Logic I (24.241)

Josh Hunt

Fall 2022

Massachusetts Institute of Technology

Course components

- Textbook

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- Textbook
- Lecture videos on D2L

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- Textbook
- Lecture videos on D2L
- Synchronous Zoom sessions

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  - WF 12:00-12:50

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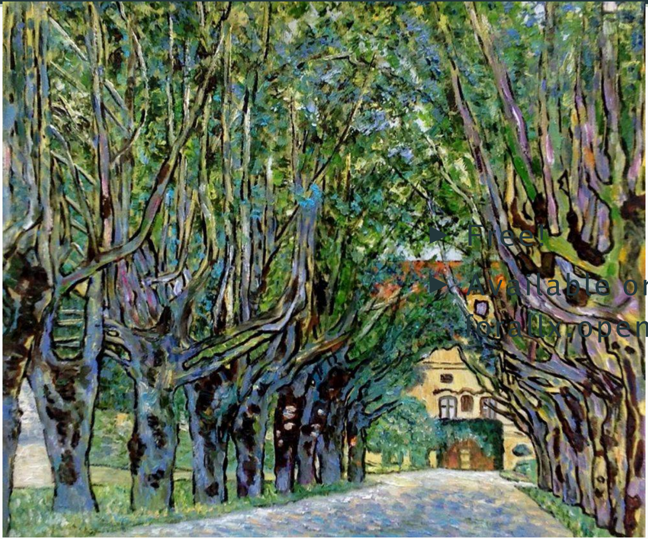
- PASS Sessions

Textbook



- Free!
- Available on D2L, and at forallx.openlogicproject.org

Problem sets

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- Should be done by Friday, will take till Monday midnight.

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- Don’t give away solutions.

Quizzes

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A typical week

Monday & Tuesday Watch videos, do reading reading.

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Friday Tutorials in the morning, lecture at noon. Finish problem set.

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Friday Tutorials in the morning, lecture at noon. Finish problem set.

Saturday–Monday Complete the quiz and timed problem.

Grading philosophy

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- Your grade depends on **how many** learning goals you complete.
- Ideally, you complete each goal during the week we cover it.
- But you get two chances to **re-do assessments**.

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- Scoring 95% or more is "complete+" (A-level, excellent performance).
- Timed problems are just complete/not complete: to meet the bar for a B, you have to submit a correct solution.

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- For C, complete at least 8/12; and for a D, 6/12 of each assessment.
- For $+$ and $-$ grade criteria, see the outline.

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- Unlimited attempts on problem set questions.
- Three attempts on quizzes, no time limit.
- Only timed problems have time limits.
- Re-do timed problems or buy more attempts at quizzes using a token.

Tokens

Six tokens to spend on:

- One more shot at a timed problem.

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- Three more attempts on a quiz.
- Up to two do-overs per assessment.

House rules

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- Don't cheat on quizzes and Timed Problems.

Read the course outline!

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I. What is logic?

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a. Arguments and validity

An easy puzzle

Where does Sanjeev live?

Sanjeev lives in Calgary or in Edmonton.
Sanjeev doesn't live in Edmonton.

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A: Obviously, in Calgary.

Arguments and sentences

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- "Therefore" ($\therefore$) indicates that the last sentence (supposedly) **follows from** the first two.
- The last sentence is called the **conclusion**.
- The others are called the **premises**.

Valid and invalid arguments

Argument 2

Mandy enjoys skiing or hiking (or both).
Mandy doesn't enjoy hiking.
$\therefore$ Mandy enjoys skiing.

Argument 3

Mandy enjoys skiing or hiking (or both).
Mandy enjoys skiing.
$\therefore$ Mandy doesn't enjoy hiking.

What's the difference?

(Deductive) Validity

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Definition

A **case** is some hypothetical scenario that makes each sentence in an argument either true or false.

Argument 2 is valid

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Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
∴ Mandy enjoys skiing.

Argument 2 is **valid**: whenever the premises are true, the conclusion is also true.

Argument 3 is not valid

Argument 3

Mandy enjoys skiing or hiking.
Mandy enjoys skiing.
$\therefore$ Mandy doesn't enjoy hiking.

Argument 3 is **invalid**: there is a possible case where the premises are true and the conclusion isn't (Mandy enjoys both skiing and hiking).

A harder puzzle

Where does Sarah live?

Sarah lives in Calgary or Edmonton.
Amir lives in Calgary unless he enjoys hiking.
If Amir lives in Calgary, Sarah doesn't.
Neither Sarah nor Amir enjoy hiking.

I. What is logic?

b. Cases and determining validity

Validity

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- E.g., imagine you have a friend, her name is Mandy, she loves hiking but hates skiing.
- That's a case where "Mandy enjoys hiking or skiing" is true.
- Some cases can be imagined even though they never happen IRL, e.g, "It is raining and the skies are clear."
- Some things you can't imagine, e.g., "There is a blizzard but there is no wind."

Determining validity

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Deductively Valid?

Some rodents have bushy tails.
All squirrels are rodents.
$\therefore$ Some squirrels have bushy tails.

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- Imagine squirrels evolving so that they no longer have bushy tails. Then conclusion is false.

Deductively Valid?

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- Imagine squirrels evolving so that they no longer have bushy tails. Then conclusion is false.
- But premises still true:

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- Imagine squirrels evolving so that they no longer have bushy tails. Then conclusion is false.
- But premises still true:
  - Imagine chinchillas still have bushy tails.

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- Imagine squirrels evolving so that they no longer have bushy tails. Then conclusion is false.
- But premises still true:
  - Imagine chinchillas still have bushy tails.
  - Imagine also that squirrels have not evolved too much—they are still rodents.

Valid?

All rodents have bushy tails.
All squirrels are rodents.
∴ All squirrels have bushy tails.

Valid?

All rodents have bushy tails.
All squirrels are rodents.
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- If it were invalid, you'd have a case that makes the conclusion false: some squirrels without bushy tails.

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- If it were invalid, you'd have a case that makes the conclusion false: some squirrels without bushy tails.
- They would have to be rodents still (otherwise premise 2 false).
- And that would require that they have bushy tails (otherwise premise 1 false).

I. What is logic?

c. Other logical notions

Logical Consistency

Definition

Sentences are (logically) **consistent** if there is a case where they are all true.

- also called ‘jointly possible’ or ‘satisfiable’

Definition

Sentences are (logically) **inconsistent** if there is no case where they are all true.

- also called ‘jointly impossible’ or ‘unsatisfiable’

Consistent?

Some carnivores have bushy tails.
All carnivores are mammals.
No mammals have bushy tails.

Consistent?

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All carnivores are mammals.
No mammals have bushy tails.

- No case makes them all true at the same time, so **inconsistent**.

Valid?

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All carnivores are mammals.
No mammals have bushy tails.
$\therefore$ All birds are carnivores.

Valid?

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No mammals have bushy tails.
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- The premises cannot all be true in the same case, so jointly impossible.
- So: no case makes all the premises true.
- So also: no case makes the premises true and the conclusion false.
- **Arguments with inconsistent premises are automatically valid**, regardless of what the conclusion is.

Tautology (logically tautology)

Definition

A sentence is a **tautology** if there is no case where it is false.
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- The number 5 is prime.
- Physical objects are extended.

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Logical equivalence

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Two sentences are (logically) **equivalent** if there is no case where one is true and the other is false.

- What can you say about an argument where one of the premises is equivalent to the conclusion? Is it automatically valid?
- Can you have two equivalent sentences that are inconsistent?

I. What is logic?

d. Symbolization and SL

Validity in virtue of form

Argument 1

Sanjeev lives in Calgary or Edmonton.
Sanjeev doesn't live in Edmonton.
∴ Sanjeev lives in Calgary.

Argument 2

Mandy enjoys skiing or hiking.
Mandy doesn't enjoy hiking.
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Form of arguments 1 & 2

X or Y.
Not Y.
$\therefore X$.

Some valid argument forms

Disjunctive syllogism

X or Y.
Not Y.
$\therefore$ X.

Modus ponens

If X then Y.
X.
$\therefore$ Y.

Hypothetical syllogism

If X then Y.
If Y then Z.
$\therefore$ If X then Z.

Symbolizing arguments

Symbolization key

S: Mandy enjoys skiing
H: Mandy enjoys hiking

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.	$(S \vee H)$
Not: Mandy enjoy hiking.	$\sim H$
$\therefore$ Mandy enjoys skiing.	$\therefore S$

The language of SL

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  & both ... and ...

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$\vee$	either ... or ...
&	both ... and ...
$\supset$	if ... then ...
$\sim$	not ...

This can get complicated, e.g.:
"Mandy enjoys skiing or hiking, and if she lives in Edmonton, she doesn't enjoy both."

$$((S \vee H) \,\&\, (E \supset \sim(S \,\&\, H)))$$

I. What is logic?

e. What are we going to learn, and why?

What is logic?

- **Logic is the science of what follows from what.**

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- Logic investigates what makes the first argument **valid** and the second **invalid**.

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- (Meta-logical properties of logical systems)

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- Logic applies to semantics of natural language (philosophy of language, linguistics).

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  - Formal proofs make mechanical **proof checking** and **proof search** possible.

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  - Theoretical computer science (theory of computational complexity, semantics of programming languages)

II. Symbolization in SL

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a. Symbolization keys and paraphrase

Symbolizing arguments

Argument 2

Mandy enjoys skiing or Mandy enjoys hiking.

Not: Mandy enjoy hiking.

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Symbolization of argument 2 in SL

$(S \vee H)$

$\sim H$

$\therefore S$

Symbolization keys

Definition

A symbolization key is a list that pairs **sentence letters** with the basic English sentences they represent.

For instance:

Symbolization key

S: Mandy enjoys skiing
H: Mandy enjoys hiking

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  For instance:

  $$A\text{: Mandy enjoys skiing or hiking}$$

  is bad.

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- Pronouns stand in for singular terms (e.g., names): replace pronouns by those.
- “or”, “both . . . and”, “neither . . . nor” can connect sentences but also noun phrases and verb phrases: paraphrase those so they connect sentences.

Pronouns

Example

If Mandy enjoys hiking, **she** also enjoys skiing.

Replace “she” by “Mandy”:
If [Mandy enjoys hiking] then [Mandy enjoys skiing].

Coordination of noun phrases

Example

Mandy and Sanjeev enjoy hiking.

Both [Mandy enjoys hiking] and [Sanjeev enjoys hiking].

Example

Sanjeev lives in **Edmonton or Calgary**.

Either [Sanjeev lives in Edmonton] or [Sanjeev lives in Calgary].

Exercise caution!

Good

Mandy and Sanjeev ate pizza.

Both [Mandy ate pizza] and [Sanjeev ate pizza].

Bad

Mandy and Sanjeev ate the whole pizza.

Both [Mandy ate the whole pizza] and [Sanjeev ate the whole pizza].

Coordination of verb phrases

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Mandy enjoys **skiing or hiking**.

Either [Mandy enjoys skiing] or [Mandy enjoys hiking].

Example

If Sanjeev enjoys **skiing and hiking**, he lives in Calgary.

If [Sanjeev enjoys skiing] and [Sanjeev enjoys hiking], then [Sanjeev lives in Calgary].

II. Symbolization in SL

b. Basic symbolization

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Even though Mandy lives in Edmonton**,** she enjoys hiking.
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Ignore the suggestion that “either . . . or . . . ” is exclusive. We’ll always treat it as inclusive unless explicitly stated.

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- **Paraphrase** using "**if** A **then** B" any sentence of the form:
  - "if A, B"
  - "B if A" (note order is reversed!)
  - "B provided A"
- **Symbolize** "if A then B" as $(A \supset B)$.

Example

- Mandy enjoys hiking **if** Sanjeev lives in Calgary.
- If [**Sanjeev lives in Calgary**] then [**Mandy enjoys hiking**].
- **If** C **then** H.

Conditional

- **Paraphrase** using "**if** A **then** B" any sentence of the form:
  - "if A, B"
  - "B if A" (note order is reversed!)
  - "B provided A"
- **Symbolize** "if A then B" as $(A \supset B)$.

Example

- Mandy enjoys hiking **if** Sanjeev lives in Calgary.
- If [**Sanjeev lives in Calgary**] then [**Mandy enjoys hiking**].
- **If** C **then** H.
- $(C \supset H)$

The parts of a conditional

- $(A \supset B)$ symbolizes:
  - “if A, B”
  - “B if A” (note order is reversed!)
  - “B provided A”
- A is the **antecedent**: it symbolizes the condition that has to be met for the “then” part to apply.
- B is the **consequent**: it symbolizes what must be true if the antecedent condition is true.

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

If [Sanjeev lives in Calgary **or** Edmonton] then [Mandy **doesn't** enjoy hiking].

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

If [Sanjeev lives in Calgary **or** Edmonton] then [Mandy **doesn't** enjoy hiking].

If [either [**Sanjeev lives in Calgary**] or [**Sanjeev lives in Edmonton**]] then [it is not the case that [**Mandy enjoys hiking**]].

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

If [Sanjeev lives in Calgary **or** Edmonton] then [Mandy **doesn't** enjoy hiking].

If [either [**Sanjeev lives in Calgary**] or [**Sanjeev lives in Edmonton**]] then [it is not the case that [**Mandy enjoys hiking**]].

If [either C or E] then [it is not the case that H].

Mix & match

Example

Mandy doesn't enjoy hiking, **provided** Sanjeev lives in Calgary or Edmonton.

If [Sanjeev lives in Calgary **or** Edmonton] then [Mandy **doesn't** enjoy hiking].

If [either [**Sanjeev lives in Calgary**] or [**Sanjeev lives in Edmonton**]] then [it is not the case that [**Mandy enjoys hiking**]].

If [either C or E] then [it is not the case that H].

$((C \vee E) \supset {\sim}H)$

II. Symbolization in SL

c. Conditionals

A logic puzzle

- Every card has a letter on one side and a number on the other side.
- You're a card inspector tasked with making sure that cards satisfy this quality standard:

  If a card has an even number on one side, then it has a vowel on the other.

A logic puzzle

Which card(s) do you have to turn over to make sure that:

> *If a card has an even number on one side, then it has a vowel on the other.*

E	K	3	4
(1)	(2)	(3)	(4)

Another logic puzzle

- At an all–ages event where everyone has a drink
- You know how old some of the people are, and you can tell what some of them are drinking
- You're tasked with making sure that the following rule is followed:

  If a person is drinking alcohol, then they are at least 18 years old.

Another logic puzzle

Which of these people do you have to check (age or drink) to ensure that:

> *If a person is drinking alcohol, then they must be at least 18 years old.*

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Another logic puzzle

Which of these people do you have to check (age or drink) to ensure that:

> *If a person is drinking alcohol, then they must be at least 18 years old.*

22 years	16 years	drinks pop	drinks beer
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Which card(s) do you have to turn over to make sure that:

> *If a card has an even number on one side, then it has a vowel on the other.*

E	K	3	4
(1)	(2)	(3)	(4)

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If *A*, then *B*" can only be **false** if:

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If A, then B" can only be **false** if:
  - A is **true**: we check age if X is drinking beer (A true), not if drinking pop; **and**

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If *A*, then *B*" can only be **false** if:
  - *A* is **true**: we check age if X is drinking beer (*A* true), not if drinking pop; **and**
  - *B* is **false**: we check drink if X underage (*B* false), not if over 18 (*B* true)

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If A, then B" can only be **false** if:
  - A is **true**: we check age if X is drinking beer (A true), not if drinking pop; **and**
  - B is **false**: we check drink if X underage (B false), not if over 18 (B true)
- "If A, then B" is true if:

Truth conditions of conditionals

$$\text{If } \underbrace{\text{X is drinking alcohol}}_{A}\text{, then } \underbrace{\text{X is over 18}}_{B}$$

- "If A, then B" can only be **false** if:
  - A is **true**: we check age if X is drinking beer (A true), not if drinking pop; **and**
  - B is **false**: we check drink if X underage (B false), not if over 18 (B true)
- "If A, then B" is true if:
  - A is **false** (we don't check people drinking pop); **or**

Truth conditions of conditionals

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- "If A, then B" can only be **false** if:
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- “If A, then B” is true if:
  - A is **false** (we don’t check people drinking pop); **or**
  - B is **true** (we don’t card if X is over 18);
  - (or both)

II. Symbolization in SL

d. “Only if” and “unless”

‘If’ and ‘only if’

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \supset B$$

‘If’ and ‘only if’

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \supset B$$

- False if Sue is underage, but drinks beer.

‘If’ and ‘only if’

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \supset B$$

- False if Sue is underage, but drinks beer.
- Sue drinks beer (A) if she is over 18 (B).

$$B \supset A$$

‘If’ and ‘only if’

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \supset B$$

- False if Sue is underage, but drinks beer.
- Sue drinks beer (A) if she is over 18 (B).

$$B \supset A$$

- False if she’s 25 but drinks pop.

‘If’ and ‘only if’

- Sue drinks beer (A) **only if** she is over 18 (B)

$$A \supset B$$

- False if Sue is underage, but drinks beer.
- Sue drinks beer (A) if she is over 18 (B).

$$B \supset A$$

- False if she’s 25 but drinks pop.
- Not false if she’s 16 and drinking beer.

Conditional

- **Paraphrase** "A only if B" as "**if A then B**".

Conditional

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- **Symbolize** "A only if B" as $(A \supset B)$.

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  - "A only if B" is $(A \supset B)$

Conditional

- **Paraphrase** "A only if B" as "**if A then B**".
- **Symbolize** "A only if B" as $(A \supset B)$.
- Note:
  - "A if B" is $(B \supset A)$
  - "A only if B" is $(A \supset B)$
- **Symbolize** "A if and only if B" as $(A \equiv B)$.

Unless

Which of these people do you have to check (age or drink) to ensure that:

People are drinking pop unless they are over 18.

22 years	16 years	drinks pop	drinks beer
(1)	(2)	(3)	(4)

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

- "*A* unless *B*" can only be **false** if:

Unless

$$\underbrace{\text{X is drinking pop}}_{A}\text{, unless }\underbrace{\text{X is over 18}}_{B}$$

- "A unless B" can only be **false** if:
  - A is **false**
    (we check age if person is drinking beer), **and**

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

- "A unless B" can only be **false** if:
  - A is **false**
    (we check age if person is drinking beer), **and**
  - B is **false**
    (we check drink if person not at least 18)

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

- "*A* unless *B*" can only be **false** if:
  - *A* is **false**
    (we check age if person is drinking beer), **and**
  - *B* is **false**
    (we check drink if person not at least 18)
- "*A* unless *B*" is true (test OK) if *A* or *B* or both are true.

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Unless

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- "*A* unless *B*" can be paraphrased and symbolized by:
  - "*A* if not *B*" ($\sim B \supset A$)

Unless

$$\underbrace{\text{X is drinking pop}}_{A}, \text{ unless } \underbrace{\text{X is over 18}}_{B}$$

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  - A is **false**
    (we check age if person is drinking beer), **and**
  - B is **false**
    (we check drink if person not at least 18)
- "A unless B" is true (test OK) if A or B or both are true.
- "A unless B" can be paraphrased and symbolized by:
  - "A if not B" ($\sim B \supset A$)
  - "either A or B" ($A \vee B$)

Unless

Treat "**unless**" the same way you would treat "or"

Example

Mandy enjoys hiking unless Sanjeev lives in Calgary.

$(H \vee C)$

II. Symbolization in SL

e. More connectives

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

Both [if S then H] and [if H then S].

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

Both [if S then H] and [if H then S].

$((S \supset H) \& (H \supset S))$

If and only if

Example

Mandy enjoys hiking if and only if she enjoys skiing.

Both [if S then H] and [if H then S].

$((S \supset H) \& (H \supset S))$

$(H \equiv S)$

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

> [either H or S]
> [it is not the case that [both H and S]].

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

[either H or S]
[it is not the case that [both H and S]].

Exclusive or

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

Exclusive or

Paraphrase sentences containing "either A or B but not both" using "**both [either A or B] and**
[it is not the case that [both A and B]]"

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

Exclusive or

Paraphrase sentences containing “either A or B but not both” using
“**both [either A or B] and**
[it is not the case that [both A and B]]”

Example

Mandy enjoys hiking or skiing but not both.

Both [either H or S] and
[it is not the case that [both H and S]].

$((H \vee S) \,\&\, {\sim}(H \,\&\, S))$

Neither ... nor ...

Example

Mandy enjoys neither hiking nor skiing.

Neither . . . nor . . .

Paraphrase sentences containing “neither A nor B” using
“**both [it is not the case that A] and**
[it is not the case that B]”

Example

Mandy enjoys neither hiking nor skiing.

Both [it is not the case that H] and
[it is not the case that S].

Neither … nor …

Paraphrase sentences containing "neither A nor B" using
"**both [it is not the case that A] and**
[it is not the case that B]"

Example

Mandy enjoys neither hiking nor skiing.

Both [it is not the case that H] and
[it is not the case that S].

$(\sim H \mathbin{\&} \sim S)$

Mix & match

Example

Sarah lives in Calgary or Edmonton.

Amir lives in Calgary unless he enjoys hiking.

If Amir lives in Calgary, Sarah doesn't.

Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Edmonton.

Mix & match

Example

Sarah lives in Calgary or Edmonton.

Either [Sarah lives in Calgary] or [Sarah lives in Edmonton].

Amir lives in Calgary unless he enjoys hiking.

If Amir lives in Calgary, Sarah doesn't.

Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Edmonton.

Mix & match

Example

Sarah lives in Calgary or Edmonton.

Either [Sarah lives in Calgary] or [Sarah lives in Edmonton].

Amir lives in Calgary unless he enjoys hiking.

Either [Amir lives in Calgary] or [Amir enjoys hiking].

If Amir lives in Calgary, Sarah doesn't.

Neither Sarah nor Amir enjoy hiking.

∴ Sarah lives in Edmonton.

Mix & match

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Sarah lives in Calgary or Edmonton.

Either [Sarah lives in Calgary] or [Sarah lives in Edmonton].

Amir lives in Calgary unless he enjoys hiking.

Either [Amir lives in Calgary] or [Amir enjoys hiking].

If Amir lives in Calgary, Sarah doesn't.

If [Amir lives in Calgary] then [it is not the case that [Sarah lives in Calgary]].

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Either [Amir lives in Calgary] or [Amir enjoys hiking].

If Amir lives in Calgary, Sarah doesn't.

If [Amir lives in Calgary] then [it is not the case that [Sarah lives in Calgary]].

Neither Sarah nor Amir enjoy hiking.

Both [it is not the case that [Sarah enjoys hiking]] and [it is not the case that [Amir enjoys hiking]].

$\therefore$ Sarah lives in Edmonton.

Mix & match

Example

Sarah lives in Calgary or Edmonton.

$(C \vee E)$

Amir lives in Calgary unless he enjoys hiking.

$(A \vee M)$

If Amir lives in Calgary, Sarah doesn't.

$(A \supset {\sim} C)$

Neither Sarah nor Amir enjoy hiking.

$({\sim} S \,\&\, {\sim} M)$

∴ Sarah lives in Edmonton.

$\therefore\ E$.

II. Symbolization in SL

f. Ambiguity

Types of ambiguity

- Lexical ambiguity: one word—many meanings
  e.g., “bank”, “crane”
- Syntactic ambiguity: one sentence—many readings
  e.g.,
  - “Flying planes can be dangerous” (Chomsky)
  - “One morning I shot an elephant in my pajamas.
    How he got in my pajamas, I don’t know.” (Groucho Marx)

The man who was hanged by a comma

- Sir Roger Casement (1864–1916)
- British consul to Congo and Peru
- Tried to recruit Irish revolutionaries in Germany during WWI
- Tried for treason

Treason Act of 1351

ITEM, Whereas divers Opinions have been before this Time in what Case Treason shall be said, and in what not; the King, at the Request of the Lords and of the Commons, hath made a Declaration in the Manner as hereafter followeth, that is to say; When a Man doth compass or imagine the Death of our Lord the King, or of our Lady his Queen or of their eldest Son and Heir; or if a Man do violate the King's Companion, or the King's eldest Daughter unmarried, or the Wife of the King's eldest Son and Heir; or **if a Man** do levy War against our Lord the King in his Realm, or **be adherent to the King's Enemies in his Realm, giving to them Aid and Comfort in the Realm or elsewhere**, and thereof be probably attainted of open Deed by the People of their Condition: . . . And it is to be understood, that in the Cases above rehearsed, that ought to be judged Treason which extends to our Lord the King, and his Royal Majesty: . . .

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R v. Casement in QL

- Symbolization key:

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  A: Casement was adherent to the King's enemies in the realm.

R v. Casement in QL

- Symbolization key:

  A: Casement was adherent to the King's enemies in the realm.

  G: Casement gave aid and comfort to the King's enemies in the realm.

R v. Casement in QL

- Symbolization key:

  A: Casement was adherent to the King's enemies in the realm.

  G: Casement gave aid and comfort to the King's enemies in the realm.

  B: Casement was adherent to the King's enemies abroad.

R v. Casement in QL

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R v. Casement in QL

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  H: Casement gave aid and comfort to the King's enemies abroad.
- Not treason:

R v. Casement in QL

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$$A \vee (G \vee H)$$

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  H: Casement gave aid and comfort to the King's enemies abroad.

- Not treason:

  $A \vee (G \vee H)$

- Treason:

  $(A \vee B) \vee (G \vee H)$

Ambiguity of $\&$ and $\vee$

- English sentences don't have parentheses.

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- In SL, symbolizations are unambiguous:

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  $(B \,\&\, (C \vee D))$

III. SL and truth tables

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a. Characteristic truth tables

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$(H \& S)$ is true if and only if H is true and S is also true.

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- Recall that a **case** makes basic sentences **true** or **false** (and never both).
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

When is $(H \& S)$ true?

$(H \& S)$ is true if and only if H is true and S is also true.
Suppose a case makes H true and S false.

Sentence letters and connectives

- Symbolization involves **sentence letters** like H and **connectives** ($\sim$, $\vee$, $\&$, $\supset$, $\equiv$)
- Recall that a **case** makes basic sentences **true** or **false** (and never both).
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

When is $(H \& S)$ true?

$(H \& S)$ is true if and only if H is true and S is also true.
Suppose a case makes H true and S false.
In that case, $(H \& S)$ would be ???.

Sentence letters and connectives

- Symbolization involves **sentence letters** like H and **connectives** ($\sim$, $\vee$, &, $\supset$, $\equiv$)
- Recall that a **case** makes basic sentences **true** or **false** (and never both).
- So if we can determine **truth conditions** for sentences involving connectives, we can assign **true** and **false** also to results of symbolization.

When is $(H \,\&\, S)$ true?

$(H \,\&\, S)$ is true if and only if H is true and S is also true.
Suppose a case makes H true and S false.
In that case, $(H \,\&\, S)$ would be **false**.

Negation $\sim$

Definition

$\sim\mathscr{A}$ is true iff $\mathscr{A}$ is false.

Characteristic truth table:

$\mathscr{A}$	$\sim\mathscr{A}$
T	F
F	T

Conjunction &

Definition

$(\mathscr{A} \& \mathscr{B})$ is true iff $\mathscr{A}$ is true and $\mathscr{B}$ is true, and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \& \mathscr{B})$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction $\vee$

Definition

$(\mathscr{A} \vee \mathscr{B})$ is true iff $\mathscr{A}$ is true or $\mathscr{B}$ is true (or both), and false otherwise.

Characteristic truth table:

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \vee \mathscr{B})$
T	T	T
T	F	T
F	T	T
F	F	F

A logic puzzle

Which card(s) do you have to turn over to make sure that:

> *If a card has an even number on one side, then it has a vowel on the other.*

E	K	3	4
(1)	(2)	(3)	(4)

The material conditional $\supset$

Definition

$(\mathscr{A} \supset \mathscr{B})$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true (or both), and false otherwise.

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \supset \mathscr{B})$
T	T	T
T	F	F
F	T	T
F	F	T

The material biconditional $\equiv$

Definition

$(\mathscr{A} \equiv \mathscr{B})$ is true iff $\mathscr{A}$ and $\mathscr{B}$ have the same truth value, and false otherwise.

$\mathscr{A}$	$\mathscr{B}$	$(\mathscr{A} \equiv \mathscr{B})$
T	T	T
T	F	F
F	T	F
F	F	T

III. SL and truth tables

b. Sentences of SL

Sentences of SL

Definition

1. Every sentence letter is a sentence.
2. If $\mathscr{A}$ is a sentence, then $\sim\mathscr{A}$ is a sentence.
3. If $\mathscr{A}$ and $\mathscr{B}$ are sentences, then
   - $(\mathscr{A} \,\&\, \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \vee \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \supset \mathscr{B})$ is a sentence.
   - $(\mathscr{A} \equiv \mathscr{B})$ is a sentence.
4. Nothing else is a sentence.

The indicated connective is called the **main connective**.

Construction of sentences

- H is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \ \& \ S)$ is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \,\&\, S)$ is a sentence.
- $\sim(H \,\&\, S)$ is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \mathbin{\&} S)$ is a sentence.
- $\sim(H \mathbin{\&} S)$ is a sentence.
- $((H \vee S) \mathbin{\&} \sim(H \mathbin{\&} S))$ is a sentence.

Construction of sentences

- H is a sentence.
- S is a sentence.
- $(H \vee S)$ is a sentence.
- $(H \mathbin{\&} S)$ is a sentence.
- $\sim(H \mathbin{\&} S)$ is a sentence.
- $((H \vee S) \mathbin{\&} \sim(H \mathbin{\&} S))$ is a sentence.

(Main connective is **highlighted**.)

Examples of non-sentences

- *HikesMandy*

  single sentence letters
- $(H \sim S)$

  $\sim$ can't go between sentences
- $(H \,\&\, S \,\&\, C)$

  $\&$ combines only two sentences
- $(\sim H)$

  no parentheses around $\sim H$
- $(H \supset (S \,\&\, C)$

  missing closing parenthesis
- $H \vee S$

  missing parentheses
- $[H \supset (S \,\&\, C)]$

  only one kind of parentheses

III. SL and truth tables

c. Valuations

Valuations

Definition

A **valuation** is an assignment of **T** or **F** to each sentence letter in a sentence or sentences.

Definition

The **truth value of a sentence** $\mathscr{S}$ on a valuation is:

1. if $\mathscr{S}$ is a sentence letter: the truth value assigned to it
2. if $\mathscr{S}$ is $\sim\mathscr{A}$: opposite of the truth value of $\mathscr{A}$
3. if $\mathscr{S}$ is $(\mathscr{A} * \mathscr{B})$: result of characteristic truth table of $*$ for truth values of $\mathscr{A}$ and $\mathscr{B}$.

Computing truth values

Valuation: H is **T**, S is **F**.

On this valuation:

- H is **T**.
- S is **F**.
- $(H \vee S)$ is **T** (because **T** $\vee$ **F** gives **T**).
- $(H \,\&\, S)$ is **F** (because **T** & **F** gives **F**).
- $\sim(H \,\&\, S)$ is **T** (because $\sim$**F** is **T**).
- $((H \vee S) \,\&\, \sim(H \,\&\, S))$ is **T** (because **T** & **T** gives **T**).

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**								

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	**T**		**F**			**T**		**F**

- Copy truth values under sentence letters.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	**T**		**F**			**T**		**F**
			$\uparrow$						

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.

Computing truth values

H	S	$(($	H	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**		**T**	**T**	**F**			**T**		**F**
				$\uparrow$						

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	T	T	F			**T**		**F**
								$\uparrow$	

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	T	T	F			**T**	**F**	**F**
								$\uparrow$	

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	**T**	**T**	**F**			**T**	**F**	**F**
						$\uparrow$			

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.
- Use computed values for larger parts.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	T	T	F		**T**	T	**F**	F
						$\uparrow$			

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.
- Use computed values for larger parts.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	**T**	**T**	**F**		**T**	**T**	**F**	**F**
					$\uparrow$				

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.
- Use computed values for larger parts.

Computing truth values

H	S	$((H$	$\vee$	$S)$	$\&$	$\sim$	$(H$	$\&$	$S))$
T	**F**	**T**	**T**	**F**	**T**	**T**	**T**	**F**	**F**
					$\uparrow$				

- Copy truth values under sentence letters.
- Compute values of parts that combine sentence letters.
- Use computed values for larger parts.
- Done when you have the value under the main connective.

III. SL and truth tables

d. Validity and truth tables

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.
- A **case** must make every **basic sentence** true or false (and not both).

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.
- A **case** must make every **basic sentence** true or false (and not both).
- In SL, **valuations** make every **sentence letter** true or false (and not both).

Validity

- In English: an argument is **valid** if there is no **case** where all premises are true and conclusion is false.
- A **case** must make every **basic sentence** true or false (and not both).
- In SL, **valuations** make every **sentence letter** true or false (and not both).
- Also: every valuation makes every **sentence** true or false (and not both), and we can compute the truth value.

Validity in SL

Definition

An argument is **valid in SL** if there is **no** valuation in which all premises are **T** and the conclusion is **F**.

An argument is **invalid in SL** if there is **at least one** valuation in which all premises are **T** and the conclusion is **F**.

Disjunctive syllogism

$$\begin{array}{rl} & H \vee S \\ & \sim S \\ \therefore & H \end{array}$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$H \vee S$
$\sim S$
$\therefore H$

H	S	$(H \vee S)$	$\sim S$	H
T	**T**			
T	**F**			
F	**T**			
F	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$H \vee S$$
$$\sim S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	**T**		**T**			
T	**F**	**T**		**F**			
F	**T**	**F**		**T**			
F	**F**	**F**		**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$H \vee S$
$\sim S$
$\therefore\ H$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	**T**	**T**	**T**			
T	**F**	**T**	**T**	**F**			
F	**T**	**F**	**T**	**T**			
F	**F**	**F**	**F**	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$\begin{array}{rl} & H \vee S \\ & \sim S \\ \therefore & H \end{array}$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	T	T	T		**T**	
T	**F**	T	T	F		**F**	
F	**T**	F	T	T		**T**	
F	**F**	F	F	F		**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
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Disjunctive syllogism

$H \vee S$

$\sim S$

$\therefore\ H$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	T	T	T	**F**	**T**	
T	**F**	T	T	F	**T**	**F**	
F	**T**	F	T	T	**F**	**T**	
F	**F**	F	F	F	**T**	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$H \vee S$$
$$\sim S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	T	T	T	F	T	**T**
T	**F**	T	T	F	T	F	**T**
F	**T**	F	T	T	F	T	**F**
F	**F**	F	F	F	T	F	**F**

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$
\begin{array}{l}
\quad H \vee S \\
\quad \sim S \\
\therefore\ H
\end{array}
$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H
T	**T**	T	**T**	T	**F**	T	**T**
T	**F**	T	**T**	F	**T**	F	**T**
F	**T**	F	**T**	T	**F**	T	**F**
F	**F**	F	**F**	F	**T**	F	**F**

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$\begin{array}{r l} & H \vee S \\ & {\sim}S \\ \therefore & H \end{array}$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	$\leftarrow$
T	**F**	T	**T**	F	**T**	F	**T**	
F	**T**	F	**T**	T	**F**	T	**F**	
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$H \vee S$$
$$\sim S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	←
F	**T**	F	**T**	T	**F**	T	**F**	
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$H \vee S$

$\sim S$

$\therefore\ H$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	$\leftarrow$
F	**F**	F	**F**	F	**T**	F	**F**	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$H \vee S$$
$$\sim S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	✓
F	**F**	F	**F**	F	**T**	F	**F**	$\leftarrow$

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

Disjunctive syllogism

$$H \vee S$$
$$\sim S$$
$$\therefore\ H$$

H	S	$(H$	$\vee$	$S)$	$\sim$	S	H	
T	**T**	T	**T**	T	**F**	T	**T**	✓
T	**F**	T	**T**	F	**T**	F	**T**	✓
F	**T**	F	**T**	T	**F**	T	**F**	✓
F	**F**	F	**F**	F	**T**	F	**F**	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- All valuations check out: valid.

An invalid argument

$$H \vee S$$
$$H$$
$$\therefore\ {\sim}S$$

H	S	$(H \vee S)$	H	${\sim} S$

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$\begin{array}{rl} & H \vee S \\ & H \\ \therefore & \sim S \end{array}$$

H	S	$(H \vee S)$	H	$\sim S$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$
H
$\therefore \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S
T	**T**	**T**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**F**	**T**	**T**	**F**
F	**T**	**F**	**T**	**T**	**F**	**F**	**T**
F	**F**	**F**	**F**	**F**	**F**	**T**	**F**

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$\begin{array}{ll} & H \vee S \\ & H \\ \therefore & {\sim}S \end{array}$$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S
T	**T**	T	**T**	T	**T**	**F**	T
T	**F**	T	**T**	F	**T**	**T**	F
F	**T**	F	**T**	T	**F**	**F**	T
F	**F**	F	**F**	F	**F**	**T**	F

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$
H
$\therefore \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	$\leftarrow$
T	**F**	T	**T**	F	**T**	**T**	F	
F	**T**	F	**T**	T	**F**	**F**	T	
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$
H
$\therefore \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	
F	**T**	F	**T**	T	**F**	**F**	T	
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$
H
$\therefore \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	$\leftarrow$
F	**T**	F	**T**	T	**F**	**F**	T	
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$
H
$\therefore\ \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	←
F	**F**	F	**F**	F	**F**	**T**	F	

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$$\begin{array}{l} H \vee S \\ H \\ \therefore\ {\sim}S \end{array}$$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	✓
F	**F**	F	**F**	F	**F**	**T**	F	←

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

An invalid argument

$H \vee S$

H

$\therefore \sim S$

H	S	$(H$	$\vee$	$S)$	H	$\sim$	S	
T	**T**	T	**T**	T	**T**	**F**	T	✗
T	**F**	T	**T**	F	**T**	**T**	F	✓
F	**T**	F	**T**	T	**F**	**F**	T	✓
F	**F**	F	**F**	F	**F**	**T**	F	✓

- List all valuations for H, S.
- Compute truth values of premises, conclusion.
- Check each valuation: one premise **F**, or conclusion **T**?
- Find a valuation with all premises **T** and conclusion **F**: invalid.

III. SL and truth tables

e. Large truth tables

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations

Large truth tables

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  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations
  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.
  - For two letters A, B: B can be **T** or **F** for every possible valuation (2) of A: $2 \times 2 = 2^2 = 4$ valuations

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations
  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.
  - For two letters A, B: B can be **T** or **F** for every possible valuation (2) of A: $2 \times 2 = 2^2 = 4$ valuations
  - For three letters A, B, C: C can be **T** or **F** for every possible valuation (4) of A and B: $2 \times 4 = 2^3 = 8$ valuations

Large truth tables

- For arguments with n sentence letters, there are 2^n possible valuations
  - A single letter A can be **T** or **F**: $2^1 = 2$ valuations.
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  - Etc.

Large truth tables

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  - For three letters A, B, C: C can be **T** or **F** for every possible valuation (4) of A and B: $2 \times 4 = 2^3 = 8$ valuations
  - Etc.
- In the ith reference column, alternate **T** and **F** every 2^{n-i} lines

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	$\dots$
1				$\dots$
2				$\dots$
3				$\dots$
4				$\dots$
5				$\dots$
6				$\dots$
7				$\dots$
8				$\dots$

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1				...
2				...
3				...
4				...
5				...
6				...
7				...
8				...

$\uparrow$

alternate every ...

4

rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**			...
2	**T**			...
3	**T**			...
4	**T**			...
5	**F**			...
6	**F**			...
7	**F**			...
8	**F**			...

$\uparrow$

alternate every ...
2
rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**	**T**		...
2	**T**	**T**		...
3	**T**	**F**		...
4	**T**	**F**		...
5	**F**	**T**		...
6	**F**	**T**		...
7	**F**	**F**		...
8	**F**	**F**		...

↑

alternate every ...

1

rows

A complex truth table

3 sentence letters A, C, E: $2^3 = 8$ lines

	A	C	E	...
1	**T**	**T**	**T**	...
2	**T**	**T**	**F**	...
3	**T**	**F**	**T**	...
4	**T**	**F**	**F**	...
5	**F**	**T**	**T**	...
6	**F**	**T**	**F**	...
7	**F**	**F**	**T**	...
8	**F**	**F**	**F**	...

alternate every . . .

1

rows

Example (simplified)

Sarah lives in Calgary or Edmonton.
Amir lives in Calgary unless he enjoys hiking.
If Amir lives in Calgary, Sarah doesn't.
Amir doesn't enjoy hiking.
$\therefore$ Sarah lives in Edmonton.

$$\begin{array}{l} C \lor E \\ A \lor M \\ A \supset {\sim}C \\ {\sim}M \\ \therefore E \end{array}$$

A	C	E	M	$C \vee E$	$A \vee M$	$A \supset \sim C$	$\sim M$	E
T	**T**	**T**	**T**					
T	**T**	**T**	**F**					
T	**T**	**F**	**T**					
T	**T**	**F**	**F**					
T	**F**	**T**	**T**					
T	**F**	**T**	**F**					
T	**F**	**F**	**T**					
T	**F**	**F**	**F**					
F	**T**	**T**	**T**					
F	**T**	**T**	**F**					
F	**T**	**F**	**T**					
F	**T**	**F**	**F**					
F	**F**	**T**	**T**					
F	**F**	**T**	**F**					
F	**F**	**F**	**T**					
F	**F**	**F**	**F**					

A	C	E	M	$C \vee E$		$A \vee M$		$A \supset \sim C$		$\sim M$	E
T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	T	T	F	T	T	F	T
T	T	F	T	T	F	T	T	T	T	T	F
T	T	F	F	T	F	T	F	T	T	F	F
T	F	T	T	F	T	T	T	T	F	T	T
T	F	T	F	F	T	T	F	T	F	F	T
T	F	F	T	F	F	T	T	T	F	T	F
T	F	F	F	F	F	T	F	T	F	F	F
F	T	T	T	T	T	F	T	F	T	T	T
F	T	T	F	T	T	F	F	F	T	F	T
F	T	F	T	T	F	F	T	F	T	T	F
F	T	F	F	T	F	F	F	F	T	F	F
F	F	T	T	F	T	F	T	F	F	T	T
F	F	T	F	F	T	F	F	F	F	F	T
F	F	F	T	F	F	F	T	F	F	T	F
F	F	F	F	F	F	F	F	F	F	F	F

A	C	E	M	C	$\vee$	E	A	$\vee$	M	A	$\supset$	$\sim$	C	$\sim$	M	E
T	T	T	T	T	T	T	T	T	T	T	F	F	T	F	T	T
T	T	T	F	T	T	T	T	T	F	T	F	F	T	T	F	T
T	T	F	T	T	T	F	T	T	T	T	F	F	T	F	T	F
T	T	F	F	T	T	F	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	T	T	T	T	T	T	T	T	F	F	T	T
T	F	T	F	F	T	T	T	T	F	T	T	T	F	T	F	T
T	F	F	T	F	F	F	T	T	T	T	T	T	F	F	T	F
T	F	F	F	F	F	F	T	T	F	T	T	T	F	T	F	F
F	T	T	T	T	T	T	F	T	T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F	F	F	F	T	F	T	T	F	T
F	T	F	T	T	T	F	F	T	T	F	T	F	T	F	T	F
F	T	F	F	T	T	F	F	F	F	F	T	F	T	T	F	F
F	F	T	T	F	T	T	F	T	T	F	T	T	F	F	T	T
F	F	T	F	F	T	T	F	F	F	F	T	T	F	T	F	T
F	F	F	T	F	F	F	F	T	T	F	T	T	F	F	T	F
F	F	F	F	F	F	F	F	F	F	F	T	T	F	T	F	F

A	C	E	M	C	$\vee$	E	A	$\vee$	M	A	$\supset$	$\sim$	C	$\sim$	M	E
T	T	T	T	T	T	T	T	T	T	T	F	F	T	F	T	T
T	T	T	F	T	T	T	T	T	F	T	F	F	T	T	F	T
T	T	F	T	T	T	F	T	T	T	T	F	F	T	F	T	F
T	T	F	F	T	T	F	T	T	F	T	F	F	T	T	F	F
T	F	T	T	F	T	T	T	T	T	T	T	T	F	F	T	T
T	F	T	F	F	T	T	T	T	F	T	T	T	F	T	F	T
T	F	F	T	F	F	F	T	T	T	T	T	T	F	F	T	F
T	F	F	F	F	F	F	T	T	F	T	T	T	F	T	F	F
F	T	T	T	T	T	T	F	T	T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F	F	F	F	T	F	T	T	F	T
F	T	F	T	T	T	F	F	T	T	F	T	F	T	F	T	F
F	T	F	F	T	T	F	F	F	F	F	T	F	T	T	F	F
F	F	T	T	F	T	T	F	T	T	F	T	T	F	F	T	T
F	F	T	F	F	T	T	F	F	F	F	T	T	F	T	F	T
F	F	F	T	F	F	F	F	T	T	F	T	T	F	F	T	F
F	F	F	F	F	F	F	F	F	F	F	T	T	F	T	F	F

III.e.4

A	C	E	M	C	$\vee$	E	A	$\vee$	M	A	$\supset$	$\sim$	C	$\sim$	M	E
T	T	T	T	T	T	T	T	T	T	T	**F**	F	T	**F**	T	**T**
T	T	T	F	T	T	T	T	T	F	T	**F**	F	T	T	F	**T**
T	T	F	T	T	T	F	T	T	T	T	**F**	F	T	**F**	T	F
T	T	F	F	T	T	F	T	T	F	T	**F**	F	T	T	F	F
T	F	T	T	F	T	T	T	T	T	T	T	T	F	**F**	T	**T**
T	F	T	F	F	**T**	T	T	**T**	F	T	**T**	T	F	**T**	F	**T**
T	F	F	T	F	**F**	F	T	T	T	T	T	T	F	**F**	T	F
T	F	F	F	F	**F**	F	T	T	F	T	T	T	F	T	F	F
F	T	T	T	T	T	T	F	T	T	F	T	F	T	**F**	T	**T**
F	T	T	F	T	T	T	F	**F**	F	F	T	F	T	T	F	**T**
F	T	F	T	T	T	F	F	T	T	F	T	F	T	**F**	T	F
F	T	F	F	T	T	F	F	**F**	F	F	T	F	T	T	F	F
F	F	T	T	F	T	T	F	T	T	F	T	T	F	**F**	T	T
F	F	T	F	F	T	T	F	**F**	F	F	T	T	F	T	F	**T**
F	F	F	T	F	**F**	F	F	T	T	F	T	T	F	**F**	T	F
F	F	F	F	F	**F**	F	F	**F**	F	F	T	T	F	T	F	F

Every valuation makes at least one premise false, or makes the conclusion true: the argument is valid. III.e.4

III. SL and truth tables

f. Entailment, equivalence, tautologies

Validity of arguments

Definition

An argument is **valid in SL** iff every valuation either makes one or more of the premises false or it makes the conclusion true.

An argument is **invalid in SL** iff at least one valuation makes all the premises true and it makes the conclusion false.

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ **entail** a sentence $\mathscr{B}$ iff every valuation either makes at least one of $\mathscr{A}_1, \ldots, \mathscr{A}_n$ false or makes $\mathscr{B}$ true.

In that case we write $\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B}$.

We have:

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \vDash \mathscr{B} \text{ iff the argument } \mathscr{A}_1, \ldots, \mathscr{A}_n \therefore \mathscr{B} \text{ is valid.}$$

Entailment

Does $\sim(\sim A \vee \sim B), A \supset \sim C \vDash A \supset (B \supset C)$?

A	B	C	$\sim$	$(\sim$	A	$\vee$	$\sim$	$B)$	A	$\supset$	$\sim$	C	A	$\supset$	$(B$	$\supset$	$C)$
T	T	T	T	F	T	F	F	T	T	F	F	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	T	F	T	F	T	F	F
T	F	T	F	F	T	T	T	F	T	F	F	T	T	T	F	T	T
T	F	F	F	F	T	T	T	F	T	T	T	F	T	T	F	T	F
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	T	T	T
F	T	F	F	T	F	T	F	T	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	T	F	F	T	F	T	F	T	F	T	T
F	F	F	F	T	F	T	T	F	F	T	T	F	F	T	F	T	F

Entailment

A	B	C	$\sim$	$(\sim$	A	$\vee$	$\sim$	$B)$	A	$\supset$	$\sim$	C	A	$\supset$	$(B$	$\supset$	$C)$	
T	T	T	T	F	T	F	F	T	T	F	F	T	T	T	T	T	T	
T	T	F	T	F	T	F	F	T	T	T	T	F	T	F	T	F	F	←
T	F	T	F	F	T	T	T	F	T	F	F	T	T	T	F	T	T	
T	F	F	F	F	T	T	T	F	T	T	T	F	T	T	F	T	F	
F	T	T	F	T	F	T	F	T	F	T	F	T	F	T	T	T	T	
F	T	F	F	T	F	T	F	T	F	T	T	F	F	T	T	F	F	
F	F	T	F	T	F	T	T	F	F	T	F	T	F	T	F	T	T	
F	F	F	F	T	F	T	T	F	F	T	T	F	F	T	F	T	F	

Equivalent sentences

Definition

Two sentences $\mathscr{A}$ and $\mathscr{B}$ are **equivalent in SL** iff every valuation either makes both $\mathscr{A}$ and $\mathscr{B}$ true or it makes both $\mathscr{A}$ and $\mathscr{B}$ false.

In other words: $\mathscr{A}$ and $\mathscr{B}$ agree in truth value, for every valuation.

Equivalent sentences

A	B	$\sim$	A	$\vee$	B	A	$\supset$	B
T	T		T		T	T		T
T	F		T		F	T		F
F	T		F		T	F		T
F	F		F		F	F		F

Equivalent sentences

A	B	$\sim$	A	$\vee$	B	A	$\supset$	B
T	T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	F	T	F	T	F	F	T	F

Equivalent sentences

A	B	$\sim$	A	$\vee$	B	A	$\supset$	B
T	T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F	F
F	T	T	F	T	T	F	T	T
F	F	T	F	T	F	F	T	F

Equivalence and entailment

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Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

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Proof

Equivalence and entailment

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If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.

Equivalence and entailment

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If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.
- If $\mathscr{A}$ is true, $\mathscr{B}$ is also true (since $\mathscr{A}$ and $\mathscr{B}$ agree in truth value on every valuation).

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.
- If $\mathscr{A}$ is true, $\mathscr{B}$ is also true (since $\mathscr{A}$ and $\mathscr{B}$ agree in truth value on every valuation).
- So if $\mathscr{A}$ is true, the valuation is also not a counterexample.

Equivalence and entailment

Fact

If $\mathscr{A}$ and $\mathscr{B}$ are equivalent, then $\mathscr{A} \vDash \mathscr{B}$ (and $\mathscr{B} \vDash \mathscr{A}$).

Proof

- Look at any valuation: it makes $\mathscr{A}$ true or false.
- If $\mathscr{A}$ is false, the valuation is not a counterexample.
- If $\mathscr{A}$ is true, $\mathscr{B}$ is also true (since $\mathscr{A}$ and $\mathscr{B}$ agree in truth value on every valuation).
- So if $\mathscr{A}$ is true, the valuation is also not a counterexample.
- So, no valuation can be a counterexample to $\mathscr{A} \vDash \mathscr{B}$.

Tautologies

Definition

A sentence $\mathscr{A}$ is a **tautology** iff it is true on every valuation.

A	A	$\supset$	A
T	T	T	T
F	F	T	F

III. SL and truth tables

g. Joint satisfiability

Joint satisfiability

Definition

Sentences $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are **jointly satisfiable** in SL if there is at least one valuation that makes all of them true.

If they are not satisfiable, we say they are **jointly unsatisfiable**.

$A \vee B$, $\sim A$, B are jointly satisfiable.

$A \vee B$, $\sim A$, $\sim B$ are unsatisfiable.

Unsatisfiability and validity

- Any argument with jointly unsatisfiable premises is valid.

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Unsatisfiability and validity

- Any argument with jointly unsatisfiable premises is valid.
  - If premises are jointly unsatisfiable, no valuation makes them all true.
  - No valuation makes them all true and the conclusion false.
  - No valuation can be a counterexample.
- An argument is valid if, and only if, the premises together with the negation of the conclusion are jointly unsatisfiable.

LSAT puzzle

A, B, C, D: Amir, Betty, Ching, Dana are in the boat.

Amir won't go without Ching.

Ching only goes if at least one of Betty and Dana goes too.

Amir and Dana can't be in the boat together.

LSAT puzzle

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$$A \supset C$$

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Amir and Dana can't be in the boat together.

$$\sim(A \,\&\, D)$$

$$A \supset \sim D$$

$$\sim A \vee \sim D$$

Dependency resolution by SAT checking

A, B, C, D: package A, B, C, D is installed.

Package A depends on package C.

Package C requires either package B or D.

Package A is incompatible with package D.

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A, B, C, D: package A, B, C, D is installed.

Package A depends on package C.

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Package C requires either package B or D.

$$C \supset (B \vee D)$$

Package A is incompatible with package D.

$$\sim(A \,\&\, D)$$

$$A \supset \sim D$$

$$\sim A \vee \sim D$$

Solution as satisfiability question

Can you send **Amir** in the boat?

Can package **A** be installed?

Same as: Are these sentences jointly satisfiable?

$$\mathbf{A}$$

$$A \supset C$$

$$C \supset (B \vee D)$$

$$\sim(A \,\&\, D)$$

More complex satisfiability questions

Can you send **Amir without Betty** in the boat?

Can package **A** be installed **without installing B**?

Same as: Are these sentences jointly satisfiable?

$$A \,\&\, {\sim}B$$

$$A \supset C$$

$$C \supset (B \vee D)$$

$$\sim(A \,\&\, D)$$

(Exercise: construct a complete truth table. Which valuations, if any, satisfy all four sentences?)

Complexity of logical testing

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Complexity of logical testing

- Testing for validity, satisfiability, tautology, etc., requires making a complete truth table
  - Testing for validity requires checking **every** valuation.
  - Testing for satisfiability requires finding **at least one** valuation.
- If there are n sentence letters, there are 2^n valuations to check.
- Computer scientists have yet to find a method that can (always) do this faster than truth tables ("P vs NP problem").

IV. Proofs in SL

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a. Why proofs?

Showing that an argument is valid

- Construct a truth table; verify there is no valuation where premises are true and conclusion is false.

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- E.g., the example argument

$$C \lor E, A \lor M, A \supset {\sim}C, {\sim}S \mathbin{\&} {\sim}M \therefore E$$

requires 32 lines and 608 individual truth values.

Showing that an argument is valid

- Construct a truth table; verify there is no valuation where premises are true and conclusion is false.
- Truth tables can get very large very quickly.
- E.g., the example argument

$$C \lor E, A \lor M, A \supset {\sim}C, {\sim}S \mathbin{\&} {\sim}M \therefore E$$

  requires 32 lines and 608 individual truth values.
- Is there a better way?

Proofs

- Idea: work our way from premises to conclusion using steps we know are entailed by the premises.

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  - From “Neither Sarah nor Amir enjoys hiking” we can conclude “Amir doesn’t enjoy hiking.”

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- For instance:
  - From “Neither Sarah nor Amir enjoys hiking” we can conclude “Amir doesn’t enjoy hiking.”
  - From “Either Amir lives in Calgary or he enjoys hiking” and “Amir doesn’t enjoy hiking” we can conclude “Amir lives in Calgary” (Disjunctive syllogism DS).

Proofs

- Idea: work our way from premises to conclusion using steps we know are entailed by the premises.
- For instance:
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  - etc.
- If we manage to work from the premises to the conclusion in this way, we know that the argument must be valid.

An informal proof

Our argument

1. Sarah lives in Calgary or Edmonton.
2. Amir lives in Calgary unless he enjoys hiking.
3. If Amir lives in Calgary, Sarah doesn't.
4. Neither Sarah nor Amir enjoy hiking.

$\therefore$ Sarah lives in Edmonton.

5. Amir doesn't enjoy hiking (from 4).

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8. Sarah lives in Edmonton (from 1 and 7).

A more formal proof

Our argument

1. $C \vee E$
2. $A \vee M$
3. $A \supset {\sim}C$
4. ${\sim}S \,\&\, {\sim}M$

$\therefore E$

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- But: what's a rule?

IV. Proofs in SL

b. Rules for &

Rules of natural deduction

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- (Role of $\mathscr{P}$ played by $\sim S$ and that of $\mathscr{Q}$ played by $\sim M$.)

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- For instance, if we have “Sarah doesn’t enjoy hiking” and also “Amir doesn’t enjoy hiking”, we can conclude “Neither Sarah nor Amir enjoys hiking”.

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- (Role of $\mathscr{P}$ played by $\sim S$ and $\mathbb{Q}$ played by $\sim M$: $\sim S, \sim M \vDash \sim S \,\&\, \sim M$.)

Rules for &

$$\begin{array}{r|ll} m & \mathscr{P} & \\ n & \mathscr{Q} & \\ & \mathscr{P} \,\&\, \mathscr{Q} & :m,\ n\ \ \&\mathrm{I} \end{array}$$

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We'll illustrate using the exercises in Carnap.

1	$A \,\&\, B$	
2	A	:1 & E
3	B	:1 & E
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3	$B \,\&\, C$	:1 & E
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IV. Proofs in SL

c. Rules for $\supset$

Eliminating $\supset$

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- We used the conditional "If Amir lives in Calgary, Sarah isn't" to justify "Sarah doesn't live in Calgary".

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- What can we **justify using** $\mathscr{P} \supset \mathbb{Q}$?
- We used the conditional "If Amir lives in Calgary, Sarah isn't" to justify "Sarah doesn't live in Calgary".
- What is the general rule? What can we justify using $\mathscr{P} \supset \mathbb{Q}$? What do we need in addition to $\mathscr{P} \supset \mathbb{Q}$?

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- (In inference from $A \supset {\sim}C$ and A to ${\sim}C$, role of $\mathscr{P}$ is played by A and role of $\mathbb{Q}$ by ${\sim}C$.)

Elimination rule for $\supset$

$$
\begin{array}{l|ll}
m & \mathscr{P} \supset \mathbb{Q} & \\
n & \mathscr{P} & \\
 & \mathbb{Q} & :m,\ n \supset\mathrm{E}
\end{array}
$$

We'll illustrate using the exercise in Carnap: we show that $A \,\&\, B, A \supset C, B \supset D \models C \,\&\, D$.

1	$A \& B$	
2	$A \supset C$	
3	$B \supset D$	
4	A	:1 & E
5	C	:2, 4 ⊃E
6	B	:1 & E
7	D	:3, 6 ⊃E
8	$C \& D$	:5, 7 & I

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- Idea: show instead that $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$.
- The conditional $\supset$ no longer appears, so this seems easier.
- It's a good move, because if $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then $\mathscr{R} \vDash \mathscr{P} \supset \mathbb{Q}$.

Justifying $\supset$I

Fact

If $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then $\mathscr{R} \vDash \mathscr{P} \supset \mathbb{Q}$.

- If $\mathscr{R}, \mathscr{P} \vDash \mathbb{Q}$ then every valuation makes one of $\mathscr{R}, \mathscr{P}$ false or it makes $\mathbb{Q}$ true.

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  2. So any valuation must make $\mathscr{R}$ false, $\mathscr{P}$ false, or $\mathbb{Q}$ true.
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  3. If it makes $\mathscr{R}$ false, it's not a counterexample to $\mathscr{R} \vDash \mathscr{P} \supset \mathbb{Q}$.
  4. If it makes $\mathscr{P}$ false, it makes $\mathscr{P} \supset \mathbb{Q}$ true, so it's not a counterexample.

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  2. So any valuation must make $\mathscr{R}$ false, $\mathscr{P}$ false, or $\mathbb{Q}$ true.
  3. If it makes $\mathscr{R}$ false, it's not a counterexample to $\mathscr{R} \vDash \mathscr{P} \supset \mathbb{Q}$.
  4. If it makes $\mathscr{P}$ false, it makes $\mathscr{P} \supset \mathbb{Q}$ true, so it's not a counterexample.
  5. If it makes $\mathbb{Q}$ true, it also makes $\mathscr{P} \supset \mathbb{Q}$ true, so it's not a counterexample.
- So, there are no counterexamples to $\mathscr{R} \vDash \mathscr{P} \supset \mathbb{Q}$.

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- Once we're done (have proved $\mathbb{Q}$), close this "subproof".
- Justification of $\mathscr{P} \supset \mathbb{Q}$ is the **entire** subproof.
- **Important**: nothing **inside** a subproof is available outside as a justification (it depends on the assumption)

Introduction rule for $\supset$

$$
\begin{array}{r|l}
m & \quad \begin{array}{|l} \mathscr{P} \\ \hline \end{array} \\
 & \quad \begin{array}{|l} \vdots \end{array} \\
n & \quad \begin{array}{|l} \mathscr{Q} \end{array} \\
 & \mathscr{P} \supset \mathscr{Q} \qquad : m\text{-}n \supset\mathrm{I}
\end{array}
$$

We'll illustrate using the exercises here

- Show: $A \supset B, B \supset C \models A \supset C$.
- Show: $A \supset (B \supset C) \models (A \,\&\, B) \supset (A \,\&\, C)$

1	$A \supset B$	
2	$B \supset C$	
3	$\quad A$	
4	$\quad B$	:1, 3 $\supset$E
5	$\quad C$	:2, 4 $\supset$E
6	$A \supset C$	:3-5 $\supset$I

1	$A \supset (B \supset C)$	
2	$\quad A \,\&\, B$	
3	$\quad A$	:2 & E
4	$\quad B \supset C$	:1, 3 ⊃E
5	$\quad B$	:2 & E
6	$\quad C$	:4, 5 ⊃E
7	$\quad A \,\&\, C$	:3, 6 & I
8	$(A \,\&\, B) \supset (A \,\&\, C)$	:2-7 ⊃I

IV. Proofs in SL

d. Use of subproofs

Reiteration

$\mathscr{P} \vDash \mathscr{P}$, so "Reiteration" R is a good rule:

$$
\begin{array}{r|l l}
m & \mathscr{P} & \\
 & & \\
k & \mathscr{P} & :m\ \mathrm{R}
\end{array}
$$

Uses of reiteration:

- Proof of $A \vDash A$.

Reiteration

$\mathscr{P} \models \mathscr{P}$, so "Reiteration" R is a good rule:

$$\begin{array}{r|ll} m & \mathscr{P} & \\ & & \\ k & \mathscr{P} & :m \text{ R} \end{array}$$

Uses of reiteration:

- Proof of $A \models A$.
- Proof that $A \supset (B \supset A)$ is a tautology.

$$
\begin{array}{l|l|ll}
1 & & \multicolumn{1}{|l}{A} & \\
\cline{3-3}
2 & & \multicolumn{1}{|l}{A} & :1\ \mathrm{R} \\
3 & \multicolumn{2}{l}{A \supset A} & :1\text{-}2\ {\supset}\mathrm{I}
\end{array}
$$

1	$\quad \mid A$ (assumption)	
2	$\quad \mid \quad \mid B$ (assumption)	
3	$\quad \mid \quad \mid A$	:1 R
4	$\quad \mid B \supset A$	:2-3 $\supset$I
5	$A \supset (B \supset A)$	:1-4 $\supset$I

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- When a rule calls for a subproof, we cite it as n–m, with first and last line numbers of the subproof.

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- Assumption line **and** last line have to match rule.
- After a subproof is done, you can only cite the whole thing, and not any individual line in it.
- Subproofs can be nested.
- When that happens, you also can't cite any subproof entirely contained inside another subproof, once the surrounding subproof is done.

Reiteration

Which are correct applications of R?

1	$\quad\vert$ A (assumption)		
2	$\quad\vert\quad\vert$ A (assumption)		
3	$\quad\vert\quad\vert$ A	:1	R
4	$\quad\vert$ A	:1	R
5	$\quad\vert$ A	:2	R
6	A	:2	R
7	A	:1	R

Reiteration

Which are correct applications of R?

1	$\quad\vert\ \vert\ A$	
2	$\quad\vert\ \vert\ \vert\ A$	
3	$\quad\vert\ \vert\ \vert\ A$	:1 ✓ R
4	$\quad\vert\ \vert\ A$	:1 R
5	$\quad\vert\ \vert\ A$	:2 R
6	$\quad\vert\ A$	:2 R
7	$\quad\vert\ A$	:1 R

Reiteration

Which are correct applications of R?

1	A	
2	A	
3	A	:1 ✓ R
4	A	:1 ✓ R
5	A	:2 R
6	A	:2 R
7	A	:1 R

Reiteration

Which are correct applications of R?

1	A	
2	A	
3	A	:1 ✓ R
4	A	:1 ✓ R
5	A	:2 ✗ R
6	A	:2 R
7	A	:1 R

Reiteration

Which are correct applications of R?

1	$\mid\ \mid\ A$	
2	$\mid\ \mid\ \mid\ A$	
3	$\mid\ \mid\ \mid\ A$	:1 ✓ R
4	$\mid\ \mid\ A$	:1 ✓ R
5	$\mid\ \mid\ A$	:2 ✗ R
6	$\mid\ A$	:2 ✗ R
7	$\mid\ A$	:1 R

Reiteration

Which are correct applications of R?

1	$\quad\mid\; A$	
2	$\quad\mid\;\quad\mid\; A$	
3	$\quad\mid\;\quad\mid\; A$	:1 ✓ R
4	$\quad\mid\; A$	:1 ✓ R
5	$\quad\mid\; A$	:2 ✗ R
6	A	:2 ✗ R
7	A	:1 ✗ R

IV. Proofs in SL

e. Rules for $\lor$

Introduction rule for $\vee$

We have $\mathscr{P} \vDash \mathscr{P} \vee \mathscr{Q}$. So:

$$\begin{array}{r|ll} m & \mathscr{P} & \\ \hline & \mathscr{P} \vee \mathscr{Q} & :m \vee\mathrm{I} \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{Q} & \\ \hline & \mathscr{P} \vee \mathscr{Q} & :m \vee\mathrm{I} \end{array}$$

1	$\quad A$	
2	$\quad B \vee A$	:1 $\vee$I
3	$A \supset (B \vee A)$	:1-2 $\supset$I

Eliminating $\vee$

- What can we justify with disjunction $\mathscr{P} \vee \mathbb{Q}$?

Eliminating $\vee$

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- Not $\mathscr{P}$ and also not $\mathbb{Q}$: neither is entailed by $\mathscr{P} \vee \mathbb{Q}$.

Eliminating $\lor$

- What can we justify with disjunction $\mathscr{P} \lor \mathbb{Q}$?
- Not $\mathscr{P}$ and also not $\mathbb{Q}$: neither is entailed by $\mathscr{P} \lor \mathbb{Q}$.
- But: if both $\mathscr{P}$ and $\mathbb{Q}$ separately entail some third sentence $\mathscr{R}$, then we know that $\mathscr{R}$ follows!

Eliminating $\vee$

- What can we justify with disjunction $\mathscr{P} \vee \mathbb{Q}$?
- Not $\mathscr{P}$ and also not $\mathbb{Q}$: neither is entailed by $\mathscr{P} \vee \mathbb{Q}$.
- But: if both $\mathscr{P}$ and $\mathbb{Q}$ separately entail some third sentence $\mathscr{R}$, then we know that $\mathscr{R}$ follows!
- To show this, we need **two** proofs that show $\mathscr{R}$, but in each proof we are allowed to use only one of $\mathscr{P}$, $\mathbb{Q}$.

Elimination rule for $\vee$

$$
\begin{array}{l|l}
m & \mathscr{P} \vee \mathscr{Q} \\
i & \quad \mathscr{P} \\
 & \quad \vdots \\
j & \quad \mathscr{R} \\
k & \quad \mathscr{Q} \\
 & \quad \vdots \\
l & \quad \mathscr{R} \\
 & \mathscr{R} \qquad\qquad : m,\ i\text{–}j,\ k\text{–}l\ \vee\mathrm{E}
\end{array}
$$

1	$A \vee B$	
2	$\quad A$	
3	$\quad B \vee A$	:2 $\vee$I
4	$\quad B$	
5	$\quad B \vee A$	:4 $\vee$I
6	$B \vee A$	:1, 2–3, 4–5 $\vee$E

1	$A \vee B$	
2	$A \supset B$	
3	$\quad \vert\ A$	
4	$\quad \vert\ B$	:2, 3 $\supset$E
5	$\quad \vert\ B$	
6	$\quad \vert\ B$	:5 R
7	B	:1, 3-4, 5-6 $\vee$E

IV. Proofs in SL

f. Contradictions

Contradictions

- In proofs, we don't just use the premises of the argument, but also sentences we've proved, and sentences we've assumed (for $\supset$I, $\vee$E).

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Contradictions

- In proofs, we don't just use the premises of the argument, but also sentences we've proved, and sentences we've assumed (for $\supset$I, $\vee$E).
- Sometimes it happens that assumptions we must make for correct applications of these rules are incompatible with the premises.
- For instance, to prove the disjunctive syllogism $A \vee B, {\sim}B \models A$ using $\vee$E, the assumption B for the second case conflicts with the premise ${\sim}B$.

Disjunctive syllogism

1	$A \vee B$	
2	$\sim B$	
3	$\quad A$	
4	$\quad A$	:3 R
5	$\quad B$	
	$\quad \vdots$	
k	$\quad A$	
$k+1$	A	:1, 3-4, 5-k $\vee$E

Contradictions: eliminating $\sim$

We highlight the situation where inside a subproof we have run into a contradictory situation by the symbol

$$\perp$$

$$
\begin{array}{r|ll}
m & \sim\mathscr{P} & \\
n & \mathscr{P} & \\
 & \perp & : m,\ n\ \sim\mathrm{E}
\end{array}
$$

Since this also eliminates a $\sim$, we'll call it $\sim$E.

Explosion

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- So whenever we can justify $\bot$ in a proof, we should be able to justify **anything**.
- "From a contradiction, anything follows."

$$\begin{array}{r|l}
m & \bot \\
 & \\
k & \mathscr{P} \quad\quad :m\ \mathrm{X}
\end{array}$$

Disjunctive syllogism

1	$A \vee B$	
2	$\sim B$	
3	$\quad \mid A$	
4	$\quad \mid A$	:3 R
5	$\quad \mid B$	
6	$\quad \mid \bot$	:2, 5 $\sim$E
7	$\quad \mid A$	:6 X
8	A	:1, 3-4, 5-7 $\vee$E

IV. Proofs in SL

g. Introducing $\sim$

Introducing $\sim$

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- An argument is valid iff the premises together with the negation of the conclusion are jointly unsatisfiable.
- For instance:
  - $\mathbb{Q} \vDash \mathscr{P}$ iff $\mathbb{Q}$ and $\sim\mathscr{P}$ are jointly unsatisfiable.
  - $\mathbb{Q} \vDash \sim\mathscr{P}$ iff $\mathbb{Q}$ and $\mathscr{P}$ are jointly unsatisfiable.

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- This last one gives us idea for $\sim$I rule: To justify $\sim\mathscr{P}$, show that $\mathscr{P}$ (together with all other premises) is unsatisfiable.

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  - $\mathbb{Q} \vDash \sim\mathscr{P}$ iff $\mathbb{Q}$ and $\mathscr{P}$ are jointly unsatisfiable.
- This last one gives us idea for $\sim$I rule: To justify $\sim\mathscr{P}$, show that $\mathscr{P}$ (together with all other premises) is unsatisfiable.
- Unsatisfiable means: a contradiction ($\bot$) follows!

Introduction rule for $\sim$

$$
\begin{array}{r|l|l}
m & & \mathscr{P} \\ \cline{3-3}
 & & \vdots \\
n & & \bot \\
 & \multicolumn{2}{l}{{\sim}\mathscr{P} \qquad : m\text{–}n \ {\sim}\mathrm{I}}
\end{array}
$$

1	$A \supset B$	
2	$\sim B$	
3	A	
4	B	:1, 3 $\supset$E
5	$\perp$	:2, 4 $\sim$E
6	$\sim A$	:3-5 $\sim$I
7	$\sim B \supset \sim A$	:2-6 $\supset$I
8	$(A \supset B) \supset (\sim B \supset \sim A)$	:1-7 $\supset$I

Indirect proof rule

$$
\begin{array}{l|l}
m & \quad \begin{array}{|l} \sim\mathscr{P} \\ \hline \vdots \\ \\ \bot \end{array} \\
n & \\
 & \mathscr{P} \qquad\qquad :m\text{–}n \text{ IP}
\end{array}
$$

1	$\vert\ \vert\ \sim A \supset \sim B$	
2	$\vert\ \vert\ \vert\ B$	
3	$\vert\ \vert\ \vert\ \vert\ \sim A$	
4	$\vert\ \vert\ \vert\ \vert\ \sim B$	:1, 3 ⊃E
5	$\vert\ \vert\ \vert\ \vert\ \bot$	:2, 4 ∼E
6	$\vert\ \vert\ \vert\ A$	:3-5 IP
7	$\vert\ \vert\ B \supset A$	:2-6 ⊃I
8	$\vert\ (\sim A \supset \sim B) \supset (B \supset A)$	:1-7 ⊃I

IV. Proofs in SL

h. Strategies and examples

The rules, one more time: &

$$\begin{array}{r|ll} m & \mathscr{P} & \\ n & \mathbb{Q} & \\ & \mathscr{P} \,\&\, \mathbb{Q} & :m,\ n\ \ \&\mathrm{I} \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \,\&\, \mathbb{Q} & \\ & \mathscr{P} & :m\ \ \&\mathrm{E} \end{array}$$

$$\begin{array}{r|ll} m & \mathscr{P} \,\&\, \mathbb{Q} & \\ & \mathbb{Q} & :m\ \ \&\mathrm{E} \end{array}$$

The rules, one more time: $\supset$

$$
\begin{array}{l|l}
m & \quad \begin{array}{|l} \underline{\mathscr{P}} \\ \vdots \\ \mathbb{Q} \end{array} \\
n & \\
 & \mathscr{P} \supset \mathbb{Q} \qquad : m\text{–}n \supset\mathrm{I}
\end{array}
$$

$$
\begin{array}{l|l}
m & \mathscr{P} \supset \mathbb{Q} \\
n & \mathscr{P} \\
 & \mathbb{Q} \qquad : m,\ n \supset\mathrm{E}
\end{array}
$$

The rules, one more time: $\vee$

$$
\begin{array}{r|l}
m & \mathscr{P} \vee \mathscr{Q} \\
i & \quad \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
j & \\
k & \quad \begin{array}{|l} \mathscr{Q} \\ \hline \vdots \\ \mathscr{R} \end{array} \\
l & \\
 & \mathscr{R} \qquad : m,\ i\text{–}j,\ k\text{–}l\ \vee\mathrm{E}
\end{array}
$$

$$
\begin{array}{r|l}
m & \mathscr{P} \\
 & \mathscr{P} \vee \mathscr{Q} \qquad : m\ \vee\mathrm{I}
\end{array}
$$

$$
\begin{array}{r|l}
m & \mathscr{Q} \\
 & \mathscr{P} \vee \mathscr{Q} \qquad : m\ \vee\mathrm{I}
\end{array}
$$

The rules, one more time: $\sim$

$$\begin{array}{r|ll} m & \sim\mathscr{P} & \\ n & \mathscr{P} & \\ & \bot & :m,\ n\ \sim\text{E} \end{array}$$

$$\begin{array}{r|l} m & \begin{array}{|l} \mathscr{P} \\ \hline \vdots \\ \bot \end{array} \\ n & \\ & \sim\mathscr{P} \quad :m\text{–}n\ \sim\text{I} \end{array}$$

The rules, one more time: R, X, and IP

$$
\begin{array}{r|ll}
m & \mathscr{P} & \\
k & \mathscr{P} & :m\ \text{R}
\end{array}
$$

$$
\begin{array}{r|ll}
m & \bot & \\
k & \mathscr{P} & :m\ \text{X}
\end{array}
$$

$$
\begin{array}{r|l|l}
m & & \underline{\sim\mathscr{P}} \\
 & & \vdots \\
n & & \bot \\
 & \mathscr{P} & :m\text{–}n\ \text{IP}
\end{array}
$$

Working forward and backward

- **Working backward** from a conclusion (goal) means:

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  - Find main connective of goal sentence

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  - Write out what else you need to apply the E rule (new goals)

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  - Find main connective of goal sentence
  - Match with conclusion of corresponding I rule
  - Write out (above the goal!) what you'd need to apply that rule
- **Working forward** from a premise, assumption, or already justified sentence means:
  - Find main connective of premise, assumption, or sentence
  - Match with top premise of corresponding E rule
  - Write out what else you need to apply the E rule (new goals)
  - If necessary, write out conclusion of the rule

Constructing a proof

- Write out premises at the top (if there are any)

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  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I, or forward using $\vee$E

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  - Work forward using $\&$E

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  - Work forward using $\&$E
  - Work forward from $\sim$E if your goal is $\perp$

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- Write out premises at the top (if there are any)
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Constructing a proof

- Write out premises at the top (if there are any)
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- Work backward & forward from goals and premises/assumptions in this order:
  - Work backward using $\&$I, $\supset$I, $\equiv$I, $\sim$I, or forward using $\lor$E
  - Work forward using $\&$E
  - Work forward from $\sim$E if your goal is $\bot$
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\lor$I

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  - Work forward using $\&$E
  - Work forward from $\sim$E if your goal is $\perp$
  - Work forward using $\supset$E, $\equiv$E
  - Work backward from $\vee$I
  - Try IP
- Repeat for each new goal from top

1	$\sim(A \vee B)$	
2	$\quad A$	
3	$\quad A \vee B$	:2 $\vee$I
4	$\quad \bot$	:1, 3 $\sim$E
5	$\sim A$	:2-4 $\sim$I
6	$\quad B$	
7	$\quad A \vee B$	:6 $\vee$I
8	$\quad \bot$	:1, 7 $\sim$E
9	$\sim B$	:6-8 $\sim$I
10	$\sim A \,\&\, \sim B$	:5, 9 &I

IV.h.8

1	$\sim A \,\&\, \sim B$	
2	$A \vee B$	
3	A	
4	$\sim A$	:1 & E
5	$\bot$	:4, 3 $\sim$E
6	B	
7	$\sim B$	:1 & E
8	$\bot$	:7, 6 $\sim$E
9	$\bot$	:2, 3-5, 6-8 $\vee$E
10	$\sim(A \vee B)$	:2-9 $\sim$I
11	$(\sim A \,\&\, \sim B) \supset \sim(A \vee B)$	:1-10 $\supset$I

$$
\begin{array}{rll}
1 & \mid\ \mid\ \sim(A \vee \sim A) & \\
\hline
2 & \mid\ \mid\ \mid\ A & \\
\hline
3 & \mid\ \mid\ \mid\ A \vee \sim A & :2\ \vee\mathrm{I} \\
4 & \mid\ \mid\ \mid\ \bot & :1,\ 3\ \sim\mathrm{E} \\
5 & \mid\ \mid\ \sim A & :2\text{-}4\ \sim\mathrm{I} \\
6 & \mid\ \mid\ A \vee \sim A & :5\ \vee\mathrm{I} \\
7 & \mid\ \mid\ \bot & :1,\ 6\ \sim\mathrm{E} \\
8 & \mid\ A \vee \sim A & :1\text{-}7\ \mathrm{IP}
\end{array}
$$

IV.h.10

V. Introduction to first-order logic

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a. The goals of QL

Limits of symbolization in SL

- Consider the argument:

Limits of symbolization in SL

- Consider the argument:

  Greta is a hero.

Limits of symbolization in SL

- Consider the argument:

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  ∴ There is a hero.

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- Consider the argument:

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    - Everyone is a hero.

The problem

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- New language: **first-order logic QL**

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- We want the symbolization to have a proof.

V. Introduction to first-order logic

b. Beginning symbolization in QL

First steps: names

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- Later on, we’ll be able to deal with other expressions that play a similar role to names, e.g., “the prime minister of Canada”
- In QL, names will be lowercase letters a–r

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H(x)&: \text{ \_\_\_\_\_}_x \text{ is a hero}
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- **Domain**: what things we're talking about

  e.g., people alive in 2022

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  - Greta is a **hero who doesn't wear a cape**.
    Greta is a hero and it's not the case that Greta wears a cape.
    $H(g) \& \sim C(g)$

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  - "Fake" and other privative adjectives ("pretend," "fictitious") imply opposite!

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  At least one: $W(g,a) \vee W(a,g)$
  Exactly one: $(W(g,a) \vee W(a,g)) \& \sim(W(g,a) \& W(a,g))$

V. Introduction to first-order logic

c. The existential quantifier

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- For instance:
  - **Someone** wears a cape.
  - **There is** a hero.
  - **Something** inspires.
- Note: can (often) go where names and pronouns also go.
- But works differently from names ("something" doesn't pick out a unique, specific object).

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- Better idea: symbolize (complex) **properties** and introduce mechanism for expressing that properties are **instantiated**.

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- Note: all contain a **single** variable x

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- Combine $\exists$ with expression for a property (e.g., $(H(x) \mathbin{\&} C(x))$) to say "something (someone) has that property"
- Put the variable that serves as a marker for the gap also after $\exists$. E.g.,

$$\exists x\, (H(x) \mathbin{\&} C(x))$$

  says "Someone is a hero and wears a cape"

Quantifiers and variables

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- In second case, one person can be the hero, and another wears a cape.

Quantifiers and variables

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$$\exists x\,(H(x)\,\&\,C(x)) \text{ to}$$
$$\exists x\,H(x)\,\&\,\exists x\,C(x)$$

- In first case, the same person must be a hero and wear a cape.
- In second case, one person can be the hero, and another wears a cape.
- Multiple $\exists x$ are independent, even if they use the same x. No difference in meaning between

$$\exists x\,H(x)\,\&\,\exists x\,C(x) \text{ and}$$
$$\exists x\,H(x)\,\&\,\exists y\,C(y).$$

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  - Now take the domain to include only Greta.
  - Relative to that domain, $\exists x\, W(x, g)$ is true iff Greta watched herself.

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- General form: “Some F is G.”

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- We'll also symbolize the plural form this way ("Some Fs are Gs").
- And more generally (most) sentences of the form: "G(some F)" or "G(something that Fs)".

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V. Introduction to first-order logic

d. The universal quantifier

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- In QL: $\forall x\, F(x)$.
- E.g.:
  - "Everyone wears a cape" $\forall x\, C(x)$
  - "Everyone watched Greta or Autumn" $\forall x(W(x,g) \vee W(x,a))$

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- Determiners with universal meaning: **all, every, any**.

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- Watch out for "any": not always universal.

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  - $\forall x(F(x) \supset G(x))$
    If x is F, x must also be G.
    (If x is not F, doesn't matter if it's G or not.)

Symbolizing “all Fs are Gs”

Symbolize the following as

$$\forall x(F(x) \supset G(x))$$

- All Fs are Gs.
- Every F is G.
- Any F is G.

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V. Introduction to first-order logic

e. No, only, a, and some & any again

No F is G

- "**No Fs are Gs**" can be paraphrased as

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V. Introduction to first-order logic

f. Mixed domains

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VI. Semantics of QL

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a. Arguments and validity in QL

Validity of arguments

Everyone is either good or evil.
Not everyone is a villain.
Only villains are evil.
$\therefore$ Some heroes are good.

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- So we want to ignore any restrictions the predicate symbols place on their **extensions**
- Hence: allow **any** extension in a potential counterexample
- An argument is **first-order valid** if there is no **interpretation** in which the premises are true and the conclusion false

Forms of arguments

Everyone is either good or evil.
Not everyone is a villain.
Only villains are evil.
∴ Some heroes are good.

$$\forall x(G(x) \vee E(x))$$
$$\sim\forall x\, V(x)$$
$$\forall x(E(x) \supset V(x))$$
$$\therefore \exists x(H(x) \,\&\, G(x))$$

(In)validity of arguments

$$\begin{array}{rl} & \forall x(G(x) \lor E(x)) \\ & {\sim}\forall x\, V(x) \\ & \forall x(E(x) \supset V(x)) \\ \therefore & \exists x(H(x)\ \&\ G(x)) \end{array}$$

Domain: the inner planets
$G(x)$: x is smaller than Earth
$E(x)$: x is inhabited
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Extensions

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Domain: Mercury, Venus, Earth, Mars
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Domain: 1, 2, 3, 4
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Domain: 1

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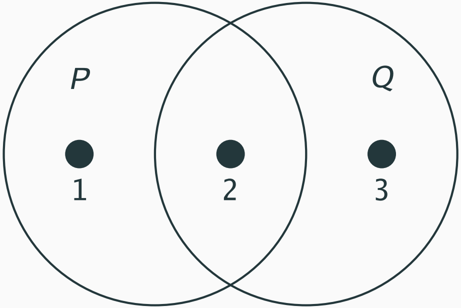
Extensions of predicates

Domain: 1, 2, 3

$P(x)$: 1, 2

$Q(x)$: 2, 3

$R(x)$: —



$R = \varnothing$

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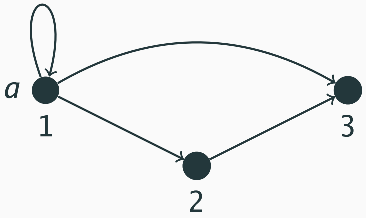
Extensions of predicates

Domain: 1, 2, 3

a: 1

$A(x, y)$: $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 1, 3\rangle$, $\langle 2, 3\rangle$



VI. Semantics of QL

c. Truth of sentences of QL

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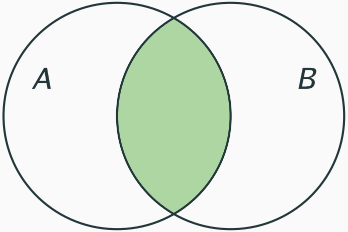
Making "Some As are Bs" true

- $\exists x\,(A(x) \,\&\, B(x))$
- Extension of A and B must have something in common.

  (Filled area must contain at least one object)
- A and B can overlap, be equal, or be contained.
- Same situations make "No As are Bs" **false**.



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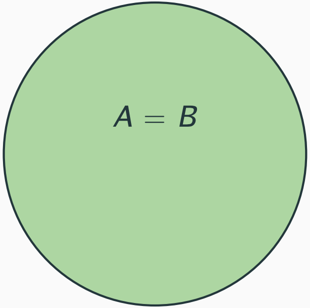

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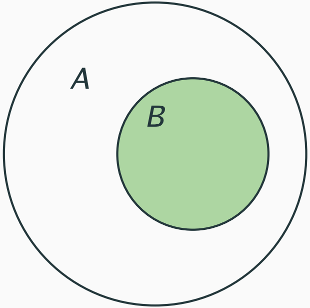
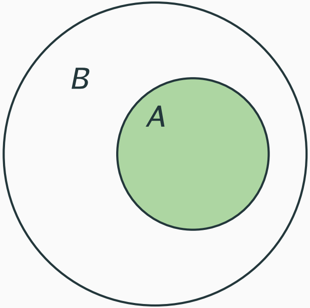


Making “Some *A*s are *B*s” true

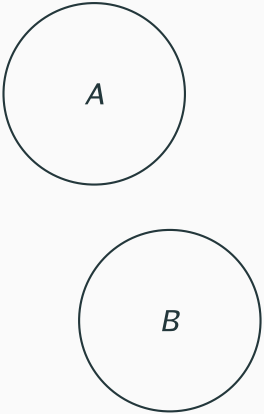
- $\exists x\,(A(x)\,\&\,B(x))$
- Extension of *A* and *B* must have something in common.

  (Filled area must contain at least one object)
- *A* and *B* can overlap, be equal, or be contained.
- Same situations make “No *A*s are *B*s” **false**.



Making “Some As are Bs” false

- $\sim\exists x\,(A(x)\,\&\,B(x))$
- Extension of A and B must have nothing in common.
- A and B don't overlap, or one or both is empty.
- Same situations make “No As are Bs” **true**.



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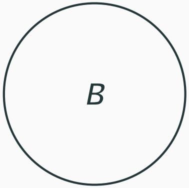

$A = \varnothing$

Making "Some As are Bs" false

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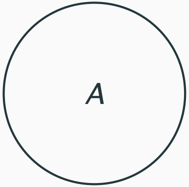
$B = \varnothing$



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- $\sim\exists x\,(A(x)\,\&\,B(x))$
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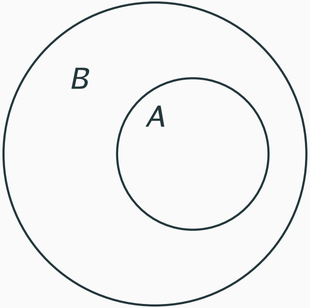
- A and B don't overlap, or one or both is empty.
- Same situations make "No As are Bs" **true**.

$$A = B = \varnothing$$

Making "All *A*s are *B*s" true

- $\forall x\,(A(x) \supset B(x))$
- Extension of A must be contained in extension of B.
- Extensions of A and B can be the same.
- Extension of A can be empty.
- Same situations make ...
  - "Only *B*s are *A*s" **true**.
  - "Some *A*s are not *B*s" **false**.



Making "All *A*s are *B*s" true

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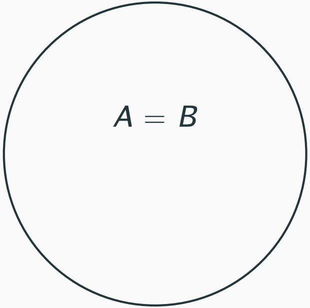

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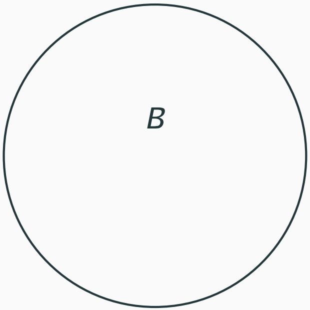

$A = \varnothing$

Making "All *A*s are *B*s" false

- $\forall x\,(A(x) \supset B(x))$
- Extension of A must contain something not in B.
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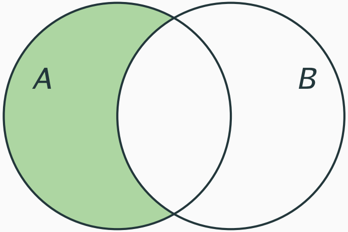
- Same situations make …
  - "Only Bs are As" **false**.
  - "Some As are not Bs" **true**.



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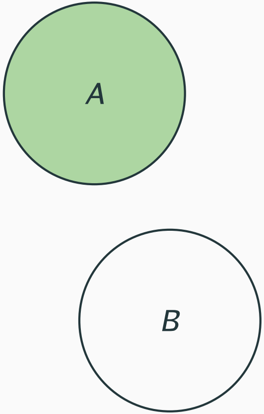

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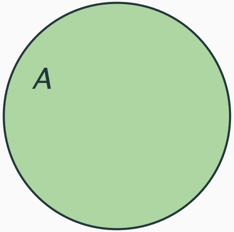

$B = \varnothing$

VI. Semantics of QL

d. Testing for validity

Arguments involving quantifiers

1. If an action x is morally wrong then A is blameworthy for freely doing x.

(John Skorupski, *Ethical Explorations*, 2000 (link)

Arguments involving quantifiers

1. If an action x is morally wrong then A is blameworthy for freely doing x.
2. If x is rationally optimal (there is no action which A has reason to think there is more reason for A to do), then A is not blameworthy for freely doing x.

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Arguments involving quantifiers

1. If an action x is morally wrong then A is blameworthy for freely doing x.
2. If x is rationally optimal (there is no action which A has reason to think there is more reason for A to do), then A is not blameworthy for freely doing x.
3. Therefore, if x is morally wrong, then x is not rationally optimal. (Principle of moral categoricity.)

(John Skorupski, *Ethical Explorations*, 2000 (link)

Symbolizing Skorupski

1. If an action x is morally wrong then A is blameworthy for freely doing x.

Domain: actions
$W(x)$: x is morally wrong
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$$\forall x(W(x) \supset B(x))$$

Symbolizing Skorupski

2. If x is rationally optimal, then A is not blameworthy for freely doing x.

Domain: actions
$W(x)$: x is morally wrong
$B(x)$: A is blameworthy for freely doing x
$O(x)$: x is rationally optimal

$$\forall x(W(x) \supset B(x))$$

Symbolizing Skorupski

2. If x is rationally optimal, then A is not blameworthy for freely doing x.

Domain: actions
$W(x)$: x is morally wrong
$B(x)$: A is blameworthy for freely doing x
$O(x)$: x is rationally optimal

$$\forall x(W(x) \supset B(x))$$
$$\forall x(O(x) \supset {\sim}B(x))$$

Symbolizing Skorupski

3. Therefore, if x is morally wrong, then x is not rationally optimal.

Domain: actions
$W(x)$: x is morally wrong
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Symbolizing Skorupski

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Domain: actions
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$$\forall x(O(x) \supset \sim B(x))$$
$$\therefore \forall x(W(x) \supset \sim O(x))$$

Symbolizing Skorupski

Domain: actions
$W(x)$: x is morally wrong
$B(x)$: A is blameworthy for freely doing x
$O(x)$: x is rationally optimal

$$\forall x(W(x) \supset B(x))$$
$$\forall x(O(x) \supset {\sim}B(x))$$
$$\therefore \forall x(W(x) \supset {\sim}O(x))$$

All Ws are Bs
No Os are Bs (iff No Bs are Os)
∴ No Ws are Os

Determining validity

- Make conclusion
  $\forall x(W(x) \supset \sim O(x))$ false.

Determining validity

- Make conclusion $\forall x(W(x) \supset \sim O(x))$ false.
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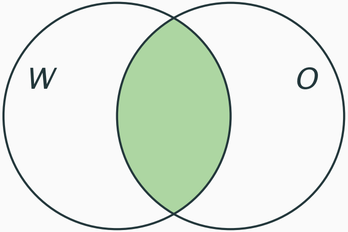

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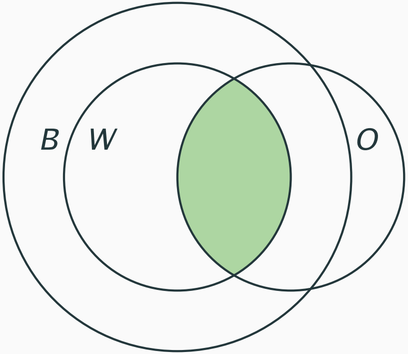

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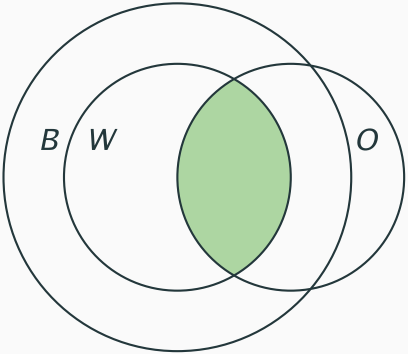
- $\exists x(O(x) \,\&\, B(x))$ is now forced to be true.



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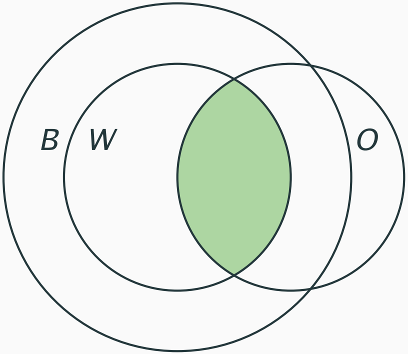

Determining validity

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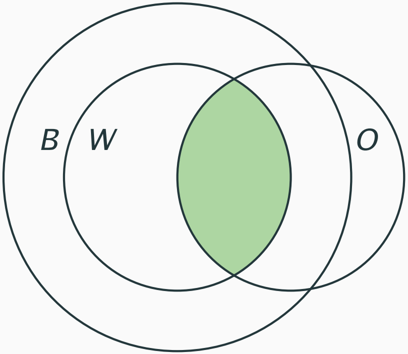
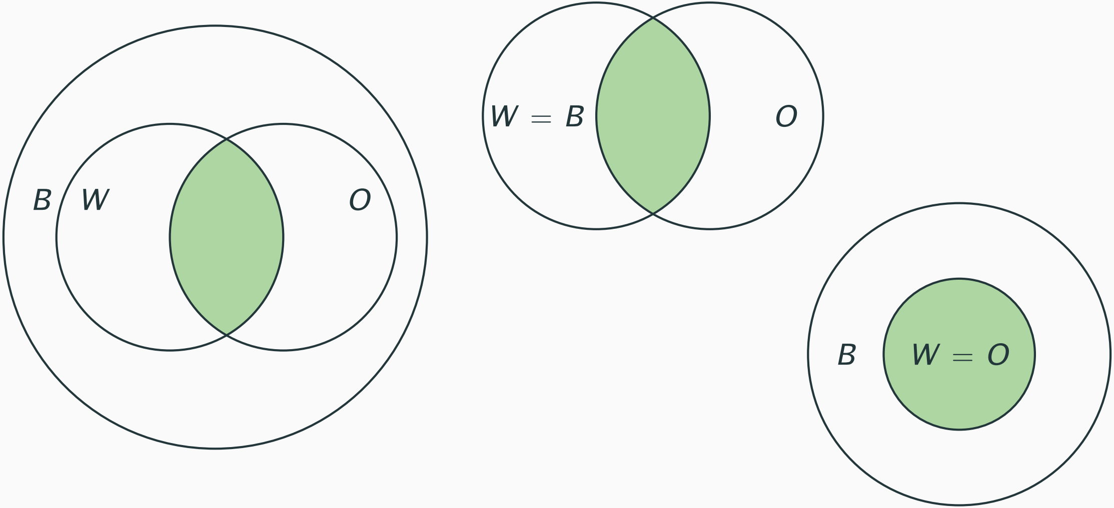
- $\exists x(O(x) \& B(x))$ is now forced to be true.
- So, $\forall x(O(x) \supset \sim B(x))$ is false.
- But those are not the only possibilities!



Other configurations



VI. Semantics of QL

e. Semantic notions in QL

Semantics notions in QL

- $\mathscr{P}_1, \ldots, \mathscr{P}_n \vDash \mathbb{Q}$ if no interpretation makes all of $\mathscr{P}_1, \ldots, \mathscr{P}_n$ true and $\mathbb{Q}$ false.

Semantics notions in QL

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- $\mathscr{P}$ and $\mathbb{Q}$ are **equivalent in QL** if no interpretation makes one true but the other false.
- $\mathscr{P}_1, \ldots, \mathscr{P}_n$ are **jointly satisfiable in QL** if some interpretation makes all of them true at the same time.

Using interpretations

- By providing one suitable interpretation we **can** show that...

Using interpretations

- By providing one suitable interpretation we **can** show that...
  - an argument is **not valid** in QL

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- By providing one suitable interpretation we **can** show that...
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- But we **cannot** show using any number of interpretations that...

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  - some sentences **are not satisfiable** in QL

Examples

- $\forall x(A(x) \vee B(x))$ and $\forall x\, A(x) \vee \forall x\, B(x)$ are not equivalent.

Test solutions on carnap.io

Examples

- $\forall x(A(x) \lor B(x))$ and $\forall x\, A(x) \lor \forall x\, B(x)$ are not equivalent.
- $\forall x(A(x) \supset B(x)), \forall x(A(x) \supset {\sim}B(x))$ are jointly satisfiable.

Test solutions on carnap.io

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- $\forall x({\sim}A(x) \supset B(x)), \exists x(B(x) \mathbin{\&} C(x, b)) \not\models \exists x({\sim}A(x) \mathbin{\&} C(x, b))$.

Test solutions on carnap.io

Examples

- $\forall x(A(x) \vee B(x))$ and $\forall x\, A(x) \vee \forall x\, B(x)$ are not equivalent.
- $\forall x(A(x) \supset B(x)), \forall x(A(x) \supset \sim B(x))$ are jointly satisfiable.
- $\forall x(\sim A(x) \supset B(x)), \exists x(B(x) \,\&\, C(x, b)) \not\models \exists x(\sim A(x) \,\&\, C(x, b))$.
- $\not\models \exists x\, A(a, x) \supset \exists x\, A(x, x)$.

Test solutions on carnap.io

VI. Semantics of QL

f. Arguing about interpretations

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- The informal argument makes use of the **truth conditions** for sentences of QL.

Arguing about Interpretations

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- That's because there is no way to inspect all possible interpretations.
- But we can show that arguments are valid, by:
  - a formal proof (next time)
  - an informal argument
- The informal argument makes use of the **truth conditions** for sentences of QL.
- Analogous to arguing about valuations in SL.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x (\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.

Example

$$\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x) \models \forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$$

- Suppose an interpretation makes premise $\forall x\, \mathscr{A}(x) \lor \forall x\, \mathscr{B}(x)$ true.
- By the truth conditions for $\lor$, it makes either $\forall x\, \mathscr{A}(x)$ or $\forall x\, \mathscr{B}(x)$ true.

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- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
    - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.

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- Suppose it's the first, i.e., $\forall x\,\mathscr{A}(x)$ is true.
    - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
    - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
    - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.

Example

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- Suppose it's the first, i.e., $\forall x\, \mathscr{A}(x)$ is true.
  - By the truth conditions for $\forall$, every object in the domain satisfies $\mathscr{A}(x)$.
  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.
- Suppose it's the second, i.e., $\forall x\, \mathscr{B}(x)$ is true: Similarly.

Example

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  - By the truth conditions for $\lor$, every object satisfies $\mathscr{A}(x) \lor \mathscr{B}(x)$
  - So, by the truth conditions for $\forall$, $\forall x(\mathscr{A}(x) \lor \mathscr{B}(x))$ is true.
- Suppose it's the second, i.e., $\forall x\, \mathscr{B}(x)$ is true: Similarly.
- These are the only possibilities: the interpretation must make the conclusion also true.

VII. Proofs in QL

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a. Rules for ∀

Rules for formal proofs

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Candidates for rules

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Candidates for rules

- Only simple sentence close to $\forall x\, \mathscr{A}(x)$ is $\mathscr{A}(c)$
- Gives simple, elegant $\forall$E rule:

k	$\forall x\, \mathscr{A}(x)$	
	$\mathscr{A}(c)$	: k $\forall$E

- This is a good rule: $\forall x\, \mathscr{A}(x) \models \mathscr{A}(c)$.

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- Diagnosis: the c in $\mathscr{A}(c)$ is a name for a **specific object**.
- We need a name for an **arbitrary, unspecified object**.
- If $\mathscr{A}(c)$ is true for whatever c **could** name, then $\mathscr{A}(x)$ is satisfied by **every** object.

Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

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Names for arbitrary objects

- When we give proofs of general claims, we often do use names for arbitrary objects (well, mathematicians do at least).

  All heroes admire Greta.
  Only people who wear capes admire Greta.
  ∴ All heroes wear capes.

  Proof: Let Carl be any hero. Since all heroes admire Greta, Carl admires Greta. Since only people who wear capes admire Greta, Carl is wears a cape. But "Carl" stands for **any** hero. So all heroes wear capes.

Universal generalization

$$\begin{array}{l|ll} k & \mathscr{A}(c) & \\ & & \\ & \forall x\, \mathscr{A}(x) & :k\ \forall\mathrm{I} \end{array}$$

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 & \vdots & \\
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- c is special: c must not appear in any premise or assumption of a subproof not already ended
- $\mathscr{A}(x)$ is obtained from $\mathscr{A}(c)$ by replacing **all** occurrences of c by x.
- In other words, c must also not occur in $\forall x\, \mathscr{A}(x)$.

General conditional proof

Proving "All As are Bs"

k	$\quad A(c)$	
l	$\quad B(c)$	
$l+1$	$A(c) \supset B(c)$	$: k\text{-}l \supset\mathrm{I}$
$l+2$	$\forall x(A(x) \supset B(x))$	$: l+1\ \forall\mathrm{I}$

Example

All heroes admire Greta.
Only people who wear capes admire Greta.
∴ All heroes wear capes.

$$\begin{array}{rl} & \forall x(H(x) \supset A(x,g)) \\ & \forall x(A(x,g) \supset C(x)) \\ \therefore & \forall x(H(x) \supset C(x)) \end{array}$$

Let's do it on carnap.io

Example

1	$\forall x(H(x) \supset A(x, g))$	
2	$\forall x(A(x, g) \supset C(x))$	
3	$\quad H(c)$	
4	$\quad H(c) \supset A(c, g)$	:1 $\forall$E
5	$\quad A(c, g)$	:4, 3 $\supset$E
6	$\quad A(c, g) \supset C(c)$	:2 $\forall$E
7	$\quad C(c)$	:6, 5 $\supset$E
8	$H(c) \supset C(c)$	:3–7 $\supset$I
9	$\forall x(H(x) \supset C(x))$	:8 $\forall$I

Example

1	$\forall x\, A(x) \vee \forall x\, B(x)$	
2	$\quad \forall x\, A(x)$	
3	$\quad A(c)$	:2 $\forall$E
4	$\quad A(c) \vee B(c)$	:3 $\vee$I
5	$\quad \forall x\, B(x)$	
6	$\quad B(c)$	:5 $\forall$E
7	$\quad A(c) \vee B(c)$	:6 $\vee$I
8	$A(c) \vee B(c)$	:1, 2–4, 5–7 $\vee$E
9	$\forall x (A(x) \vee B(x))$	:8 $\forall$I

VII. Proofs in QL

b. Rules for ∃

Rules for $\exists$

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Arbitrary objects again

- Problem: corresponding "elim rule" isn't valid:

$$\begin{array}{r|l}
k & \exists x\, \mathscr{A}(x) \\
 & \\
 & \mathscr{A}(c) \qquad\qquad : k \textbf{ doesn't follow from}
\end{array}$$

- If we know that $\exists x\, \mathscr{A}(x)$ is true, we know that **some** objects satisfy $\mathscr{A}(x)$, but not which ones.
- To use this information, we have to introduce a temporary name that stands for any one of the objects that satisfy $\mathscr{A}(x)$.

Reasoning from existential information

- To use $\exists x\, \mathscr{A}(x)$, pretend the x has a name c, and reason from $\mathscr{A}(c)$.
- This is what we'd do if we reason informally from existential information, e.g.,

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
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  Proof: We know there are heroes who wear capes. Let Cate be an arbitrary one of them. So Cate wears a cape. Since anyone who wears a cape admires Greta, Cate admires Greta. Since Cate is a hero who admires Greta, some heroes admire Greta.

Existential elimination

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 & \mathscr{B} \qquad\qquad :k,\ m\text{-}n\ \exists\text{E}
\end{array}
$$

- c is special: c must not appear outside subproof

Example

  There are heroes who wear capes.
  Anyone who wears a cape admires Greta.
$\therefore$ Some heroes admire Greta.

$$\exists x(H(x) \,\&\, C(x))$$
$$\forall x(C(x) \supset A(x, g))$$
$$\therefore \exists x(H(x) \,\&\, A(x, g))$$

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2	$\forall x(C(x) \supset A(x, g))$	
3	$H(c) \,\&\, C(c)$	
4	$C(c)$	:3 & E
5	$C(c) \supset A(c, g)$	:2 $\forall$E
6	$A(c, g)$	:4, 5 $\supset$E
7	$H(c)$	:3 & E
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VIII. Multiple quantifiers

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a. Two quantifiers

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- It's true iff **every pair** of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$.
- $\exists x \exists y\, \mathscr{B}(x, y)$ is a sentence.
- It's true iff **at least one pair** of objects α, β stand in the relation expressed by $\mathscr{B}(x, y)$.

Multiple uses of a single quantifier: $\forall$

- $A(x, y)$ . . . x admires y.

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- In other words: everyone admires everyone.

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- So: $\forall x \forall y\, A(x, y)$ does **not** symbolize “everyone admires everyone **else**.”

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- $\exists x \exists y\, A(x, y)$ ... for at least one pair $\langle \alpha, \beta \rangle$, α admires β.

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- Note: includes pairs $\langle \alpha, \alpha \rangle$, i.e.,
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- That means, we could just have one person admiring themselves.
- So: $\exists x \exists y\, A(x, y)$ does **not** symbolize "someone admires someone **else**."

Alternating quantifiers

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   (not necessarily the same person)
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3. $\exists x \forall y\, A(x, y)$
   Someone admires everyone
   (including themselves)
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Alternating quantifiers

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Convergence vs. uniform convergence

- A function f **point-wise continuous** if

$$\forall \epsilon \forall x \forall y \exists \delta (|x - y| < \delta \supset |f(x) - f(y)| < \epsilon)$$

Convergence vs. uniform convergence

- A function f **point-wise continuous** if

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- A function f **uniformly continuous** if

$$\forall\epsilon\exists\delta\forall x\forall y(|x-y|<\delta\supset|f(x)-f(y)|<\epsilon)$$

VIII. Multiple quantifiers

b. Using quantifiers to express properties

Our symbolization key

Domain: people alive in 2022 and items of clothing

a: Autumn

g: Greta

$P(x)$: $\_\_\_\_\_\__x$ is a person

$L(x)$: $\_\_\_\_\_\__x$ is an item of clothing.

$E(x)$: $\_\_\_\_\_\__x$ is a cape

$R(x, y)$: $\_\_\_\_\_\__x$ wears $\_\_\_\_\_\__y$

$H(x)$: $\_\_\_\_\_\__x$ is a hero

$I(x)$: $\_\_\_\_\_\__x$ inspires

$Y(x, y)$: $\_\_\_\_\_\__x$ is younger than $\_\_\_\_\_\__y$

$A(x, y)$: $\_\_\_\_\_\__x$ admires $\_\_\_\_\_\__y$

$O(x, y)$: $\_\_\_\_\_\__x$ owns $\_\_\_\_\_\__y$

Expressing properties, revisited

- One-place predicates express properties, e.g.,

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  $H(x) \,\&\, C(x)$ expresses "x is a hero who wears a cape"

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- Using quantifiers, we can express even more complex properties, e.g.,

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  $A(x, g)$ expresses "x admires Greta"
  $H(x) \& C(x)$ expresses "x is a hero who wears a cape"

- Using quantifiers, we can express even more complex properties, e.g.,

  $\exists y(P(y) \& A(x, y))$ expresses "x admires someone"

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.

$E(x)$: $\_\_\_\_\__{x}$ is a cape
$R(x, y)$: $\_\_\_\_\__{x}$ wears $\_\_\_\_\__{y}$

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  - x admires a hero.

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    $\exists y(H(y) \,\&\, A(g, y))$
  - x admires a hero.
    $\exists y(H(y) \,\&\, A(x, y))$

$E(x)$: $\_\_\_\_\__x$ is a cape
$R(x, y)$: $\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
  - Greta admires a hero.
    $\exists y(H(y) \,\&\, A(g,y))$
  - x admires a hero.
    $\exists y(H(y) \,\&\, A(x,y))$
- If you can say it for x, you can say it for Greta.

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- If you can say it for x, you can say it for Greta.
  - x wears a cape.

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Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
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    $\exists y(H(y) \mathbin{\&} A(g, y))$
  - x admires a hero.
    $\exists y(H(y) \mathbin{\&} A(x, y))$
- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $\exists y(E(y) \mathbin{\&} R(x, y))$

$E(x)$: $\underline{\quad\quad}_x$ is a cape
$R(x, y)$: $\underline{\quad\quad}_x$ wears $\underline{\quad\quad}_y$

Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
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- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $\exists y(E(y) \,\&\, R(x,y))$
  - Greta wears a cape.

$E(x)$: $\_\_\_\__x$ is a cape
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Finding, using properties expressed

- If you can say it for Greta, you can say it for x.
  - Greta admires a hero.
    $\exists y(H(y) \,\&\, A(g, y))$
  - x admires a hero.
    $\exists y(H(y) \,\&\, A(x, y))$
- If you can say it for x, you can say it for Greta.
  - x wears a cape.
    $\exists y(E(y) \,\&\, R(x, y))$
  - Greta wears a cape.
    $\exists y(E(y) \,\&\, R(g, y))$

$E(x)$: $\_\_\_\__x$ is a cape
$R(x, y)$: $\_\_\_\__x$ wears $\_\_\_\__y$

Examples

- x wears a cape.

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \& R(x, y))$

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \,\&\, R(x, y))$
- x is admired by everyone.

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\_\__x$ wears $\_\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \,\&\, R(x, y))$
- x is admired by everyone.
  $\forall y(P(y) \supset A(y, x))$

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- x wears a cape.
  $\exists y(E(y) \,\&\, R(x,y))$
- x is admired by everyone.
  $\forall y(P(y) \supset A(y,x))$
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  $\exists y(H(y) \,\&\, A(x,y))$

$P(x)$	$\_\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\_\__x$ is an item of clothing
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- x wears a cape.
  $\exists y(E(y) \,\&\, R(x,y))$
- x is admired by everyone.
  $\forall y(P(y) \supset A(y,x))$
- x admires a hero.
  $\exists y(H(y) \,\&\, A(x,y))$
- x admires only heroes.

$P(x)$	$\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\__x$ is a cape	$R(x,y)$	$\_\_\_\__x$ wears $\_\_\_\__y$

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- x wears a cape.
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- x is admired by everyone.
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- x admires a hero.
  $\exists y(H(y) \,\&\, A(x, y))$
- x admires only heroes.
  $\forall y(A(x, y) \supset H(y))$
- x is naked.

$P(x)$	$\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\__x$ is a cape	$R(x, y)$	$\_\_\_\__x$ wears $\_\_\_\__y$

Examples

- x wears a cape.
  $\exists y(E(y) \,\&\, R(x,y))$
- x is admired by everyone.
  $\forall y(P(y) \supset A(y,x))$
- x admires a hero.
  $\exists y(H(y) \,\&\, A(x,y))$
- x admires only heroes.
  $\forall y(A(x,y) \supset H(y))$
- x is naked.
  $\sim\exists y(L(y) \,\&\, R(x,y))$
  $\forall y(L(y) \supset \sim R(x,y))$

$P(x)$	$\_\_\_\__x$ is a person	$L(x)$	$\_\_\_\__x$ is an item of clothing
$E(x)$	$\_\_\_\__x$ is a cape	$R(x,y)$	$\_\_\_\__x$ wears $\_\_\_\__y$

VIII. Multiple quantifiers

c. Multiple determiners

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?
- Deal with each determiner separately!

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- Think of determiner phrase as replaced with name or variable—result has one less determiner.

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- When you're down to one determiner, apply known methods for single quantifiers.

Symbolizing multiple determiners

- What if your sentence contains more than one determiner phrase?
- Deal with each determiner separately!
- Think of determiner phrase as replaced with name or variable—result has one less determiner.
- When you're down to one determiner, apply known methods for single quantifiers.
- This results in formulas that express properties or relations, but themselves contain quantifiers.

Two separate determiner phrases

- All heroes wear a cape

Two separate determiner phrases

- All heroes wear a cape
- All heroes satisfy "x wears a cape"

$$\forall x(H(x) \supset \text{“}x \text{ wears a cape”})$$

Two separate determiner phrases

- All heroes wear a cape
- All heroes satisfy "x wears a cape"

$$\forall x(H(x) \supset \text{"}x \text{ wears a cape"})$$

- x wears a cape

$$\exists y(E(y) \,\&\, R(x,y))$$

Two separate determiner phrases

- All heroes wear a cape
- All heroes satisfy “x wears a cape”

$$\forall x(H(x) \supset \text{“}x \text{ wears a cape”})$$

- x wears a cape

$$\exists y(E(y) \mathbin{\&} R(x, y))$$

- Together:

$$\forall x(H(x) \supset \exists y(E(y) \mathbin{\&} R(x, y)))$$

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.
- All things that satisfy "x is a hero who wears a cape" admire Greta.

$$\forall x(\text{“x is a hero who wears a cape”} \supset A(x, g))$$

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- All heroes who wear a cape admire Greta.
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$$\forall x(\text{“x is a hero who wears a cape”} \supset A(x, g))$$

- x is a hero who wears a cape

$$H(x) \& \exists y(E(y) \& R(x, y))$$

Determiner within determiner phrase

- All heroes who wear a cape admire Greta.
- All things that satisfy "x is a hero who wears a cape" admire Greta.

$$\forall x(\text{"x is a hero who wears a cape"} \supset A(x,g))$$

- x is a hero who wears a cape

$$H(x) \mathbin{\&} \exists y(E(y) \mathbin{\&} R(x,y))$$

- Together:

$$\forall x((H(x) \mathbin{\&} \exists y(E(y) \mathbin{\&} R(x,y))) \supset A(x,g))$$

Mary Astell, 1666–1731

- British political philosopher
- *Some Reflections upon Marriage* (1700)
- In preface to 3rd ed. 1706 reacts to William Nicholls' claim (in *The Duty of Inferiors towards their Superiors, in Five Practical Discourses* (London 1701), Discourse IV: The Duty of Wives to their Husbands), that women are naturally inferior to men.

Astell TL;DR

- What can Nicholls possibly mean by "women are naturally inferior to men"?
- It can't be that some women is inferior to some man, since that's "no great discovery."
- After all, surely some men are inferior to some women.
- The obviously intended meaning must be: **all** women are inferior to **all** men.
- But that can't be right, for then "the greatest Queen ought not to command but to obey her Footman."
- It can't even be just: **all** women are inferior to **some** men.
- Since "had they been pleased to remember their Oaths of Allegiance and Supremacy, they might have known that *One* Women is superior to *All* the Men in these Nations."

Symbolizing Astell

- Some woman is superior to every man

Symbolizing Astell

- Some woman is superior to every man
- Some woman satisfies "x is superior to every man"

$$\exists x(W(x)\ \&\ \text{“}x\text{ is superior to every man”})$$

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$$\forall y(M(y) \supset S(x, y))$$

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Formalizing Astell

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- Some woman is superior to some man.

  $\exists x(W(x) \,\&\, \exists y(M(y) \,\&\, S(x,y)))$

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \,\&\, \exists y(M(y) \,\&\, S(x, y)))$
- Every woman is superior to every man.

Formalizing Astell

- Some woman is superior to some man.

  $\exists x(W(x) \& \exists y(M(y) \& S(x, y)))$

- Every woman is superior to every man.

  $\forall x(W(x) \supset \forall y(M(y) \supset S(x, y)))$

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- Some woman is superior to some man.

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- Every woman is superior to some man.

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- Some woman is superior to every man.

  $\exists x(W(x) \& \forall y(M(y) \supset S(x,y)))$

“Any”

- Any (every) cape is worn by a hero.

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$$\forall x(E(x) \supset \exists y(H(y) \,\&\, R(y, x)))$$

"Any"

- Any (every) cape is worn by a hero.

$$\forall x(E(x) \supset \exists y(H(y) \mathbin{\&} R(y,x)))$$

- No hero wears any cape.

$$\forall x(H(x) \supset {\sim}\exists y(E(y) \mathbin{\&} R(x,y)))$$

"Any"

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VIII. Multiple quantifiers

d. Quantifier scope ambiguity

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.
- Autumn admires Isra or Luisa, and so does Greta.

$$(A(a, i) \vee A(a, l))\ \&$$
$$(A(g, i) \vee A(g, l))$$

More scope ambiguity

- Autumn and Greta admire Isra or Luisa.
- Autumn admires Isra or Luisa, and so does Greta.

$$(A(a, i) \vee A(a, l)) \,\&$$
$$(A(g, i) \vee A(g, l))$$

- Autumn and Greta both admire Isra, or they both admire Luisa.

$$(A(a, i) \,\&\, A(g, i)) \vee$$
$$(A(a, l) \,\&\, A(g, l))$$

Negation and the quantifiers

- “All heroes don’t inspire”

Negation and the quantifiers

- “All heroes don’t inspire”
  - Denial of “all heroes inspire”
    (“Do all heroes inspire? No, all heroes don’t inspire”)

$$\sim\forall x(H(x) \supset I(x))$$
$$\exists x(H(x) \mathbin{\&} \sim I(x))$$

Negation and the quantifiers

- "All heroes don't inspire"
  - Denial of "all heroes inspire"
    ("Do all heroes inspire? No, all heroes don't inspire")

$$\sim\forall x(H(x) \supset I(x))$$
$$\exists x(H(x) \,\&\, \sim I(x))$$

  - All heroes are: not inspiring, i.e.,
    No heroes inspire

$$\forall x(H(x) \supset \sim I(x))$$
$$\sim\exists x(H(x) \,\&\, I(x))$$

Multiple quantifiers and ambiguity

- “All heroes wear a cape”

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e., "For every hero, there is a cape they wear"

$$\forall x(H(x) \supset \exists y(E(y) \,\&\, R(x, y)))$$

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e., "For every hero, there is a cape they wear"

$$\forall x(H(x) \supset \exists y(E(y) \mathbin{\&} R(x, y)))$$

  - "All heroes" in scope of "a cape", i.e., "There is a cape which every hero wears"

$$\exists y(E(y) \mathbin{\&} \forall x(H(x) \supset R(x, y)))$$

Multiple quantifiers and ambiguity

- "All heroes wear a cape"
  - "A cape" in the scope of "all heroes", i.e., "For every hero, there is a cape they wear"

$$\forall x(H(x) \supset \exists y(E(y) \mathbin{\&} R(x, y)))$$

  - "All heroes" in scope of "a cape", i.e., "There is a cape which every hero wears"

$$\exists y(E(y) \mathbin{\&} \forall x(H(x) \supset R(x, y)))$$

- Compare the joke: "Every day, a tourist is mugged on the streets of New York. We will interview him tonight."

VIII. Multiple quantifiers

e. Donkey sentences

Happy farmers

"Every farmer who owns a donkey is happy"

- Step-by-step symbolization: "All As are Bs"

Happy farmers

"Every farmer who owns a donkey is happy"

- Step-by-step symbolization: "All *A*s are *B*s"
- x is a farmer who owns a donkey ...

$$F(x) \,\&\, \exists y(D(y) \,\&\, O(x, y))$$

Happy farmers

"Every farmer who owns a donkey is happy"

- Step-by-step symbolization: "All As are Bs"
- x is a farmer who owns a donkey . . .

$$F(x) \,\&\, \exists y(D(y) \,\&\, O(x, y))$$

- Every farmer who owns a donkey is happy

$$\forall x((F(x) \,\&\, \exists y(D(y) \,\&\, O(x, y))) \supset H(x))$$

Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step-by-step symbolization: "All As are Bs"

Unhappy donkeys

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Unhappy donkeys

"Every farmer who owns a donkey beats it"

- Step-by-step symbolization: "All As are Bs"
- x is a farmer who owns a donkey ...

$$F(x) \,\&\, \exists y(D(y) \,\&\, O(x,y))$$

- Every farmer who owns a donkey beats it

$$\forall x((F(x) \,\&\, \exists y(D(y) \,\&\, O(x,y))) \supset B(x,y))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\sim\exists x(F(x) \,\&\, \exists y(D(y) \,\&\, O(x,y) \,\&\, \sim B(x,y)))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

$$\sim\exists x(F(x) \mathbin{\&} \exists y(D(y) \mathbin{\&} O(x,y) \mathbin{\&} \sim B(x,y)))$$

- For every farmer and every donkey they own: the farmer beats the donkey.

$$\forall x \forall y((F(x) \mathbin{\&} (D(y) \mathbin{\&} O(x,y))) \supset B(x,y)))$$

Symbolizing donkey sentences

"Every farmer who owns a donkey beats it"

- When is it false that every farmer who owns a donkey beats it? If there's a farmer who owns a donkey but doesn't beat it. Deny that!

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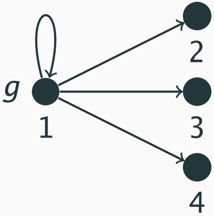
IX. Identity

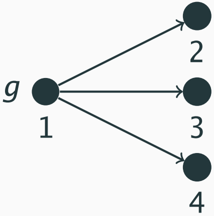
IX. Identity

a. The identity predicate

Greta admires everyone (else)

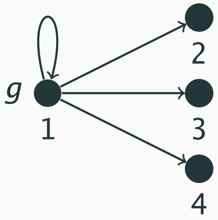


Greta admires everyone.



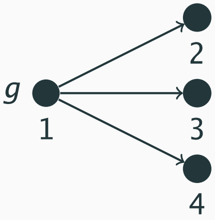
Greta admires everyone **else**.

Greta admires everyone (else)



Greta admires everyone.

$\forall x\, A(g, x)$



Greta admires everyone **else**.

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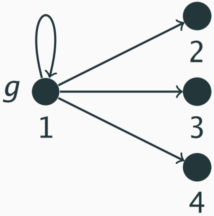
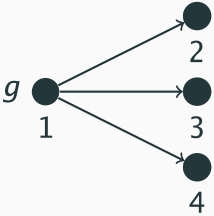


Greta admires everyone.

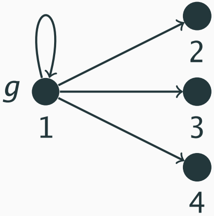
$\forall x\, A(g, x)$



Greta admires everyone **else**.

$\forall x(\text{“}x \text{ is not Greta”} \supset A(g, x))$

Greta admires everyone (else)



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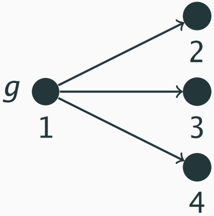

Greta admires everyone **else**.

$\forall x(\text{“}x \text{ is not Greta”} \supset A(g, x))$

$\forall x(\sim x = g \supset A(g, x))$

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Something else/everything else

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- The closest quantifier (typically) determines if you should use $\&$ or $\supset$:

$$\forall x \exists y(\sim x = y \mathbin{\&} A(x, y)) \qquad \exists x \forall y(\sim x = y \supset A(x, y))$$

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Other than, except

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Singular “only”

- “**No-one other than Greta** is a hero”:

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- “**No-one other than Greta** is a hero”:

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  - Or more succinctly: $\exists x \forall y(H(y) \equiv x = y)$

IX. Identity

b. Numerical quantification

Numerical Quantification

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$$\exists x(H(x) \,\&\, I(x))$$

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- At least 2 heroes are inspiring:

$$\exists x \exists y(\sim x = y \,\&\, ((H(x) \,\&\, I(x)) \,\&\, (H(y) \,\&\, I(y))))$$

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$$\exists x \exists y(\sim x = y \,\&\, ((H(x) \,\&\, I(x)) \,\&\, (H(y) \,\&\, I(y))))$$

- At least 3 heroes are inspiring:

$$\begin{aligned}\exists x \exists y \exists z((&\sim x = y \,\&\, (\sim y = z \,\&\, \sim x = z)) \,\&\, \\ &((H(x) \,\&\, I(x)) \,\&\, ((H(y) \,\&\, I(y)) \,\&\, (H(z) \,\&\, I(z)))))\end{aligned}$$

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- There are at least n As ("$\exists^{\geq n} x\, A(x)$")::

$$\exists x_1 \ldots \exists x_n$$

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(A(x_1) \mathbin{\&} (A(x_2) \mathbin{\&} \ldots \mathbin{\&} A(x_n)) \ldots))
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At least n

- Note: must state that **every pair** of variables is different, e.g.,

$$\exists x_1 \exists x_2 \exists x_3 ((\sim x_1 = x_2 \mathbin{\&} \sim x_2 = x_3) \mathbin{\&} \\ (H(x_1) \mathbin{\&} (H(x_2) \mathbin{\&} H(x_3))))$$

only says "There are at least two heroes"!

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  - Then 1 can play role of x_1 and x_3, 2 role of x_2.
  - Both "$\sim 1 = 2$" and "$\sim 2 = 3$" are true.
- At least n Bs are Cs: take $B(x) \,\&\, C(x)$ for $A(x)$:

$$\exists^{\geq n} x (B(x) \,\&\, C(x))$$

Exactly one

- There is exactly one hero:

$$\exists x(H(x) \,\&\, \sim\exists y(H(y) \,\&\, \sim x = y))$$

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- In general: "x has property A **uniquely**":

$$A(x) \,\&\, \forall y(A(y) \supset x = y)$$

or just: $$\forall y(A(y) \equiv x = y)$$

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- Exactly n Bs are Cs:

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At most n

- There are **at most n** As $\Leftrightarrow$ There are **not at least $n+1$** As

$$\exists^{\leq n} x\, A(x) \Leftrightarrow {\sim}\exists^{\geq (n+1)} x\, A(x)$$

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- For instance: There are at most two heroes:

$${\sim}\exists x \exists y \exists z ((H(x) \,\&\, (H(y) \,\&\, H(z))) \,\&\, ({\sim}x = y \,\&\, ({\sim}x = z \,\&\, {\sim}y = z)))$$

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- $\sim\exists^{\geq(n+1)} x\, A(x)$ is equivalent to:

$$\begin{aligned} &\forall x_1 \ldots \forall x_{n+1}((A(x_1) \,\&\, \ldots \,\&\, A(x_{n+1})) \supset \\ &\quad (x_1 = x_2 \lor (x_1 = x_3 \lor \cdots \lor (x_1 = x_{n+1} \lor \\ &\qquad (x_2 = x_3 \lor \cdots \lor (x_2 = x_{n+1} \lor \\ &\qquad\qquad \ddots \\ &\qquad\qquad x_n = x_{n+1}) \ldots))) \end{aligned}$$

IX. Identity

c. “The”, “both”, “neither”

Definite descriptions

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Definite descriptions

- Definite description: **the so-and-so**
- Russell's analysis of definite description: to say

  "The A is B"

  is to say:
- There is something, which:
  - is A,
  - is the only A,
  - is B.
- In QL:

$$\exists x(A(x) \,\&\, \forall y(A(y) \supset x = y) \,\&\, B(x))$$

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$$\exists x(A(x) \,\&\, \forall y(A(y) \supset x = y) \,\&\, B(x))$$

- or more succinctly:

$$\exists x(\forall y(A(y) \equiv x = y) \,\&\, B(x))$$

“The” vs. “exactly one”

- Compare:

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- Compare:
  1. The hero inspires:

$$\exists x(H(x) \,\&\, \forall y(H(y) \supset x = y) \,\&\, I(x))$$

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- Compare:
  1. The hero inspires:

$$\exists x(H(x) \mathbin{\&} \forall y(H(y) \supset x = y) \mathbin{\&} I(x))$$

  2. There is exactly one inspiring hero:

$$\exists x(H(x) \mathbin{\&} \forall y((H(y) \mathbin{\&} I(y)) \supset x = y) \mathbin{\&} I(x))$$

“The” vs. “exactly one”

- Compare:
  1. The hero inspires:
  $$\exists x(H(x) \,\&\, \forall y(H(y) \supset x = y) \,\&\, I(x))$$
  2. There is exactly one inspiring hero:
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- (2) can be true without (1), but not vice versa.

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- (Namely when there is exactly one inspiring hero, but also a non-inspiring hero.)

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- (2) can be true without (1), but not vice versa.
- (Namely when there is exactly one inspiring hero, but also a non-inspiring hero.)
- So (1) entails (2), but not vice versa.

Strawson's analysis

- According to Russell, "The hero wears a cape" is false if there is no hero, or if there is more than one.
- P. F. Strawson disagrees: we only succeed in making a statement if there is a unique hero.
- "There is a unique hero" is not part of what is **said**, but is only **presupposed**.

Singular possessive

- Singular possessives make noun phrases, e.g., “Joe’s cape”

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$$\exists x[$$

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$$\exists x[((E(x) \& O(j,x)) \& \\ \forall y((E(y) \& O(j,y)) \supset x = y)) \& \\ W(a,x)]$$

Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:

Singular vs. plural possessive

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Singular vs. plural possessive

- Compare **plural** possessives: those are $\forall$'s:
  - "Autumn wears Joe's cape**s**" symbolizes the same as:
    "Autumn wears every cape that Joe owns":

$$\forall x[(E(x) \mathbin{\&} O(j,x)) \supset W(a,x)]$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\exists x \exists y$$

Both

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$$\exists x \exists y [((\sim x = y \, \& \, (H(x) \, \& \, H(y)))$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\exists x \exists y[((\sim x = y \,\&\, (H(x) \,\&\, H(y))) \,\&$$
$$\forall z(H(z) \supset (z = x \vee z = y)))$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\begin{aligned}\exists x \exists y[((\sim x = y \,\&\, (H(x) \,\&\, H(y))) \,\&\\ \forall z(H(z) \supset (z = x \vee z = y))) \,\&\\ (I(x) \,\&\, I(y))]\end{aligned}$$

Both

- "Both heroes inspire": There are exactly 2 heroes, and both inspire:

$$\begin{aligned} \exists x \exists y[((\sim x = y \mathbin{\&} (H(x) \mathbin{\&} H(y))) \mathbin{\&} \\ \forall z(H(z) \supset (z = x \lor z = y))) \mathbin{\&} \\ (I(x) \mathbin{\&} I(y))] \end{aligned}$$

- Note: "Both heroes inspire" implies "There are exactly two inspiring heroes", but not vice versa!

Neither

- "Neither hero inspires": There are exactly 2 heroes, and neither of them inspires:

$$
\begin{aligned}
\exists x \exists y [((\sim x = y \,\&\, (H(x) \,\&\, H(y))) \,\& \\
\forall z (H(z) \supset (z = x \vee z = y))) \,\& \\
(\sim I(x) \,\&\, \sim I(y))]
\end{aligned}
$$

X. Proofs for full QL

X. Proofs for full QL

a. Proofs with multiple quantifiers

Eliminating $\forall$

$$
\begin{array}{r|ll}
m & \forall x \mathscr{A}(\ldots x \ldots c \ldots) & \\
 & \mathscr{A}(\ldots c \ldots c \ldots) & : m \ \forall\mathrm{E}
\end{array}
$$

- No restriction on c:
  - May be in an assumption.

Eliminating $\forall$

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\begin{array}{r|ll}
m & \forall x \mathscr{A}(\ldots x \ldots c \ldots) & \\
 & \mathscr{A}(\ldots c \ldots c \ldots) & : m\ \forall\text{E}
\end{array}
$$

- No restriction on c:
  - May be in an assumption.
  - May also be in $\forall x \mathscr{A}(\ldots x \ldots x \ldots)$ already!

Working forward from $\forall$

- If you **have** $\forall x \mathscr{A}(x)$, replace every x by the same c.

Working forward from $\forall$

- If you **have** $\forall x \mathscr{A}(x)$, replace every x by the same c.
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- If you **have** $\forall x \mathscr{A}(x)$, replace every x by the same c.
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Working forward from $\forall$

- If you **have** $\forall x \mathscr{A}(x)$, replace every x by the same c.
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- You can pick any c.
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- You may need to try multiple candidates.

Introducing $\forall$

$$
\begin{array}{l|ll}
m & \mathscr{A}(\ldots c \ldots c \ldots) & \\
 & \forall x \mathscr{A}(\ldots x \ldots x \ldots) & :m\ \forall\mathrm{I}
\end{array}
$$

- Restrictions on c:

Introducing $\forall$

$$
\begin{array}{r|ll}
m & \mathscr{A}(\ldots c \ldots c \ldots) & \\
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\end{array}
$$

- Restrictions on c:
  - must not occur in any undischarged assumption.

Introducing $\forall$

$$\begin{array}{r|l l}
m & \mathscr{A}(\ldots c \ldots c \ldots) & \\
 & & \\
 & \forall x \mathscr{A}(\ldots x \ldots x \ldots) & : m \ \forall\mathrm{I}
\end{array}$$

- Restrictions on c:
  - must not occur in any undischarged assumption.
  - must not occur in $\forall x \mathscr{A}(\ldots x \ldots x)$.

Working backward from $\forall$

- If you **want** $\forall x \mathscr{A}(x)$, replace every x by the same c.

Working backward from $\forall$

- If you **want** $\forall x \mathscr{A}(x)$, replace every x by the same c.
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- As long as c is fresh, this will work if you can prove $\forall x \mathscr{A}(x)$ at all.

Introducing $\exists$

$$
\begin{array}{r|l}
m & \mathscr{A}(\ldots c \ldots c \ldots) \\
\\
& \exists x \mathscr{A}(\ldots x \ldots c \ldots) \qquad : m\ \exists\mathrm{I}
\end{array}
$$

- No restriction on c:

Introducing $\exists$

$$\begin{array}{r|l}
m & \mathscr{A}(\ldots c \ldots c \ldots) \\
 & \exists x \mathscr{A}(\ldots x \ldots c \ldots) \qquad : m\ \exists\mathrm{I}
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- No restriction on c:
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m & \mathscr{A}(\ldots c \ldots c \ldots) \\
 & \\
 & \exists x \mathscr{A}(\ldots x \ldots c \ldots) \qquad : m \ \exists\mathrm{I}
\end{array}
$$

- No restriction on c:
  - May be in an assumption.
  - May also be in $\exists x \mathscr{A}(\ldots x \ldots c \ldots)$.

Introducing $\exists$

$$\begin{array}{r|ll} m & \mathscr{A}(\ldots c \ldots c \ldots) & \\ & & \\ & \exists x \mathscr{A}(\ldots x \ldots c \ldots) & : m\ \exists\mathrm{I} \end{array}$$

- No restriction on c:
  - May be in an assumption.
  - May also be in $\exists x \mathscr{A}(\ldots x \ldots c \ldots)$.
  - So you can also justify $\exists x \mathscr{A}(\ldots x \ldots x \ldots)$ or $\exists x \mathscr{A}(\ldots c \ldots x \ldots)$.

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- Good candidates: c which occur in assumptions or in the sentences you're trying to prove.

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- Try to prove this **above** $\exists x \mathscr{A}(x)$
- Justify $\exists x \mathscr{A}(x)$ by $\exists I$.
- You can pick any c.
- Good candidates: c which occur in assumptions or in the sentences you're trying to prove.
- That includes $\exists x \mathscr{A}(x)$!

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- You may need to try multiple candidates.

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- This may not work (especially at the beginning, or if you need IP)!

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- Justify $\exists x \mathscr{A}(x)$ by $\exists I$.
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- That includes $\exists x \mathscr{A}(x)$!
- You may need to try multiple candidates.
- This may not work (especially at the beginning, or if you need IP)!
- Try other strategies first, especially strategies that put more c into play ($\exists E$).

Eliminating $\exists$

$$
\begin{array}{l|l}
m & \exists x \mathscr{A}(\ldots x \ldots x \ldots) \\
i & \quad \mathscr{A}(\ldots c \ldots c \ldots) \\
j & \quad \mathscr{B} \\
 & \mathscr{B} \qquad\qquad :m,\ i\text{–}j\ \exists\text{E}
\end{array}
$$

- Restrictions on c:

Eliminating $\exists$

$$
\begin{array}{r|l}
m & \exists x \mathscr{A}(\ldots x \ldots x \ldots) \\
i & \quad \begin{array}{|l} \mathscr{A}(\ldots c \ldots c \ldots) \\ \hline \end{array} \\
j & \quad \begin{array}{|l} \mathscr{B} \end{array} \\
 & \mathscr{B} \qquad\qquad\qquad\qquad :m,\ i\text{–}j\ \exists\text{E}
\end{array}
$$

- Restrictions on c:
  - must not occur in any assumption still open when you apply $\exists E$.

Eliminating $\exists$

$$
\begin{array}{l|l}
m & \exists x \mathscr{A}(\ldots x \ldots x \ldots) \\
i & \quad \mid \mathscr{A}(\ldots c \ldots c \ldots) \\
 & \quad \mid \overline{\phantom{\mathscr{A}(\ldots c \ldots c \ldots)}} \\
j & \quad \mid \mathscr{B} \\
 & \mathscr{B} \qquad\qquad\qquad\qquad :m,\ i\text{–}j\ \exists\mathrm{E}
\end{array}
$$

- Restrictions on c:
  - must not occur in any assumption still open when you apply $\exists E$.
  - must not occur in $\exists x \mathscr{A}(\ldots x \ldots x)$.

Eliminating ∃

$$
\begin{array}{l|l}
m & \exists x \mathscr{A}(\ldots x \ldots x \ldots) \\
i & \quad \mathscr{A}(\ldots c \ldots c \ldots) \\
j & \quad \mathscr{B} \\
 & \mathscr{B} \qquad\qquad\qquad\qquad : m,\ i\text{–}j\ \exists\text{E}
\end{array}
$$

- Restrictions on c:
  - must not occur in any assumption still open when you apply $\exists E$.
  - must not occur in $\exists x \mathscr{A}(\ldots x \ldots x)$.
  - must not occur in $\mathscr{B}$.

Working forward from $\exists$

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Working forward from $\exists$

- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
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- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.

Working forward from $\exists$

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  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.
  - Prove $\mathscr{B}$ on its last line

Working forward from $\exists$

- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.
  - Prove $\mathscr{B}$ on its last line
  - Justify $\mathscr{B}$ after the subproof using $\exists E$.

Working forward from $\exists$

- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.
  - Prove $\mathscr{B}$ on its last line
  - Justify $\mathscr{B}$ after the subproof using $\exists E$.
- You must pick a **new** c not already in the proof constructed so far.

Working forward from $\exists$

- If you **have** $\exists x \mathscr{A}(x)$, and you **want** $\mathscr{B}$:
  - replace every x by the same c.
  - The result is $\mathscr{A}(c)$.
  - Start a subproof with this.
  - Prove $\mathscr{B}$ on its last line
  - Justify $\mathscr{B}$ after the subproof using $\exists E$.
- You must pick a **new** c not already in the proof constructed so far.
- As long as c is fresh, this will work if you can prove $\mathscr{B}$ from $\exists x \mathscr{A}(x)$ at all.

Admirers and admired

Someone is admired by everyone

$\therefore$ Everyone admires someone

$$\begin{array}{|l} \exists y \forall x\, A(x, y) \\ \hline \forall x \exists y\, A(x, y) \end{array}$$

Let's do it on carnap.io

Admirers and admired

1	$\exists y \forall x\, A(x, y)$	
2	$\quad \forall x\, A(x, c)$	
3	$\quad A(d, c)$	:2 $\forall$E
4	$\quad \exists y\, A(d, y)$	:3 $\exists$I
5	$\quad \forall x \exists y\, A(x, y)$	:4 $\forall$I
6	$\forall x \exists y\, A(x, y)$	:1, 2-5 $\exists$E

All hail Queen Anne

Some woman is superior to every man.
$\therefore$ Every man is inferior to some woman.

$$\begin{array}{|l} \exists y(W(y) \mathbin{\&} \forall x(M(x) \supset S(y,x))) \\ \hline \forall x(M(x) \supset \exists y(W(y) \mathbin{\&} S(y,x))) \end{array}$$

1	$\exists y(W(y) \,\&\, \forall x(M(x) \supset S(y,x)))$	
2	$\quad W(c) \,\&\, \forall x(M(x) \supset S(c,x))$	
3	$\quad\quad M(d)$	
4	$\quad\quad \forall x(M(x) \supset S(c,x))$	:2 & E
5	$\quad\quad M(d) \supset S(c,d))$	
6	$\quad\quad S(c,d)$	
7	$\quad\quad W(c)$	:2 & E
8	$\quad\quad W(c) \,\&\, S(c,d)$	:6, 7 & I
9	$\quad\quad \exists y(W(y) \,\&\, S(y,x))$	:8 ∃I
10	$\quad M(d) \supset \exists y(W(y) \,\&\, S(y,d))$	
11	$\quad \forall x(M(x) \supset \exists y(W(y) \,\&\, S(y,x)))$	:10 ∀I
12	$\forall x(M(x) \supset \exists y(W(y) \,\&\, S(y,x)))$	:1, 2-11 ∃E

X.a.12

X. Proofs for full QL

b. Proofs with identity

Everybody loves my baby

Everybody loves my baby.
But my baby don't love nobody but me.
$\therefore$ My baby is me.

$$\forall x\, L(x, b)$$

$$\forall x(L(b, x) \supset x = i)$$

$$\overline{}$$

$$b = i$$

- "Everybody Loves my Baby" on YouTube

My baby is me

1	$\forall x\, L(x, b)$	
2	$\forall x(L(b, x) \supset x = i)$	
3	$L(b, b)$	:1 $\forall$E
4	$L(b, b) \supset b = i$	:2 $\forall$E
5	$b = i$	:3, 4 $\supset$E

I am my baby

1	$\forall x\, L(x, b)$	
2	$\forall x (L(b, x) \supset x = i)$	
3	$L(b, b)$	:1 $\forall$E
4	$L(b, b) \supset b = i$	:2 $\forall$E
5	$i = b$	?

Proofs with identity

	$\mathscr{c} = \mathscr{c}$	=I

m	$\mathscr{a} = \mathscr{b}$	
n	$\mathscr{A}(\ldots \mathscr{a} \ldots \mathscr{a} \ldots)$	
	$\mathscr{A}(\ldots \mathscr{b} \ldots \mathscr{a} \ldots)$	: m, n =E

m	$\mathscr{a} = \mathscr{b}$	
n	$\mathscr{A}(\ldots \mathscr{b} \ldots \mathscr{b} \ldots)$	
	$\mathscr{A}(\ldots \mathscr{a} \ldots \mathscr{b} \ldots)$	: m, n =E

I am my baby

- We symbolized "My baby is me" as $b = i$.
- But it's equivalent to "I am my baby," $i = b$.
- $=$I and $=$E let us prove this:

1	$b = i$	
2	$b = b$	=I
3	$i = b$	:1, 2 =E

Different properties, different things

- Two names d, e may name the same thing.

Different properties, different things

- Two names d, e may name the same thing.
- In that case, $d = e$ would be true.

Different properties, different things

- Two names d, e may name the same thing.
- In that case, $d = e$ would be true.
- And then anything that's true about d is also true about e.

Different properties, different things

- Two names d, e may name the same thing.
- In that case, $d = e$ would be true.
- And then anything that's true about d is also true about e.
- In other words:

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- In other words:

$$P(d), \sim P(e) \models \sim d = e$$

Different properties, different things

1	$P(d)$	
2	$\sim P(e)$	
3	$\quad d = e$	
4	$\quad P(e)$	:1, 3 =E
5	$\quad \bot$	:2, 4 ∼E
6	$\sim d = e$	:2–5 ∼I

Uniqueness, again

The two symbolizations of “there is exactly one hero” are equivalent:

$$\exists x(H(x) \,\&\, \forall y(H(y) \supset x = y))$$

$$\exists x \forall y(H(y) \equiv x = y)$$

Uniqueness, again

1	$\exists x(H(x) \,\&\, \forall y(H(y) \supset x = y))$	
2	$\quad H(a) \,\&\, \forall y(H(y) \supset a = y)$	
3	$\quad\quad H(c)$	
4	$\quad\quad \forall y(H(y) \supset a = y)$	:2 & E
5	$\quad\quad H(c) \supset a = c$	:4 $\forall$E
6	$\quad\quad a = c$	:3, 5 $\supset$E
7	$\quad\quad a = c$	
8	$\quad\quad H(a)$	:2 & E
9	$\quad\quad H(c)$	:7, 8 =E
10	$\quad H(c) \equiv a = c$	:3-6, 7-9 $\equiv$I
11	$\quad \forall y(H(y) \equiv a = y)$	:10 $\forall$I
12	$\quad \exists x \forall y(H(y) \equiv x = y)$	:11 $\exists$I
13	$\exists x \forall y(H(y) \equiv x = y)$	:1, 2-12 $\exists$E

Uniqueness, again

1	$\exists x \forall y (H(y) \equiv x = y)$	
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3	$H(a) \equiv a = a$	:2 $\forall$E
4	$a = a$	$=$I
5	$H(a)$	:3, 4 $\equiv$E
6	$H(c)$	
7	$H(c) \equiv a = c$	:2 $\forall$E
8	$a = c$	:6, 7 $\equiv$E
9	$H(c) \supset a = c$	:6-8 $\supset$I
10	$\forall y (H(y) \supset a = y)$	:9 $\forall$I
11	$H(a) \,\&\, \forall y (H(y) \supset a = y)$	:5, 10 & I
12	$\exists x (H(x) \,\&\, \forall y (H(y) \supset x = y))$	:11 $\exists$I
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Proofs with numerical claims

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Proofs with numerical claims

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8	$a = c$	:6, 7 $\supset$E
9	$P(c) \supset a = c$	:4-8 $\supset$I
10	$\forall y(P(y) \supset a = y)$	
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12	$\exists x(P(x) \,\&\, \forall y(P(y) \supset x = y))$	:11 $\exists$I
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XI. Interpretations for full QL

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a. Interpretations and truth, revisited

Interpretations

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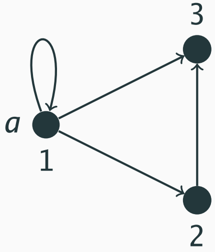
  - **Extension** of each 2-place predicate symbol: which pairs of objects standing in the relation
  - Extension of $=$ is all pairs $\langle \alpha, \alpha \rangle$.

Extensions of predicates

Domain: 1, 2, 3

a: 1

$A(x, y)$: $\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 3\rangle$



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- $\mathscr{A} \mathbin{\&} \mathscr{B}$ is true iff both $\mathscr{A}$, $\mathscr{B}$ are true
- $\mathscr{A} \supset \mathscr{B}$ is true iff $\mathscr{A}$ is false or $\mathscr{B}$ is true

Satisfaction of formulas

- Suppose $\mathscr{A}(x)$ has only the variable x free.

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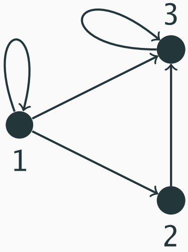
- $\exists x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is satisfied by **at least one** object in the domain.
- $\forall x\, \mathscr{A}(x)$ is true iff $\mathscr{A}(x)$ is satisfied by **every** object in the domain

Satisfaction

- 1 satisfies $\exists y\, A(x, y)$...



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- 1 satisfies $\exists y\, A(x, y)$...
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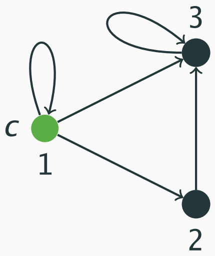

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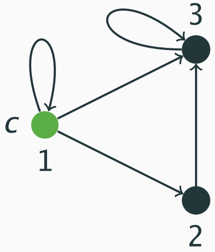

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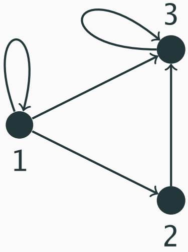

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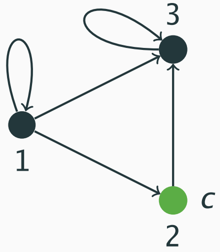

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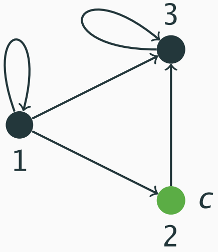

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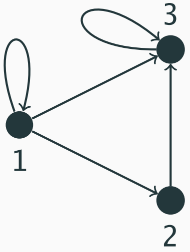

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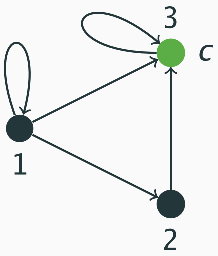

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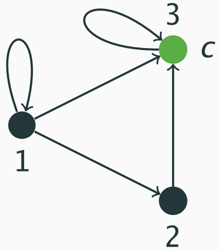

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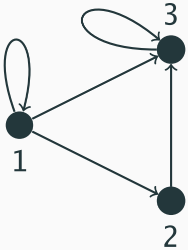

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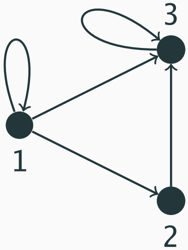
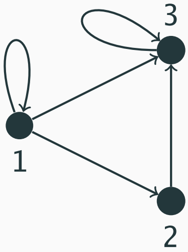
- So every object satisfies $\exists y\, A(x, y)$
- So $\forall x \exists y\, A(x, y)$ is true.



Satisfaction with $=$

- 1 satisfies $\exists y\,(\sim x = y \,\&\, A(x, y))\ldots$



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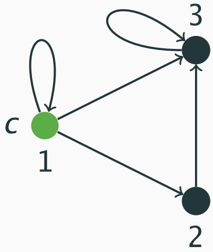

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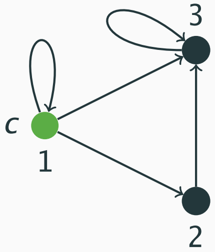

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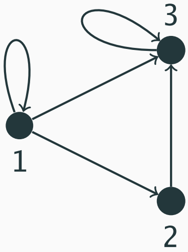

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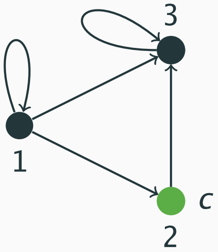

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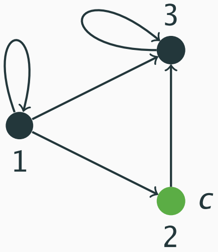

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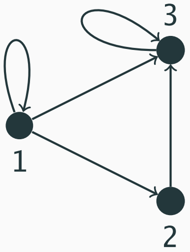

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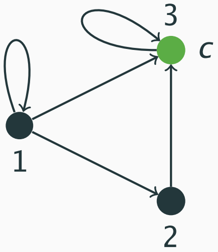

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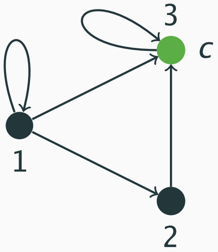
- 3 **doesn't** satisfy $\exists y\,(\sim x = y \,\&\, A(x, y))\ldots$
  - because $\exists y\,(\sim c = y \,\&\, A(c, y))$ is false...
  - because nothing satisfies $\sim c = y \,\&\, A(c, y)$.



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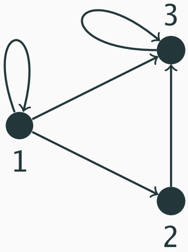
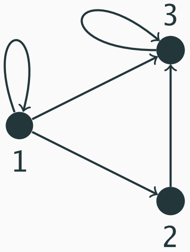
  - because $\exists y\,(\sim c = y \,\&\, A(c,y))$ is true ...
  - because 3 satisfies $\sim c = y \,\&\, A(c,y)$.
- 3 **doesn't** satisfy $\exists y\,(\sim x = y \,\&\, A(x,y))$...
  - because $\exists y\,(\sim c = y \,\&\, A(c,y))$ is false...
  - because nothing satisfies $\sim c = y \,\&\, A(c,y)$.
- So **not** every object satisfies $\exists y\,(\sim x = y \,\&\, A(x,y))$



Satisfaction with $=$

- 1 satisfies $\exists y\,(\sim x = y \,\&\, A(x,y))\ldots$
  - because $\exists y\,(\sim c = y \,\&\, A(c,y))$ is true...
  - because 2 (and 3) satisfies $\sim c = y \,\&\, A(c,y)$.
- 2 satisfies $\exists y\,(\sim x = y \,\&\, A(x,y))\ldots$
  - because $\exists y\,(\sim c = y \,\&\, A(c,y))$ is true ...
  - because 3 satisfies $\sim c = y \,\&\, A(c,y)$.
- 3 **doesn't** satisfy $\exists y\,(\sim x = y \,\&\, A(x,y))\ldots$
  - because $\exists y\,(\sim c = y \,\&\, A(c,y))$ is false...
  - because nothing satisfies $\sim c = y \,\&\, A(c,y)$.
- So **not** every object satisfies $\exists y\,(\sim x = y \,\&\, A(x,y))$
- So $\forall x \exists y\,(\sim x = y \,\&\, A(x,y))$ is **false**.



XI. Interpretations for full QL

b. Constructing (counter)examples

Counterexamples

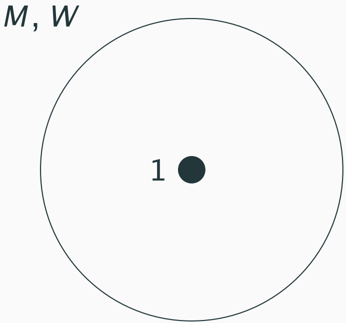
$$\begin{aligned} & \exists x\, M(x), \exists y\, W(y), \\ & \forall x(M(x) \supset \exists y(W(y) \mathbin{\&} S(y,x))) \\ \not\models\quad & \exists y(W(y) \mathbin{\&} \forall x(M(x) \supset S(y,x))) \end{aligned}$$

$M = W = \varnothing$

1 ●

Counterexamples

$$\exists x\, M(x), \exists y\, W(y),$$
$$\forall x (M(x) \supset \exists y (W(y) \,\&\, S(y,x)))$$
$$\not\models \quad \exists y (W(y) \,\&\, \forall x (M(x) \supset S(y,x)))$$



Counterexamples

$$\exists x\, M(x), \exists y\, W(y),$$
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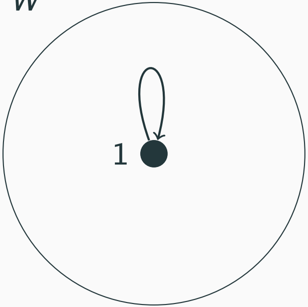
$$\not\models \quad \exists y(W(y)\,\&\,\forall x(M(x) \supset S(y,x)))$$

M, W



Counterexamples

$$\exists x\, M(x), \exists y\, W(y),$$
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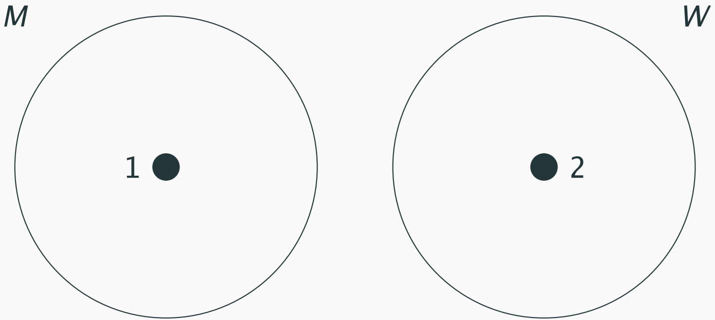

Counterexamples

$$\exists x\, M(x), \exists y\, W(y),$$
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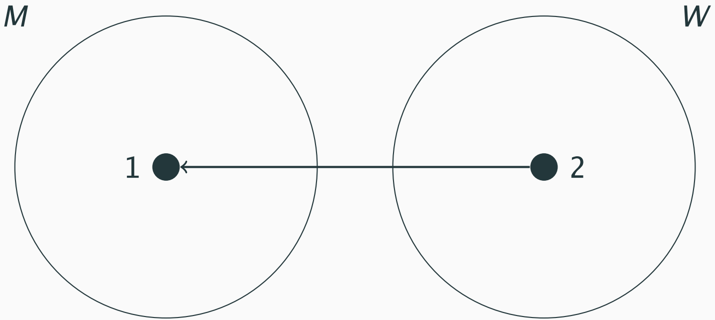

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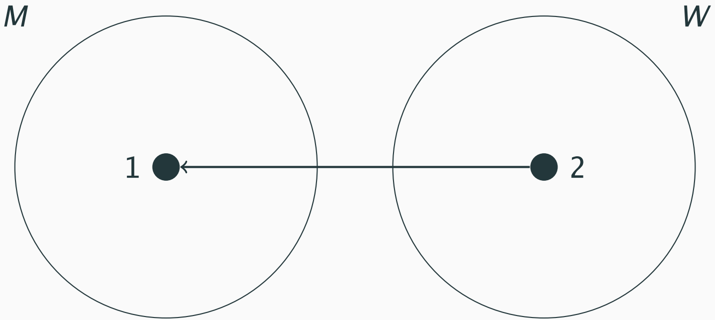

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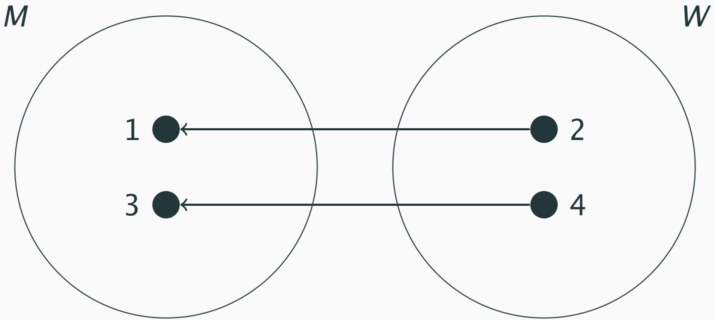

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \,\&\, A(x, y))$ true.

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet?

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \;\&\; A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now?

1

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
- Add another object instead.

Constructing an interpretation

- We want to make $\forall x \exists y (\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
- Add another object instead.
- Add an arrow. How about now?

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
- Add another object instead.
- Add an arrow. How about now?
  - 1 satisfies $\exists y\, (\sim x = y \,\&\, A(x, y))$.

Constructing an interpretation

- We want to make $\forall x \exists y\, (\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
- Add another object instead.
- Add an arrow. How about now?
    - 1 satisfies $\exists y\, (\sim x = y \,\&\, A(x, y))$.
    - But 2 does not.

Constructing an interpretation

- We want to make $\forall x \exists y\,(\sim x = y \,\&\, A(x, y))$ true.
- Start with one object. Is it true yet? No.
- Add an arrow. How about now? Still no.
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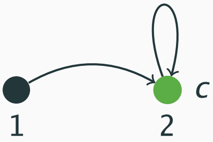

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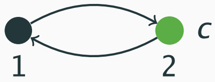
- Add an arrow. How about now? Still no.
- Add another object instead.
- Add an arrow. How about now?
  - 1 satisfies $\exists y\,(\sim x = y \,\&\, A(x, y))$.
  - But 2 does not.
- Add another arrow.
- Ok, a different arrow then.



Constructing an interpretation

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- Add an arrow. How about now?
  - 1 satisfies $\exists y\,(\sim x = y \,\&\, A(x, y))$.
  - But 2 does not.
- Add another arrow.
- Ok, a different arrow then.
- Now 2 satisfies $\exists y\,(\sim x = y \,\&\, A(x, y))$ also.

Constructing an interpretation

- We want to make $\forall x \exists y\,(\sim x = y \,\&\, A(x,y))$ true.
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  - But 2 does not.
- Add another arrow.
- Ok, a different arrow then.
- Now 2 satisfies $\exists y\,(\sim x = y \,\&\, A(x,y))$ also.
- So $\forall x \exists y\,(\sim x = y \,\&\, A(x,y))$ is true.

Constructing another interpretation

- Let's make $\exists y \forall x\, (\sim x = y \supset A(x, y))$ true.

Constructing another interpretation

- Let's make $\exists y \forall x\, (\sim x = y \supset A(x, y))$ true.
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Constructing another interpretation

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- Start with one object. Is it true yet? Yes!
- Add another object. Still true?

Constructing another interpretation

- Let's make $\exists y \forall x\, (\sim x = y \supset A(x, y))$ true.
- Start with one object. Is it true yet? Yes!
- Add another object. Still true? No!
  - 1 does not satisfy $\forall x\, (\sim x = y \supset A(x, y))$ ...

Constructing another interpretation

- Let's make $\exists y \forall x\,(\sim x = y \supset A(x, y))$ true.
- Start with one object. Is it true yet? Yes!
- Add another object. Still true?
  - 1 does not satisfy $\forall x\,(\sim x = y \supset A(x, y))$ ...
  - because $\forall x\,(\sim x = c \supset A(x, c))$ is false...

Constructing another interpretation

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- Start with one object. Is it true yet? Yes!
- Add another object. Still true?
  - 1 does not satisfy $\forall x\, (\sim x = y \supset A(x, y))$ . . .
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  - because $\forall x\, (\sim x = c \supset A(x, c))$ is false...
  - because 2 does not satisfy $\sim x = c \supset A(x, c)$
  - 2 also doesn't satisfy $\forall x\, (\sim x = y \supset A(x, y))$.

Constructing another interpretation

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- Add an arrow. How about now?
  - 2 satisfies $\forall x\, (\sim x = y \supset A(x, y))$ ...
  - because $\forall x\, (\sim x = c \supset A(x, c))$ is true...
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- Add an arrow. How about now?
  - 2 satisfies $\forall x\,(\sim x = y \supset A(x, y))$ ...
  - because $\forall x\,(\sim x = c \supset A(x, c))$ is true...
  - because 1 satisfies $\sim x = c \supset A(x, c)$,
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  - because 2 does not satisfy $\sim x = c \supset A(x, c)$
  - 2 also doesn't satisfy $\forall x\,(\sim x = y \supset A(x, y))$.
- Add an arrow. How about now?
  - 2 satisfies $\forall x\,(\sim x = y \supset A(x, y))$ ...
  - because $\forall x\,(\sim x = c \supset A(x, c))$ is true...
  - because 1 satisfies $\sim x = c \supset A(x, c)$,
  - and 2 satisfies $\sim x = c \supset A(x, c)$.
- So $\exists y \forall x\,(\sim x = y \supset A(x, y))$ is true.

Examples and counterexamples

- $\exists y \forall x\, (\sim x = y \supset A(x, y)) \not\models \forall x \exists y\, (\sim x = y \,\&\, A(x, y))$.

Examples and counterexamples

- $\exists y \forall x\, (\sim x = y \supset A(x, y)) \not\models \forall x \exists y\, (\sim x = y \,\&\, A(x, y))$.

or

Examples and counterexamples

- $\exists y \forall x\, ({\sim}x = y \supset A(x, y)) \not\models \forall x \exists y\, ({\sim}x = y \,\&\, A(x, y))$.

$$\begin{tikzcd} \underset{1}{\bullet} & \text{or} & \underset{1}{\bullet} \arrow[r, bend left] & \underset{2}{\bullet} \end{tikzcd}$$

- Compare: $\exists y \forall x\, A(x, y) \models \forall x \exists y\, A(x, y)$!

Examples and counterexamples

- $\exists y \forall x\, (\sim x = y \supset A(x, y)) \not\models \forall x \exists y\, (\sim x = y \,\&\, A(x, y))$.

- Compare: $\exists y \forall x\, A(x, y) \models \forall x \exists y\, A(x, y)$!
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XI. Interpretations for full QL

c. Properties of relations

Relations and extensions

- 2–place predicate symbols express relations.

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- Extension of a 2-place predicate symbol is a set of ordered pairs.

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- This is exactly how mathematicians think of relations.

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- 2-place predicate symbols express relations.
- Extension of a 2-place predicate symbol is a set of ordered pairs.
- This is exactly how mathematicians think of relations.
- Let's think of properties relations can have, and categorize relations by these properties.

Reflexivity

Definition

A relation R is **reflexive** if every object stands in the relation R to itself.

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- x and y share a parent.

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- Not: x and y are siblings.
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Expressing reflexivity in QL

The extension of P in an interpretation is reflexive if and only if $\forall x\, P(x, x)$ is true.

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Definition

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Expressing symmetry in QL

The extension of P in an interpretation is symmetric if and only if $\forall x \forall y(P(x, y) \supset P(y, x))$ is true.

Transitivity

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- Not: x and y are siblings.
- $x \leq y$ and $x < y$.
- $x \mid y$.

Expressing transitivity in QL

The extension of P in an interpretation is transitive if and only if $\forall x \forall y \forall z((P(x, y) \,\&\, P(y, z) \supset P(x, z))$ is true.

Anti-Symmetry

Definition

A relation R is **anti-symmetric** if it never holds in both directions, except possibly for things being R-related to themselves.

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Anti-Symmetry

Definition

A relation R is **anti-symmetric** if it never holds in both directions, except possibly for things being R-related to themselves.

- Not: x is the same age as y.
- Not: x and y share a parent.
- Not: x and y are siblings.
- $x \leq y$.

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Expressing anti-symmetry in QL

The extension of P in an interpretation is anti-symmetric if and only if $\forall x \forall y((P(x, y) \,\&\, P(y, x)) \supset x = y)$ is true.

QL and properties of relations

- A relation is **universal** iff $\forall x \forall y\, P(x, y)$.

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  - symmetric: $\forall x \forall y\, P(x, y) \models \forall x \forall y (P(x, y) \supset P(y, x))$
  - transitive: $\forall x \forall y\, P(x, y) \models \forall x \forall y \forall z ((P(x, y) \,\&\, P(y, z)) \supset P(x, z))$.

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  - transitive: $\forall x \forall y\, P(x, y) \models \forall x \forall y \forall z ((P(x, y) \,\&\, P(y, z)) \supset P(x, z))$.
- But not vice versa!

QL and properties of relations

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$$\forall x \forall y ((P(x,y) \mathbin{\&} P(y,x)) \supset x = y)$$

- Relations can be transitive and symmetric without being reflexive:
- We have:

$$\begin{aligned} & \forall x \forall y \forall z ((P(x,y) \mathbin{\&} P(y,z) \supset P(x,z)) \\ & \forall x \forall y (P(x,y) \supset P(y,x)) \\ \not\models \quad & \forall x\, P(x,x) \end{aligned}$$

XII. Functional completeness and normal forms

XII. Functional completeness and normal forms

a. Functional completeness

Truth functions

Definition

An (n–place) **truth function** t is a mapping of n–tuples of **T** and **F** to either **T** or **F**.

n–place truth functions correspond to truth tables of sentence S with n sentence letters $A_1, \ldots, A_n$.

		$t_{\&}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

		$t_{\vee}$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

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A_1	A_2	$t_\&$	S
T	**T**	**T**	?
T	**F**	**F**	?
F	**T**	**F**	?
F	**F**	**F**	?

A_1	A_2	$t_\vee$	S
T	**T**	**T**	?
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A_1	A_2	$t_\&$	$A_1 \& A_2$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**F**	**F**
F	**F**	**F**	**F**

A_1	A_2	$t_\vee$	S
T	**T**	**T**	?
T	**F**	**T**	?
F	**T**	**T**	?
F	**F**	**F**	?

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n–place truth functions correspond to truth tables of sentence S with n sentence letters $A_1, \ldots, A_n$.

A_1	A_2	$t_{\&}$	$A_1 \& A_2$
T	**T**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**F**	**F**
F	**F**	**F**	**F**

A_1	A_2	$t_{\vee}$	$A_1 \vee A_2$
T	**T**	**T**	**T**
T	**F**	**T**	**T**
F	**T**	**T**	**T**
F	**F**	**F**	**F**

Definition

A sentence S containing the sentence letters $A_1, \ldots, A_n$ **expresses** the truth function t iff the truth value of S on the valuation which assigns v_i to A_i is $t(v_1, \ldots, v_n)$.

An n-place truth function is **expressible** if there is a sentence containing sentence letters $A_1, \ldots, A_n$ that expresses it.

Examples

		t_1
T	**T**	**T**
T	**F**	**T**
F	**T**	**F**
F	**F**	**F**

		t_{xor}
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Examples

A_1	A_2	t_1	S?
T	T	T	
T	F	T	
F	T	F	
F	F	F	

A_1	A_2	t_{xor}	S?
T	T	F	
T	F	T	
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Examples

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A_1 or: $A_1 \,\&\, (A_2 \vee {\sim} A_2)$

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F	T	T	
F	F	F	

Examples

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F	**T**	**F**	
F	**F**	**F**	

A_1 or: $A_1 \,\&\, (A_2 \vee {\sim}A_2)$

A_1	A_2	t_{xor}	S?
T	**T**	**F**	
T	**F**	**T**	
F	**T**	**T**	
F	**F**	**F**	

$(A_1 \vee A_2) \,\&\, {\sim}(A_1 \,\&\, A_2)$ or: ${\sim}(A_1 \equiv A_2)$

Functional completeness

Definition

A collection of connectives is **functionally complete** if every truth function is expressible by a sentence containing only those connectives.

Functional completeness: results

- Functionally complete are:

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  - Connectives we know:

$$\& + \sim \qquad \vee + \sim \qquad \supset + \sim \qquad \supset + \bot$$

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$$\& + \sim \qquad \vee + \sim \qquad \supset + \sim \qquad \supset + \bot$$

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  - Two two-place connectives by themselves: neither-not (NOR) and not-both (NAND).

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  - Two two-place connectives by themselves: neither-not (NOR) and not-both (NAND).
- No other (sets of) one and two-place connectives are functionally complete.

Functional completeness: results

- Functionally complete are:
  - Connectives we know:

$$\& + \sim \qquad \lor + \sim \qquad \supset + \sim \qquad \supset + \bot$$

  - Any other set of connectives containing one of those.
  - Two two-place connectives by themselves: neither-not (NOR) and not-both (NAND).
- No other (sets of) one and two-place connectives are functionally complete.
- We'll prove this for $\& + \lor$.

XII. Functional completeness and normal forms

b. Proving connectives are functionally complete

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	**T**	**T**	**T**	
T	**T**	**F**	**F**	
T	**F**	**T**	**F**	
T	**F**	**F**	**T**	
F	**T**	**T**	**F**	
F	**T**	**F**	**T**	
F	**F**	**T**	**T**	
F	**F**	**F**	**F**	

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3))$
T	T	F	F	
T	F	T	F	
T	F	F	T	
F	T	T	F	
F	T	F	T	
F	F	T	T	
F	F	F	F	

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3))$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \& (\sim A_2 \& \sim A_3))$
F	T	T	F	
F	T	F	T	
F	F	T	T	
F	F	F	F	

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3))$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \& (\sim A_2 \& \sim A_3))$
F	T	T	F	
F	T	F	T	$(\sim A_1 \& (A_2 \& \sim A_3))$
F	F	T	T	
F	F	F	F	

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3))$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \& (\sim A_2 \& \sim A_3))$
F	T	T	F	
F	T	F	T	$(\sim A_1 \& (A_2 \& \sim A_3))$
F	F	T	T	$(\sim A_1 \& (\sim A_2 \& A_3))$
F	F	F	F	

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3)) \vee$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \& (\sim A_2 \& \sim A_3)) \vee$
F	T	T	F	
F	T	F	T	$(\sim A_1 \& (A_2 \& \sim A_3)) \vee$
F	F	T	T	$(\sim A_1 \& (\sim A_2 \& A_3))$
F	F	F	F	

$\& + \vee + \sim$ are functionally complete

- Each line makes one, and only one, conjunction true, e.g.,

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- $\sim A_1 \,\&\, A_2 \,\&\, \sim A_3$ is true in, and only in, line **F T F**.

$\& + \vee + \sim$ are functionally complete

- Each line makes one, and only one, conjunction true, e.g.,
- $\sim A_1 \,\&\, A_2 \,\&\, \sim A_3$ is true in, and only in, line **F T F**.
- Combine using $\vee$: make S true in all (and only) the lines where it is supposed to be true.

The “neither…nor …” connective: $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

$\downarrow$ is functionally complete

- We already know that $\sim + \ \& \ + \vee$ is functionally complete, i.e.,

$\downarrow$ is functionally complete

- We already know that $\sim$ + & + $\vee$ is functionally complete, i.e.,
- Every truth function can be expressed using only $\vee$, &, $\sim$.

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- To show $\downarrow$ is functionally complete, suffices to show that **every sentence containing only $\sim$, $\vee$, $\&$ is equivalent to one containing only $\downarrow$.**

$\downarrow$ is functionally complete

- We already know that $\sim + \ \& \ + \vee$ is functionally complete, i.e.,
- Every truth function can be expressed using only $\vee$, $\&$, $\sim$.
- To show $\downarrow$ is functionally complete, suffices to show that **every sentence containing only $\sim$, $\vee$, $\&$ is equivalent to one containing only $\downarrow$.**
- For that, it suffices to show that any negated sentence, conjunction, disjunction, can be expressed using only $\downarrow$.

Expressing $\sim$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- Note how $P \downarrow Q$ is **F** in the first line and **T** in the last (when P and Q have same truth value).

Expressing $\sim$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- Note how $P \downarrow Q$ is **F** in the first line and **T** in the last (when P and Q have same truth value).
- So $P \downarrow P$ is **F** if P is **T**, and **T** if P is **F**, i.e.,

$$\sim P \Leftrightarrow (P \downarrow P).$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\sim(P \vee Q)$, i.e.,

$$\sim(P \vee Q) \Leftrightarrow P \downarrow Q$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	**T**	**F**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\sim(P \vee Q)$, i.e.,

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- Negate both sides:

$$P \vee Q \Leftrightarrow \sim(P \downarrow Q)$$

Expressing $\vee$ using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\sim(P \vee Q)$, i.e.,

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- Negate both sides:

$$P \vee Q \Leftrightarrow \sim(P \downarrow Q)$$

- Apply what we figured out in last slide:

$$P \vee Q \Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$$

Expressing & using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
F	T	F
F	F	T

- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\sim P \,\&\, \sim Q$, i.e.,

$$(\sim P \,\&\, \sim Q) \Leftrightarrow P \downarrow Q$$

Expressing & using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
T	F	F
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- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as ${\sim}P \,\&\, {\sim}Q$, i.e.,

$$({\sim}P \,\&\, {\sim}Q) \Leftrightarrow P \downarrow Q$$

- Equivalence holds for **all sentences** P, Q, so also if we replace P by ${\sim}R$ and Q by ${\sim}S$:

$$\sim\sim R \,\&\, \sim\sim S \Leftrightarrow ({\sim}R \downarrow {\sim}S)$$

Expressing & using $\downarrow$

P	Q	$(P \downarrow Q)$
T	T	F
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- $P \downarrow Q$ is the "neither ... nor" connective, which can also be expressed as $\sim P \,\&\, \sim Q$, i.e.,

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- Equivalence holds for **all sentences** P, Q, so also if we replace P by $\sim R$ and Q by $\sim S$:

$$\sim\sim R \,\&\, \sim\sim S \Leftrightarrow (\sim R \downarrow \sim S)$$

- Express $\sim$ using $\downarrow$:

$$R \,\&\, S \Leftrightarrow (R \downarrow R) \downarrow (S \downarrow S)$$

Functionally complete connectives

- De Morgan's Law: $\&$ can be expressed by $\vee$ and $\sim$.

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- $\supset, \perp$ is functionally complete (HW).
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- No other sets of connectives that don't contain one of these sets are functionally complete.
- "Neither ... nor" (NOR) is functionally complete by itself.
- "Not both" (NAND) connective is functionally complete by itself.
- No other 2-place connectives are functionally complete by themselves.

XII. Functional completeness and normal forms

c. Proving connectives aren’t functionally complete

$\& + \vee$ not functionally complete

- $\& + \vee$ is not functionally complete.

$\& + \vee$ not functionally complete

- $\& + \vee$ is not functionally complete.
- Remember: To be functionally complete, **every** truth function would have to be expressible using only $\&$ and $\vee$.

& + ∨ not functionally complete

- & + ∨ is not functionally complete.
- Remember: To be functionally complete, **every** truth function would have to be expressible using only & and ∨.
- Which 2-place truth-functions can be expressed using & and ∨?

$\& + \vee$ not functionally complete

- $\& + \vee$ is not functionally complete.
- Remember: To be functionally complete, **every** truth function would have to be expressible using only $\&$ and $\vee$.
- Which 2-place truth-functions can be expressed using $\&$ and $\vee$?
- Not this one:

		t_{xor}
T	**T**	**F**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Proof by induction

- Sometimes need to prove something for **all** sentences.

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  - Now show that it then also holds for $(\mathscr{P} \,\&\, \mathbb{Q})$, $(\mathscr{P} \vee \mathbb{Q})$, etc.

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- Why does this work?

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  - Now show that it then also holds for $(\mathscr{P} \mathbin{\&} \mathbb{Q})$, $(\mathscr{P} \vee \mathbb{Q})$, etc.
- Why does this work?
- This is how we form sentences (involving only $\&$, $\vee$).

Proof by induction

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  - Show that it holds for **sentence letters** (and $\bot$).
  - Suppose sentences $\mathscr{P}$, $\mathbb{Q}$ have the property.
  - Now show that it then also holds for $(\mathscr{P} \,\&\, \mathbb{Q})$, $(\mathscr{P} \lor \mathbb{Q})$, etc.
- Why does this work?
- This is how we form sentences (involving only $\&$, $\lor$).
- Property "S is a sentence expressing a truth function other than t_{xor}" propagates from atomic sentences to all sentences.

Proof by induction: example

Theorem

Every sentence contains an even number of parentheses.

- Every atomic sentence contains an even number of parentheses:

$$B \qquad \bot$$

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Every sentence contains an even number of parentheses.

- Every atomic sentence contains an even number of parentheses:

$$B \qquad \bot$$

- If $\mathscr{P}$ contains an even number of parentheses, so does $\sim\mathscr{P}$.
- If $\mathscr{P}$ and $\mathbb{Q}$ both contain an even number of parentheses, so do

$$(\mathscr{P} \,\&\, \mathbb{Q}), (\mathscr{P} \vee \mathbb{Q}), (\mathscr{P} \supset \mathbb{Q}), (\mathscr{P} \equiv \mathbb{Q}).$$

$\&$ + $\vee$ not functionally complete

Theorem

Any sentence containing only A_1, A_2, $\&$, $\vee$ has a **T** *in the first line of its truth table.*

- Sentence letters: truth table of A_i is just copy of column under A_i, so has **T** in first line where valuation assigns **T** to A.

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Theorem

Any sentence containing only A_1, A_2, &, $\vee$ expresses a truth function t with $t(\mathbf{T}, \mathbf{T}) = \mathbf{T}$.

- Sentence letters: A_1, A_2: express t_1, t_2.

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- Suppose $\mathscr{P}$, $\mathbb{Q}$ are sentences which contain only A_1, A_2, $\&$, $\vee$ and express truth functions t, t' with $t(\mathbf{T}, \mathbf{T}) = t'(\mathbf{T}, \mathbf{T}) = \mathbf{T}$

$\& + \vee$ not functionally complete

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- $(\mathscr{P} \& \mathbb{Q})$ expresses truth function s with

$$s(\mathbf{T}, \mathbf{T}) = t_{\&}(t(\mathbf{T}, \mathbf{T}), t'(\mathbf{T}, \mathbf{T})) = \mathbf{T}$$

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  - Functionally complete:

$$\sim + \vee \qquad \sim + \& \qquad \sim + \supset$$

Non-functionally complete connectives

- We've shown that $\&$ + $\vee$ are not functionally complete.
- Same idea shows that $\supset$ and $\equiv$ not functionally complete.
- When we add $\sim$ things get interesting:
  - Functionally complete:

$$\sim + \vee \qquad \sim + \ \& \qquad \sim + \supset$$

  - Not functionally complete:

$$\sim + \equiv$$

    (harder to prove).

XII. Functional completeness and normal forms

d. Normal forms

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- Negations apply only to sentence letters.
- Alternation between $\&$ and $\lor$ is minimal.
- Useful for applications:
  - Combinational circuits.
  - SAT solvers and theorem provers need inputs in CNF.
  - Complexity theory talks about problems involving sentences in normal form.

Scope of a connective

Definition

The **scope** of an occurrence of a connective in a sentence is that sub-sentence of which the connective is the main connective.

$$(\underbrace{\boxed{\sim(A \vee B)}}_{\text{scope of }\sim} \vee \underbrace{\boxed{((A \supset B) \,\&\, (B \supset C))}}_{\text{scope of }\&})$$

Disjunctive normal form

DNF

A sentence is in **disjunctive normal form** (DNF) iff it:

- contains only $\&$, $\vee$, $\sim$;
- only sentence letters are in scope of $\sim$;
- only sentence letters, $\&$, and $\sim$ are in scope of $\&$.

In other words: DNF are disjunctions of conjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \mathbin{\&} \sim B) \vee ((\sim A \mathbin{\&} C) \vee (B \mathbin{\&} C))$$
$$\sim A \vee (B \mathbin{\&} C)$$
$$A \mathbin{\&} (B \mathbin{\&} C)$$
$$A \vee (B \vee C)$$

DNF theorem

Theorem

Every sentence is equivalent to one in disjunctive normal form.

Proof.

- Construct truth table.

□

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- Apply method we used to show $\& + \vee + \sim$ is functionally complete.

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- This gives us a sentence involving only $\&$, $\vee$, $\sim$ with same truth table, i.e., is equivalent in SL.

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DNF theorem

Theorem

Every sentence is equivalent to one in disjunctive normal form.

Proof.

- Construct truth table.
- Apply method we used to show $\& + \vee + \sim$ is functionally complete.
- This gives us a sentence involving only $\&$, $\vee$, $\sim$ with same truth table, i.e., is equivalent in SL.
- That sentence is always in DNF.

$\square$

$\& + \vee + \sim$ are functionally complete

A_1	A_2	A_3	t_{odd}	S
T	T	T	T	$(A_1 \& (A_2 \& A_3))\vee$
T	T	F	F	
T	F	T	F	
T	F	F	T	$(A_1 \& (\sim A_2 \& \sim A_3))\vee$
F	T	T	F	
F	T	F	T	$(\sim A_1 \& (A_2 \& \sim A_3))\vee$
F	F	T	T	$(\sim A_1 \& (\sim A_2 \& A_3))$
F	F	F	F	

Conjunctive normal form

CNF

A sentence is in **conjunctive normal form** (CNF) if it:

- contains only $\&$, $\vee$, $\sim$;
- only sentence letters are in scope of $\sim$;
- only sentence letters, $\vee$, and $\sim$ are in scope of $\vee$.

In other words: CNF are conjunctions of disjunctions of sentence letters and negated sentence letters, e.g.:

$$(A \vee \sim B) \,\&\, ((\sim A \vee C) \,\&\, (B \vee C))$$

$$\sim A \,\&\, (B \vee C)$$

$$A \vee (B \vee C)$$

$$A \,\&\, (B \,\&\, C)$$

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Every sentence is equivalent to one in conjunctive normal form.

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CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:

□

CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A if A is assigned **F**.

□

CNF theorem

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Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A if A is assigned **F**.
  - Write $\sim A$ if A is assigned **T**.

□

CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A if A is assigned **F**.
  - Write $\sim A$ if A is assigned **T**.
- Put $\&$'s between all of them.

$\square$

CNF theorem

Theorem

Every sentence is equivalent to one in conjunctive normal form.

Proof.

- Construct truth table.
- For each line where sentence is **F**, write a disjunction of sentence letters and negated sentence letters:
  - Write A if A is assigned **F**.
  - Write $\sim A$ if A is assigned **T**.
- Put & 's between all of them.
- Resulting is true iff the original sentence is true, and is in CNF.

$\square$

CNF from truth table

A_1	A_2	A_3	S	CNF
T	T	T	T	
T	T	F	F	$(\sim A_1 \vee (\sim A_2 \vee A_3))$ &
T	F	T	F	$(\sim A_1 \vee (A_2 \vee \sim A_3))$ &
T	F	F	T	
F	T	T	F	$(A_1 \vee (\sim A_2 \vee \sim A_3))$ &
F	T	F	T	
F	F	T	T	
F	F	F	F	$(A_1 \vee (A_2 \vee A_3))$

XII. Functional completeness and normal forms

e. Equivalent transformations

Transformation equivalences

Defining $\supset$, $\equiv$ (Cond, Bicond)

$$(\mathscr{P} \supset \mathbb{Q}) \Leftrightarrow (\sim\mathscr{P} \vee \mathbb{Q})$$

$$\sim(\mathscr{P} \supset \mathbb{Q}) \Leftrightarrow (\mathscr{P} \mathbin{\&} \sim\mathbb{Q})$$

$$(\mathscr{P} \equiv \mathbb{Q}) \Leftrightarrow (\mathscr{P} \supset \mathbb{Q}) \mathbin{\&} (\mathbb{Q} \supset \mathscr{P})$$

Double negation (DN)

$$\sim\sim\mathscr{P} \Leftrightarrow \mathscr{P}$$

Transformation equivalences

De Morgan's Laws (DeM):

$$\sim(\mathscr{P} \vee \mathscr{Q}) \Leftrightarrow (\sim\mathscr{P} \,\&\, \sim\mathscr{Q})$$

$$\sim(\mathscr{P} \,\&\, \mathscr{Q}) \Leftrightarrow (\sim\mathscr{P} \vee \sim\mathscr{Q})$$

Commutativity (Comm):

$$\mathscr{P} \vee \mathscr{Q} \Leftrightarrow \mathscr{Q} \vee \mathscr{P}$$

$$\mathscr{P} \,\&\, \mathscr{Q} \Leftrightarrow \mathscr{Q} \,\&\, \mathscr{P}$$

Distributivity (Dist):

$$\mathscr{P} \vee (\mathscr{Q} \,\&\, \mathscr{R}) \Leftrightarrow (\mathscr{P} \vee \mathscr{Q}) \,\&\, (\mathscr{P} \vee \mathscr{R})$$

$$\mathscr{P} \,\&\, (\mathscr{Q} \vee \mathscr{R}) \Leftrightarrow (\mathscr{P} \,\&\, \mathscr{Q}) \vee (\mathscr{P} \,\&\, \mathscr{R})$$

Transforming sentences into DNF/CNF

- Replace any subsentence of the form $(\mathscr{P} \supset \mathbb{Q})$, $(\mathscr{P} \equiv \mathbb{Q})$ by its equivalent.

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- Use De Morgan's laws to place $\sim$'s in front of sentence letters

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- Replace any subsentence of the form $(\mathscr{P} \supset \mathbb{Q})$, $(\mathscr{P} \equiv \mathbb{Q})$ by its equivalent.
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  - DNF: no $\vee$ is in the scope of $\&$.

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- Replace any subsentence of the form $(\mathscr{P} \supset \mathbb{Q})$, $(\mathscr{P} \equiv \mathbb{Q})$ by its equivalent.
- Use De Morgan's laws to place $\sim$'s in front of sentence letters
- Remove double negations.
- Use distributivity and commutativity to ensure
  - DNF: no $\vee$ is in the scope of $\&$.
  - CNF: no $\&$ is in the scope of $\vee$.

Transforming sentences into CNF/DNF

$$\sim[(A \equiv B) \vee \sim(B \supset C)]$$

Transforming sentences into CNF/DNF

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Transforming sentences into CNF/DNF

$$\sim[(A \equiv B) \vee \sim(B \supset C)]$$

$$\sim[((A \supset B) \mathbin{\&} (B \supset A)) \vee \sim(B \supset C)]$$ Bicond

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$\sim\underbrace{((A \supset B) \,\&\, (B \supset A)_{\mathbb{Q}})}_{\mathscr{P}} \,\&\, \sim\underbrace{\sim(B \supset C)}_{\mathbb{Q}}$ DeM

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$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim(\underbrace{(A \supset B)}_{\mathscr{P}} \,\&\, \underbrace{(B \supset A)}_{\mathbb{Q}}) \,\&\, \sim\sim(B \supset C)$	DeM

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$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim(\,\underbrace{(A \supset B)}_{\mathscr{P}} \,\&\, \underbrace{(B \supset A)}_{\mathbb{Q}}\,) \,\&\, \sim\sim(B \supset C)$	DeM
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$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
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$\sim((A \supset B) \& (B \supset A)) \& \sim\sim(B \supset C)$	DeM
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$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\boxed{\sim(A \supset B)} \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\boxed{(A \,\&\, \sim B)} \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond

Transforming sentences into CNF/DNF

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$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
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$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \sim\sim(B \supset C)$	Cond

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \boxed{\sim\sim}(B \supset C)$	Cond

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, (B \supset C)$	DN

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \boxed{(B \supset C)}$	DN

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \boxed{(B \supset C)}$	DN
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \boxed{(\sim B \vee C)}$	Cond

Transforming sentences into CNF/DNF

$\sim[(A \equiv B) \vee \sim(B \supset C)]$	
$\sim[((A \supset B) \,\&\, (B \supset A)) \vee \sim(B \supset C)]$	Bicond
$\sim((A \supset B) \,\&\, (B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$(\sim(A \supset B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	DeM
$((A \,\&\, \sim B) \vee \sim(B \supset A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, \sim\sim(B \supset C)$	Cond
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, (B \supset C)$	DN
$((A \,\&\, \sim B) \vee (B \,\&\, \sim A)) \,\&\, (\sim B \vee C)$	Cond

Transforming sentences into DNF

$$[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \vee C)$$

Transforming sentences into DNF

$$\underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}} \,\&\, (\underbrace{{\sim}B}_{\mathscr{Q}} \vee \underbrace{C}_{\mathscr{R}})$$

Transforming sentences into DNF

$$\underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}} \,\&\, (\underbrace{{\sim}B}_{\mathscr{Q}} \vee \underbrace{C}_{\mathscr{R}})$$

$$(\underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}} \,\&\, \underbrace{{\sim}B}_{\mathscr{Q}}) \vee (\underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}} \,\&\, \underbrace{C}_{\mathscr{R}}) \qquad \text{Dist}$$

Transforming sentences into DNF

$[(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, (\sim B \vee C)$

$(\underbrace{[(A \,\&\, \sim B) \vee (B \,\&\, \sim A)]}_{\mathscr{P}} \,\&\, \underbrace{\sim B}_{\mathbb{Q}}) \vee ([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, C)$ Dist

Transforming sentences into DNF

$$[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \vee C)$$

$$(\underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}} \,\&\, \underbrace{{\sim}B}_{\mathbb{Q}}) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C) \qquad \text{Dist}$$

$$(\underbrace{{\sim}B}_{\mathbb{Q}} \,\&\, \underbrace{[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]}_{\mathscr{P}}) \,\&\, ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C) \qquad \text{Comm}$$

Transforming sentences into DNF

$[(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, (\sim B \vee C)$

$([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, \sim B) \vee ([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, C)$ Dist

$(\underbrace{\sim B}_{\mathscr{P}} \,\&\, [\underbrace{(A \,\&\, \sim B)}_{\mathscr{Q}} \vee \underbrace{(B \,\&\, \sim A)}_{\mathscr{R}}]) \,\&\, ([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, C)$ Comm

Transforming sentences into DNF

$[(A \,\&\, {\sim}B) \lor (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \lor C)$

$([(A \,\&\, {\sim}B) \lor (B \,\&\, {\sim}A)] \,\&\, {\sim}B) \lor ([(A \,\&\, {\sim}B) \lor (B \,\&\, {\sim}A)] \,\&\, C)$ Dist

$(\underbrace{{\sim}B}_{\mathscr{P}} \,\&\, [\underbrace{(A \,\&\, {\sim}B)}_{\mathbb{Q}} \lor \underbrace{(B \,\&\, {\sim}A)}_{\mathscr{R}}]) \,\&\, ([(A \,\&\, {\sim}B) \lor (B \,\&\, {\sim}A)] \,\&\, C)$ Comm

$([\underbrace{{\sim}B}_{\mathscr{P}} \,\&\, \underbrace{(A \,\&\, {\sim}B)}_{\mathbb{Q}}] \lor [\underbrace{{\sim}B}_{\mathscr{P}} \,\&\, \underbrace{(B \,\&\, {\sim}A)}_{\mathscr{R}}]) \lor ([(A \,\&\, {\sim}B) \lor (B \,\&\, {\sim}A)] \,\&\, C)$ Dist

Transforming sentences into DNF

$[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \vee C)$

$([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, {\sim}B) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$ Dist

$({\sim}B \,\&\, [(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]) \,\&\, ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$ Comm

$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \vee [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$ Dist

Transforming sentences into DNF

$$[(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, (\sim B \vee C)$$

$$([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, \sim B) \vee ([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, C) \qquad \text{Dist}$$

$$(\sim B \,\&\, [(A \,\&\, \sim B) \vee (B \,\&\, \sim A)]) \,\&\, ([(A \,\&\, \sim B) \vee (B \,\&\, \sim A)] \,\&\, C) \qquad \text{Comm}$$

$$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([\boxed{(A \,\&\, \sim B)}_{\mathbb{Q}} \vee \boxed{(B \,\&\, \sim A)}_{\mathscr{R}}] \,\&\, \boxed{C}_{\mathscr{P}}) \qquad \text{Dist}$$

Transforming sentences into DNF

$[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \vee C)$

$([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, {\sim}B) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$ Dist

$({\sim}B \,\&\, [(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]) \,\&\, ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$ Comm

$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \vee [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \vee ([\,\boxed{(A \,\&\, {\sim}B)}_{\mathbb{Q}} \vee \boxed{(B \,\&\, {\sim}A)}_{\mathscr{R}}\,] \,\&\, \boxed{C}_{\mathscr{P}})$ Dist

$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \vee [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \vee ([\,\boxed{(A \,\&\, {\sim}B)}_{\mathbb{Q}} \,\&\, \boxed{C}_{\mathscr{P}}\,] \vee [\,\boxed{(B \,\&\, {\sim}A)}_{\mathscr{R}} \,\&\, \boxed{C}_{\mathscr{P}}\,])$
Dist

Transforming sentences into DNF

$[(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, ({\sim}B \vee C)$	
$([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, {\sim}B) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$	Dist
$({\sim}B \,\&\, [(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)]) \,\&\, ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$	Comm
$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \vee [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \vee ([(A \,\&\, {\sim}B) \vee (B \,\&\, {\sim}A)] \,\&\, C)$	Dist
$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \vee [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \vee ([(A \,\&\, {\sim}B) \,\&\, C] \vee [(B \,\&\, {\sim}A) \,\&\, C])$	Dist

XII. Functional completeness and normal forms

f. Simplification

Simplification equivalences

Associativity (Assoc):

$$\mathscr{P} \vee (\mathbb{Q} \vee \mathscr{R}) \Leftrightarrow (\mathscr{P} \vee \mathbb{Q}) \vee \mathscr{R} \qquad \mathscr{P} \,\&\, (\mathbb{Q} \,\&\, \mathscr{R}) \Leftrightarrow (\mathscr{P} \,\&\, \mathbb{Q}) \,\&\, \mathscr{R}$$

Idempotence (Id):

$$(\mathscr{P} \vee \mathscr{P}) \Leftrightarrow \mathscr{P} \qquad (\mathscr{P} \,\&\, \mathscr{P}) \Leftrightarrow \mathscr{P}$$

Absorption (Abs):

$$\mathscr{P} \,\&\, (\mathscr{P} \vee \mathbb{Q}) \Leftrightarrow \mathscr{P} \qquad \mathscr{P} \vee (\mathscr{P} \,\&\, \mathbb{Q}) \Leftrightarrow \mathscr{P}$$

Simplification (Simp):

$$\mathscr{P} \,\&\, (\mathbb{Q} \vee {\sim}\mathbb{Q}) \Leftrightarrow \mathscr{P} \qquad \mathscr{P} \vee (\mathbb{Q} \,\&\, {\sim}\mathbb{Q}) \Leftrightarrow \mathscr{P}$$

$$\mathscr{P} \vee (\mathbb{Q} \vee {\sim}\mathbb{Q}) \Leftrightarrow (\mathbb{Q} \vee {\sim}\mathbb{Q}) \qquad \mathscr{P} \,\&\, (\mathbb{Q} \,\&\, {\sim}\mathbb{Q}) \Leftrightarrow (\mathbb{Q} \,\&\, {\sim}\mathbb{Q})$$

Simplifying sentences

$$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$$

Simplifying sentences

$$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$$

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, {\sim}B)] \vee [\sim B \,\&\, (B \,\&\, {\sim}A)]) \vee ([(A \,\&\, {\sim}B) \,\&\, C] \vee [(B \,\&\, {\sim}A) \,\&\, C])$

$([\sim B \,\&\, ({\sim}B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, {\sim}A)]) \vee ([(A \,\&\, {\sim}B) \,\&\, C] \vee [(B \,\&\, {\sim}A) \,\&\, C])$ Comm

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$Comm

Simplifying sentences

$$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$$

$$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])\text{Comm}$$

$$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])\text{Assoc}$$

Simplifying sentences

$([\sim B \;\&\; (A \;\&\; \sim B)] \vee [\sim B \;\&\; (B \;\&\; \sim A)]) \vee ([(A \;\&\; \sim B) \;\&\; C] \vee [(B \;\&\; \sim A) \;\&\; C])$

$([\sim B \;\&\; (\sim B \;\&\; A)] \vee [\sim B \;\&\; (B \;\&\; \sim A)]) \vee ([(A \;\&\; \sim B) \;\&\; C] \vee [(B \;\&\; \sim A) \;\&\; C])$ Comm

$([(\sim B \;\&\; \sim B) \;\&\; A] \vee [\sim B \;\&\; (B \;\&\; \sim A)]) \vee ([(A \;\&\; \sim B) \;\&\; C] \vee [(B \;\&\; \sim A) \;\&\; C])$ Assoc

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, B]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Simp

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, B]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Simp

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \lor [\sim B \,\&\, (B \,\&\, \sim A)]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	
$([\sim B \,\&\, (\sim B \,\&\, A)] \lor [\sim B \,\&\, (B \,\&\, \sim A)]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Comm
$([(\sim B \,\&\, \sim B) \,\&\, A] \lor [\sim B \,\&\, (B \,\&\, \sim A)]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \lor [\sim B \,\&\, (B \,\&\, \sim A)]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Id
$([\sim B \,\&\, A] \lor [(\sim B \,\&\, B) \,\&\, \sim A]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \lor [\sim B \,\&\, B]) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Simp
$(\sim B \,\&\, A) \lor ([(A \,\&\, \sim B) \,\&\, C] \lor [(B \,\&\, \sim A) \,\&\, C])$	Simp

Simplifying sentences

$([\sim B \& (A \& \sim B)] \lor [\sim B \& (B \& \sim A)]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	
$([\sim B \& (\sim B \& A)] \lor [\sim B \& (B \& \sim A)]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Comm
$([(\sim B \& \sim B) \& A] \lor [\sim B \& (B \& \sim A)]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Assoc
$([\sim B \& A] \lor [\sim B \& (B \& \sim A)]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Id
$([\sim B \& A] \lor [(\sim B \& B) \& \sim A]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Assoc
$([\sim B \& A] \lor [\sim B \& B]) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Simp
$(\sim B \& A) \lor ([(A \& \sim B) \& C] \lor [(B \& \sim A) \& C])$	Simp

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	
$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Comm
$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Id
$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \vee [\sim B \,\&\, B]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Simp
$(\sim B \,\&\, A) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Simp

Simplifying sentences

$([{\sim}B \,\&\, (A \,\&\, {\sim}B)] \lor [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$

$([{\sim}B \,\&\, ({\sim}B \,\&\, A)] \lor [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Comm

$([({\sim}B \,\&\, {\sim}B) \,\&\, A] \lor [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Assoc

$([{\sim}B \,\&\, A] \lor [{\sim}B \,\&\, (B \,\&\, {\sim}A)]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Id

$([{\sim}B \,\&\, A] \lor [({\sim}B \,\&\, B) \,\&\, {\sim}A]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Assoc

$([{\sim}B \,\&\, A] \lor [{\sim}B \,\&\, B]) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Simp

$({\sim}B \,\&\, A) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Simp

$(A \,\&\, {\sim}B) \lor ([(A \,\&\, {\sim}B) \,\&\, C] \lor [(B \,\&\, {\sim}A) \,\&\, C])$ Comm

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$

$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Id

$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Assoc

$([\sim B \,\&\, A] \vee [\sim B \,\&\, B]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Simp

$(\sim B \,\&\, A) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Simp

$(A \,\&\, \sim B) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$ Comm

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	
$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Comm
$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Id
$([\sim B \,\&\, A] \vee [(\sim B \,\&\, B) \,\&\, \sim A]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \vee [\sim B \,\&\, B]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Simp
$(\sim B \,\&\, A) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Simp
$(A \,\&\, \sim B) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Comm
$((A \,\&\, \sim B) \vee [(A \,\&\, \sim B) \,\&\, C]) \vee [(B \,\&\, \sim A) \,\&\, C]$	Assoc

Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	
$([\sim B \,\&\, (\sim B \,\&\, A)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Comm
$([(\sim B \,\&\, \sim B) \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Assoc
$([\sim B \,\&\, A] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	Id
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Simplifying sentences

$([\sim B \,\&\, (A \,\&\, \sim B)] \vee [\sim B \,\&\, (B \,\&\, \sim A)]) \vee ([(A \,\&\, \sim B) \,\&\, C] \vee [(B \,\&\, \sim A) \,\&\, C])$	
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XIII. Further topics

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a. History of logic

The beginnings

- Rules of debate & rhetoric

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- Ancient India: Gautama, **Nyayasutra** (600 BCE–200 CE)
- Ancient Greece: Aristotle (384-322 BCE)
- Cataloged valid arguments ("syllogisms"), e.g.,
- All ungulates have hooves.
  No fish have hooves.
  ∴ No fish are ungulates.

The middle ages

- Ibn Sīnā (Avicenna)

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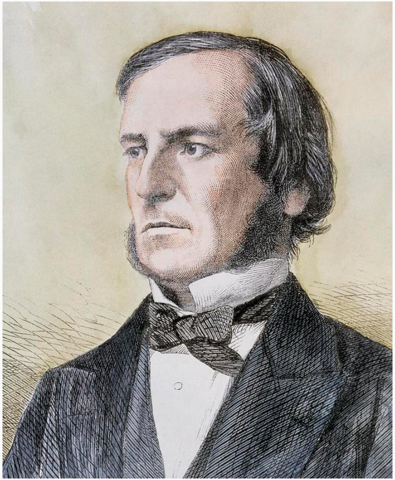
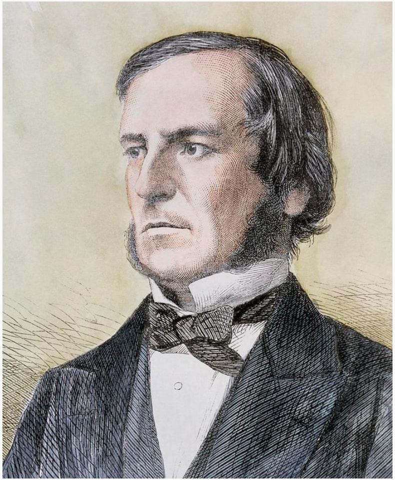


- George Boole

Mathematical logic



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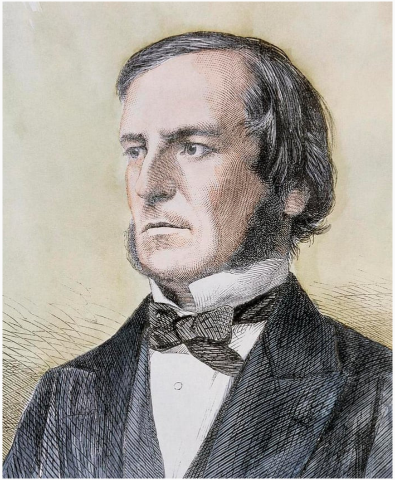

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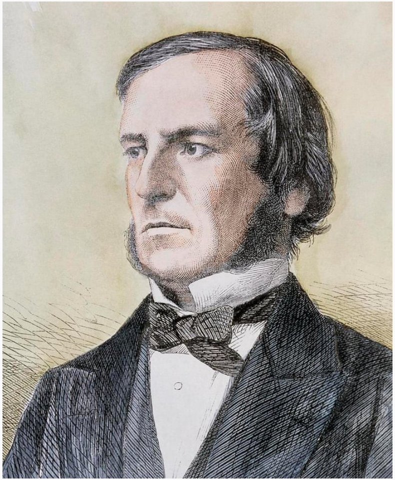

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- John Venn
- Augustus De Morgan
- Charles Lutwidge Dodgson (aka Lewis Caroll)

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- Christine Ladd Franklin

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- Ernst Schröder

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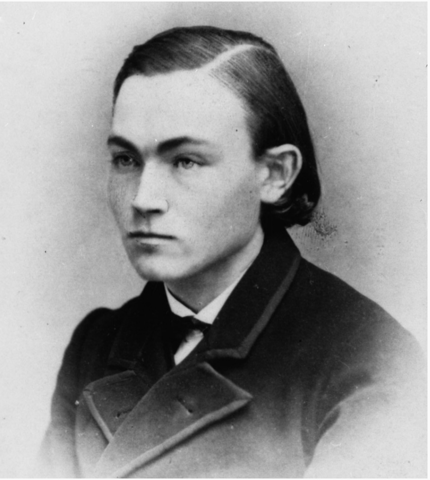

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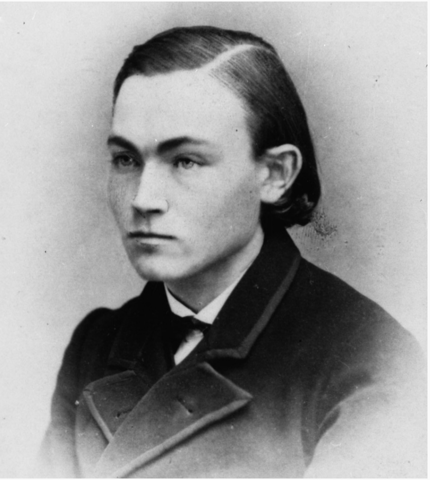

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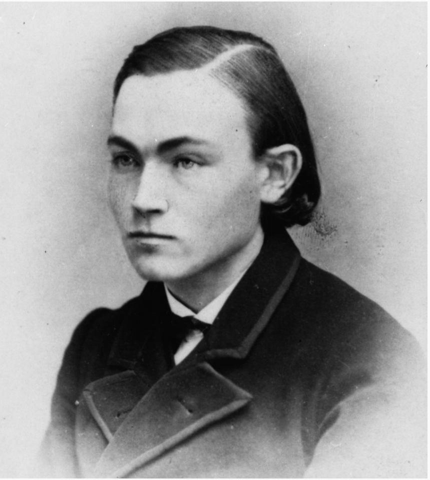

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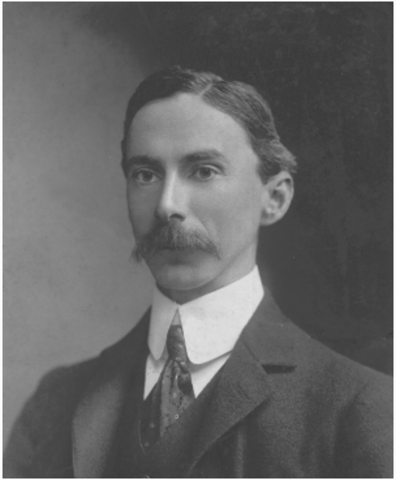

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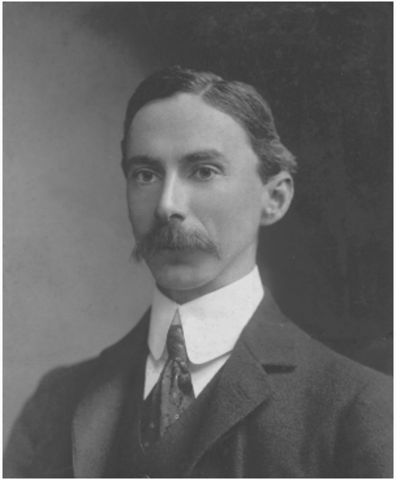

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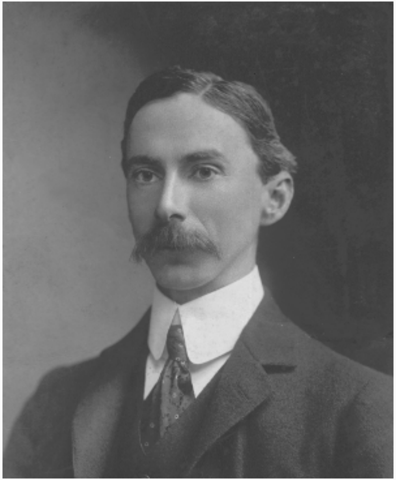

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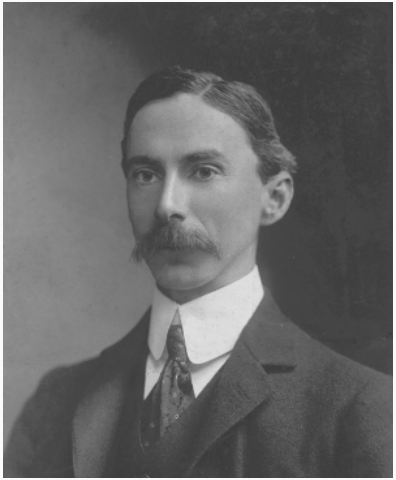

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- Rudolf Carnap, Saul Kripke, Ruth Barcan Marcus

XIII. Further topics

b. Philosophy and nonstandard logics

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- Difficulty: What logically possible circumstances are there?

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  - Every argument valid in QL is valid
  - Every argument with a formal proof is valid (soundness!)

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- Non-standard logics: expand SL, QL to deal with these

Many-valued logic

- Add to the truth-values T and F, e.g.,
  - "Undetermined": neither true nor false

P	$\sim P$
T	F
U	U
F	T

P	Q	$(P \,\&\, Q)$
T	T	T
T	U	U
T	F	F
U	T	U
U	U	U
U	F	F
F	T	F
F	U	F
F	F	F

P	Q	$(P \vee Q)$
T	T	T
T	U	T
T	F	T
U	T	T
U	U	U
U	F	U
F	T	T
F	U	U
F	F	F

  - "Inconsistent": both true and false
  - Fuzzy truth values: any number between 0 and 1

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Definition

A connective $*$ is **truth functional** iff the truth value of $*A$ depends only on the truth value of A.

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- "If … then": iffy.

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- “Richard believes that”, “Richard knows that”

Possibly

- "It is possible that . . . ", "Possibly, . . . "

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- Otherwise (1) and (2) would have to have the same truth value.

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- Indicative conditional is (plausibly) **truth-functional**: truth value of "If P, then Q" depends **only on truth values** of P and Q.

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- Temporal logic
  "It was true that" (P), "It will be true that" (F)

$$\mathbf{F}\,\mathbf{P}\, A \supset (\mathbf{P}\, A \vee A \vee \mathbf{F}\, A)$$

- “Always A”: $\Box A$

Temporal logic

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XIII. Further topics

c. Metalogic and applications

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- A validity is a sentence that’s true in all interpretations

Soundness and completeness

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  Arguments have formal proofs **only if** they are valid
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  Proved by Kurt Gödel (1929)

Church–Turing Theorem

Instance: Sentence *A* of QL
Problem: Is *A* a validity/provable?

- Undecidable: no computer program can answer this question correctly for all *A*.
- Proved independently by Alonzo Church and Alan Turing in 1935

Cook's Theorem

Instance: Sentence *A* of SL
Problem: Is *A* a tautology?

- Decidable: write a computer program that checks all valuations for *A*.
- But: it's hard: "co-NP complete"
- Proved independently by Stephen Cook (1971) and Leonid Levin (1973)

Decidable classes

- The decision problem **in general** is undecidable
- But special cases **can** be decided, e.g.:

Instance: Sentence A with only 1-place predicate symbols
Problem: Is A a validity?

- Decidable
- Proved by Leopold Löwenheim (1915)
- Complexity is NEXPTIME-complete.

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  - KR classification systems, e.g., SNOMED-CT
  - Mereology, theories of truth, scientific theories

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- Paradigm of axiomatic method: geometry (Euclid)

Examples of theories: linear orders

A relation $\preceq$ on a set O is a **linear order** iff it makes following axioms true:

$$\forall x \forall y ((x \preceq y \,\&\, y \preceq x) \supset x = y) \qquad \text{Antisymmetry}$$

$$\forall x \forall y \forall z ((x \preceq y \,\&\, y \preceq z) \supset x \preceq z) \qquad \text{Transitivity}$$

$$\forall x \forall y (x \preceq y \lor y \preceq x) \qquad \text{Totality}$$

Every total relation is reflexive:

$$LO \models \forall x \, x \preceq x$$

Examples of theories: Robinson's Q

Theories of arithmetic, such as Robinson's theory Q:

$$\sim\exists x\,(x+1)=0$$

$$\forall x(x=0 \vee \exists y\,(y+1)=x)$$

$$\forall x\forall y((x+1)=(y+1)\supset x=y)$$

$$\forall x\,(x+0)=x$$

$$\forall x\forall y\,(x+(y+1))=((x+y)+1)$$

$$\forall x\,(x\times 0)=0$$

$$\forall x\forall y\,(x\times(y+1))=((x\times y)+x)$$

Examples of theories: SNOMED-CT

```
bacterial pneumonia =
         is-a|bacterial infectious disease
         is-a|infective pneumonia
         causative agent|bacteria
         finding site|lung structure
```

$$\begin{aligned}
&\forall x(\mathit{BacterialPneumonia}(x) \equiv \\
&\quad \mathit{BacterialInfectiousDisease}(x)\ \&\\
&\quad \mathit{InfectivePneumonia}(x)\ \&\\
&\quad \exists y(\mathit{HasCausativeAgent}(x,y)\ \&\ \mathit{Bacteria}(y))\ \&\\
&\quad \exists y(\mathit{HasFindingSite}(x,y)\ \&\ \mathit{LungStructure}(y)))
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- SNOMED-CT is decidable

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- Some axioms:

$\forall x\, Pt(x, x)$	Reflexivity
$\forall x \forall y \forall z((Pt(x, y) \,\&\, Pt(y, z)) \supset Pt(x, z))$	Transitivity
$\forall x \forall y((Pt(x, y) \,\&\, Pt(y, x)) \supset x = y)$	Antisymmetry

Examples of Theories: Mereology

- Defined properties and relations

$$PP(x, y) \equiv (Pt(x, y) \,\&\, {\sim}x = y)$$
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  - Is everything made of atomless "gunk"?

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- **Gödel's Incompleteness Theorem (1930)**
  Arithmetic, set theory, mereology are incompleteable
- Philosophical upshot of this: truth in the intended interpretation(s) of the theory outstrips provability from the theory