

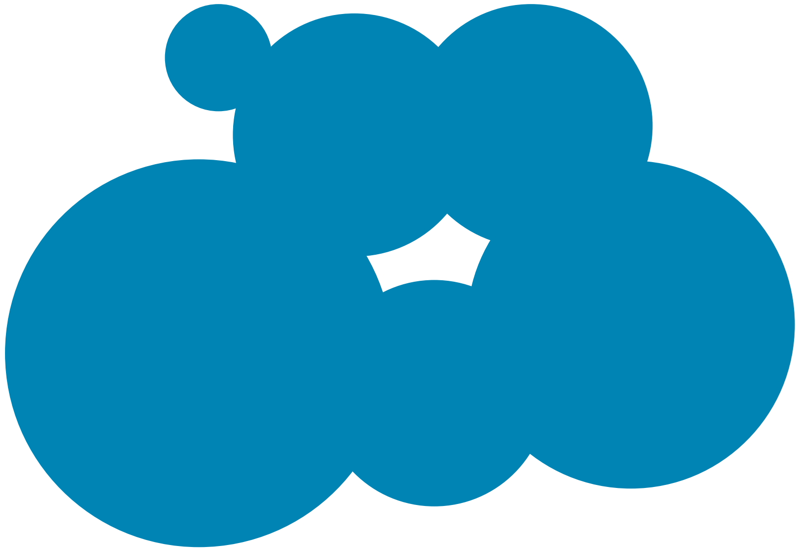
$x_i y_{i+1} - y_i x_{i+1}$

[redacted]

"shoelace formula"

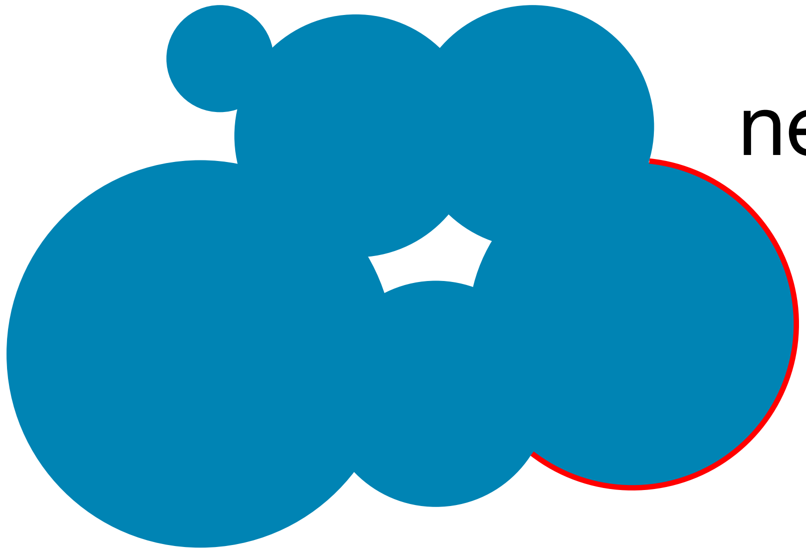
Example 2: Union of Balls

Calculate the area of a union of balls of different radii:



Example 2: Union of Balls

Calculate the area of a union of balls of different radii:

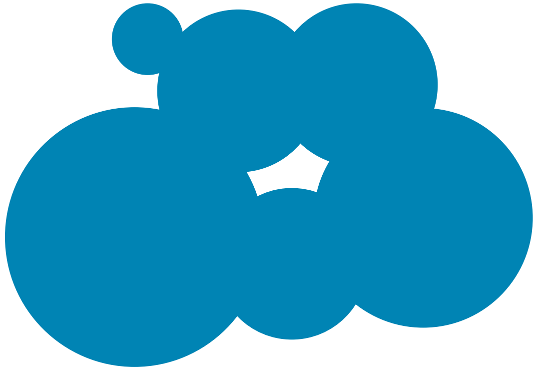


need a “shoelace formula”
for circular arcs

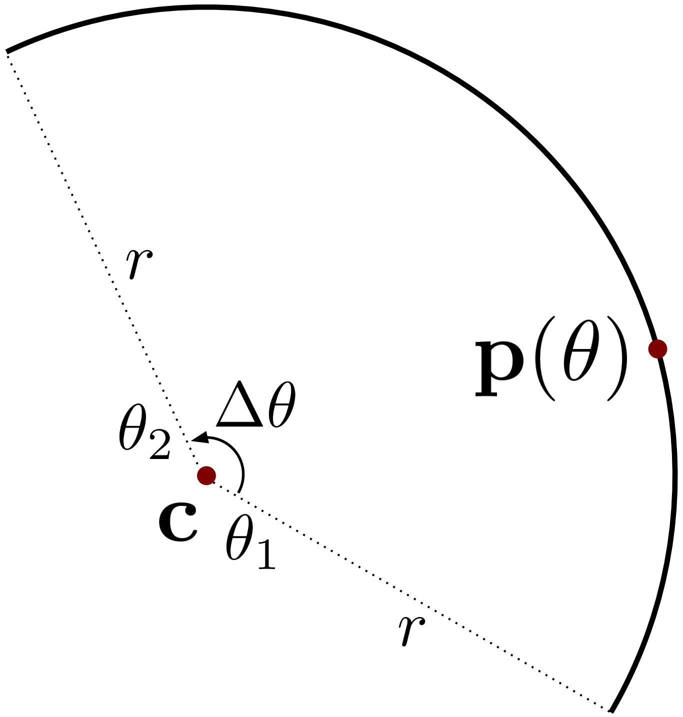
We already know that

[redacted]

Example 2: Union of Balls



Given a circular arc:

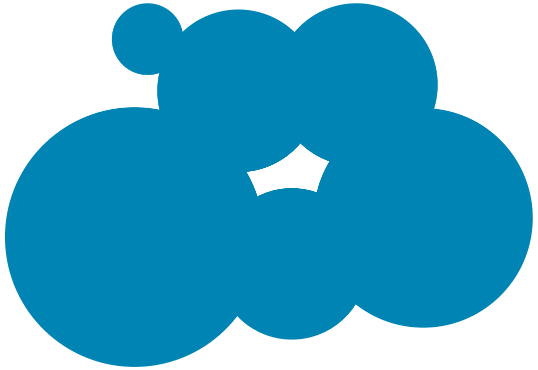


We need to compute

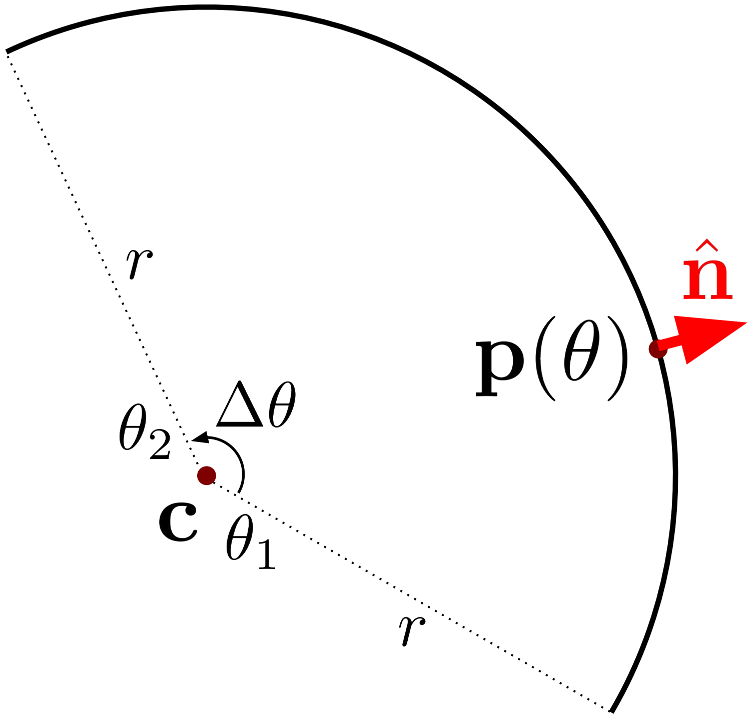
$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

idea: parameterize boundary by angle θ

Example 2: Union of Balls



Given a circular arc:

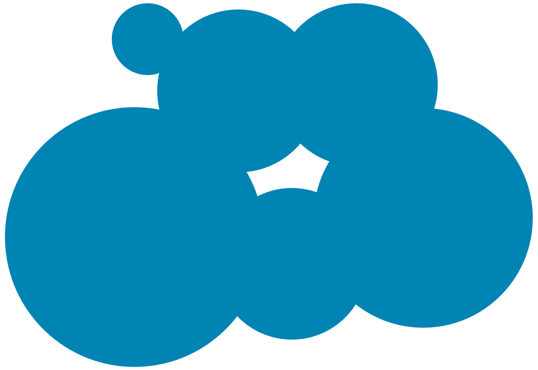


We need to compute $\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$

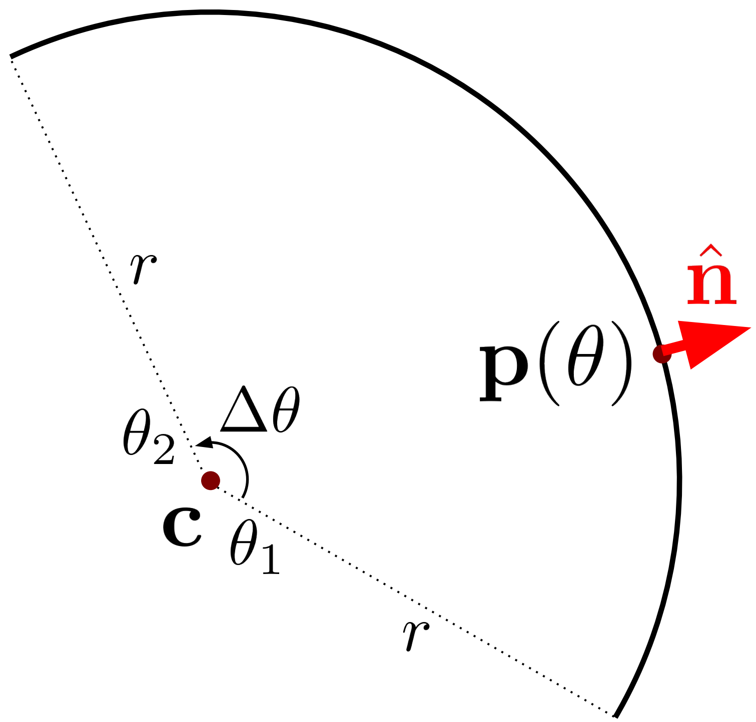
idea: parameterize boundary by angle θ

$$\mathbf{p}(\theta) = (c_x + r\cos\theta, c_y + r\sin\theta)$$

Example 2: Union of Balls



Given a circular arc:



We need to compute

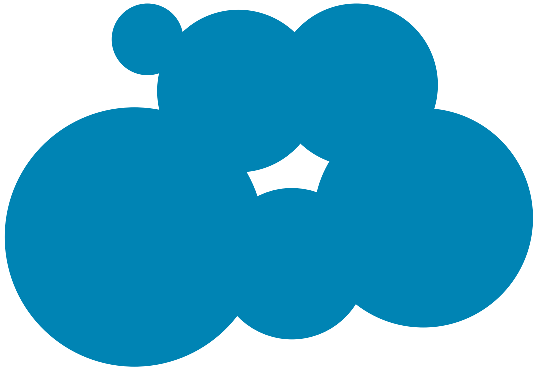
$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

idea: parameterize boundary by angle θ

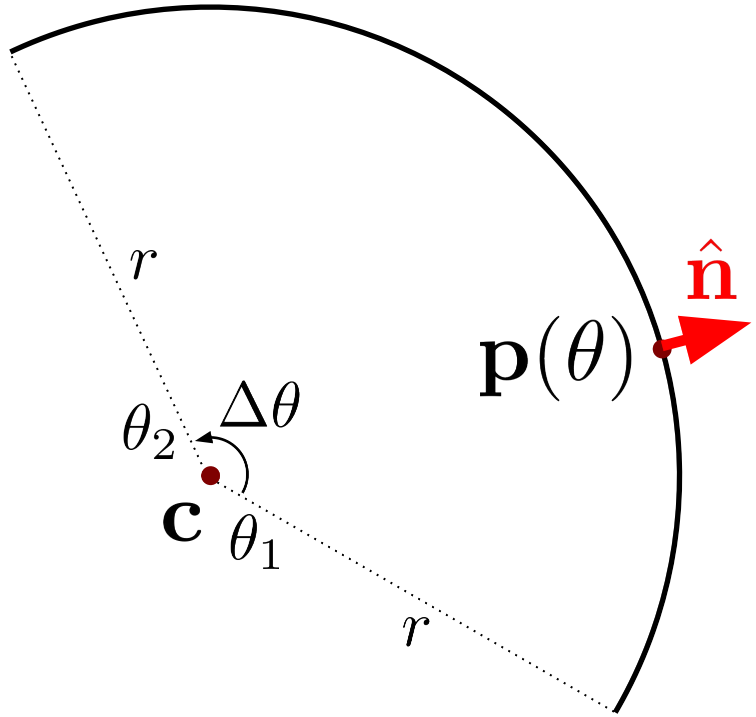
$$\mathbf{p}(\theta) = (c_x + r\cos\theta, c_y + r\sin\theta)$$

$$\hat{\mathbf{n}}(\theta) = (\cos\theta, \sin\theta)$$

Example 2: Union of Balls



Given a circular arc:



We need to compute

$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

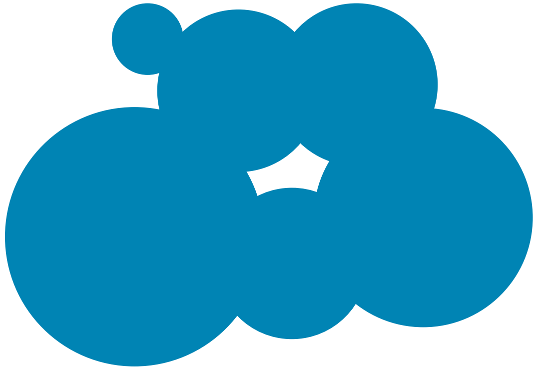
idea: parameterize boundary by angle θ

$$\mathbf{p}(\theta) = (c_x + r\cos\theta, c_y + r\sin\theta)$$

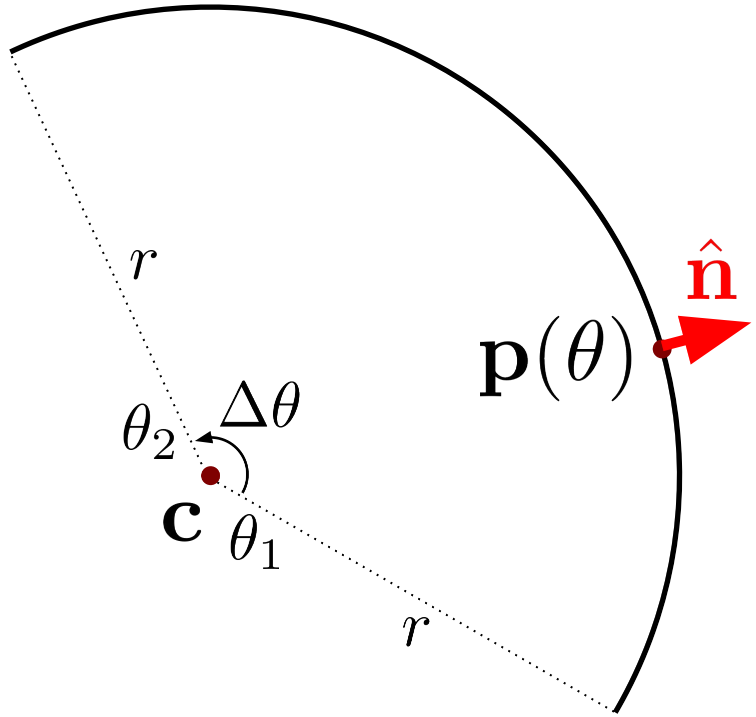
$$\hat{\mathbf{n}}(\theta) = (\cos\theta, \sin\theta)$$

$$ds = r\, d\theta$$

Example 2: Union of Balls



Given a circular arc:



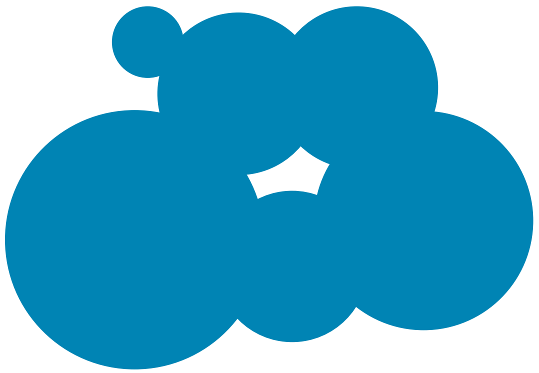
We need to compute

$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

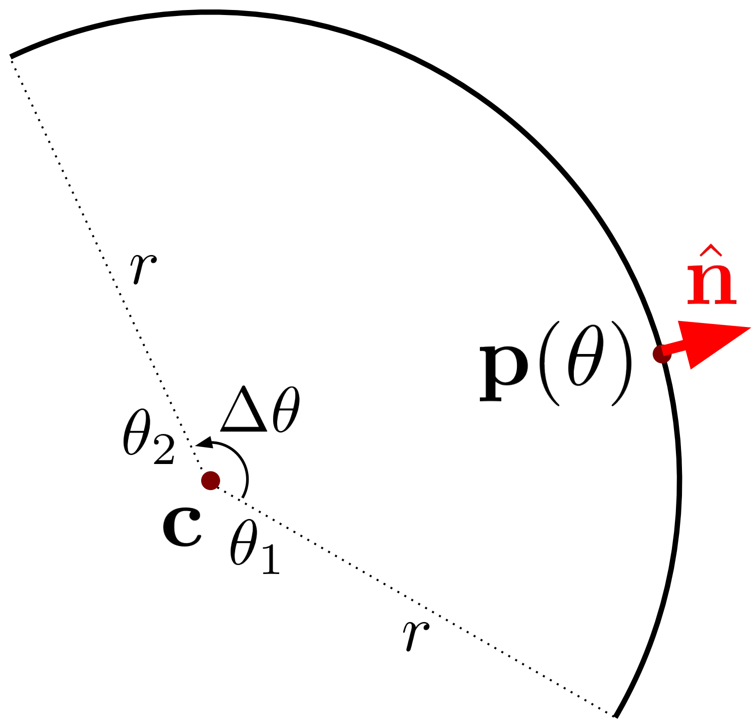
[redacted]

$$\mathbf{p}(\theta) = (c_x + r\cos\theta, c_y + r\sin\theta)$$
$$\hat{\mathbf{n}}(\theta) = (\cos\theta, \sin\theta)$$
$$ds = r\, d\theta$$

Example 2: Union of Balls



Given a circular arc:



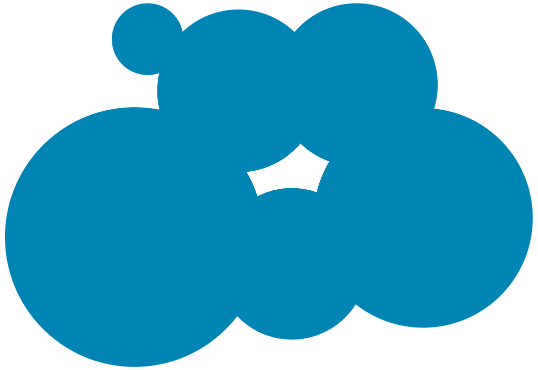
We need to compute

$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

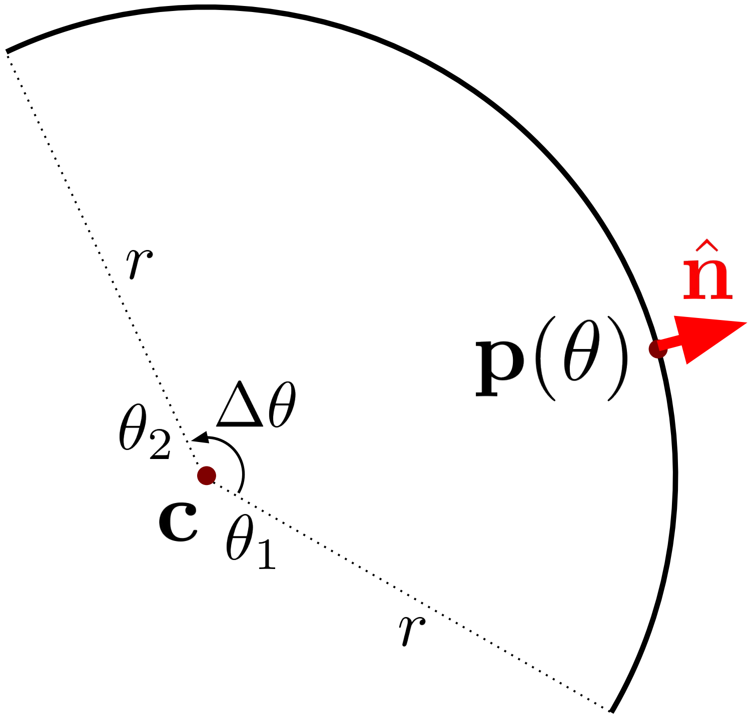
[redacted]

$$= r \int_{\theta_1}^{\theta_2} (c_x \cos\theta + c_y \sin\theta + r)\, d\theta$$

Example 2: Union of Balls



Given a circular arc:



We need to compute

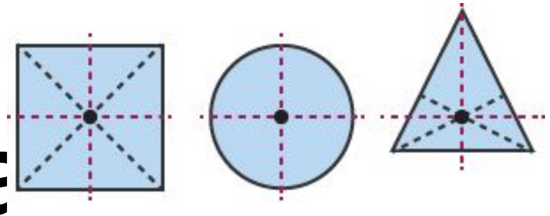
$$\int_{\partial\Omega} (x, y) \cdot \hat{\mathbf{n}}\, ds$$

[redacted]

$$= r \int_{\theta_1}^{\theta_2} (c_x \cos\theta + c_y \sin\theta + r)\, d\theta$$

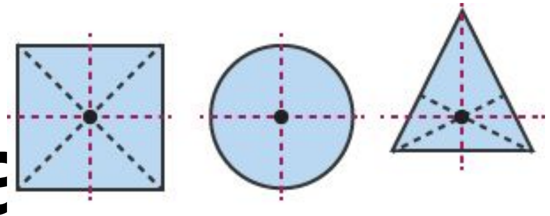
[redacted]

Example 3: Center of Ma

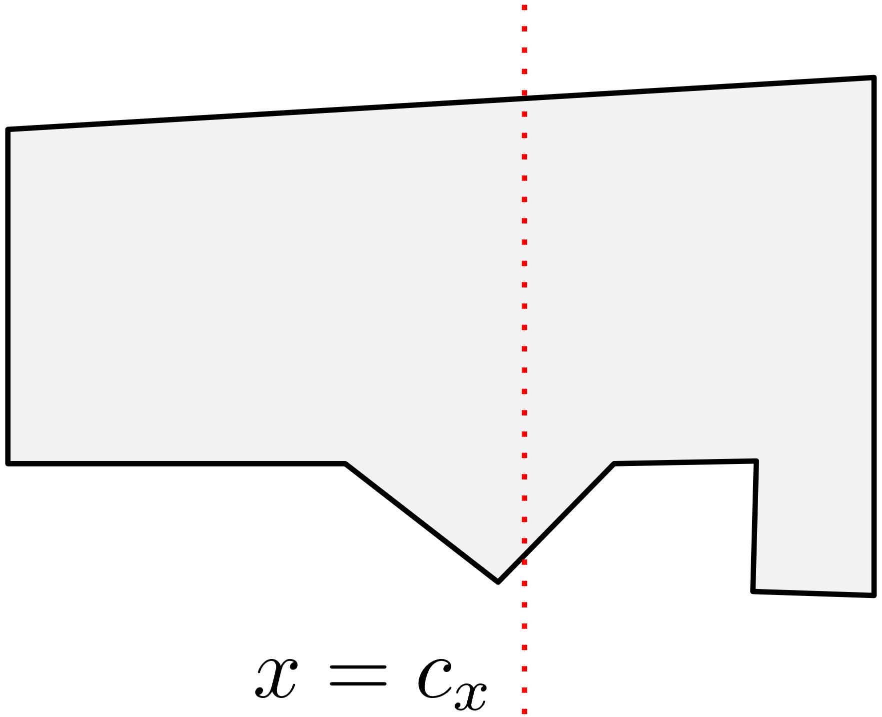


Given a polygon, find the vertical line $x = c_x$ that divides the area exactly in half

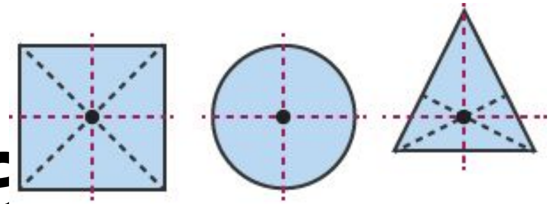
Example 3: Center of Ma



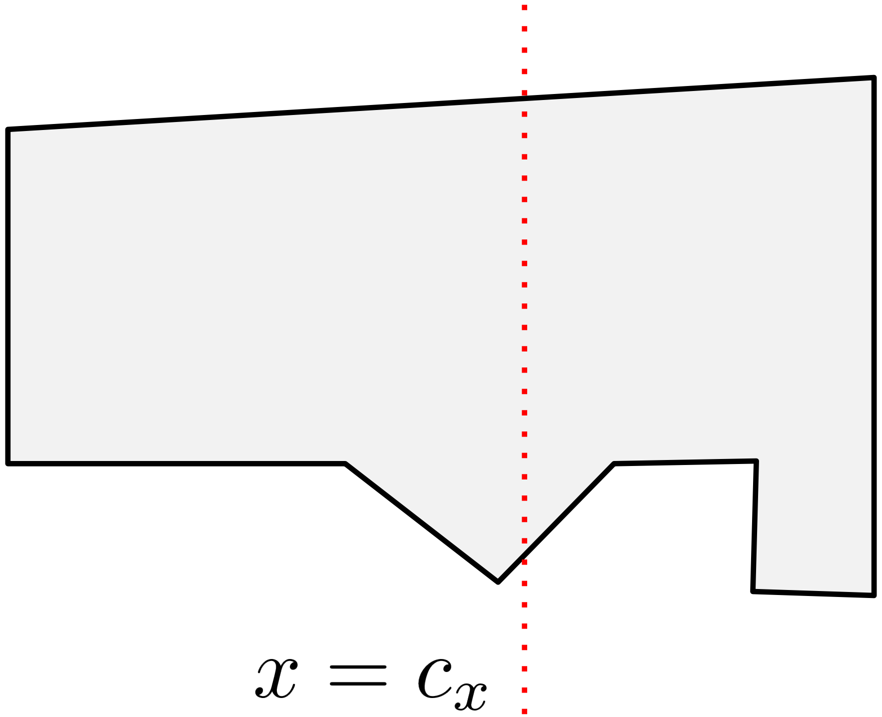
Given a polygon, find the vertical line $x = c_x$ that divides the area exactly in half



Example 3: Center of Ma

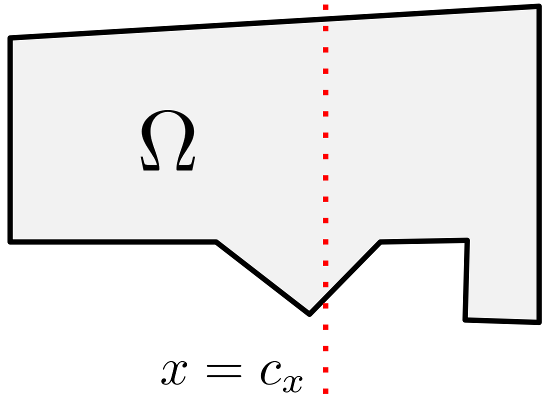


Given a polygon, find the vertical line $x = c_x$ that divides the area exactly in half



Several methods possible:

- linear sweep
- binary search
- Stokes's Theorem



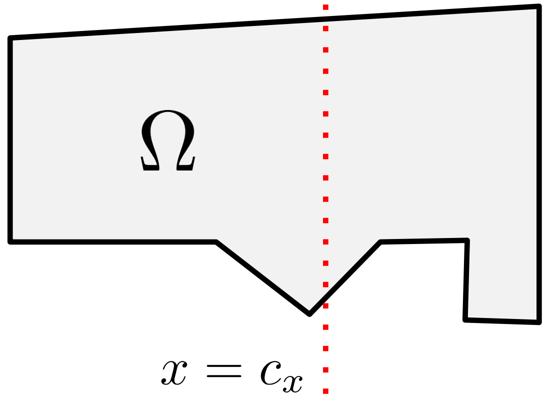
Example 3: Center of Mass

Given a polygon, find the vertical line $x = c_x$ that divides the area exactly in half

Solution strategy:

1. Express c_x as integral over the **interior** of polygon
2. Write the integrand as divergence of a vector field
3. Use Stokes's Theorem to move integral to boundary
4. Simplify the boundary integral

Example 3: Center of Mass



Given a polygon, find the vertical line $x = c_x$ that divides the area exactly in half

$$c_x = \frac{1}{A}\int_{\Omega} x\, dA$$

Solution strategy:

1. Express c_x as integral over the **interior** of polygon
2. Write the integrand as divergence of a vector field
3. Use Stokes's Theorem to move integral to boundary
4. Simplify the boundary integral

$$c_x = \frac{1}{A}\int_\Omega x\,dA$$

Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x,y)$ and $v_y(x,y)$ satisfying $\nabla \cdot \mathbf{v} = x$

$$c_x = \frac{1}{A}\int_\Omega x\,dA$$

Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x, y)$ and $v_y(x, y)$ satisfying $\nabla \cdot \mathbf{v} = x$

- one solution: $\mathbf{v}(x, y) = (x^2/2, 0)$

$$c_x = \frac{1}{A}\int_\Omega x\,dA$$

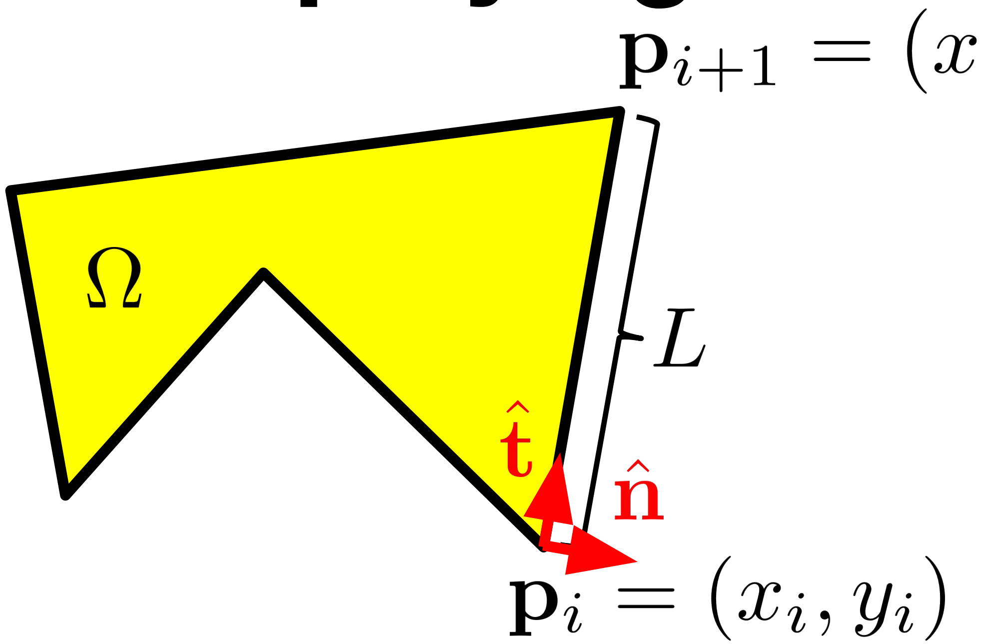
Reverse-Engineering the Vector Field

Recall: $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$

We need to find arbitrary functions $v_x(x,y)$ and $v_y(x,y)$ satisfying $\nabla \cdot \mathbf{v} = x$

- one solution: $\mathbf{v}(x,y) = (x^2/2, 0)$
- another: $\mathbf{v}(x,y) = (0, xy)$

Simplifying the Boundary Integral

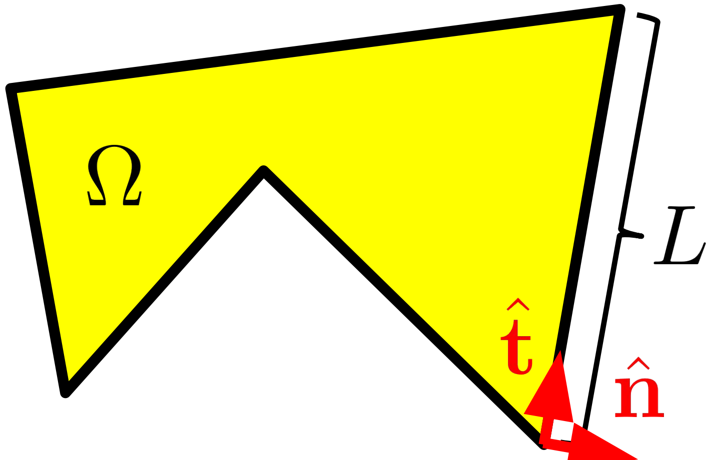


$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



$$\mathbf{p}_i = (x_i, y_i)$$

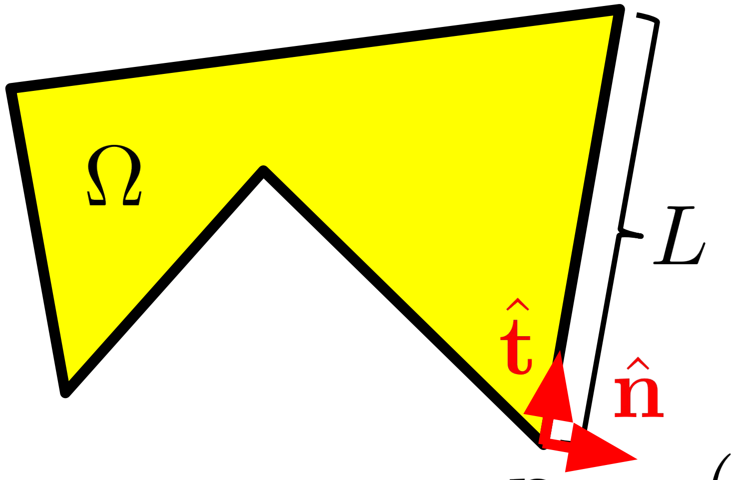
$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

$$\underbrace{\int_0^L \left([p_i]_x + s\hat{t}_x\right)^2 \hat{n}_x \, ds}$$

[redacted]

Simplifying the Boundary Integral

$$\mathbf{p}_{i+1} = (x_{i+1}, y_{i+1})$$



$$\mathbf{p}_i = (x_i, y_i)$$

$$\hat{\mathbf{t}} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{L}$$

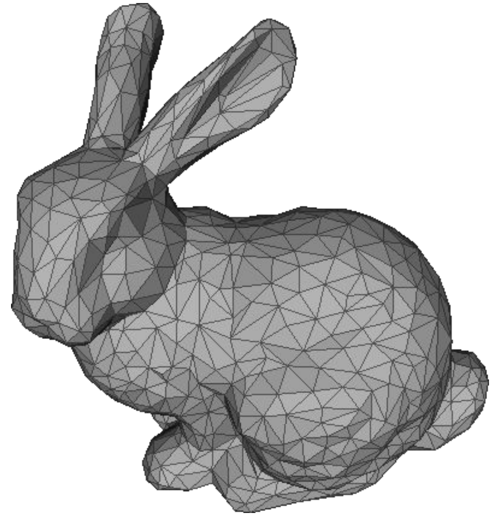
$$\hat{n}_x \left([p_i]_x^2 \, L + [p_i]_x \, \hat{t}_x L^2 + \frac{\hat{t}_x^2 L^3}{3} \right)$$

[redacted]

Other Applications of Stokes's Theorem

Calculating volumes, center of mass, etc. in 3D

- "shoelace formula" for polyhedral



Determining if a point is inside or outside a polygon/polyhedron

- by computing the **winding number**