MATH1103 FALL 2022
PROBLEM SET 3

This problem set is due on Wednesday, September 21 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.4.1) Find
$$\int \sqrt{2+x}\,dx.$$
(Don't forget to add $+C$ to your final answer.)

(b) (Strang 5.4.14) Find
$$\int t^3\sqrt{1-t^2}\,dt.$$
Hint: Starting with $u = 1 - t^2$ worked for me.

(c) (Strang 5.4.20) Find
$$\int \sin^3 x\,dx.$$
Hint: The identity $\sin^2 x = 1 - \cos^2 x$ may be useful.

(d) (Strang 7.1.2) Find
$$\int xe^{4x}\,dx.$$

(e) (Strang 7.1.27) Find
$$\int_0^1 \ln(x)\,dx.$$

(f) (Strang 7.3.3) Using trigonometric substitution, find
$$\int \sqrt{4-x^2}\,dx$$
and use this to calculate
$$\int_{-2}^{2} \sqrt{4-x^2}\,dx.$$
Could you have found this definite integral using geometry instead?

Problem 2.

(a) Use software such as Desmos or WolframAlpha to find an approximation to the following definite integral:

$$\int_0^4 e^{(x-2)^4}\,dx.$$

Do the same for the following definite integral:

$$\int_0^4 xe^{(x-2)^4}\,dx.$$

If you divide the bigger number by the smaller, what do you get?

Hint: If you didn't get a positive integer, you probably inputted one or more integrals wrong somehow.

(b) Prove that your observation is in fact exactly true. You may find it useful to use u-substitution along with a hefty dose of symmetry.

Hint: The values of the definite integrals in part (a) have no known closed forms! This suggests that trying to get exact values for the integrals will be a dead end.

Hint 2: If you are still stuck, see the footnote.[1] Try plotting the first term in the right hand side of the footnote and see if you notice anything.

Problem 3. Let

$$f(x) = \int_1^x \frac{1}{t}\,dt.$$

For example, this means that $f(3) = \int_1^3 \frac{1}{t}\,dt$, $f(s) = \int_1^s \frac{1}{t}\,dt$, and $f(xy) = \int_1^{xy} \frac{1}{t}\,dt$.

Put yourself in the mind of someone who does not know yet that f is the natural logarithm function and wants to prove that f is the natural logarithm function. One way to start proving this is to show that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y. That is, f turns multiplication into addition. This is one of the defining properties of the natural logarithm.

Using properties of integrals and u-substitution, prove that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y.

Optional challenge: The other ingredient needed to prove that f is the natural logarithm is to prove that f is continuous and that $f(e) = 1$. Continuity (in fact, differentiability) follows from the fact that f is an area function whose derivative is $1/x$. Can you prove that $f(e) = 1$ from first principles? Of course, you'll need a working definition of e. Here is one:

$$e \coloneqq \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

[1] $xe^{(x-2)^4} = (x-2)e^{(x-2)^4} + 2e^{(x-2)^4}$.

Problem 4. Recall that I mentioned in passing that areas are also invariant under rotations and reflections. In particular, areas are invariant under a reflection across the line $y = x$. This is interesting because the graph of the inverse of a function f is also the reflection across the line $y = x$ of the graph of f. We will apply this to show a surprising relationship between the definite integrals of a function and its inverse, using the example of the inverse pair e^x and $\ln x$.

(a) First, show using the integration by parts technique that, for any $b > 1$,

$$\int_1^b \ln t \, dt = b \ln b - b + 1.$$

(b) Now we will show the same identity without using any integration techniques! Do the following:

(i) Draw on coordinate axes the graph of e^x and shade the region under this curve from $x = 0$ to $x = \ln b$. Also find the area of this region.

(ii) Reflect everything across the line $y = x$ and show the result on a new set of coordinate axes. Also draw the rectangle with corners $(0, 0)$, $(b, 0)$, $(b, \ln b)$, and $(0, \ln b)$.

(iii) If you drew everything right, the unshaded region within the rectangle should correspond exactly to the definite integral

$$\int_1^b \ln t \, dt.$$

Deduce the formula from part (a) by combining this observation with previous observations.