MATH1103 FALL 2022
EXAM 2

WEDNESDAY, NOVEMBER 30, 2022

Name: ______________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1. Convergence tests. Any proof will do. You may use results from class and homework.

(a) (5 points) Find, with proof, whether $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{314159^k}$ converges or diverges.

(b) (5 points) Find, with proof, whether $\displaystyle\sum_{n=314159}^{\infty} \frac{1}{n-1}$ converges or diverges.

Problem 2. Power series. You may freely use without proof the power series for e^x we established in class:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

You do not need to write any series in sigma notation.

(a) (5 points) What is the power series of e^{-5x}?

(b) (5 points) The below series converges. (You do not need to prove it.)

$$\frac{1}{0!} - \frac{10}{1!} + \frac{100}{2!} - \frac{1000}{3!} + \cdots .$$

What is its exact value? This problem does not depend on part (a).

Problem 3 (10 points)**.** You are freely given (thanks to Euler) that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

(a) (5 points) Prove, from the above equation, that

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{24}.$$

(b) (5 points) Using the result of part (a), find the value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots.$$

Problem 4 (10 points)**.** What are the first 10 digits of $\dfrac{1}{98}$?

Note: A solution using long division will be awarded at most 5 points.

Hint: First prove that

$$\frac{1}{98} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n}{100^n},$$

then use that result to figure out the digits.

Problem 5 (10 points)**.** For $n \geq 1$, let

$$b_n = \begin{cases} 4^n & \text{if } n < 314159 \\ 271828 & \text{if } n \geq 314159. \end{cases}$$

What is $\lim_{n\to\infty} b_n$? Prove it using the ε definition of limit.