M. D. GUNZBURGER
L. S. HOU
TH. P. SVOBODNY

Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with Dirichlet controls

<http://www.numdam.org/item?id=M2AN_1991__25_6_711_0>

ANALYSIS AND FINITE ELEMENT APPROXIMATION OF OPTIMAL CONTROL PROBLEMS FOR THE STATIONARY NAVIER-STOKES EQUATIONS WITH DIRICHLET CONTROLS (*)

M. D. GUNZBURGER ([1]), L. S. HOU ([2]) and Th. P. SVOBODNY ([3])

Communicated by R. TEMAM

Abstract. — *Optimal control problems for the stationary Navier-Stokes equations are examined from analytical and numerical points of view. The controls considered are of Dirichlet type, that is, control is effected through the velocity field on (or the mass flux through) the boundary ; the functionals minimized are either the viscous dissipation or the L^4-distance of candidate flows to some desired flow. We show that optimal solutions exist and justify the use of Lagrange multiplier techniques to derive a system of partial differential equations from which optimal solutions may be deduced. We study the regularity of solutions of this system. Then, finite element approximations of solutions of the optimality system are defined and optimal error estimates are derived.*

Résumé. — *On examine quelques problèmes de contrôle optimal des équations de Navier-Stokes du point de vue à la fois analytique et numérique. Le contrôle est du type condition de Dirichlet, c'est-à-dire qu'on choisit le champ de vecteurs vitesses sur la frontière pour minimiser une fonctionnelle. On considère ici des fonctionnelles de type fonction de dissipation qui mesurent l'effet de la traînée et une distance dans l'espace L^4. On démontre l'existence de solutions optimales et on utilise la méthode des multiplicateurs de Lagrange pour obtenir des conditions nécessaires d'optimalité. Après avoir établi quelques résultats concernant la régularité des solutions optimales, on définit des approximations par des espaces d'éléments finis et on présente les majorations d'erreur optimales.*

([1]) Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA.

([2]) Département de Mathématiques et de Statistiques, Université Laval, Québec, G1K 7P4, Canada. Present address : Department of mathematics and statistics, York University, North York, M3J 1P3, Canada.

([3]) Department of Mathematics and Statistics, Wright State University, Dayton, OH 45435, USA.

1. INTRODUCTION

Let $\mathbf{u}$, p and $\mathbf{g}$ denote the velocity, pressure, and control fields, respectively. Consider the functionals

$$\mathfrak{I}(\mathbf{u}, \mathbf{g}) = \frac{1}{4} \int_\Omega |\mathbf{u} - \mathbf{u}_0|^4 \, d\Omega + \frac{\nu}{2} \int_{\Gamma_c} (|\operatorname{grad}_s \mathbf{g}|^2 + |\mathbf{g}|^2) \, d\Gamma \quad (1.1)$$

and

$$\mathscr{K}(\mathbf{u}, p, \mathbf{g}) = \frac{\nu}{2} \int_\Omega |(\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T|^2 \, d\Omega$$

$$- \int_\Omega p \operatorname{div} \mathbf{u} \, d\Omega - \int_\Omega \mathbf{f} \cdot \mathbf{u} \, d\Omega + \frac{\nu}{2} \int_{\Gamma_c} (|\operatorname{grad}_s \mathbf{g}|^2 + |\mathbf{g}|^2) \, d\Gamma \quad (1.2)$$

where grad_s denotes the surface gradient operator. The first of these effectively measures the difference between the velocity field $\mathbf{u}$ and a prescribed field $\mathbf{u}_0$. The use of the $\mathbf{L}^4(\Omega)$-norm in (1.1) is discussed in Section 5. Except for the last term, the right hand side of (1.2) is the drag exerted by the fluid on the bounding surface of Ω. For a discussion of the relation between (1.2) and the drag, see [19]. Note that for incompressible flows, the term in (1.2) involving p vanishes, so that we could omit it. We choose to include it because it provides for a slight simplification in some of the considerations below.

The appearance of the control $\mathbf{g}$ in (1.1) and (1.2) is necessary since we will not impose any *a priori* constraints on the size of these controls. Reasons for our use of first derivatives of $\mathbf{g}$ in (1.1) and (1.2) are discussed in Sections 3.2 and 4.1. Problems such that the controls are constrained to belong to closed, convex, bounded sets of the underlying control spaces, including cases in which the control may be omitted from the functional to be minimized, are treated in [14].

Control problems in fluid mechanics are also considered by Abergel and Temam [1], wherein time dependent problems are treated. Their goal is to minimize the L^2-norm, in space and time, of the vorticity; the controls considered are of the distributed type as well as boundary velocities or temperatures.

The optimization problems we study are to seek state pairs $(\mathbf{u}, p)$ and controls $\mathbf{g}$ such that either one of $\mathfrak{I}(.,.)$ or $\mathscr{K}(.,.,.)$ is minimized, subject to the constraints

$$\nu \operatorname{div} ((\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T) + \mathbf{u} \cdot \operatorname{grad} \mathbf{u} + \operatorname{grad} p = \mathbf{f} \quad \text{in } \Omega, \quad (1.3)$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.4)$$

$$\mathbf{u} = \mathbf{b} \quad \text{on } \Gamma_u \quad (1.5)$$

and

$$\mathbf{u} = \mathbf{b} + \mathbf{g} \qquad \text{on } \Gamma_c, \tag{1.6}$$

i.e., $\mathbf{u}, p$ and $\mathbf{g}$ satisfy the Navier-Stokes equations (1.3), the incompressibility condition (1.4), and the inhomogeneous boundary conditions (1.5) and (1.6).

In (1.1)-(1.6), Ω denotes a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 with a boundary Γ; Γ_u and Γ_c are portions of Γ such that $\bar{\Gamma}_u \cup \bar{\Gamma}_c = \bar{\Gamma}$ and $\Gamma_u \cap \Gamma_c = \varnothing$. When finite element approximations are considered, we will assume that Ω is a convex polyhedral domain; otherwise, we will assume that either Ω is convex or Γ is of class $C^{1,1}$. In (1.3)-(1.6), v denotes the (constant) kinematic viscosity, $\mathbf{f}$ a given body force and $\mathbf{b}$ a given velocity field defined on the boundary. Thus Γ_c and Γ_u denote the portions of Γ where velocity controls are and are not applied, respectively. In (1.3) we have absorbed the constant density into the pressure and the body force. If the variables in (1.1)-(1.3) are nondimensionalized, then v is simply the inverse of the Reynolds number Re. Also note that since the density is a constant, the boundary conditions (1.5)-(1.6) also specify the mass flux at the boundary.

Some constraints are placed on candidate controls. Most notably, we will require that

$$\int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, d\Gamma = - \int_{\Gamma} \mathbf{b} \cdot \mathbf{n} \, d\Gamma = 0 \tag{1.7}$$

and, if Γ_c has a boundary,

$$\mathbf{g} = 0 \quad \text{on} \quad \partial \Gamma_c, \tag{1.8}$$

where $\partial \Gamma_c$ denotes the boundary of Γ_c, the latter viewed as a subset of Γ. The incompressibility constraint (1.4) necessitates the imposition of the compatibility condition given by the left equality in (1.7); we impose the right inequality only for the sake of simplifying the exposition. All our results hold equally well if the right equality in (1.7) is not assumed. The relation (1.8) is imposed in order to ensure that solutions of our optimization problems are « sufficiently » regular.

The only type of controls we allow are the velocity (or mass flux) on the boundary. Such a situation is common, e.g., one often attempts, through the suction or injection of fluid through orifices on the boundary, to reduce the viscous drag on a body moving through a fluid. Control may be effected in other ways, e.g., through the body force or the stress vector on the boundary. Such cases are treated in [15] and the results of that paper and the present one may be combined to deal with problems wherein more than one

type of control mechanism is employed. The treatment of the various types of controls is sufficiently different, both analytically and algorithmically, to warrant separate discussion.

In practical situations it is likely that the boundary condition (1.5) is imposed on only part of Γ_u. Thus, for example, one may also want to consider problems such that on part of Γ_u one specifies the stress force, or more generally, some components of the velocity and complementary components of the stress. In principle, there is no difficulty extending the results of this paper to such cases, provided the necessary existence, regularity and approximation results for analogous boundary-value problems for the Navier-Stokes equations are available. For example, for some combinations of velocity and stress boundary conditions, some care must be exercised in defining finite element approximations ; see [22]. In any case, the exposition is greatly simplified if we stick to the boundary condition (1.5).

The plan of the paper is as follows. In the remainder of this section we introduce the notation that will be used throughout the paper. Then, in section 2, we give a precise statement of the optimization problem for the functional (1.2) and prove that an optimal solution exists. In section 3, we prove the existence of Lagrange multipliers and then use the method of Lagrange multipliers to derive an optimality system. In that section we also study the regularity of solutions of the optimality system. In section 4, we consider finite element approximations and derive error estimates. In section 5, we briefly consider the optimization of the functional (1.1).

1.1. Notation

Throughout, C will denote a positive constant whose meaning and value changes with context. Also, $H^s(\mathcal{D})$, $s \in \mathbb{R}$, denotes the standard Sobolev space of order s with respect to the set $\mathcal{D}$, where $\mathcal{D}$ is either the flow domain Ω, or its boundary Γ, or part of that boundary. Of course, $H^0(\mathcal{D}) = L^2(\mathcal{D})$. Corresponding Sobolev spaces of vector-valued functions will be denoted by $\mathbf{H}^s(\mathcal{D})$, e.g., $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^d$. Dual spaces will be denoted by $(.)^*$.

Of particular interest will be the space

$$\mathbf{H}^1(\Omega) = \left\{ v_j \in L^2(\Omega) \, \middle| \, \frac{\partial v_j}{\partial x_k} \in L^2(\Omega) \text{ for } j, k = 1, ..., d \right\}$$

and the subspaces

$$\mathbf{H}_0^1(\Omega) = \{ \mathbf{v} \in \mathbf{H}^1(\Omega) \, | \, \mathbf{v} = \mathbf{0} \text{ on } \Gamma \}$$

and

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \middle| \int_\Omega q \, d\Omega = 0 \right\}.$$

For functions defined on Γ_c we will use the subspaces

$$\mathbf{W}(\Gamma_c) = \begin{cases} \mathbf{H}_0^1(\Gamma_c) & \text{if } \Gamma_c \text{ has a boundary} \\ \mathbf{H}_0^1(\Gamma_c) & \text{otherwise} \end{cases}$$

and

$$\mathbf{W}_n(\Gamma_c) = \begin{cases} \mathbf{H}_n^1(\Gamma_c) \cap \mathbf{H}_0^1(\Gamma_c) & \text{if } \Gamma_c \text{ has a boundary} \\ \mathbf{H}_n^1(\Gamma_c) & \text{otherwise}, \end{cases}$$

where

$$\mathbf{H}_n^1(\Gamma_c) = \left\{ \mathbf{g} \in \mathbf{H}^1(\Gamma_c) \middle| \int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, d\Gamma = 0 \right\},$$

and, whenever Γ_c has a boundary,

$$\mathbf{H}_0^1(\Gamma_c) = \left\{ \mathbf{g} \in \mathbf{H}^1(\Gamma_c) \middle| \mathbf{g} = \mathbf{0} \text{ on } \partial\Gamma_c \right\}.$$

Norms of functions belonging to $H^s(\Omega)$, $H^s(\Gamma)$ and $H^s(\Gamma_c)$ are denoted by $\|\cdot\|_s$, $\|\cdot\|_{s,\Gamma}$ and $\|\cdot\|_{s,\Gamma_c}$, respectively. Of particular interest are the $L^2(\Omega)$-norm $\|\cdot\|_0$ and the semi-norm

$$|v|_1^2 = \sum_{j=1}^d \left\| \frac{\partial v}{\partial x_j} \right\|_0^2$$

and norm

$$\|v\|_1^2 = |v|_1^2 + \|v\|_0^2$$

defined for functions belonging to $H^1(\Omega)$. Note that $|v|_1^2 + \|v\|_{0,\Gamma}^2$ defines a norm equivalent to $(\|v\|_1^2)$. Norms for spaces of vector valued functions will be denoted by the same notation as that used for their scalar counterparts. For example,

$$\|\mathbf{v}\|_{L^r(\Omega)}^r = \sum_{j=1}^d \|v_j\|_{L^r(\Omega)}^r \qquad \text{and} \qquad \|\mathbf{v}\|_1^2 = \sum_{j=1}^d \|v_j\|_1^2,$$

where v_j, $j = 1, ..., d$, denote the components of v. We note that the semi-norm $|\cdot|_1$, defined by either

$$|v|_1^2 = \sum_{j=1}^d |v_j|_1^2 \quad \text{or} \quad |v|_1^2 = \frac{1}{2} \int_\Omega |\operatorname{grad} v + (\operatorname{grad} v)^T|^2 \, d\Omega \, ,$$

defines a norm, equivalent to $\|\cdot\|_1$, for functions belonging to $\mathbf{H}_0^1(\Omega)$. Also, the semi-norm $|\cdot|_{1,\Gamma_c}$, defined by

$$|g|_{1,\Gamma_c}^2 = \int_{\Gamma_c} |\operatorname{grad}_s g|^2 \, d\Gamma \, ,$$

defines an norm on $\mathbf{H}_0^1(\Gamma_c)$ equivalent to $\|g\|_{1,\Gamma_c}^2 = |g|_{1,\Gamma_c}^2 + \|g\|_{0,\Gamma_c}^2$.

We define, for $(pq) \in L^1(\Omega)$ and $(\mathbf{u} \cdot \mathbf{v}) \in L^1(\Omega)$,

$$(p, q) = \int_\Omega pq \, d\Omega \qquad \text{and} \qquad (\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \, d\Omega \, , \qquad (1.9)$$

respectively, for $(pq) \in L^1(\Gamma)$ and $(\mathbf{u} \cdot \mathbf{v}) \in L^1(\Gamma)$,

$$(p, q)_\Gamma = \int_\Gamma pq \, d\Gamma \qquad \text{and} \qquad (\mathbf{u}, \mathbf{v})_\Gamma = \int_\Gamma \mathbf{u} \cdot \mathbf{v} \, d\Gamma \, , \qquad (1.10)$$

respectively, and, for $(pq) \in L^1(\Gamma_c)$ and $(\mathbf{u} \cdot \mathbf{v}) \in L^1(\Gamma_c)$,

$$(p, q)_{\Gamma_c} = \int_{\Gamma_c} pq \, d\Gamma \qquad \text{and} \qquad (\mathbf{u}, \mathbf{v})_{\Gamma_c} = \int_{\Gamma_c} \mathbf{u} \cdot \mathbf{v} \, d\Gamma \, , \qquad (1.11)$$

respectively. Thus, the inner products in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are both denoted by $(\cdot, \cdot)$, those in $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ by $(\cdot, \cdot)_\Gamma$, and those in $L^2(\Gamma_c)$ and $\mathbf{L}^2(\Gamma_c)$ by $(\cdot, \cdot)_{\Gamma_c}$. Since, in general, we will use L^2-spaces as pivot spaces, the notation of (1.9)-(1.11) will also be employed to denote pairings between Sobolev spaces and their duals.

We will use the two bilinears forms

$$a(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_\Omega ((\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T) :$$

$$((\operatorname{grad} \mathbf{v}) + (\operatorname{grad} \mathbf{v})^T) \, d\Omega \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$$

and

$$b(\mathbf{v}, q) = - \int_\Omega q \operatorname{div} \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \quad \text{and} \quad \forall p \in L^2(\Omega)$$

and the trilinear form

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{u} \cdot \operatorname{grad} \mathbf{v} \cdot \mathbf{w} \, d\Omega \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega) \, .$$

These forms are continuous in the sense that there exist constants c_a, c_b and $c_c > 0$ such that

$$|a(\mathbf{u}, \mathbf{v})| \leqslant c_a \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{1.12}$$

$$|b(\mathbf{v}, q)| \leqslant c_b \|\mathbf{v}\|_1 \|q\|_0 \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \text{ and } q \in L^2(\Omega) \tag{1.13}$$

and

$$|c(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leqslant c_c \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}^1(\Omega). \tag{1.14}$$

Moreover, we have the coercivity properties

$$a(\mathbf{v}, \mathbf{v}) \geqslant C_a \|\mathbf{v}\|_1^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \tag{1.15}$$

and

$$\sup_{0 \neq \mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \geqslant C_b \|q\|_0 \quad \forall q \in L_0^2(\Omega), \tag{1.16}$$

for some constants C_a and $C_b > 0$.

For details concerning the notation employed and/or for (1.12)-(1.16), one may consult [2], [11], [12] and [20].

2. THE OPTIMIZATION PROBLEM AND THE EXISTENCE OF OPTIMAL SOLUTIONS

We begin by giving a precise statement of the optimization problem we consider. Let $\mathbf{g} \in \mathbf{W}_n(\Gamma_c)$ denote the boundary control and let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $p \in L_0^2(\Omega)$ denote the state, i.e., the velocity and pressure fields, respectively. The state and control variables are constrained to satisfy the system (1.3)-(1.6), which we recast into the following particular weak form (see, e.g., [3], [11], [12] or [20]) :

$$\nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - (\mathbf{v}, \mathbf{t})_\Gamma = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \tag{2.1}$$

$$b(\mathbf{u}, q) = 0 \quad \forall q \in L_0^2(\Omega) \tag{2.2}$$

and

$$(\mathbf{u}, \mathbf{s})_\Gamma - (\mathbf{g}, \mathbf{s})_{\Gamma_c} = (\mathbf{b}, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma), \tag{2.3}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{b} \in \mathbf{H}^1(\Gamma)$ are given functions. One may show that, in a distributional sense,

$$\mathbf{t} = [-p\mathbf{n} + \nu (\operatorname{grad} \mathbf{u} + (\operatorname{grad} \mathbf{u})^T) . \mathbf{n}]_\Gamma,$$

i.e., $\mathbf{t}$ is the stress force on the boundary.

Remark : We make some comments on the use, in the weak formulation (2.1)-(2.3), of the Lagrange multiplier **t** to enforce the boundary condition on the velocity. In the first place, there are technical reasons for this choice, the most important one appearing in the proof of the error estimates for finite element approximations. We will remark on this point further in Section 4. From a practical point of view, the introduction of the Lagrange multiplier **t** does not introduce any new difficulties. It was shown in [13], in the context of finite element approximations of solutions of the Navier-Stokes equations, that one may in fact uncouple the computation of the multiplier **t** from that of the velocity and pressure fields. Indeed, one may devise schemes such that one may solve (a discretization) of (2.3) for the velocity on the boundary, and then solve for **u** and *p* from (discretizations of) (2.1)-(2.3) by using (subspaces of) $\mathbf{H}_0^1(\Omega)$ in (a discretization of) (2.3). Subsequently, one may compute (an approximation to) **t**, if one so desires. (See [13] for details.) Moreover, since **t** is the stress on the boundary, this method provides a systematic mechanism for computing this interesting variable.

The functional (1.2), using the notation introduced in Section 1.1, is given by

$$\mathcal{K}(\mathbf{u}, p, \mathbf{g}) = \frac{\nu}{2} a(\mathbf{u}, \mathbf{u}) + b(\mathbf{u}, p) - (\mathbf{f}, \mathbf{u}) + \frac{\nu}{2} \|\mathbf{g}\|_{1, \Gamma_c}^2. \qquad (2.4)$$

(If Γ_c has a boundary we may replace the term $(\nu/2)\|\mathbf{g}\|_{1, \Gamma_c}^2$ by $(\nu/2)|\mathbf{g}|_{1, \Gamma_c}^2$.) Optimization problems involving the functional (1.1) will be considered in Section 5.

The *admissibility set* $\mathcal{U}_{ad}$ is defined by

$$\mathcal{U}_{ad} = \{ (\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c) :$$
$$\mathcal{K}(\mathbf{u}, p, \mathbf{g}) < \infty, \text{ and there exists a } \mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma) \qquad (2.5)$$
$$\text{such that } (2.1)\text{-}(2.3) \text{ are satisfied } \}.$$

Then, $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \in \mathcal{U}_{ad}$ is called an *optimal solution* if there exists $\varepsilon > 0$ such that

$$\mathcal{K}(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \leq \mathcal{K}(\mathbf{u}, p, \mathbf{g}) \ \forall (\mathbf{u}, p, \mathbf{g}) \in \mathcal{U}_{ad} \text{ satisfying}$$

$$\|\mathbf{u} - \hat{\mathbf{u}}\|_1 + \|p - \hat{p}\|_0 + \|\mathbf{g} - \hat{\mathbf{g}}\|_{1, \Gamma_c} \leq \varepsilon. \qquad (2.6)$$

We first show that an optimal solution exists and prove a preliminary regularity result.

THEOREM 2.1 : *There exists an optimal solution* $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \in \mathcal{U}_{ad}$. *Moreover, any optimal solution satisfies* $\hat{\mathbf{u}} \in \mathbf{H}^{3/2}(\Omega)$ *and* $\hat{p} \in H^{1/2}(\Omega) \cap L_0^2(\Omega)$ *and if* $\hat{\mathbf{t}} \in \mathbf{H}^{-1/2}(\Gamma)$ *is such that* $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{t}})$ *is a solution of* (2.1)-(2.3), *then* $\mathbf{t} \in \mathbf{L}^2(\Gamma)$.

Proof : We first claim that $\mathcal{U}_{ad}$ is not empty. Let $\mathbf{g} \equiv \mathbf{0}$ and then let $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ be a solution of (2.1)-(2.3); note that with $\mathbf{g} = \mathbf{0}$, (2.1)-(2.3) is equivalent to

$$\nu a(\tilde{\mathbf{u}}, \mathbf{v}) + c(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega),$$

$$b(\tilde{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega),$$

$$\tilde{\mathbf{u}} = \mathbf{b} \quad \text{on} \quad \Gamma$$

and

$$\tilde{\mathbf{t}} = [-\tilde{p}\mathbf{n} + \nu (\text{grad } \tilde{\mathbf{u}} + (\text{grad } \tilde{\mathbf{u}})^T) \cdot \mathbf{n}]_\Gamma.$$

Since $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{b} \in \mathbf{H}^1(\Gamma)$, it is well known ([11] or [20]) that such $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}})$ exists. Moreover, we have $\mathcal{K}(\tilde{\mathbf{u}}, \tilde{p}, \mathbf{0}) \leqslant C(\|\tilde{\mathbf{u}}\|_1 + \|\tilde{p}\|_0 + \|\mathbf{f}\|_0)\|\tilde{\mathbf{u}}\|_1 < \infty$. Thus, $(\tilde{\mathbf{u}}, \tilde{p}, \mathbf{0}) \in \mathcal{U}_{ad}$.

Now, let $\{\mathbf{u}^{(k)}, p^{(k)}, \mathbf{g}^{(k)}\}$ be a sequence in $\mathcal{U}_{ad}$ such that

$$\lim_{k \to \infty} \mathcal{K}(\mathbf{u}^{(k)}, p^{(k)}, \mathbf{g}^{(k)}) = \inf_{(\mathbf{u}, p, \mathbf{g}) \in \mathcal{U}_{ad}} \mathcal{K}(\mathbf{u}, p, \mathbf{g}).$$

Then, using (2.4) and (2.5), we have that $|\mathbf{g}^{(k)}|_{1, \Gamma_c}$, $\|p^{(k)}\|_0$, and $|\mathbf{u}^{(k)}|_1$ are uniformly bounded. Then, since the first of these defines a norm on $\mathbf{W}_n(\Gamma_c)$ and since $|\mathbf{u}|_1 + \|\mathbf{u}\|_{0, \Gamma}$ defines a norm on $\mathbf{H}^1(\Omega)$, we have that $(\mathbf{u}^{(k)}, p^{(k)}, \mathbf{g}^{(k)})$ is uniformly bounded in $\mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c)$. Also, for some $\mathbf{t}^{(k)} \in \mathbf{H}^{-1/2}(\Gamma)$,

$$\nu a(\mathbf{u}^{(k)}, \mathbf{v}) + c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) + b(\mathbf{v}, p^{(k)}) - (\mathbf{v}, \mathbf{t}^{(k)})_\Gamma$$
$$= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad (2.7)$$

$$b(\mathbf{u}^{(k)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \quad (2.8)$$

and

$$(\mathbf{u}^{(k)}, \mathbf{s})_\Gamma - (\mathbf{g}^{(k)}, \mathbf{s})_{\Gamma_c} = (\mathbf{b}, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma). \quad (2.9)$$

One easily concludes that $\|\mathbf{t}^{(k)}\|_{-1/2, \Gamma}$ is uniformly bounded. We may then extract subsequences such that

$$\mathbf{g}^{(k)} \rightharpoonup \hat{\mathbf{g}} \quad \text{in} \quad \mathbf{W}_n(\Gamma_c)$$
$$\mathbf{u}^{(k)} \rightharpoonup \hat{\mathbf{u}} \quad \text{in} \quad \mathbf{H}^1(\Omega)$$
$$p^{(k)} \rightharpoonup \hat{p} \quad \text{in} \quad L_0^2(\Omega)$$
$$\mathbf{t}^{(k)} \rightharpoonup \hat{\mathbf{t}} \quad \text{in} \quad \mathbf{H}^{-1/2}(\Gamma)$$
$$\mathbf{u}^{(k)} \to \hat{\mathbf{u}} \quad \text{in} \quad \mathbf{L}^2(\Omega)$$
$$\mathbf{u}^{(k)}|_\Gamma \to \hat{\mathbf{u}}|_\Gamma \quad \text{in} \quad \mathbf{L}^2(\Gamma)$$

for some $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{t}}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c) \times \mathbf{H}^{-1/2}(\Gamma)$. The last two convergence results above follow from the compact imbeddings $\mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma) \subset \mathbf{L}^2(\Gamma)$. We may then pass to the limit in (2.7)-(2.9) to determine that $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}, \hat{\mathbf{t}})$ satisfies (2.1)-(2.3). Indeed, the only troublesome term when one passes to the limit is the nonlinearity $c(.\,,\,.\,,\,.\,)$. However, note that

$$c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = \int_\Gamma (\mathbf{u}^{(k)} \cdot \mathbf{n}) \, \mathbf{u}^{(k)} \cdot \mathbf{v} \, d\Gamma$$

$$- \int_\Omega \mathbf{u}^{(k)} \cdot \mathrm{grad}\, \mathbf{v} \cdot \mathbf{u}^{(k)} \, d\Omega \quad \forall \mathbf{v} \in C^\infty(\bar{\Omega}) \,.$$

Then, since $\mathbf{u}^{(k)} \to \hat{\mathbf{u}}$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{u}^{(k)}|_\Gamma \to \hat{\mathbf{u}}|_\Gamma$ in $\mathbf{L}^2(\Gamma)$, we have that

$$\lim_{n \to \infty} c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = \int_\Gamma (\hat{\mathbf{u}} \cdot \mathbf{n}) \, \hat{\mathbf{u}} \cdot \mathbf{v} \, d\Gamma$$

$$- \int_\Omega \hat{\mathbf{u}} \cdot \mathrm{grad}\, \mathbf{v} \cdot \hat{\mathbf{u}} \, d\Omega = c(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in C^\infty(\bar{\Omega}) \,.$$

Then, since $C^\infty(\bar{\Omega})$ is dense in $\mathbf{H}^1(\Omega)$, we also have that

$$\lim_{k \to \infty} c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{v}) = c(\hat{\mathbf{u}}, \hat{\mathbf{u}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \,.$$

Finally, by the weak lower semicontinuity of $\mathscr{K}(.\,,\,.\,,\,.\,)$, we conclude that $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$ is an optimal solution, i.e.,

$$\mathscr{K}(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) = \inf_{(\mathbf{u}, p, \mathbf{g}) \in \mathscr{U}_{ad}} \mathscr{K}(\mathbf{u}, p, \mathbf{g}) \,.$$

Thus we have shown that an optimal solution belonging to $\mathscr{U}_{ad}$ exists.

Next, note that any optimal solution $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$ satisfies, by definition,

$$\nu a(\hat{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \hat{p}) = (\tilde{\mathbf{f}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \,, \tag{2.10}$$

$$b(\hat{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega) \tag{2.11}$$

and

$$\hat{\mathbf{u}} = \begin{cases} \tilde{\mathbf{g}} + \mathbf{b} & \text{on} \quad \Gamma_c \\ \mathbf{b} & \text{on} \quad \Gamma_u \end{cases} \tag{2.12}$$

where $\tilde{\mathbf{f}} = \mathbf{f} - \hat{\mathbf{u}} \cdot \mathrm{grad}\, \hat{\mathbf{u}}$. Due to (1.7), (1.8) and (2.12), we have that $\hat{\mathbf{u}}|_{\Gamma_c} \in \mathbf{H}^1(\Gamma)$. Moreover, since $\hat{\mathbf{u}} \in \mathbf{H}^1(\Omega)$, we have that $\hat{\mathbf{u}} \in \mathbf{L}^6(\Omega)$ and

$\partial\hat{\mathbf{u}}/\partial x_j \in \mathbf{L}^2(\Omega)$ for $j = 1, ..., d$, so that $\hat{\mathbf{u}} \cdot \operatorname{grad} \hat{\mathbf{u}} \in \mathbf{L}^{3/2}(\Omega)$ and therefore $\tilde{\mathbf{f}} \in \mathbf{L}^{3/2}(\Omega)$. Then, it follows from results of [8] (see also [11] and [20]) that the solution of the Stokes problem (2.10)-(2.12) is such that $\hat{\mathbf{u}} \in \mathbf{H}^{3/2}(\Omega)$, $\hat{p} \in H^{1/2}(\Omega) \cap L_0^2(\Omega)$, and

$$\hat{\mathbf{t}} = [-\hat{p}\mathbf{n} + v(\operatorname{grad} \hat{\mathbf{u}} + (\operatorname{grad} \hat{\mathbf{u}})^T) \cdot \mathbf{n}]_{\Gamma} \in \mathbf{L}^2(\Gamma) . \qquad \blacksquare$$

3. THE EXISTENCE OF LAGRANGE MULTIPLIERS AND AN OPTIMALITY SYSTEM

3.1. Existence of Lagrange multipliers

We wish to use the method of Lagrange multipliers to turn the constrained optimization problem (2.5) into an unconstrained one. We first show that suitable Lagrange multipliers exist.

Let $B_1 = \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c) \times \mathbf{H}^{-1/2}(\Gamma)$ and $B_2 = (\mathbf{H}^1(\Omega))^* \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$ and let the nonlinear mapping $M : B_1 \to B_2$ denote the (generalized) constraint equations, i.e., $M(\mathbf{u}, p, \mathbf{g}, \mathbf{t}) = (\mathbf{f}, z, \mathbf{b})$ for $(\mathbf{u}, p, \mathbf{g}, \mathbf{t}) \in B_1$ and $(\mathbf{f}, z, \mathbf{b}) \in B_2$ if and only if

$$va(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - (\mathbf{v}, t)_{\Gamma} = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad (3.1)$$

$$b(\mathbf{u}, q) = (z, q) \quad \forall q \in L_0^2(\Omega) \qquad (3.2)$$

and

$$(\mathbf{u}, \mathbf{s})_{\Gamma} - (\mathbf{g}, \mathbf{s})_{\Gamma_c} = (\mathbf{b}, \mathbf{s})_{\Gamma} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma) . \qquad (3.3)$$

Thus, the constraints (2.1)-(2.3) can be expressed as $M(\mathbf{u}, p, \mathbf{g}, \mathbf{t}) = (\mathbf{f}, 0, \mathbf{b})$.

Given $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the operator $M'(\mathbf{u}) \in \mathcal{L}(B_1 ; B_2)$ may be defined as follows : $M'(\mathbf{u}) \cdot (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) = (\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}})$ for $(\mathbf{w}, r, \mathbf{k}, \mathbf{y}) \in B_1$ and $(\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}}) \in B_2$ if and only if

$$va(\mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{v}, r) - (\mathbf{v}, \mathbf{y})_{\Gamma}$$
$$= (\bar{\mathbf{f}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad (3.4)$$

$$b(\mathbf{w}, q) = (\bar{z}, q) \quad \forall q \in L^2(\Omega) \qquad (3.5)$$

and

$$(\mathbf{w}, \mathbf{s})_{\Gamma} - (\mathbf{k}, \mathbf{s})_{\Gamma_c} = (\bar{\mathbf{b}}, \mathbf{s})_{\Gamma} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma) . \qquad (3.6)$$

LEMMA 3.1 : *For $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the operator $M'(\mathbf{u})$ from B_1 into B_2 has closed range.*

Proof: It is easily seen, for $\mathbf{u} \in \mathbf{H}^1(\Omega)$, that $M'(\mathbf{u})$ is a compact perturbation of the operator $S \in \mathscr{L}(B_1 ; B_2)$, where the latter is defined as follows : $S \cdot (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) = (\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}})$ for $(\mathbf{w}, r, \mathbf{k}, \mathbf{y}) \in B_1$ and $(\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}}) \in B_2$ if and only if

$$va(\mathbf{w}, \mathbf{v}) + b(\mathbf{v}, r) - (\mathbf{v}, \mathbf{y})_\Gamma = (\bar{\mathbf{f}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$b(\mathbf{w}, q) = (\bar{z}, q) \quad \forall q \in L^2(\Omega)$$

and

$$(\mathbf{w}, \mathbf{s})_\Gamma - (\mathbf{k}, \mathbf{s})_{\Gamma_c} = (\bar{\mathbf{b}}, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma).$$

The adjoint operator to S can be shown to be a semi-Fredholm operator, i.e., to have a closed range and a finite-dimensional kernel. Then it follows that S itself, and any compact perturbation of S, has closed range ; see [18]. ∎

LEMMA 3.2 : *For* $\mathbf{u} \in \mathbf{H}^1(\Omega)$, *the operator* $M'(\mathbf{u})$ *from* B_1 *into* B_2 *is onto.*

Proof: Assume that $M'(\mathbf{u})$ is not onto. Then, the image of $M'(\mathbf{u})$ is strictly contained in B_2 and, by Lemma 3.1, is closed, so that there exists a nonzero $(\boldsymbol{\mu}, \phi, \boldsymbol{\tau}) \in (B_2)^* = \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ such that

$$\left\langle (\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}}), (\boldsymbol{\mu}, \phi, \boldsymbol{\tau}) \right\rangle = 0 \quad \forall (\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}}) \text{ belonging to the range of } M'(\mathbf{u}),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between B_2 and B_2^*; this equation may be rewritten in the form

$$(\tilde{\mathbf{f}}, \boldsymbol{\mu}) + (\bar{z}, \phi) + (\bar{\mathbf{b}}, \boldsymbol{\tau})_\Gamma$$
$$= 0 \quad \forall (\bar{\mathbf{f}}, \bar{z}, \bar{\mathbf{b}}) \text{ belonging to the range of } M'(\mathbf{u}).$$

Then, using (3.4)-(3.6), we conclude that there exists a nonzero $(\boldsymbol{\mu}, \phi, \boldsymbol{\tau}) \in (B_2)^* = \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ such that

$$va(\mathbf{w}, \boldsymbol{\mu}) + c(\mathbf{w}, \mathbf{u}, \boldsymbol{\mu}) + c(\mathbf{u}, \mathbf{w}, \boldsymbol{\mu}) + b(\mathbf{w}, \phi)$$
$$+ (\mathbf{w}, \boldsymbol{\tau})_\Gamma = 0 \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega), \quad (3.7)$$

$$b(\boldsymbol{\mu}, r) = 0 \quad \forall r \in L_0^2(\Omega), \quad (3.8)$$

$$(\boldsymbol{\mu}, \mathbf{y})_\Gamma = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma) \quad (3.9)$$

and

$$(\mathbf{k}, \boldsymbol{\tau})_{\Gamma_c} = 0 \quad \forall \mathbf{k} \in \mathbf{W}_n(\Gamma_c). \quad (3.10)$$

The system (3.7)-(3.10) is a weak form of the boundary value problem

$$- \nu \, \text{div} \, ((\text{grad} \, \boldsymbol{\mu}) + (\text{grad} \, \boldsymbol{\mu})^T) + \boldsymbol{\mu} \, . \, (\text{grad} \, \mathbf{u})^T - \mathbf{u} \, . \, \text{grad} \, \boldsymbol{\mu}$$
$$+ \, \text{grad} \, \phi = 0 \quad \text{in} \quad \Omega \, ,$$

$$\text{div} \, \boldsymbol{\mu} = 0 \quad \text{in} \quad \Omega \, , \tag{3.11}$$

$$\boldsymbol{\mu} = 0 \quad \text{on} \quad \Gamma \tag{3.12}$$

and

$$\boldsymbol{\tau} = \phi \mathbf{n} - \nu \, (\text{grad} \, \boldsymbol{\mu} + (\text{grad} \, \boldsymbol{\mu})^T) \, . \, \mathbf{n} - (\mathbf{u} \, . \, \mathbf{n}) \, \boldsymbol{\mu} = C \mathbf{n} \quad \text{on} \quad \Gamma_c \, .$$

for some constant C. Letting $\tilde{\boldsymbol{\tau}} = \boldsymbol{\tau} - C \mathbf{n}$ and $\tilde{\phi} = \phi - C$, we are easily led to (3.11)-(3.12) and

$$- \nu \, \text{div} \, ((\text{grad} \, \boldsymbol{\mu}) + (\text{grad} \, \boldsymbol{\mu})^T) + \boldsymbol{\mu} \, . \, (\text{grad} \, \boldsymbol{u})^T$$
$$- \mathbf{u} \, . \, \text{grad} \, \boldsymbol{\mu} + \text{grad} \, \tilde{\phi} = \mathbf{0} \quad \text{in} \, \Omega \, , \tag{3.13}$$

and

$$\tilde{\boldsymbol{\tau}} = \tilde{\phi} \mathbf{n} - \nu \, (\text{grad} \, \boldsymbol{\mu} + (\text{grad} \, \boldsymbol{\mu})^T) \cdot \mathbf{n} - (\mathbf{u} \cdot \mathbf{n}) \, \boldsymbol{\mu} = \mathbf{0} \quad \text{on} \, \Gamma_c. \tag{3.14}$$

Now, let the domains Ω' and Ω_e be constructed as indicated in Figure 3.1, i.e., as a smooth expansion of Ω such that $\Gamma \cap \Gamma' \subset \Gamma_c$ and $\Omega_e = \Omega \cup \Omega' \cup (\Gamma \cap \Gamma')$, where Γ' denotes the boundary of Ω'. Let $\mathbf{u}_e$ denote a fixed extension of $\mathbf{u}$ such that $\mathbf{u}_e \in \mathbf{H}^1(\Omega_e)$. The boundary conditions (3.12) and (3.14) then allows us to define extensions $\boldsymbol{\mu}_e$ and $\tilde{\phi}_e$ such that $\boldsymbol{\mu}_e = \boldsymbol{\mu}$ and $\tilde{\phi}_e = \tilde{\phi}$ on Ω, $\boldsymbol{\mu}_e = 0$ and $\tilde{\phi}_e = 0$ on Ω', and such that the differential equations (3.11) and (3.13) hold (in the appropriate weak senses) on Ω_e, i.e.,

$$- \nu \, \text{div} \, ((\text{grad} \, \boldsymbol{\mu}_e) + (\text{grad} \, \boldsymbol{\mu}_e)^T) + \boldsymbol{\mu}_e \, . \, (\text{grad} \, \boldsymbol{u}_e)^T$$
$$- \mathbf{u}_e \, . \, \text{grad} \, \boldsymbol{\mu}_e + \text{grad} \, \tilde{\phi}_e = \mathbf{0} \quad \text{in} \, \Omega_e \, , \tag{3.15}$$

and

$$\text{div} \, \boldsymbol{\mu}_e = 0 \quad \text{in} \, \Omega_e \, . \tag{3.16}$$

Furthermore, (3.12) and the facts that $\boldsymbol{\mu}_e = \boldsymbol{\mu}$ on Ω and $\boldsymbol{\mu}_e = \mathbf{0}$ on Ω' imply that

$$\boldsymbol{\mu}_e = \mathbf{0} \quad \text{on} \quad \Gamma_e \, , \tag{3.17}$$

where Γ_e denotes the boundary of Ω_e. For a fixed domain Ω_e, it is possible for (3.15)-(3.17) to have a nontrivial solution $(\boldsymbol{\mu}_e, \tilde{\phi}_e)$, i.e., for $1/\nu$ to be an eigenvalue of the problem (3.15)-(3.17). However, this problem involves a
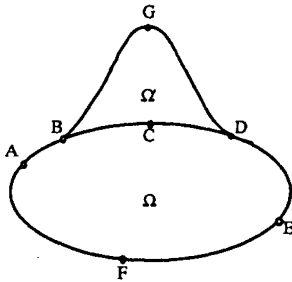
Figure 3.1. — **The domains** Ω **and** $\Omega' \cdot \Gamma = ABCDEFA$,

$\Gamma_c = ABCDE, \Gamma_u = EFA, \Gamma' = BGDB, \Gamma_e = ABGDEFA.$

compact perturbation of the Stokes operator, and thus its spectrum is discrete. Then, by appropriately choosing the extended domain Ω_e, we can guarantee that $1/\nu$ is not an eigenvalue of (3.15)-(3.17), and that therefore these homogeneous, linear equations have only the trivial solution $\boldsymbol{\mu}_e = \mathbf{0}$ and $\tilde{\phi}_e = 0$ in Ω_e. (Note that from (3.15)-(3.17), we first conclude that $\tilde{\phi}_e = \text{constant}$, but since $\tilde{\phi}_e = 0$ on Ω' we can then conclude that this constant vanishes.) It then follows that $\boldsymbol{\mu} = \mathbf{0}$ and $\phi = C$ in Ω. But ϕ has zero mean over Ω, so that necessarily $C = 0$, and therefore $\phi = 0$ in Ω. It then follows that $\boldsymbol{\tau} = \mathbf{0}$ on Γ. This, of course, provides a contradiction, and thus the operator $M'(\mathbf{u})$ from B_1 into B_2 is onto.   ∎

For fixed $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and given $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $p \in L_0^2(\Omega)$, and $\mathbf{g} \in \mathbf{H}^1(\Gamma_c)$, we have that the operator $\mathcal{K}'(\mathbf{u}, p, \mathbf{g}) \in \mathcal{L}(B_1 ; \mathbb{R})$ may be defined as follows : $\mathcal{K}'(\mathbf{u}, p, \mathbf{g}) . (w, r, \mathbf{k}, \mathbf{y}) = \tilde{a}$ for $(w, r, \mathbf{k}, \mathbf{y}) \in B_1$ and $\tilde{a} \in \mathbb{R}$ if and only if

$$\nu a(\mathbf{w}, \mathbf{u}) + b(\mathbf{w}, p) + b(\mathbf{u}, r) - (\mathbf{f}, \mathbf{w})$$
$$+ \nu(\text{grad}_s \, \mathbf{g}, \text{grad}_s \, \mathbf{k})_{\Gamma_c} + \nu(\mathbf{g}, \mathbf{k})_{\Gamma_c} = \tilde{a} . \quad (3.18)$$

Let $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \in \mathbf{H}^1(\Omega) \times \mathbf{W}_n(\Gamma_c)$ denote an optimal solution in the sense of (2.8). Then, consider the nonlinear operator $N : B_1 \to \mathbb{R} \times B_2$ defined by

$$N(\mathbf{u}, p, \mathbf{g}, \mathbf{t}) = \begin{pmatrix} \mathcal{K}(\mathbf{u}, p, \mathbf{g}) - \mathcal{K}(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \\ M(\mathbf{u}, p, \mathbf{g}, \mathbf{t}) \end{pmatrix} .$$

Then, for $(\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma_c)$, the operator $N'(\mathbf{u}, p, \mathbf{g})$ from B_1 into $\mathbb{R} \times B_2$ may be defined as follows : $N'(\mathbf{u}, p, \mathbf{g}) . (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) = (\tilde{a}, \mathbf{f}, \tilde{z}, \tilde{\mathbf{b}})$ for $(\mathbf{w}, r, \mathbf{k}, \mathbf{y}) \in B_1$ and $(\tilde{a}, \tilde{\mathbf{f}}, \tilde{z}, \tilde{\mathbf{b}}) \in \mathbb{R} \times B_2$ if and only if

$$\nu a(\mathbf{w}, \mathbf{u}) + b(\mathbf{w}, p) + b(\mathbf{u}, r) - (\mathbf{f}, \mathbf{w}) + \nu(\text{grad}_s \, \mathbf{g}, \text{grad}_s \, \mathbf{k})_{\Gamma_c}$$
$$+ \nu(\mathbf{g}, \mathbf{k})_{\Gamma_c} = \tilde{a} , \quad (3.19)$$

$$\nu a(\mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{w}, \mathbf{v}) + b(\mathbf{v}, r) - (\mathbf{v}, \mathbf{y})_\Gamma$$

$$= (\tilde{\mathbf{f}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) , \quad (3.20)$$

$$b(\mathbf{w}, q) = (\tilde{z}, q) \quad \forall q \in L_0^2(\Omega) \quad (3.21)$$

and

$$(\mathbf{w}, \mathbf{s})_\Gamma - (\mathbf{k}, \mathbf{s})_{\Gamma_c} = (\tilde{\mathbf{b}}, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma) . \quad (3.22)$$

LEMMA 3.3. — *For* $(\mathbf{u}, p, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma_c)$, *the operator* $N'(\mathbf{u}, p, \mathbf{g})$ *from* B_1 *into* $\mathbb{R} \times B_2$ *has closed range but is not onto.*

Proof : From Lemma 3.1, we have that $M'(\mathbf{u})$ has a closed range. Also, the continuity of the various bilinear and trilinear forms, i.e., (1.12)-(1.14), and of the inner products appearing in the definition of $M'(\mathbf{u})$, imply that this operator belongs to $\mathscr{L}(B_1, B_2)$ and therefore the kernel of $M'(\mathbf{u})$ is a closed subspace. Now, $\mathscr{K}'(\mathbf{u}, p, \mathbf{g})$ acting on the kernel of $M'(\mathbf{u})$ is either identically zero or onto $\mathbb{R}$. (This follows from the obvious result that whenever ψ is a linear functional on a Banach space X, then either $\psi \equiv 0$ or the range of ψ is $\mathbb{R}$.) Thus, we have shown that $\mathscr{K}'(\mathbf{u}, p, \mathbf{g})$ acting on the kernel of $M'(\mathbf{u})$ has a closed range, and therefore the operator $N'(\mathbf{u}, p, \mathbf{g})$ has a closed range in B_2. [This follows from the following well known result. Let X, Y, Z be Banach spaces and $A : X \to Y$ and $B : X \to Z$ be linear continuous operators. If the range of B is closed in Z and the subspace $A \ker(B)$ is closed in Y, then, if we define $C : X \to Y \times Z$ by $Cx = (Ax, Bx)$, the range of C is closed in $Y \times Z$.]

The operator $N'(\mathbf{u}, p, \mathbf{g})$ is not onto because if it were, by the Implicit Function Theorem, we would have $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{g}}) \in \mathcal{U}_{ad}$ such that $\|\hat{\mathbf{u}} - \tilde{\mathbf{u}}\|_1 + \|\hat{p} - \tilde{p}\|_0 + \|\hat{\mathbf{g}} - \tilde{\mathbf{g}}\|_{1, \Gamma_c} \le \varepsilon$ and $\mathscr{K}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{g}}) < \mathscr{K}(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$, contradicting the hypothesis that $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$ is an optimal solution. ∎

We are now prepared to show the existence of Lagrange multipliers.

THEOREM 3.4. — *Let* $(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) \in \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Gamma_c)$ *denote an optimal solution in the sense of* (2.8). *Then there exists a nonzero Lagrange multiplier* $(\hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ *satisfying the Euler equations*

$$- \mathscr{K}'(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}) . (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) + \left\langle (\hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}), M'(\hat{\mathbf{u}}) . (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) \right\rangle = 0$$

$$\forall (\mathbf{w}, r, \mathbf{k}, \mathbf{y}) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c) \times \mathbf{H}^{-1/2}(\Gamma) , \quad (3.24)$$

where $\langle ., . \rangle$ *denotes the duality pairing between* $\mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ *and* $\mathbf{H}^1(\Omega)^* \times L_0^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$.

Proof : From Lemma 3.3, we have that the range of $N'(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$ is a closed, proper subspace of $\mathbb{R} \times B_2$. Then, the Hahn-Banach theorem

implies that there exists a nonzero element of $\mathbb{R} \times (B_2)^* = \mathbb{R} \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ that annihilates the range of $N'(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$, i.e., there exists $(\hat{\alpha}, \hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \in \mathbb{R} \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$ such that

$$\left\langle (\tilde{a}, \tilde{\mathbf{f}}, \tilde{z}, \tilde{\mathbf{b}}), (\hat{\alpha}, \hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \right\rangle = 0$$

$$\forall (\tilde{a}, \tilde{\mathbf{f}}, \tilde{z}, \tilde{\mathbf{b}}) \quad \text{belonging to the range of } N'(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}}), \qquad (3.25)$$

where in this instance $\langle ., . \rangle$ denotes the duality pairing between $\mathbb{R} \times B_2$ and its dual $\mathbb{R} \times (B_2)^*$. Note that $\hat{\alpha} \neq 0$ since otherwise we would have, using Lemna 3.2, that $\left\langle (\tilde{\mathbf{f}}, \tilde{z}, \tilde{\mathbf{b}}), (\hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \right\rangle = 0$ for all $(\tilde{\mathbf{f}}, \tilde{z}, \tilde{\mathbf{b}}) \in B_2$ so that $(\hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \equiv 0$, contradicting the fact that $(\hat{\alpha}, \hat{\boldsymbol{\mu}}, \hat{\phi}, \hat{\boldsymbol{\tau}}) \neq 0$. We may, without any loss of generality, set $\hat{\alpha} = -1$. Clearly, using the definition of the operator $N'(\hat{\mathbf{u}}, \hat{p}, \hat{\mathbf{g}})$, (3.24) and (3.25) are equivalent. ∎

3.2. The optimality system

Using (3.4)-(3.6), setting $\hat{\alpha} = -1$, and dropping the (ˆ) notation for optimal solutions, we may rewrite (3.25) in the form

$$va(\mathbf{w}, \boldsymbol{\mu}) + c(\mathbf{w}, \mathbf{u}, \boldsymbol{\mu}) + c(\mathbf{u}, \mathbf{w}, \boldsymbol{\mu}) + b(\mathbf{w}, \phi) + (\mathbf{w}, \boldsymbol{\tau})_\Gamma$$
$$= va(\mathbf{u}, \mathbf{w}) + b(\mathbf{w}, p) - (\mathbf{f}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega), \quad (3.26)$$

$$b(\boldsymbol{\mu}, r) = b(\mathbf{u}, r) = 0 \quad \forall r \in L_0^2(\Omega), \qquad (3.27)$$

$$(\boldsymbol{\mu}, \mathbf{y})_\Gamma = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma) \qquad (3.28)$$

and

$$v(\text{grad}_s \, \mathbf{g}, \text{grad}_s \, \mathbf{k})_{\Gamma_c} + v(\mathbf{g}, \mathbf{k})_{\Gamma_c} = -(\mathbf{k}, \boldsymbol{\tau})_{\Gamma_c} \quad \forall \mathbf{k} \in \mathbf{W}_n(\Gamma_c), \quad (3.29)$$

where in (3.27) we have used (2.2).

Since for some $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma)$ optimal solutions satisfy the constraints (2.1)-(2.3), we see necessary conditions for an optimum are that (2.1)-(2.3) and (3.26)-(3.29) are satisfied. This system of equations will be called the *optimality system*.

Using (2.1), we may replace (3.26) by

$$va(\mathbf{w}, \boldsymbol{\mu}) + c(\mathbf{w}, \mathbf{u}, \boldsymbol{\mu}) + c(\mathbf{u}, \mathbf{w}, \boldsymbol{\mu}) + b(\mathbf{w}, \phi) - (\mathbf{w}, \boldsymbol{\theta})_\Gamma$$
$$= -c(\mathbf{u}, \mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega) \quad (3.30)$$

where $\boldsymbol{\theta} = \mathbf{t} - \boldsymbol{\tau}$. The replacement of the right hand side of (3.26) by the right hand side of (3.30) facilitates the derivation of error estimates in Section 4.

Thus, the optimality system in terms of the variables $\mathbf{u}$, p, $\mathbf{t}$, $\mathbf{g}$, $\boldsymbol{\mu}$, $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ is given by (2.1)-(2.3) and (3.27)-(3.30). Integrations by parts may be used to show that this system constitutes a weak formulation of the boundary value problem

$$- \nu \operatorname{div} ((\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T) + \mathbf{u} \cdot \operatorname{grad} \mathbf{u} + \operatorname{grad} p = \mathbf{f} \quad \text{in } \Omega, \tag{3.31}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.32}$$

$$\mathbf{u} = \begin{cases} \mathbf{g} + \mathbf{b} & \text{on } \Gamma_c \\ \mathbf{b} & \text{on } \Gamma_u, \end{cases} \tag{3.33}$$

$$\nu(- \Delta_s \mathbf{g} + \mathbf{g}) + \beta \mathbf{n} = \phi \mathbf{n} - \nu ((\operatorname{grad} \boldsymbol{\mu}) + (\operatorname{grad} \boldsymbol{\mu})^T) \cdot \mathbf{n} - (\mathbf{u} \cdot \mathbf{n}) \boldsymbol{\mu} - p \mathbf{n}$$
$$+ \nu((\operatorname{grad} \mathbf{u}) + (\operatorname{grad} \mathbf{u})^T) \cdot \mathbf{n} \quad \text{on} \quad \Gamma_c, \tag{3.34}$$

$$\int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, d\Gamma = 0 \quad \text{and, if } \Gamma_c \text{ has a boundary}, \quad \mathbf{g} = 0 \quad \text{on } \partial \Gamma_c, \tag{3.35}$$

$$- \nu \operatorname{div}((\operatorname{grad} \boldsymbol{\mu}) + (\operatorname{grad} \boldsymbol{\mu})^T) + \boldsymbol{\mu} \cdot (\operatorname{grad} \mathbf{u})^T - \mathbf{u} \cdot \operatorname{grad} \boldsymbol{\mu} + \operatorname{grad} \phi$$
$$= - \mathbf{u} \cdot \operatorname{grad} \mathbf{u} \quad \text{in } \Omega, \tag{3.36}$$

$$\operatorname{div} \boldsymbol{\mu} = 0 \quad \text{in } \Omega \tag{3.37}$$

and

$$\boldsymbol{\mu} = 0 \quad \text{on } \Gamma. \tag{3.38}$$

Note that in (3.34) Δ_s denotes the surface Laplacian and in (3.36)

$$(\mathbf{u} \cdot \operatorname{grad} \boldsymbol{\mu})_i = \sum_{j=1}^{d} u_j \frac{\partial \mu_i}{\partial x_j} \quad \text{and} \quad (\boldsymbol{\mu} \cdot (\operatorname{grad} \mathbf{u})^T)_i$$

$$= \sum_{j=1}^{d} \mu_j \frac{\partial u_j}{\partial x_i} \quad \text{for} \quad i = 1, ..., d.$$

Also, in (3.34), $\beta \in \mathbb{R}$ is an additional unknown constant that accounts for the single integral constraint of (3.35).

The optimality system (3.31)-(3.38) consists of the Navier-Stokes system (3.31)-(3.33), the system (3.36)-(3.38) whose left hand side is the adjoint of Navier-Stokes operator linearized about $\mathbf{u}$, and the surface Laplacian system (3.34)-(3.35).

Remark : Note that (3.29) may be expressed in the form

$$\nu(\operatorname{grad}_s \mathbf{g}, \operatorname{grad}_s \mathbf{k})_{\Gamma_c} + \nu (\mathbf{g}, \mathbf{k})_{\Gamma_c} + \beta \int_{\Gamma_c} \mathbf{k} \cdot \mathbf{n} \, d\Gamma$$
$$= (\mathbf{k}, \boldsymbol{\theta} - \mathbf{t})_{\Gamma_c} \quad \forall \mathbf{k} \in W (\Gamma_c) \tag{3.39}$$

and

$$\int_{\Gamma_c} \mathbf{g} \cdot \mathbf{n} \, d\Gamma = 0 \, . \tag{3.40}$$

Although (3.29) and (3.39)-(3.40) are equivalent, the latter is more easily discretized. Also, note the relation between (3.34)-(3.35) and (3.39)-(3.40).

Remark : The use of the $\mathbf{H}^1(\Gamma_c)$-norm of $\mathbf{g}$ in the functional (1.2), or, equivalently, in (2.4), results in the appearance of the surface Laplacian in (3.34). The use of the more « natural » $\mathbf{H}^{1/2}(\Gamma_c)$-norm would have resulted in a much less attractive problem relating the control $\mathbf{g}$ to the variables $\mathbf{t}$ and $\boldsymbol{\theta}$. On the other hand, the use, in (1.2), of the weaker $\mathbf{L}^2(\Gamma_c)$-norm for $\mathbf{g}$ would not allow us to derive the regularity results of the following subsection. From a practical point of view, the use of the $\mathbf{H}^1(\Gamma_c)$-norm of $\mathbf{g}$ in the functional (1.2) results in less oscillatory optimal controls $\mathbf{g}$.

Remark : Our notion of an optimal solution is a local one ; see (2.6). Moreover, there is no reason to believe that, in general, optimal solutions are unique. This is to be expected since the uncontrolled stationary Navier-Stokes equations are known to have multiple solutions for sufficiently large values of the Reynolds number. However, just as in the Navier-Stokes case ([11], [12], [20] or [21]), for sufficiently small values of the Reynolds number, i.e., for « small enough » data or « large enough » viscosity, one can guarantee that optimal solutions are unique.

3.3. Regularity of solutions of the optimality system

We now examine the regularity of solutions of the optimality system (2.1)-(2.3) and (3.27)-(3.30), or equivalently, (3.31)-(3.38). Note that if Γ_c has a boundary, we can only conclude that, for arbitrary $\varepsilon > 0$, $\mathbf{u}|_\Gamma \in \mathbf{H}^{3/2-\varepsilon}(\Gamma)$, and in this case we cannot obtain the following results. Thus, throughout this section, we assume that Γ_c does not have a boundary.

THEOREM 3.5 : *Suppose that Γ_c does not have a boundary $\partial\Gamma_c$ and that the given data satisfies $\mathbf{b} \in \mathbf{H}^{3/2}(\Gamma)$ and $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Suppose that Ω is of class $C^{1,1}$. Then, if $(\mathbf{u}, p, \mathbf{g}, \boldsymbol{\mu}, \phi) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{W}_n(\Gamma_c) \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ denotes a solution of the optimality system (2.1)-(2.3) and (2.27)-(2.30), or equivalently, (3.31)-(3.38), we have that $(\mathbf{u}, p, \mathbf{g}, \boldsymbol{\mu}, \phi) \in \mathbf{H}^2(\Omega) \times H^1(\Omega) \times \mathbf{H}^{3/2}(\Gamma_c) \times \mathbf{H}^2(\Omega) \times H^1(\Omega)$. If the boundary is sufficiently smooth, we also may conclude that $\mathbf{g} \in \mathbf{H}^{5/2}(\Gamma_c)$.*

Proof : Consider the Stokes problem

$$-\nu \, \mathrm{div} \, ((\mathrm{grad} \, \boldsymbol{\mu}) + (\mathrm{grad} \, \boldsymbol{\mu})^T) + \mathrm{grad} \, \phi$$
$$= -\boldsymbol{\mu} \cdot (\mathrm{grad} \, \mathbf{u})^T + \mathbf{u}.\mathrm{grad} \, \boldsymbol{\mu} - \mathbf{u} \cdot \mathrm{grad} \, \mathbf{u} \quad \text{in } \Omega, \tag{3.41}$$

$$\text{div } \boldsymbol{\mu} = 0 \quad \text{in} \quad \Omega, \tag{3.42}$$

and

$$\boldsymbol{\mu} = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{3.43}$$

From Theorem 2.1 we already know that any solution $\mathbf{u}$ of the optimality system satisfies $\mathbf{u} \in \mathbf{H}^{3/2}(\Omega)$. Then, since $\boldsymbol{\mu} \in \mathbf{H}^1(\Omega)$, we may conclude that the right hand side of (3.41) certainly belongs to $\mathbf{H}^{-1/2}(\Omega)$, and thus standard results for the regularity of solutions of the Stokes problem ([8], [11] and [20]) then yield that $\boldsymbol{\mu} \in \mathbf{H}^{3/2}(\Omega)$ and $\phi \in H^{1/2}(\Omega)$. Moreover, we have that $[-\phi\mathbf{n} + \nu(\text{grad }\boldsymbol{\mu} + (\text{grad }\boldsymbol{\mu})^T) \cdot \mathbf{n}]_\Gamma \in \mathbf{L}^2(\Gamma)$. Then, the problem (3.34)-(3.35) has a solution $\mathbf{g} \in \mathbf{H}^2(\Gamma_c)$ and, if Γ_c has no boundary and $\mathbf{b} \in \mathbf{H}^{3/2}(\Gamma)$, we have that the right hand side of (3.33) belongs to $\mathbf{H}^{3/2}(\Gamma)$ as well.

Next, consider the Stokes problem

$$-\nu \text{ div }((\text{grad }\mathbf{u}) + (\text{grad }\mathbf{u})^T) + \text{grad } p = -\mathbf{u} \cdot \text{grad }\mathbf{u} + \mathbf{f} \text{ in } \Omega, \tag{3.44}$$

$$\text{div } \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{3.45}$$

and

$$\mathbf{u} = \begin{cases} \mathbf{g} + \mathbf{b} & \text{on} \quad \Gamma_c \\ \mathbf{b} & \text{on } \Gamma_u. \end{cases} \tag{3.46}$$

With $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{u} \in \mathbf{H}^{3/2}(\Omega)$, the right hand side of (3.44) belongs to $\mathbf{L}^2(\Omega)$. We have just concluded that the right hand side of (3.46) belongs to $\mathbf{H}^{3/2}(\Gamma)$. Then, standard results for the Stokes problem yield that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $p \in H^1(\Omega)$.

With the knowledge that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $\boldsymbol{\mu} \in \mathbf{H}^{3/2}(\Omega)$, we return to the Stokes problem (3.41)-(3.43). Now the right hand side of (3.41) belongs to $\mathbf{L}^2(\Omega)$ and we can then conclude that $\boldsymbol{\mu} \in \mathbf{H}^2(\Omega)$ and $\phi \in H^1(\Omega)$. Finally, for sufficiently smooth domains, we have that the data in (3.34) belongs to $\mathbf{H}^{1/2}(\Gamma_c)$ so that $\mathbf{g} \in \mathbf{H}^{5/2}(\Gamma_c)$.

Remark : The above result also holds for convex regions of $\mathbb{R}^2$, provided $\Gamma_c = \Gamma$. In general, we may show that if $\mathbf{f} \in \mathbf{H}^m(\Omega)$ and $\mathbf{b} \in \mathbf{H}^{m+\frac{3}{2}}(\Gamma)$ and Ω is sufficiently smooth, then $(\mathbf{u}, p, \mathbf{g}, \boldsymbol{\mu}, \phi) \in \mathbf{H}^{m+2}(\Omega) \times H^{m+1}(\Omega) \times \mathbf{H}^{m+\frac{5}{2}}(\Gamma_c) \times \mathbf{H}^{m+2}(\Omega) \times H^{m+1}(\Omega)$. In particular, if $\mathbf{f}$ and $\mathbf{b}$ are all of class $C^\infty(\bar{\Omega})$ and Ω is of class C^∞, then $\mathbf{u}, p, \mathbf{g}, \boldsymbol{\mu}$ and ϕ are all $C^\infty(\bar{\Omega})$ functions as well.

Remark : The hypotheses of Theorem 3.5 imply that $t = [-p\mathbf{n} + \nu(\text{grad }\mathbf{u} + (\text{grad }\mathbf{u})^T) \cdot \mathbf{n}]_\Gamma \in \mathbf{H}^{1/2}(\Gamma)$ and $\theta = t + [-\phi\mathbf{n} + \nu(\text{grad }\boldsymbol{\mu} + (\text{grad }\boldsymbol{\mu})^T) \cdot \mathbf{n}]_\Gamma + (\mathbf{u} \cdot \mathbf{n})\boldsymbol{\mu} \in \mathbf{H}^{1/2}(\Gamma)$.

4. FINITE ELEMENT APPROXIMATIONS

4.1. Finite element discretizations

A finite element discretization of the optimality system (2.1)-(2.3) and (3.27)-(3.30) is defined as follows. First, one chooses families of finite dimensional subspaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega)$, $S^h \subset L^2(\Omega)$. These families are parametrized by the parameter h that tends to zero ; commonly, this parameter is chosen to be some measure of the grid size in a subdivision of Ω into finite elements. We let $S_0^h = S^h \cap L_0^2(\Omega)$ and $\mathbf{V}_0^h = \mathbf{V}^h \cap \mathbf{H}_0^1(\Omega)$.

One may choose any pair of subspaces $\mathbf{V}^h$ and S^h that can be used for finding finite element approximations of solutions of the Navier-Stokes equations. Thus, concerning these subspaces, we make the following standard assumptions which are exactly those employed in well-known finite element methods for the Navier-Stokes equations. First, we have the approximation properties : there exist an integer k and a constant C, independent of h, $\mathbf{v}$ *and* q, such that

$$\inf_{\mathbf{v}^h \in \mathbf{V}^h} \|\mathbf{v} - \mathbf{v}^h\|_1 \leqslant Ch^m \|\mathbf{v}\|_{m+1} \quad \forall \mathbf{v} \in \mathbf{H}^{m+1}(\Omega) , \quad 1 \leqslant m \leqslant k \quad (4.1)$$

and

$$\inf_{q^h \in S_0^h} \|q - q^h\|_0 \leqslant Ch^m \|q\|_m \quad \forall q \in H^m(\Omega) \cap L_0^2(\Omega), \quad 1 \leqslant m \leqslant k . \quad (4.2)$$

next, we assume the *inf-sup condition*, or *Ladyzhenskaya-Babuska-Brezzi condition* : there exists a constant C, independent of h, such that

$$\inf_{0 \neq q^h \in S_0^h} \sup_{0 \neq \mathbf{v}^h \in \mathbf{V}^h} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \geqslant C . \quad (4.3)$$

This condition assures the stability of finite element discretizations of the Navier-Stokes equations. For thorough discussions of the approximation properties (4.1)-(4.2), see, e.g., [4] or [9], and for like discussions of the stability condition (4.3), see, e.g., [11] or [12]. The latter references may also be consulted for a catalogue of finite element subspaces that meet the requirements of (4.1)-(4.3).

Next, let $\mathbf{P}^h = \mathbf{V}^h|_\Gamma$, i.e., $\mathbf{P}^h$ consists of the restriction, to the boundary Γ, of functions belonging to $\mathbf{P}^h$. For all choices of conforming finite element spaces $\mathbf{V}^h$, e.g., Lagrange type finite element spaces, we then have that $\mathbf{P}^h \subset \mathbf{H}^{-1/2}(\Gamma)$. For the subspaces $\mathbf{P}^h = \mathbf{V}^h|_\Gamma$, we assume the approximation property : there exist an integer k and a constant C, independent of h and $\mathbf{s}$, such that

$$\inf_{s^h \in \mathbf{P}^h} \|\mathbf{s} - \mathbf{s}^h\|_{-1/2, \Gamma} \leqslant Ch^m \|\mathbf{s}\|_{m - \frac{1}{2}} \quad \forall \mathbf{s} \in \mathbf{H}^{m - \frac{1}{2}}(\Gamma), \quad 1 \leqslant m \leqslant k, \quad (4.4)$$

and the inverse assumption : there exists a constant C, independent of h and $\mathbf{s}^h$ such that

$$\|\mathbf{s}^h\|_{s, \Gamma} \leqslant Ch^{s-q} \|\boldsymbol{\mu}\|_{q, \Gamma} \quad \forall \mathbf{s}^h \in \mathbf{P}^h, \quad -1/2 \leqslant q \leqslant s \leqslant 1/2. \quad (4.5)$$

See [3] or [9] for details concerning (4.4) and (4.5).

Now, let $\mathbf{Q}^h = \mathbf{V}^h|_{\Gamma_c}$, i.e., $\mathbf{Q}^h$ consists of the restriction, to the boundary segment Γ_c, of functions belonging to $\mathbf{V}^h$. Again, for all choices of conforming finite element spaces $\mathbf{V}^h$ we then have that $\mathbf{Q}^h \subset \mathbf{H}^1(\Gamma_c)$. Let $\mathbf{Q}_0^h = \mathbf{Q}^h \cap \mathbf{W}(\Gamma_c)$. We assume the approximation property : there exist an integer k and a constant C, independent of h and $\mathbf{k}$, such that

$$\inf_{\mathbf{k}^h \in \mathbf{Q}_0^h} \|\mathbf{k} - \mathbf{k}^h\|_{s, \Gamma_c} \leqslant Ch^{m - s + \frac{1}{2}} \|\mathbf{k}\|_{m + \frac{1}{2}}$$

$$\forall \mathbf{k} \in \mathbf{W}(\Gamma_c), \quad 1 \leqslant m \leqslant k, \quad 0 \leqslant s \leqslant 1. \quad (4.6)$$

This property follows from (4.1), once one notes that the same type of polynomials are used in $\mathbf{Q}_0^h$ as are used in $\mathbf{V}^h$.

Once the approximating subspaces have been chosen we seek $\mathbf{u}^h \in \mathbf{V}^h$, $p^h \in S_0^h$, $\mathbf{t}^h \in \mathbf{P}^h$, $\mathbf{g}^h \in \mathbf{Q}_0^h$, $\boldsymbol{\mu}^h \in \mathbf{V}^h$, $\phi^h \in S_0^h$, $\boldsymbol{\theta}^h \in \mathbf{P}^h$ and $\beta^h \in \mathbb{R}$ such that

$$va(\mathbf{u}^h, \mathbf{v}^h) + c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + b(\mathbf{v}^h, p^h) - (\mathbf{v}^h, \mathbf{t}^h)_\Gamma$$
$$= (\mathbf{f}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \quad (4.7)$$

$$b(\mathbf{u}^h, q^h) = 0 \quad \forall q^h \in S_0^h, \quad (4.8)$$

$$(\mathbf{u}^h, \mathbf{s}^h)_\Gamma - (\mathbf{g}^h, \mathbf{s}^h)_{\Gamma_c} = (\mathbf{b}, \mathbf{s}^h)_\Gamma \quad \forall \mathbf{s}^h \in \mathbf{P}^h, \quad (4.9)$$

$$v(\mathrm{grad}, \mathbf{g}^h, \mathrm{grad}, \mathbf{k}^h)_{\Gamma_c} + v(\mathbf{g}^h, \mathbf{k}^h)_{\Gamma_c} + \beta^h \int_{\Gamma_c} \mathbf{k}^h \cdot \mathbf{n} \, d\Gamma$$
$$= (\boldsymbol{\theta}^h - \mathbf{t}^h, \mathbf{k}^h)_{\Gamma_c} \quad \forall \mathbf{k}^h \in \mathbf{Q}_0^h, \quad (4.10)$$

$$\int_{\Gamma_c} \mathbf{g}^h \cdot \mathbf{n} \, d\Gamma = 0. \quad (4.11)$$

$$va(\mathbf{w}^h, \boldsymbol{\mu}^h) + c(\mathbf{w}^h, \mathbf{u}^h, \boldsymbol{\mu}^h) + c(\mathbf{u}^h, \mathbf{w}^h, \boldsymbol{\mu}^h) + b(\mathbf{w}^h, \phi^h) - (\mathbf{w}^h, \boldsymbol{\theta}^h)_\Gamma$$
$$= -c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{w}^h) \quad \forall \mathbf{w}^h \in \mathbf{V}^h, \quad (4.12)$$

$$b(\boldsymbol{\mu}^h, r^h) = 0 \quad \forall r^h \in S_0^h \quad (4.13)$$

and

$$(\boldsymbol{\mu}^h, \mathbf{y}^h) = 0 \quad \forall \mathbf{y}^h \in \mathbf{P}^h \, . \tag{4.14}$$

From a computational standpoint, this is a formidable system. Therefore, how one solves this system is a rather important question. However, in this paper we concern ourselves only with questions related to the accuracy of finite element approximations ; questions about efficient solution methods and implementation techniques, as well as computational examples, will be addressed in another paper.

Remark : The use of the $\mathbf{H}^1(\Gamma_c)$-norm of $\mathbf{g}$ in the functional (1.2), or, equivalently, in (2.4), results in the need to solve the surface problem (4.10)-(4.11). Had we used the $\mathbf{H}^{1/2}(\Gamma_c)$-norm instead, we would be faced with an undesirable computational problem involving the $\mathbf{H}^{1/2}(\Gamma_c)$-inner product. The avoidance of such a happenstance is the main motivation for using the $\mathbf{H}^1(\Gamma_c)$-norm of $\mathbf{g}$. In addition, the regularity brought to us through the use of the $\mathbf{H}^1(\Gamma_c)$-norm of $\mathbf{g}$ turns out to be an asset in deriving error estimates.

Remark : In (4.7)-(4.14), the control $\mathbf{g}$ and the multiplier variables $\mathbf{t}$ and $\boldsymbol{\theta}$ are approximated by functions belonging to the velocity space $\mathbf{V}^h$, restricted to the boundary, i.e., by functions belonging to $\mathbf{P}^h$ and $\mathbf{Q}_0^h$. We could instead choose subspaces $\mathbf{P}^{h_2} \subset \mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{Q}^{h_3} \subset \mathbf{H}^1(\Gamma_c)$ parametrized by parameters h_2 and h_3 that tend to zero ; these parameters may be chosen to be some measure of appropriate boundary grid sizes. Thus, the approximating spaces for the control and the Lagrange multiplier variables could be defined independently of the approximating subspace for the velocity field. We may then define, in analogy to (4.7)-(4.14), a finite element discretization based on these new choices of approximating spaces. Note that necessarily the dimension of $\mathbf{P}^{h_2}$ cannot be larger than that of $\mathbf{V}^h|_\Gamma$, since otherwise the analogous equations to (4.9) and (4.14) would involve a matrix with more rows than columns, rendering the whole discrete problem singular. However, with the *caveat* that h_2 be sufficiently « larger » than h (see [3] or [4]), all of the results below remain valid for such independent choices of approximating subspaces ; one merely need replace h by max (h, h_2, h_3).

Remark : We have chosen to find approximate optimal controls and associated states by approximating the optimality system ; this approach falls under the category of *direct* methods. Other approaches are possible. For example, we could use methods based on finding saddle points of a suitable Lagrangian ; see, e.g., [17]. However, in the end, all methods will essentially require the same types of computations, e.g., at each step of an

iterative process, one must solve the state equations, a set of linearized adjoint equations, as well as some optimality condition for the controls.

4.2. Quotation of results concerning the approximation of a class of nonlinear problems

The error estimate to be derived in Section 4.3 makes use of results of [7] and [10] (see also [11]) concerning the approximation of a class of nonlinear problems, and of [13] for the approximation of the Stokes equations with inhomogeneous velocity boundary conditions. Here, for the sake of completeness, we will state the relevant results, specialized to our needs.

The nonlinear problems considered in [7], [10], and [11] are of the type

$$F(\lambda, \psi) \equiv \psi + TG(\lambda, \psi) = 0 \qquad (4.15)$$

where $T \in \mathscr{L}(Y; X)$, G is a C^2 mapping from $\Lambda \times X$ into Y, X and Y are Banach spaces and Λ is a compact interval of $\mathbb{R}$. We say that $\{(\lambda, \psi(\lambda)) : \lambda \in \Lambda\}$ is a branch of solutions of (4.14) if $\lambda \to \psi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda, \psi(\lambda)) = 0$. The branch is called a *nonsingular branch* if we also have that $D_\psi F(\lambda, \psi(\lambda))$ is an isomorphism from X into X for all $\lambda \in \Lambda$. (Here, $D_\psi F(.,.)$ denotes the Frechet derivative of $F(.,.)$ with respect to the second argument.)

Approximations are defined by introducing a subspace $X^h \subset X$ and an approximating operator $T^h \in \mathscr{L}(Y; X^h)$. Then, we seek $\psi^h \in X^h$ such that

$$F^h(\lambda, \psi^h) \equiv \psi^h + T^h G(\lambda, \psi^h) = 0 . \qquad (4.16)$$

We will assume that there exists another Banach space Z, contained in Y, with continuous imbedding, such that

$$D_\psi G(\lambda, \psi) \in \mathscr{L}(X; Z) \quad \forall \lambda \in \Lambda \quad \text{and} \quad \psi \in X . \qquad (4.17)$$

Concerning the operator T^h, we assume the approximation properties

$$\lim_{h \to 0} \| (T^h - T) y \|_X = 0 \quad \forall y \in Y \qquad (4.18)$$

and

$$\lim_{h \to 0} \| (T^h - T) \|_{\mathscr{L}(Z; X)} = 0 . \qquad (4.19)$$

Note that (4.17) and (4.19) imply that the operator $D_\psi G(\lambda, \psi) \in \mathscr{L}(X; X)$ is compact. Moreover, (4.19) follows from (4.18) whenever the imbedding $Z \subset Y$ is compact.

We can now state the first result of [7] and [10] that will be used in the sequel. In the statement of the theorem, $D^2 G$ represents any and all second Frechet derivatives of G.

THEOREM 4.1 : *Let X and Y be Banach spaces and Λ a compact subset of $\mathbb{R}$. Assume that G is a C^2 mapping from $\Lambda \times X$ into Y and that $D^2 G$ is bounded on all bounded sets of $\Lambda \times X$. Assume that (4.17)-(4.19) hold and that $\{(\lambda, \psi(\lambda)) ; \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (4.15). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and, for $h \leq h_0$ small enough, a unique C^2 function $\lambda \to \psi^h(\lambda) \in X^h$ such that $\{(\lambda, \psi^h(\lambda)) ; \lambda \in \Lambda\}$ is a branch of nonsingular solutions of (4.16) and $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$, independent of h and λ, such that*

$$\| \psi^h(\lambda) - \psi(\lambda) \|_X \leq C \| (T^h - T) G(\lambda, \psi(\lambda)) \|_X \quad \forall \lambda \in \Lambda. \blacksquare \quad (4.20)$$

For the second result, we have to introduce two other Banach spaces H and W, such that $W \subset X \subset H$, with continuous imbeddings, and assume that

for all $w \in W$, the operator $D_\psi G(\lambda, w)$ may be
extended as a linear operator of $\mathcal{L}(H ; Y)$, $\qquad$ (4.21)

the mapping $w \to D_\psi G(\lambda, w)$ being continuous from W onto $\mathcal{L}(H ; Y)$. We also suppose that

$$\lim_{h \to 0} \| T^h - T \|_{\mathcal{L}(Y ; H)} = 0. \qquad (4.22)$$

Then we may state the following additional result.

THEOREM 4.2 : *Assume the hypotheses of Theorem 4.1 and also assume that (4.21) and (4.22) hold. Assume in addition that*

for each $\lambda \in \Lambda$, $\psi(\lambda) \in W$ and the function
$\qquad \lambda \to \psi(\lambda)$ *is continuous from Λ into W* $\quad$ (4.23)

and

for each $\lambda \in \Lambda$, $D_\psi F(\lambda, \psi(\lambda))$ is an isomorphism of H. $\quad$ (4.24)

Then, for $h \leq h_1$ sufficiently small, there exists a constant C, independent of h and λ, such that

$$\| \psi^h(\lambda) - \psi(\lambda) \|_H \leq C \| (T^h - T) G(\lambda, \psi(\lambda)) \|_H +$$
$$+ \| \psi^h(\lambda) - \psi^h(\lambda) \|_X^2 \quad \forall \lambda \in \Lambda. \quad \blacksquare \quad (4.25)$$

We now turn to the results of [13] concerning the approximation of the Stokes problem with the use of Lagrange multipliers to enforce velocity boundary conditions.

THEOREM 4.3 : *The Stokes problem : seek* $\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega)$, $\tilde{p} \in L_0^2(\Omega)$ *and* $\tilde{\mathbf{t}} \in \mathbf{H}^{-1/2}(\Gamma)$ *such that*

$$a(\tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) - (\mathbf{v}, \tilde{\mathbf{t}})_\Gamma = (\tilde{\mathbf{f}}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$b(\tilde{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega)$$

and

$$(\tilde{\mathbf{u}}, \mathbf{s})_\Gamma = (\tilde{\mathbf{b}}, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma)$$

has a unique solution. Let (4.1)-(4.5) hold. Then, the discrete Stokes problem : seek $\tilde{\mathbf{u}}^h \in \mathbf{V}^h$, $\tilde{p}^h \in S_0^h$ *and* $\tilde{\mathbf{t}}^h \in \mathbf{P}^h$ *such that*

$$a(\tilde{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \tilde{p}^h) - (\mathbf{v}^h, \tilde{\mathbf{t}}^h)_\Gamma = (\tilde{\mathbf{f}}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h,$$

$$b(\tilde{\mathbf{u}}^h, q^h) = 0 \quad \forall q^h \in S_0^h$$

and

$$(\tilde{\mathbf{u}}^h, \mathbf{s}^h)_\Gamma = (\tilde{\mathbf{b}}, \mathbf{s}^h)_\Gamma \quad \forall \mathbf{s}^h \in \mathbf{P}^h$$

also has a unique solution. Moreover, as $h \to 0$,

$$\left\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \right\|_1 + \left\| \tilde{p} - \tilde{p}^h \right\|_0 + \left\| \tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h \right\|_{-1/2, \Gamma} \to 0$$

and, if $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \times \mathbf{H}^{m-\frac{1}{2}}(\Gamma)$, *then there exists a constant C, independent of h, such that*

$$\left\| \tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h \right\|_1 + \left\| \tilde{p} - \tilde{p}^h \right\|_0 + \left\| \tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h \right\|_{-1/2, \Gamma} \leqslant Ch^m (\left\| \tilde{\mathbf{u}} \right\|_{m+1} + \left\| \tilde{p} \right\|_m). \quad \blacksquare$$

4.3. Error estimates

We begin by recasting the optimality system (2.1), (2.2), (2.3), (3.27), (3.28), (3.30), (3.39) and (3.40) and its discretization (4.7)-(4.14) into a form that fits into the framework of Section 4.2. Let $\lambda = 1/\nu$; thus, if our governing system has been non-dimensionalized, λ is the Reynolds number. Let

$$X = \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma) \times \mathbf{W}(\Gamma_c) \times \mathbb{R} \times \mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\Gamma)$$

$$\dot{Y} = (\mathbf{H}^1(\Omega))^* \times \mathbf{H}^{\frac{1}{2}}(\Gamma) \times (\mathbf{H}^1(\Omega))^*$$
$$Z = \mathbf{L}^{3/2}(\Omega) \times \mathbf{H}^1(\Gamma) \times \mathbf{L}^{3/2}(\Omega)$$
$$X^h = \mathbf{V}^h \times S_0^h \times \mathbf{P}^h \times Q_0^h \times \mathbb{R} \times \mathbf{V}^h \times S_0^h \times \mathbf{P}^h$$

where $(\mathbf{H}^1(\Omega))^*$ denotes the dual space of $\mathbf{H}^1(\Omega)$. Note that $Z \subset Y$ with a compact imbedding.

Let the operator $T \in \mathscr{L}(Y; X)$ be defined in the following manner : $T(\zeta, \kappa, \eta) = (\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}, \tilde{\mathbf{g}}, \tilde{\beta}, \tilde{\mu}, \tilde{\phi}, \tilde{\theta})$ for $(\zeta, \kappa, \eta) \in Y$ and $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}, \tilde{\mathbf{g}}, \tilde{\beta}, \tilde{\mu}, \tilde{\phi}, \tilde{\theta}) \in X$ if and only if

$$a(\tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) - (\mathbf{v}, \tilde{\mathbf{t}})_\Gamma = (\zeta, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \qquad (4.26)$$

$$b(\tilde{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega), \qquad (4.27)$$

$$(\tilde{\mathbf{u}}, \mathbf{s})_\Gamma = (\kappa, \mathbf{s})_\Gamma \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma), \qquad (4.28)$$

$$(\mathrm{grad}_s\, \tilde{\mathbf{g}}, \mathrm{grad}_s\, \mathbf{k})_{\Gamma_c} + (\tilde{\mathbf{g}}, \mathbf{k})_{\Gamma_c} + \tilde{\beta} \int_{\Gamma_c} \mathbf{k} \cdot \mathbf{n}\, d\Gamma$$
$$- (\mathbf{k}, \tilde{\theta} - \tilde{\mathbf{t}})_{\Gamma_c} = 0 \quad \forall \mathbf{k} \in \mathbf{W}(\Gamma_c), \quad (4.29)$$

$$\int_{\Gamma_c} \tilde{\mathbf{g}} \cdot \mathbf{n}\, d\Gamma = 0, \qquad (4.30)$$

$$a(\bar{\mathbf{w}}, \tilde{\mu}) + b(\bar{\mathbf{w}}, \tilde{\phi}) - (\mathbf{w}, \tilde{\theta})_\Gamma = (\eta, \mathbf{w}) \quad \forall \mathbf{w} \in \bar{\mathbf{H}}^1(\Omega), \qquad (4.31)$$

$$b(\tilde{\mu}, r) = 0 \quad \forall r \in L_0^2(\Omega), \qquad (4.32)$$

and

$$(\tilde{\mu}, \mathbf{y})_\Gamma = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma), \qquad (4.33)$$

Note that this system is *weakly coupled*. First, one may separately solve the Stokes problems (4.26)-(4.28) for $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\mathbf{t}}$ and (4.31)-(4.33) for $\tilde{\mu}$, $\tilde{\phi}$ and $\tilde{\theta}$; then, one may solve the surface Laplacian problem (4.29)-(4.30) for $\tilde{\mathbf{g}}$ and $\tilde{\beta}$.

Remark : The need for introducing the Lagrange multiplier $\mathbf{t}$ in order to enforce the velocity boundary condition can now be made clear. Note the appearance of this multiplier in (4.29). Formally, we could eliminate $\tilde{\mathbf{t}}$ (and, similarly, $\tilde{\theta}$) by using the relation

$$\tilde{\mathbf{t}} = [-\tilde{p}\mathbf{n} + \nu(\mathrm{grad}\,\tilde{\mathbf{u}} + (\mathrm{grad}\,\tilde{\mathbf{u}})^T) \cdot \mathbf{n}]_\Gamma.$$

However, then we would not have that T is well defined from all of Y into X (as is required in Theorems 4.1 and 4.2) since the right hand side of

the above expression does not make sense for general $\tilde{\mathbf{u}} \in \mathbf{H}^1(\Omega)$ and $p \in L_0^2(\Omega)$.

Analogously, the operator $T^h \in \mathscr{L}(Y; X)$ is defined as follows: $T^h(\zeta, \kappa, \eta) = (\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{t}}^h, \tilde{\mathbf{g}}^h, \tilde{\beta}^h, \tilde{\mu}^h, \tilde{\phi}^h, \tilde{\theta}^h)$ for $(\zeta, \kappa, \eta) \in Y$ and $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{t}}^h, \tilde{\mathbf{g}}^h, \tilde{\beta}^h, \tilde{\mu}^h, \tilde{\phi}^h, \tilde{\theta}^h) \in X^h$ if and only if

$$a(\tilde{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \tilde{p}^h) - (\mathbf{v}^h, \tilde{\mathbf{t}}^h)_\Gamma = (\zeta, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h, \qquad (4.34)$$

$$b(\tilde{\mathbf{u}}^h, q^h) = 0 \quad \forall q^h \in S_0^h, \qquad (4.35)$$

$$(\tilde{\mathbf{u}}^h, \mathbf{s}^h)_\Gamma = (\kappa, \mathbf{s}^h)_\Gamma \quad \forall \mathbf{s}^h \in \mathbf{P}^h, \qquad (4.36)$$

$$(\operatorname{grad}_s \tilde{\mathbf{g}}^h, \operatorname{grad}_s \mathbf{k}^h)_{\Gamma_c} + (\tilde{\mathbf{g}}^h, \mathbf{k}^h)_{\Gamma_c} + \tilde{\beta}^h \int_{\Gamma_c} \mathbf{k}^h \cdot \mathbf{n} \, d\Gamma$$

$$- (\mathbf{k}^h, \tilde{\theta}^h - \tilde{\mathbf{t}}^h)_{\Gamma_c} = 0 \quad \forall \mathbf{k}^h \in \mathbf{Q}_0^h, \quad (4.37)$$

$$\int_{\Gamma_c} \tilde{\mathbf{g}}^h \cdot \mathbf{n} \, d\Gamma = 0, \qquad (4.38)$$

$$a(\mathbf{w}^h, \tilde{\mu}^h) + b(\mathbf{w}^h, \tilde{\phi}^h) - (\mathbf{w}^h, \tilde{\theta}^h)_\Gamma = (\mathbf{w}^h, \eta)_\Gamma \quad \forall \mathbf{w}^h \in \mathbf{V}^h, \quad (4.39)$$

$$b(\tilde{\mu}^h, r^h) = 0 \quad \forall r^h \in S_0^h \qquad (4.40)$$

and

$$(\tilde{\mu}^h, \mathbf{y}^h) = 0 \quad \forall \mathbf{y}^h \in \mathbf{P}^h. \qquad (4.41)$$

The system (4.34)-(4.41) is weakly coupled in the same sense as the system (4.26)-(4.33).

Let Λ denote a compact subset of $\mathbb{R}_+$. Next, we define the *nonlinear* mapping $G: \Lambda \times X \to Y$ as follows: $G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \mu, \phi, \theta)) = (\zeta, \kappa, \eta)$ for $\lambda \in \Lambda$, $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \mu, \phi, \theta) \in X$ and $(\zeta, \kappa, \eta) \in Y$ if and only if

$$(\zeta, \mathbf{v}) = \lambda c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda(\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \qquad (4.42)$$

$$(\kappa, \mathbf{s})_\Gamma = -(\mathbf{b}, \mathbf{s})_\Gamma - (\mathbf{g}, \mathbf{s})_{\Gamma_c} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma) \qquad (4.43)$$

and

$$(\eta, \mathbf{w}) = \lambda c(\mathbf{w}, \mathbf{u}, \mu) + \lambda c(\mathbf{u}, \mathbf{w}, \mu) + \lambda c(\mathbf{u}, \mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega) \quad (4.44)$$

It is easily seen that the optimality system (2.1), (2.2), (2.3), (3.27), (3.28), (3.30), (3.39) and (3.40) is equivalent to

$$(\mathbf{u}, \lambda p, \lambda \mathbf{t}, \mathbf{g}, \lambda \beta, \mu, \lambda \phi, \lambda \theta) + TG(\lambda, (\mathbf{u}, \lambda p, \lambda \mathbf{t}, \mathbf{g}, \lambda \beta, \mu, \lambda \phi, \lambda \theta)) = 0$$

$$(4.45)$$

and that the discrete optimality system (4.7)-(4.14) is equivalent to

$$(\mathbf{u}^h, \lambda p^h, \lambda t^h, \mathbf{g}^h, \lambda \beta^h, \boldsymbol{\mu}^h, \lambda \phi^h, \lambda \theta^h) + $$
$$+ TG(\lambda, (\mathbf{u}^h, \lambda p^h, \lambda t^h, \mathbf{g}^h, \lambda \beta^h, \boldsymbol{\mu}^h, \lambda \phi^h, \lambda \theta^h)) = 0. \quad (4.46)$$

We have thus recast our continuous and discrete optimality problems into a form that enables us to apply Theorems 4.1 and 4.2.

A solution $(\mathbf{u}(\lambda), p(\lambda), t(\lambda), \mathbf{g}(\lambda), \beta(\lambda), \boldsymbol{\mu}(\lambda), \phi(\lambda), \theta(\lambda))$ of the problem (2.1), (2.2), (2.3), (3.27), (3.28), (3.30), (3.39) and (3.40), or equivalently, of (4.45), is nonsingular if the linear system

$$a(\tilde{\mathbf{u}}, \mathbf{v}) + \lambda c(\tilde{\mathbf{u}}, \mathbf{u}, \mathbf{v}) + \lambda c(\mathbf{u}, \tilde{\mathbf{u}}, \mathbf{v}) + \lambda b(\mathbf{v}, \tilde{p}) - \lambda(\mathbf{v}, \tilde{\mathbf{t}})_\Gamma$$
$$= (\mathbf{f}_1, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$b(\tilde{\mathbf{u}}, q) = 0 \quad \forall q \in L_0^2(\Omega),$$

$$(\tilde{\mathbf{u}}, \mathbf{s})_\Gamma = (\tilde{\mathbf{g}}, \mathbf{s})_{\Gamma_c} = 0 \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma),$$

$$\nu(\operatorname{grad}_s \tilde{\mathbf{g}}, \operatorname{grad}_s \mathbf{k})_{\Gamma_c} + \nu(\tilde{\mathbf{g}}, \mathbf{k})_{\Gamma_c} + \tilde{\beta} \int_{\Gamma_c} \mathbf{k} \cdot \mathbf{n} \, d\Gamma - (\mathbf{k}, \tilde{\theta} - \tilde{\mathbf{t}})_{\Gamma_c}$$
$$= 0 \quad \forall \mathbf{k} \in \mathbf{W}(\Gamma_c),$$

$$\int_{\Gamma_c} \tilde{\mathbf{g}} \cdot \mathbf{n} \, d\Gamma = 0,$$

$$a(\mathbf{w}, \tilde{\boldsymbol{\mu}}) + \lambda c(\mathbf{w}, \tilde{\mathbf{u}}, \boldsymbol{\mu}) + \lambda c(\mathbf{w}, \mathbf{u}, \tilde{\boldsymbol{\mu}}) + \lambda c(\mathbf{u}, \mathbf{w}, \tilde{\boldsymbol{\mu}}) + \lambda c(\tilde{\mathbf{u}}, \mathbf{w}, \boldsymbol{\mu})$$
$$+ \lambda b(\mathbf{w}, \tilde{\phi}) - (\mathbf{w}, \tilde{\theta})_\Gamma + \lambda c(\tilde{\mathbf{u}}, \mathbf{u}, \mathbf{w}) + \lambda c(\mathbf{u}, \tilde{\mathbf{u}}, \mathbf{w}) = (\mathbf{f}_2, \mathbf{v}) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega),$$

$$b(\tilde{\boldsymbol{\mu}}, r) = 0 \quad \forall r \in L_0^2(\Omega),$$

and

$$(\tilde{\boldsymbol{\mu}}, \mathbf{y})_\Gamma = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma)$$

has a unique solution $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}, \tilde{\mathbf{g}}, \tilde{\beta}, \tilde{\boldsymbol{\mu}}, \tilde{\phi}, \tilde{\theta}) \in X$ for every $\mathbf{f}_k \in (\mathbf{H}^1(\Omega))^*$, $k = 1, 2$. An analogous definition holds for nonsingular solutions of the discrete optimality system (4.7)-(4.14), or equivalently, (4.46).

Remark : It can be shown, using techniques similar to those employed for the Navier-Stokes equations (see [21] and the references cited therein) that for almost all values of the Reynolds number, i.e., for almost all data and values of the viscosity ν, that the optimality system (2.1), (2.2), (2.3), (3.27), (3.28), (3.30), (3.39) and (3.40), or equivalently, of (4.45), is nonsingular, i.e., is locally unique. Thus, it is reasonable to assume that the

optimality system has branches of nonsingular solutions. (However, we note that, just as in the Navier-Stokes case, it is impossible to *predict*, except in very simple settings, exactly at what values of the Reynolds number singularities, e.g., bifurcations, appear.)

In order to apply the results of Section 4.1, we need to estimate the approximation properties of the operator T^h.

PROPOSITION 4.4 : *The problem (4.26)-(4.33) has a unique solution belonging to X. Assume that (4.1)-(4.6) hold. Then, the problem (4.34)-(4.41) has a unique solution belonging to X^h. Let $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}, \tilde{\mathbf{g}}, \tilde{\beta}, \tilde{\boldsymbol{\mu}}, \tilde{\phi}, \tilde{\theta})$ and $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{t}}^h, \tilde{\mathbf{g}}^h, \tilde{\beta}^h, \tilde{\boldsymbol{\mu}}^h, \tilde{\phi}^h, \tilde{\theta}^h)$ denote the solutions of (4.26)-(4.33) and (4.34)-(4.41), respectively. Then, we also have that*

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h\|_{-\frac{1}{2}, \Gamma} + \|\tilde{\mathbf{g}} - \tilde{\mathbf{g}}^h\|_{1, \Gamma_c} + |\tilde{\beta} - \tilde{\beta}^h| +$$

$$+ \|\tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 + \|\tilde{\theta} - \tilde{\theta}^h\|_{-\frac{1}{2}, \Gamma} \to 0 \quad (4.47)$$

as $h \to 0$. If, in addition, $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}, \tilde{\mathbf{g}}, \tilde{\beta}, \tilde{\boldsymbol{\mu}}, \tilde{\phi}, \tilde{\theta}) \in \mathbf{H}^{m+1}(\Omega) \times$

$$H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{m-\frac{1}{2}}(\Gamma) \times \mathbf{H}^{m+1}(\Gamma_c) \times \mathbb{R} \times \mathbf{H}^{m+1}(\Omega) \times H^m \cap L_0^2(\Omega)$$

$$\times \mathbf{H}^{m-\frac{1}{2}}(\Gamma), \text{ then}$$

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h\|_{-\frac{1}{2}, \Gamma} + \|\tilde{\mathbf{g}} - \tilde{\mathbf{g}}^h\|_{1, \Gamma_c} + |\tilde{\beta} - \tilde{\beta}^h| +$$

$$+ \|\tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 + \|\tilde{\theta} - \tilde{\theta}^h\|_{-\frac{1}{2}, \Gamma}$$

$$\leqslant Ch^m(\|\tilde{\mathbf{u}}\|_{m+1} + \|\tilde{p}\|_m + \|\tilde{\boldsymbol{\mu}}\|_{m+1} + \|\tilde{\phi}\|_m). \quad (4.48)$$

Proof : First, it follows from Theorem 4.3 that the two Stokes problems (4.26)-4.28) and (4.31)-(4.33) each have a unique solution $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}})$ and $(\tilde{\boldsymbol{\mu}}, \tilde{\phi}, \tilde{\theta})$ belonging to $\mathbf{H}^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}^{-1/2}(\Gamma)$, respectively. Also, the discrete Stokes problems (4.34)-(4.36) and (4.39)-(4.41) each have a unique solution $(\tilde{\mathbf{u}}^h, \tilde{p}^h, \tilde{\mathbf{t}}^h)$ and $(\tilde{\boldsymbol{\mu}}^h, \tilde{\phi}^h, \tilde{\theta}^h)$ belonging to $\mathbf{V}^h \times S_0^h \times \mathbf{P}^h$, respectively. Moreover, we have that

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h\|_{-\frac{1}{2}, \Gamma} \to 0 \qquad (4.49)$$

and

$$\|\tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 + \|\tilde{\theta} - \tilde{\theta}^h\|_{-\frac{1}{2}, \Gamma} \to 0 \qquad (4.50)$$

as $h \to 0$, and, if in addition $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\mathbf{t}}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times$
$\mathbf{H}^{m-\frac{1}{2}}(\Gamma)$ and $(\tilde{\boldsymbol{\mu}}, \tilde{\phi}, \tilde{\boldsymbol{\theta}}) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{m-\frac{1}{2}}(\Gamma,)$, we
have that

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}^h\|_1 + \|\tilde{p} - \tilde{p}^h\|_0 + \|\tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h\|_{-\frac{1}{2}, \Gamma} \leqslant Ch^m(\|\tilde{\mathbf{u}}\|_{m+1} + \|\tilde{p}\|_m) \quad (4.51)$$

$$\|\tilde{\boldsymbol{\mu}} - \tilde{\boldsymbol{\mu}}^h\|_1 + \|\tilde{\phi} - \tilde{\phi}^h\|_0 + \|\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^h\|_{-\frac{1}{2}, \Gamma} \leqslant Ch^m(\|\tilde{\boldsymbol{\mu}}\|_{m+1} + \|\tilde{\phi}\|_m) \,. \quad (4.52)$$

Next, consider the problems (4.29)-(4.30) and (4.37)-(4.38). One may
easily show that

$$(\operatorname{grad}_s \mathbf{k}, \operatorname{grad}_s \mathbf{k})_{\Gamma_c} + (\mathbf{k}, \mathbf{k})_{\Gamma_c}$$
$$\geqslant C \, \|\mathbf{k}\|_{1, \Gamma_c}^2 \quad \forall \mathbf{k} \in \mathbf{W}(\Gamma_c) \cap \mathbf{H}_n^1(\Gamma_c) = \mathbf{W}_n(\Gamma_c) \quad (4.53)$$

so that we also have that

$$(\operatorname{grad}_s \mathbf{k}^h, \operatorname{grad}_s \mathbf{k}^h)_{\Gamma_c} + (\mathbf{k}^h, \mathbf{k}^h)_{\Gamma_c}$$
$$\geqslant C \, \|\mathbf{k}^h\|_{1, \Gamma_c}^2 \quad \forall \mathbf{k}^h \in \mathbf{Q}_0^h \cap \mathbf{H}_n^1(\Gamma_c) \subset \mathbf{W}_n(\Gamma_c) \,. \quad (4.54)$$

Also, we have that for all $\beta \in \mathbf{R}$,

$$\sup_{0 \neq \mathbf{k} \in \mathbf{w}(\Gamma_c)} \frac{\beta \displaystyle\int_{\Gamma_c} \mathbf{k} \cdot \mathbf{n} \, d\Gamma}{\|\mathbf{k}\|_{1, \Gamma_c}} \geqslant C \, |\beta| \quad (4.55)$$

and

$$\sup_{0 \neq \mathbf{k}^h \in \mathbf{Q}_0^h} \frac{\beta \displaystyle\int_{\Gamma_c} \mathbf{k}^h \cdot \mathbf{n} \, d\Gamma}{\|\mathbf{k}^h\|_{1, \Gamma_c}} \geqslant C \, |\beta| \,. \quad (4.56)$$

Furthermore, we have that

$$(\mathbf{k}, \tilde{\mathbf{t}} - \tilde{\boldsymbol{\theta}})_{\Gamma_c} \leqslant \left(\|\tilde{\mathbf{t}}\|_{-1/2, \Gamma_c} + \|\tilde{\boldsymbol{\theta}}\|_{-1/2, \Gamma_c} \right) \|\mathbf{k}\|_{1, \Gamma_c} \quad \forall \mathbf{k} \in \mathbf{W}(\Gamma_c) \quad (4.57)$$

and

$$(\mathbf{k}^h, \tilde{\mathbf{t}}^h - \tilde{\boldsymbol{\theta}}^h)_{\Gamma_c} \leqslant \left(\|\tilde{\mathbf{t}}^h\|_{-1/2, \Gamma_c} + \|\tilde{\boldsymbol{\theta}}^h\|_{-1/2, \Gamma_c} \right) \|\mathbf{k}^h\|_{1, \Gamma_c} \quad \forall \mathbf{k}^h \in \mathbf{Q}_0^h \quad (4.58)$$

so that $(\mathbf{k}, \tilde{\mathbf{t}} - \tilde{\boldsymbol{\theta}})_{\Gamma_c}$ and $(\mathbf{k}^h, \tilde{\mathbf{t}}^h - \tilde{\boldsymbol{\theta}}^h)_{\Gamma_c}$ define bounded linear functionals on $\mathbf{W}(\Gamma_c)$ and $\mathbf{Q}_0^h$, respectively.

The results (4.53)-(4.58) are exactly those invoked in the Brezzi theory for mixed finite element methods (see [5] or [6]), applied to the problems (4.29)-(4.30) and (4.37)-(4.38). Thus, using that theory, we may conclude that these problems both have unique solutions, that

$$\| \tilde{\mathbf{g}} \|_{m+1, \Gamma_c} \leqslant C \left\| \tilde{\mathbf{t}} - \tilde{\boldsymbol{\theta}} \right\|_{m - \frac{1}{2}, \Gamma_c}$$

$$\leqslant C \left(\| \tilde{\mathbf{u}} \|_{m+1} + \| \tilde{p} \|_m + \| \tilde{\boldsymbol{\mu}} \|_{m+1} + \| \tilde{\boldsymbol{\phi}} \|_m \right), \quad (4.59)$$

and

$$\| \tilde{\mathbf{g}} - \tilde{\mathbf{g}}^h \|_{1, \Gamma_c} + | \tilde{\beta} - \tilde{\beta}^h |$$

$$\leqslant C \left(\| \tilde{\mathbf{g}} - \hat{\mathbf{g}}^h \|_{1, \Gamma_c} + \| \tilde{\mathbf{t}} - \tilde{\mathbf{t}}^h \|_{-1/2, \Gamma_c} + \| \tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^h \|_{-1/2, \Gamma_c} \right) \quad \forall \hat{\mathbf{g}}^h \in \mathbf{Q}_0^h,$$

where the last two terms in the right hand side arise from the fact that the right hand sides of (4.29) and (4.37) involve different data, i.e., $\tilde{\boldsymbol{\theta}} - \tilde{\mathbf{t}}$ for (4.29) and $\tilde{\boldsymbol{\theta}}^h - \tilde{\mathbf{t}}^h$ for (4.37). Using (4.6), (4.49) and (4.50) we then have that

$$\| \tilde{\mathbf{g}} - \tilde{\mathbf{g}}^h \|_{1, \Gamma_c} + | \tilde{\beta} - \tilde{\beta}^h | \to 0 \quad (4.60)$$

as $h \to 0$, and using (4.6), (4.51), (4.52) and (4.59), we conclude that

$$\| \tilde{\mathbf{g}} - \tilde{\mathbf{g}}^h \|_{1, \Gamma_c} + | \tilde{\beta} - \tilde{\beta}^h | \leqslant C h^m (\| \tilde{\mathbf{u}} \|_{m+1} + \| \tilde{p} \|_m + \| \tilde{\boldsymbol{\mu}} \|_{m+1} + \| \tilde{\boldsymbol{\phi}} \|_m).$$

$$(4.61)$$

Then, (4.49), (4.50) and (4.60) yield (4.47), and (4.51), (4.52) and (4.61) yield (4.48). ■

Using this proposition and Theorem 4.1, we are led to the following result.

THEOREM 4.5 : *Assume that Λ is a compact interval of $\mathbb{R}_+$ and that there exists a branch $\{ (\lambda, \psi(\lambda) = (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\phi}, \boldsymbol{\theta})) \in X : \lambda \in \Lambda \}$ of nonsingular solutions of the optimality system (2.1)-(2.3), (3.27)-(3.28), (3.3) and (3.39)-(3.40). Assume that the finite element spaces $\mathbf{V}^h$, S^h, $\mathbf{P}^h$ and $\mathbf{Q}_0^h$ satisfy the conditions (4.1)-(4.6). Then, there exists a neighborhood $\mathcal{O}$ of the origin in X and, for $h \leqslant h_0$ small enough, a unique branch $\{ (\lambda, \psi^h(\lambda) = (\mathbf{u}^h, p^h, \mathbf{t}^h, \mathbf{g}^h, \beta^h, \boldsymbol{\mu}^h, \boldsymbol{\phi}^h, \boldsymbol{\theta}^h) \in X^h) : \lambda \in \Lambda \}$ of solutions of*

the discrete optimality system (4.7)-(4.14) *such that* $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ *for all* $\lambda \in \Lambda$. *Moreover,*

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X = \|\mathbf{u} - \mathbf{u}^h\|_1$$

$$+ \|p - p^h\|_0 + \|\mathbf{t} - \mathbf{t}^h\|_{-1/2, \Gamma} + \|\mathbf{g} - \mathbf{g}^h\|_{1, \Gamma_c} |\beta - \beta^h|$$

$$+ \|\boldsymbol{\mu} - \boldsymbol{\mu}^h\|_1 + \|\phi - \phi^h\|_0 + \|\theta - \theta^h\|_{-1, 2, \Gamma - 0} \to 0 \qquad (4.62)$$

as $h \to 0$, *uniformly in* $\lambda \in \Lambda$.

If, in addition, the solution of the optimality system satisfies

$$(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \boldsymbol{\mu}, \phi, \theta) \in \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times$$

$$\mathbf{H}^{m - \frac{1}{2}}(\Gamma) \times \mathbf{H}^{m+1}(\Gamma_c) \times \mathbf{H}^{m+1}(\Omega) \times H^m(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{m - \frac{1}{2}}(\Gamma)$$

for $\lambda \in \Lambda$, *then there exists a constant C, independent of h, such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\mathbf{t} - \mathbf{t}^h\|_{-1/2, \Gamma} + \|\mathbf{g} - \mathbf{g}^h\|_{1, \Gamma_c} + |\beta - \beta^h| +$$

$$+ \|\boldsymbol{\mu} - \boldsymbol{\mu}^h\|_1 + \|\phi - \phi^h\|_0 + \|\theta - \theta^h\|_{-1, 2, \Gamma}$$

$$\leqslant C h^m (\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\boldsymbol{\mu}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m), \quad (4.63)$$

uniformly in $\lambda \in \Lambda$.

Proof: Clearly, G is a C^∞ polynomial map from $\mathbb{R}_+ \times X$ into Y. Therefore, using (1.12)-(1.14), $D^2 G(\lambda, .)$ is easily shown to be bounded on all bounded sets of X. Now, given $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \theta) \in X$, a direct computation yields that $(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\kappa}}, \tilde{\boldsymbol{\eta}}) \in Y$ satisfies

$$(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\kappa}}, \tilde{\boldsymbol{\eta}}) = D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \theta))(\mathbf{v}, q, \mathbf{s}, \mathbf{k}, \alpha, \mathbf{w}, r, y)$$

for $(\mathbf{v}, q, \mathbf{s}, \mathbf{k}, \alpha, \mathbf{w}, r, y) \in X$ if and only if

$$(\tilde{\boldsymbol{\zeta}}, \bar{\mathbf{v}}) = \lambda c(\mathbf{u}, \mathbf{v}, \bar{\mathbf{v}}) + \lambda c(\mathbf{v}, \mathbf{u}, \bar{\mathbf{v}}) \quad \forall \bar{\mathbf{v}} \in \mathbf{H}^1(\Omega),$$

$$(\tilde{\boldsymbol{\kappa}}, \bar{\mathbf{s}})_\Gamma = - (\mathbf{k}, \bar{\mathbf{s}})_{\Gamma_c} \quad \forall \bar{\mathbf{s}} \in \mathbf{H}^{-1/2}(\Gamma)$$

and

$$(\tilde{\boldsymbol{\eta}}, \bar{\mathbf{w}}) = \lambda c(\bar{\mathbf{w}}, \mathbf{v}, \boldsymbol{\mu}) + \lambda c(\bar{\mathbf{w}}, \mathbf{u}, \mathbf{w}) + \lambda c(\mathbf{v}, \bar{\mathbf{w}}, \boldsymbol{\mu})$$

$$+ \lambda c(\mathbf{u}, \bar{\mathbf{w}}, \mathbf{w}) + \lambda c(\mathbf{u}, \mathbf{v}, \bar{\mathbf{w}}) + \lambda c(\mathbf{v}, \mathbf{u}, \bar{\mathbf{w}}) \quad \forall \bar{\mathbf{w}} \in \mathbf{H}^1(\Omega).$$

Thus, for given $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \theta) \in X$, it follows from (1.12)-(1.14) that $D_\phi G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \theta)) \in \mathcal{L}(X; Y)$. On the other hand, since $(\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \theta) \in X$ and $(\mathbf{v}, q, \mathbf{s}, \mathbf{k}, \alpha, \mathbf{w}, r, y) \in X$, by the

Sobolev imbedding theorem, $\mathbf{u}, \mathbf{v}, \boldsymbol{\mu}$ and $\mathbf{w} \in \mathbf{L}^6(\Omega)$, and $\partial u/\partial x_j$, $\partial v/\partial x_j$, $\partial \mu/\partial x_j$ and $\partial w/\partial x_j \in \mathbf{L}^2(\Omega)$ for $j = 1, ..., d$. Then, it follows that $(\tilde{\boldsymbol{\zeta}}, \tilde{\boldsymbol{\kappa}}, \tilde{\boldsymbol{\eta}}) \in Z$ and that

$$D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \boldsymbol{\theta}\,\,)) \in \mathscr{L}(X; Z)$$

$$\text{for } (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \boldsymbol{\theta}\,\,) \in X.$$

Of course, Z is continuously imbedded into Y; moreover, the imbedding $Z \subset Y$ is compact.

Next, we turn to the approximation properties of the operator T^h. From Proposition 4.4, we have that (4.18) holds. Since the imbedding of Z into Y is compact, (4.19) follows from (4.18), and then (4.62) follows from (4.20). From Proposition (4.4) we also may conclude that there exists a constant C, independent of h, such that

$$\left\| (T - T^h) G(\lambda, \psi(\lambda)) \right\|_X \leq C h^m (\|\mathbf{u}\|_{m+1} + \|p\|_m + \|\boldsymbol{\mu}\|_{m+1} + \|\phi\|_m).$$

Then (4.63) follows from (4.20). ∎

Using Theorem 4.2, we now derive an estimate for the error of $\mathbf{u}^h$ and $\boldsymbol{\mu}^h$ in the $\mathbf{L}^2(\Omega)$-norm and of $\mathbf{g}^h$ in the $\mathbf{H}^{1/2}(\Gamma_c)$-norm. At this point it is convenient to examine (4.42)-(4.44) and noting that $G(\lambda, \psi) = G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{g}, \beta, \boldsymbol{\mu}, \phi, \boldsymbol{\theta}\,\,))$ does not depend on p, $\mathbf{t}$, β, ϕ or $\boldsymbol{\theta}$. Therefore, we now redefine $X = \mathbf{H}^1(\Omega) \times \mathbf{W}(\Gamma_c) \times \mathbf{H}^1(\Omega)$ and $X^h = \mathbf{V}^h \times \mathbf{Q}_0^h \times \mathbf{V}^h$; Y and Z remain as before. We also restrict our view of the various mappings to these spaces. We introduce the spaces $H = \mathbf{L}^2(\Omega) \times \mathbf{H}^{1/2}(\Gamma_c) \times \mathbf{L}^2(\Omega)$ and $W = \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Gamma_c) \times \mathbf{H}^2(\Omega)$.

THEOREM 4.6 : *Assume the hypotheses of Theorems 3.5 and 4.5. Then there exists a constant C, independent of h such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_0 + \|\boldsymbol{\mu} - \boldsymbol{\mu}^h\|_0 + \|\mathbf{g} - \mathbf{g}^h\|_{1/2, \Gamma_c} \leq$$

$$\leq C h^{m + \frac{1}{2}} (\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\boldsymbol{\mu}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m). \quad (4.64)$$

Proof: We need only verify that (4.21)-(4.24) hold in our setting; then, the approximation properties (4.1) and (4.6) and the results (4.25) and (4.63) easily leads to (4.64).

From Theorem 3.5 we have that $\mathbf{u}, \boldsymbol{\mu} \in \mathbf{H}^2(\Omega)$ and that $\mathbf{g} \in \mathbf{H}^2(\Gamma_c)$; then one can easily show that

$$\left| c(\mathbf{u}, \mathbf{v}, \bar{\mathbf{v}}) + c(\mathbf{v}, \mathbf{u}, \bar{\mathbf{v}}) \right| \leq C \|\mathbf{u}\|_2 \|\mathbf{v}\|_0 \|\bar{\mathbf{v}}\|_1$$

$$\forall \mathbf{u} \in \mathbf{H}^2(\Omega), \mathbf{v} \in \mathbf{L}^2(\Omega), \bar{\mathbf{v}} \in \mathbf{H}^1(\Omega), \quad (4.65)$$

and

$$\left| c(\bar{\mathbf{w}}, \mathbf{v}, \boldsymbol{\mu}) + c(\bar{\mathbf{w}}, \mathbf{u}, \mathbf{w}) + c(\mathbf{v}, \bar{\mathbf{w}}, \boldsymbol{\mu}) + c(\mathbf{u}, \bar{\mathbf{w}}, \mathbf{w}) \right.$$

$$\left. + c(\mathbf{u}, \mathbf{v}, \bar{\mathbf{w}}) + c(\mathbf{v}, \mathbf{u}, \bar{\mathbf{w}}) \right| \leqslant C \left(\|\mathbf{u}\|_2 \|\mathbf{w}\|_0 + \|\boldsymbol{\mu}\|_2 \|\mathbf{v}\|_0 \right) \|\bar{\mathbf{w}}\|_1$$

$$\forall \mathbf{u}, \boldsymbol{\mu} \in \mathbf{H}^2(\Omega), \mathbf{v}, \mathbf{w} \in \mathbf{L}^2(\Omega), \bar{\mathbf{v}}, \bar{\mathbf{w}} \in \mathbf{H}^1(\Omega). \tag{4.66}$$

Then, we have that for $(\mathbf{u}, \mathbf{g}, \boldsymbol{\mu}) \in W$ and $(\mathbf{v}, \mathbf{k}, \mathbf{w}) \to D_\psi G(\lambda, (\mathbf{u}, \mathbf{g}, \boldsymbol{\mu}))(\mathbf{v}, \mathbf{k}, \mathbf{w})$ belongs to $\mathscr{L}(H, Y)$. The continuity of the mapping $D_\psi G(\lambda, (.)) \in \mathscr{L}(W, \mathscr{L}(H, Y))$ is an easy consequence of (4.65) and (4.66). Thus we have verified (4.21). Next, (4.22) follows from (4.18) and the fact that X is compactly imbedded into H. The results of Section 3.3 and the fact that λ belongs to Λ, a compact interval of $\mathbb{R}_+$, easily yield (4.23), where $\psi(\lambda) = (\mathbf{u}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\mu}(\lambda))$. Finally, (4.24) follows from the well known properties of the solution operator for the Stokes problem, the continuity of the mapping $D_\psi G(\lambda, (\mathbf{u}, \mathbf{g}, \boldsymbol{\mu}))$ and the fact that we have assumed that $(\mathbf{u}(\lambda), \mathbf{g}(\lambda), \boldsymbol{\mu}(\lambda))$, $\lambda \in \Lambda$, defines a nonsingular branch of solutions.

Remark : By other means, it can be shown that actually

$$\|\mathbf{u} - \mathbf{u}^h\|_0 + \|\boldsymbol{\mu} - \boldsymbol{\mu}^h\|_0 + \|\mathbf{g} - \mathbf{g}^h\|_{0, \Gamma_c}$$

$$\leqslant C h^{m+1} (\|\mathbf{u}(\lambda)\|_{m+1} + \|p(\lambda)\|_m + \|\boldsymbol{\mu}(\lambda)\|_{m+1} + \|\phi(\lambda)\|_m).$$

See [16]. Note that in all cases the error in the approximation to the control is 1/2-order higher than that obtainable from the error estimates for the velocity approximation and an application of trace theorems.

5. THE TRACKING FUNCTIONAL

We now consider the minimization of the functional (1.1). In terms of the notation introduced in Section 1.1, this functional is given by

$$\mathfrak{J}(\mathbf{u}, \mathbf{g}) = \frac{1}{4} \|\mathbf{u} - \mathbf{u}_0\|_{L^4(\Omega)}^4 + \frac{\nu}{2} |\mathbf{g}|_{1, \Gamma_c}^2. \tag{5.1}$$

The *admissibility set* $\mathscr{V}_{ad}$ is now defined by

$$\mathscr{V}_{ad} = \{ (\mathbf{u}, \mathbf{g}) \in \mathbf{H}^1(\Omega) \times \mathbf{W}_n(\Gamma_c) : \mathfrak{J}(\mathbf{u}, \mathbf{g}) \leqslant \infty,$$

and there exist $p \in L_0^2(\Omega)$ and $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma)$

such that (2.1)-(2.3) are satisfied $\}$.

Then, $(\hat{\mathbf{u}}, \hat{\mathbf{g}}) \in \mathcal{V}_{ad}$ is called an *optimal solution* if there exists $\varepsilon > 0$ such that

$$\mathfrak{I}(\hat{\mathbf{u}}, \hat{\mathbf{g}}) < \mathfrak{I}(\mathbf{u}, \mathbf{g}) \quad \forall\, (\mathbf{u}, \mathbf{g}) \in \mathcal{V}_{ad} \text{ satisfying } \|\mathbf{u} - \hat{\mathbf{u}}\|_1 + \|\mathbf{g} - \hat{\mathbf{g}}\|_{1, \Gamma_c} \leqslant \varepsilon .$$

The optimality system corresponding to the minimization of (1.1), or equivalently, (5.1), is now given by (2.1)-(2.3), (3.27)-(3.29), and, instead of (3.26),

$$va(\mathbf{w}, \boldsymbol{\mu}) + c(\mathbf{w}, \mathbf{u}, \boldsymbol{\mu}) + c(\mathbf{u}, \mathbf{w}, \boldsymbol{\mu}) + b(\mathbf{w}, \phi) + (\mathbf{w}, \boldsymbol{\tau})_\Gamma$$
$$= ([\mathbf{u} - \mathbf{u}_0]^3, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega) ,$$

where $[\mathbf{u} - \mathbf{u}_0]^3$ denotes componentwise exponentiation, i.e., $([\mathbf{u} - \mathbf{u}_0]^3)_j = (\mathbf{u} - \mathbf{u}_0)_j^3, \, j = 1, ..., d$. In terms of differential equations, the optimality systems is therefore given by (3.31)-(3.38) with (3.36) replaced by

$$- v \operatorname{div} ((\operatorname{grad} \boldsymbol{\mu}) + (\operatorname{grad} \boldsymbol{\mu})^T) + \boldsymbol{\mu} \cdot (\operatorname{grad} \mathbf{u})^T$$
$$- \mathbf{u} \cdot \operatorname{grad} \boldsymbol{\mu} + \operatorname{grad} \phi = [\mathbf{u} - \mathbf{u}_0]^3 \quad \text{in } \Omega .$$

All results of Sections 2-4 hold in the present setting, and, for the most part, the same methods of proof may be employed as well. In the latter regard, one exception is the proof of the existence of optimal solutions. In the proof of Theorem 2.1, the facts that $\mathcal{K}(\mathbf{u}^{(k)}, p^{(k)}, \mathbf{g}^{(k)})$ is bounded for $(\mathbf{u}^{(k)}, p^{(k)}, \mathbf{g}^{(k)}) \in \mathcal{U}_{ad}$ and that $\mathbf{u}^{(k)} = \mathbf{g}^{(k)} + \mathbf{b}$ on Γ immediately imply that $\|\mathbf{u}^{(k)}\|_1$ is bounded. To obtain the latter result for the present case we proceed as follows. First, the fact that $\mathfrak{I}(\mathbf{u}^{(k)}, \mathbf{g}^{(k)})$ is bounded for a minimizing sequence $(\mathbf{u}^{(k)}, \mathbf{g}^{(k)}) \in \mathcal{V}_{ad}$ implies that $(\mathbf{u}^{(k)}, \mathbf{g}^{(k)})$ is uniformly bounded in $\mathbf{L}^4(\Omega) \times \mathbf{H}^{1/2}(\Gamma_c)$ and that for some $p^{(k)} \in L_0^2(\Omega)$ and $\mathbf{t}^{(k)} \in \mathbf{H}^{-1/2}(\Gamma)$, (2.7)-(2.9) hold. Now, for $\mathbf{g}^{(k)} \in \mathbf{W}_n(\Gamma_c)$, choose $\mathbf{w}^{(k)} \in \mathbf{H}^1(\Omega)$ and $r^{(k)} \in L_0^2(\Omega)$ to satisfy the Stokes problem

$$va(\mathbf{w}^{(k)}, \mathbf{v}) + b(\mathbf{v}, r^{(k)}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) , \tag{5.2}$$

$$b(\mathbf{w}^{(k)}, q) = 0 \quad \forall q \in L_0^2(\Omega) \tag{5.3}$$

and

$$\mathbf{w}^{(k)} = \begin{cases} \mathbf{g}^{(k)} + \mathbf{b} & \text{on } \Gamma_c \\ \mathbf{b} & \text{on } \Gamma_u . \end{cases} \tag{5.4}$$

Such $\mathbf{w}^{(k)}$ and $r^{(k)}$ exist ; moreover, the estimate

$$\|\mathbf{w}^{(k)}\|_1 \leqslant C (\|\mathbf{f}\|_0 + \|\mathbf{b}\|_{1/2, \Gamma} + \|\mathbf{g}^{(k)}\|_{1, \Gamma_c}) \tag{5.5}$$

holds ; see [20]. Note that (2.9) and (5.4) imply that $(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}) = \mathbf{0}$ on Γ so that we may let $\mathbf{v} = \mathbf{u}^{(k)} - \mathbf{w}^{(k)}$ in (2.7) and (5.2). Then, subtracting these two results and using (2.8) and (5.3), we obtain

$$v a(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}, \mathbf{u}^{(k)} - \mathbf{w}^{(k)}) = - c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)}, \mathbf{u}^{(k)} - \mathbf{w}^{(k)})$$
$$= c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)} - \mathbf{w}^{(k)}, \mathbf{u}^{(k)}) \,. \quad (5.6)$$

Note that

$$\left| c(\mathbf{u}^{(k)}, \mathbf{u}^{(k)} - \mathbf{w}^{(k)}, \mathbf{u}^{(k)}) \right|$$

$$= \frac{1}{2} \left| \int_\Omega \mathbf{u}^{(k)} \cdot ((\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)})) + (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}))^T) \cdot \mathbf{u}^{(k)} \, d\Omega \right|$$

$$\leqslant C \left\| (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)})) + (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}))^T \right\|_0 \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^2$$

$$\leqslant \frac{v}{4} \left\| (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)})) + (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}))^T \right\|_0^2 + C_v \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^4$$

so that, using (5.6), we have that

$$\frac{v}{4} \left\| (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)})) + (\operatorname{grad}(\mathbf{u}^{(k)} - \mathbf{w}^{(k)}))^T \right\|_0^2 \leqslant C_v \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^4 \,.$$

Then, by (5.5) and the triangle inequality, we have that

$$\left\| (\operatorname{grad} \mathbf{u}^{(k)}) + (\operatorname{grad} \mathbf{u}^{(k)})^T \right\|_0$$
$$\leqslant C \left(\|\mathbf{f}\|_0 + \|\mathbf{b}\|_{1/2, \Gamma} + \|\mathbf{g}^{(k)}\|_{1, \Gamma_c} + \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^2 \right) \,.$$

Then,

$$\left\| (\operatorname{grad} \mathbf{u}^{(k)}) + (\operatorname{grad} \mathbf{u}^{(k)})^T \right\|_0 + \|\mathbf{u}^{(k)}\|_{0, \Gamma}$$
$$\leqslant \left\| (\operatorname{grad} \mathbf{u}^{(k)}) + (\operatorname{grad} \mathbf{u}^{(k)})^T \right\|_0 + \|\mathbf{b}\|_{0, \Gamma} + \|\mathbf{g}^{(k)}\|_{0, \Gamma_c}$$
$$\leqslant C \left(\|\mathbf{f}\|_0 + \|\mathbf{b}\|_{1/2, \Gamma} + \|\mathbf{g}^{(k)}\|_{1, \Gamma_c} + \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^2 \right) \,. \quad (5.7)$$

Since the left hand side of (5.7) is a norm on $\mathbf{H}^1(\Omega)$, we have that

$$\|\mathbf{u}^{(k)}\|_1 \leqslant C \left(\|\mathbf{f}\|_0 + \|\mathbf{b}\|_{1/2, \Gamma} + \|\mathbf{g}^{(k)}\|_{1, \Gamma_c} + \|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}^2 \right) ,$$

and, since $\|\mathbf{u}^{(k)}\|_{\mathbf{L}^4(\Omega)}$ and $\|\mathbf{g}^{(k)}\|_{1, \Gamma_c}$ are uniformly bounded, we concluded that $\|\mathbf{u}^{(k)}\|_1$ is uniformly bounded as well. From this point, the proof of the existence of optimal solutions can proceed as in the proof of Theorem 2.1.

Remark : The reason for the use of the $\mathbf{L}^4(\Omega)$-norm of $\mathbf{u}$ in the functional (1.1), or, equivalently, in (5.1), is now evident. If we had instead used the

less cumbersome $L^2(\Omega)$-norm, we would not have been able to find a uniform bound for the $H^1(\Omega)$-norm of the elements of the minimizing sequence $u^{(k)}$.

REFERENCES

[1] F. ABERGEL and R. TEMAM, *On some control problems in fluid mechanics.* Theoret. Comput. Fluid Dynamics. To appear.

[2] R. ADAMS, *Sobolev Spaces.* Academic, New York, 1975.

[3] I. BABUŠKA, *The finite element method with Lagrange multipliers.* Numer. Math. 16 179-192, 1973.

[4] I. BABUŠKA and A. AZIZ, *Survey lectures on the mathematical foundations of the finite element method.* The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (Ed. by A. Aziz), Academic, New York, 3-359, 1973.

[5] F. BREZZI, *On the existence, uniqueness, and approximation of saddle-point problems arising form Lagrange multipliers.* RAIRO Model. Math. Anal. Numér. 8-32, 129-151, 1974.

[6] F. BREZZI, *A survey of mixed finite element methods.* Finite Elements, Theory and Application (Ed. by D. Dwoyer, M. Hussaini and R. Voigt), Springer, New York, 34-49, 1988.

[7] F. BREZZI, J. RAPPAZ and P.-A. RAVIART, *Finite-dimensional approximation of nonlinear problems. Part I : branches of nonsingular solutions.* Numer. Math. 36 1-25, 1980.

[8] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes.* Rend. Sem. Mat. Univ. Padova 31 308-340, 1961.

[9] P. CIARLET, *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[10] M. CROUZEIX, *Approximation des problèmes faiblement non linéaires.* To appear.

[11] V. GIRAULT and P.-A. RAVIART, *Finite Element Methods for Navier-Stokes Equations.* Springer, Berlin, 1986.

[12] M. GUNZBURGER, *Finite Element Methods for Incompressible Viscous Flows : A Guide to Theory, Practice and Algorithms.* Academic, Boston, 1989.

[13] M. GUNZBURGER and L. HOU, *Treating inhomogeneous essential boundary conditions in finite element methods.* SIAM J. Num. Anal., To appear.

[14] M. GUNZBURGER, L. HOU and T. SVOBODNY, *Boundary velocity control of incompressible flow with an application to viscous drag reduction.* SIAM J. Opt. Contr., To appear.

[15] M. GUNZBURGER, L. HOU and T. SVOBODNY, *Analysis and finite element approximation of optimal control problems for the stationary Navier-Stokes equations with distributed and Neumann controls.* Math. Comp. 57, 123-151, 1991.

[16] L. HOU, *Analysis and finite element approximation of some optimal control problems associated with the Navier-Stokes equations.* Ph. D. Thesis, Carnegie Mellon University, Pittsburgh, 1989.

[17] J. LIONS, *Some Aspects of the Optimal Control of Distributed Parameter Systems.* SIAM, Philadelphia, 1972.

[18] M. SCHECTER, *Principles of Functional Analysis.* Academic, New York, 1971.

[19] J. SERRIN, *Mathematical principles of classical fluid mechanics.* Handbüch der Physik VIII/1 (ed. by S. Flügge and C. Truesdell) Springer, Berlin, 125-263, 1959.

[20] R. TEMAM, *Navier-Stokes Equations.* North-Holland, Amsterdam, 1979.

[21] R. TEMAM, *Navier-Stokes Equations and Nonlinear Functional Analysis.* SIAM, Philadelphia, 1983.

[22] R. VERFÜRTH, *Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition.* Numer. Math. 50 697-621, 1987.