MATH1103 FALL 2022
PROBLEM SET 3

This problem set is due on Wednesday, September 21 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.4.1) Find

$$\int \sqrt{2+x}\,dx.$$

(Don't forget to add $+C$ to your final answer.)

Solution

We make the u-substitution $u = 2 + x$ and $du = dx$, to get $\int \sqrt{2+x}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(2+x)^{\frac{3}{2}} + C$.

(b) (Strang 5.4.14) Find

$$\int t^3\sqrt{1-t^2}\,dt.$$

Hint: Starting with $u = 1 - t^2$ worked for me.

Solution

We make the u-substitution $u = 1 - t^2$, $du = -2t\,dt$. The trick is to split up t^3 into $t^2 \cdot t$ and then use $t^2 = 1 - u$ to write $t^3\sqrt{1-t^2}$ as $(1-u)\sqrt{u}\,du$. Then

$$\begin{aligned}
\int t^3\sqrt{1-t^2}\,dt &= \int (1-u)u^{\frac{1}{2}}\,du \\
&= \int u^{\frac{1}{2}}\,du - \int u^{\frac{3}{2}}\,du \\
&= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + C \\
&= \frac{2}{3}(1-t^2)^{\frac{3}{2}} - \frac{2}{5}(1-t^2)^{\frac{5}{2}} + C.
\end{aligned}$$

(c) (Strang 5.4.20) Find

$$\int \sin^3 x \, dx.$$

Solution

We can write $\sin^3 x$ as $\sin x \cdot \sin^2 x = \sin x(1 - \cos^2 x) = \sin x - \sin x \cdot \cos^2 x$. The first term is integrated directly and the second term is integrated by u-substitution with $u = \cos x$ and $du = -\sin x \, dx$. Then

$$\begin{aligned} \int \sin^3 x \, dx &= \int \sin x \, dx - \int \sin x \cdot \cos^2 x \, dx \\ &= -\cos x - \int -u^2 \, du \\ &= -\cos x + \frac{1}{3}u^3 + C \\ &= -\cos x + \frac{1}{3}\cos^3 x + C. \end{aligned}$$

Hint: The identity $\sin^2 x = 1 - \cos^2 x$ may be useful.

(d) (Strang 7.1.2) Find

$$\int xe^{4x} \, dx.$$

Solution

We can integrate by parts, letting $u = x$ and $v = \frac{1}{4}e^{4x}$ so that $dv = e^{4x} \, dx$. Then

$$\int xe^{4x} \, dx = \frac{1}{4}xe^{4x} - \int \frac{1}{4}e^{4x} \, dx = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C.$$

(e) (Strang 7.1.27) Find

$$\int_0^1 \ln(x) \, dx.$$

Solution

We use FTC2 to evaluate the definite integral, and to use this we first must find an antiderivative of $\ln(x)$. We can use integration by parts for this, letting $u = \ln(x)$ and $v = x$, so that $dv = dx$. Then

$$\int \ln(x) \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C.$$

Let's pick the function $x \ln x - x$ as our antiderivative. Then using FTC2, we have

$$\int_0^1 \ln(x)\, dx = (x \ln x - x)\Big|_0^1 = (1 \ln 1 - 1) - (0 \ln 0 - 0).$$

Note: $0 \ln 0$ is an indeterminate form because $\ln x \to -\infty$ as $x \to 0$. Therefore one should properly write $\lim_{x \to 0^+} x \ln x$ instead. This limit evaluates to 0, by using L'Hopital's rule on the expression $\ln(x)/(1/x)$. But since we have not talked about improper integrals in class formally yet, for this problem set it is okay to simply write that $0 \ln 0 = 0$.
So we get as our answer, -1.

(f) (Strang 7.3.3) Using trigonometric substitution, find

$$\int \sqrt{4 - x^2}\, dx$$

and use this to calculate

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx.$$

Could you have found this definite integral using geometry instead?

Solution

Let $x = 2\sin(\theta)$, so that $dx = 2\cos(\theta)\, d\theta$. Then

$$\begin{aligned} \int \sqrt{4 - x^2}\, dx &= \int \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta\, d\theta \\ &= \int (2\cos\theta)^2\, d\theta \\ &= 4 \int \cos^2\theta\, d\theta. \end{aligned}$$

In class we computed $\int \cos^2\theta$ as $\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$. Therefore, our integral simplifies to $2\theta + \sin(2\theta) = 2\theta + 2\sin\theta\cos\theta$. Now to express this back in terms of x, we notice that $x = 2\sin\theta$ implies that $\theta = \sin^{-1}(\frac{x}{2})$. So the integral evaluates to $2\sin^{-1}(\frac{x}{2}) + 2(\frac{x}{2})\sqrt{1 - (\frac{x}{2})^2} + C$.

To evaluate $\int_{-2}^{2} \sqrt{4-x^2}\,dx$, we can pick $2\sin^{-1}(\frac{x}{2}) + 2(\frac{x}{2})\sqrt{1-(\frac{x}{2})^2}$ as our antiderivative and evaluate

$$\begin{aligned}&\left(2\sin^{-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\sqrt{1-\left(\frac{x}{2}\right)^2}\right)\Big|_{-2}^{2}\\&= (2\sin^{-1}(1) + 2(1)(0)) - (2\sin^{-1}(-1) + 2(-1)(0))\\&= 2\frac{\pi}{2} - 2\left(-\frac{\pi}{2}\right) = 2\pi.\end{aligned}$$

Geometrically the area under the curve is the interior of the top half of a circle of radius 2, which we know has area 2π.

Problem 2.

(a) Use software such as Desmos or WolframAlpha to find an approximation to the following definite integral:

$$\int_0^4 e^{(x-2)^4}\,dx.$$

Do the same for the following definite integral:

$$\int_0^4 xe^{(x-2)^4}\,dx.$$

If you divide the bigger number by the smaller, what do you get?

Hint: If you didn't get a positive integer, you probably inputted one or more integrals wrong somehow.

Solution

We have

$$\int_0^4 e^{(x-2)^4}\,dx \approx 584926.209\ldots$$

and

$$\int_0^4 xe^{(x-2)^4}\,dx \approx 1169852.418\ldots.$$

The ratio beteen the two numbers looks like it might be exactly 2.

(b) Prove that your observation is in fact exactly true. You may find it useful to use u-substitution along with a hefty dose of symmetry.

Hint: The values of the definite integrals in part (a) have no known closed forms! This suggests that trying to get exact values for the integrals will be a dead end.

Hint 2: If you are still stuck, see the footnote.[1] Try plotting the first term in the right hand side of the footnote and see if you notice anything.

Solution

Our task, according to what we got in part (a), is to prove that $\int_0^4 xe^{(x-2)^4}\,dx = 2\int_0^4 e^{(x-2)^4}\,dx$.
The footnote says $xe^{(x-2)^4} = (x-2)e^{(x-2)^4} + 2e^{(x-2)^4}$, which follows from simple algebra. We can integrate both sides from 0 to 4 to get

$$\int_0^4 xe^{(x-2)^4}\,dx = \int_0^4 (x-2)e^{(x-2)^4}\,dx + 2\int_0^4 e^{(x-2)^4}\,dx.$$

It therefore suffices to show that

$$\int_0^4 (x-2)e^{(x-2)^4}\,dx = 0.$$

Make the u-substitution $u = x-2$, with $du = dx$, so the integral becomes

$$\int_{-2}^2 ue^{u^4}\,dx.$$

We claim that $f(u) \coloneqq ue^{u^4}$ is an odd function in u. Indeed, for any real number u, we have $f(-u) = (-u)e^{(-u)^4} = -ue^{u^4} = -f(u)$. This shows that $\int_{-2}^2 ue^{u^4}\,dx = 0$, because the integral of any odd function from -2 to 2 equals 0. This completes the proof.

Problem 3. Let

$$f(x) = \int_1^x \frac{1}{t}\,dt.$$

For example, this means that $f(3) = \int_1^3 \frac{1}{t}\,dt$, $f(s) = \int_1^s \frac{1}{t}\,dt$, and $f(xy) = \int_1^{xy} \frac{1}{t}\,dt$.

Put yourself in the mind of someone who does not know yet that f is the natural logarithm function and wants to prove that f is the natural logarithm function. One way to start proving this is to show that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y. That is, f turns multiplication into addition. This is one of the defining properties of the natural logarithm.

Using properties of integrals and u-substitution, prove that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y.

Optional challenge: The other ingredient needed to prove that f is the natural logarithm is to prove that f is continuous and that $f(e) = 1$. Continuity (in fact, differentiability)

[1] $xe^{(x-2)^4} = (x-2)e^{(x-2)^4} + 2e^{(x-2)^4}$.

follows from the fact that f is an area function whose derivative is $1/x$. Can you prove that $f(e) = 1$ from first principles? Of course, you'll need a working definition of e. Here is one:

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Solution

Recall in class I proved that $f(6) = f(2)+f(3)$ using u-substitution. The appropriate generalization of this idea in order to prove that $f(xy) = f(x) + f(y)$ for all x, y is as follows: First, for the integral $\int_1^x \frac{1}{t}\,dt$ make the substitution $u = yt$, so $du = y\,dt$. We therefore have

$$f(x) = \int_1^x \frac{1}{t}\,dt = \int_y^{xy} \frac{y}{u}\frac{1}{y}\,du = \int_y^{xy} \frac{1}{u}\,du.$$

And so

$$f(x) + f(y) = \int_y^{xy} \frac{1}{u}\,du + \int_1^y \frac{1}{t}\,dt = \int_1^{xy} \frac{1}{t}\,dt = f(xy)$$

by the interval addition property of integrals. Thus we have proven that $f(x)+f(y) = f(xy)$ for all x and y.
Let me know if you want to see a proof that $f(e) = 1$!

Problem 4. Recall that I mentioned in passing that areas are also invariant under rotations and reflections. In particular, areas are invariant under a reflection across the line $y = x$. This is interesting because the graph of the inverse of a function f is also the reflection across the line $y = x$ of the graph of f. We will apply this to show a surprising relationship between the definite integrals of a function and its inverse, using the example of the inverse pair e^x and $\ln x$.

(a) First, show using the integration by parts technique that, for any $b > 1$,

$$\int_1^b \ln t\,dt = b\ln b - b + 1.$$

Solution

We have already found in 1(e) that an antiderivative of $\ln x$ is $x\ln x - x$. Let us now evaluate this from 1 to b:

$$(x\ln x - x)\Big|_1^b = (b\ln b - b) - (1\ln 1 - 1) = b\ln b - b + 1.$$

(b) Now we will show the same identity without using any integration techniques! Do the following:

(i) Draw on coordinate axes the graph of e^x and shade the region under this curve from $x = 0$ to $x = \ln b$. Also find the area of this region.

(ii) Reflect everything across the line $y = x$ and show the result on a new set of coordinate axes. Also draw the rectangle with corners $(0, 0)$, $(b, 0)$, $(b, \ln b)$, and $(0, \ln b)$.

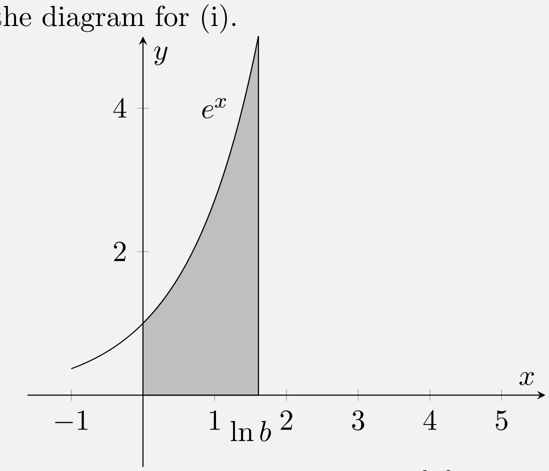
(iii) If you drew everything right, the unshaded region within the rectangle should correspond exactly to the definite integral

$$\int_1^b \ln t \, dt.$$

Deduce the formula from part (a) by combining this observation with previous observations.

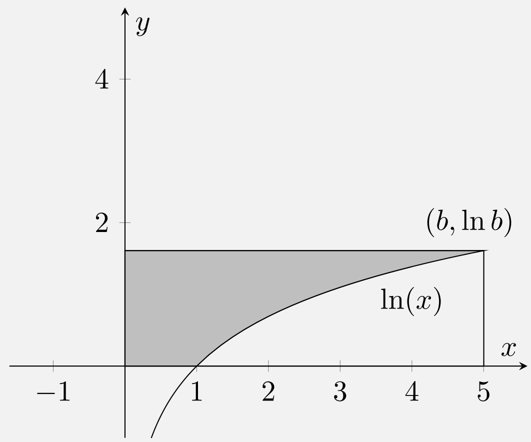
Solution

Below, I draw the diagram for (i).



The area of the shaded region is $\int_0^{\ln b} e^x \, dx = e^x \Big|_0^{\ln b} = e^{\ln b} - e^0 = b - 1$.

Below, I draw the diagram for (ii). Notice that reflecting the graph of e^x across the line $y = x$ produces the graph of $\ln(x)$.



Now we answer (iii). The area of the shaded region in the second diagram is still $b - 1$ because all we did to it was a reflection. The area of the rectangle is $b \cdot \ln b$. Therefore the unshaded region within the rectangle has area $b \ln b - (b - 1) = b \ln b - b + 1$. We can also see that the area of the unshaded part of the rectangle can be seen as the area under the graph of $\ln(x)$ from 1 to b, which is represented by the definite integral $\int_1^b \ln(x)\,dx$. Putting everything together we get

$$\int_1^b \ln(x)\,dx = b \ln b - b + 1,$$

as desired.