MATH1103 FALL 2022
PROBLEM SET 10 SOLUTIONS

This problem set is due on Wednesday, November 16 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Comparison and/or ratio test practice. Determine whether each of the following series converges. General tip: Think about comparison before thinking about ratio test. Of course, think about both strategies in case one of them doesn't seem to be leading anywhere useful.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{7+5^k}$.

Solution

The series diverges. We outline both methods.
Method 1: Comparison. We would like to compare the series with the series $\sum_{k=1}^{\infty} \frac{10^k}{5^k} = \sum_{k=1}^{\infty} 2^k$ which diverges because it is a geometric series with common ratio greater than 1. However, $\frac{10^k}{7+5^k} < \frac{10^k}{5^k}$, from which we cannot logically deduce that $\sum_{k=1}^{\infty} \frac{10^k}{7+5^k}$ diverges. However, we can solve this issue by ignoring the first term of the series in the problem, which doesn't affect whether the series converges or diverges, and then proving that $\frac{10^{k+1}}{7+5^{k+1}} > 2^k$ for all k. Here are the details. Ignoring the first term is equivalent to looking at the series

$$\sum_{k=2}^{\infty} \frac{10^k}{7+5^k} = \sum_{k=1}^{\infty} \frac{10^{k+1}}{7+5^{k+1}}.$$

Now we compare this series term-by-term with the series $\sum_{k=1}^{\infty} 2^k$, that is, we are comparing $\frac{10^{k+1}}{7+5^{k+1}}$ with 2^k. Let's prove that $\frac{10^{k+1}}{7+5^{k+1}} > 2^k$. We have the chain of logical equivalences

$$\begin{aligned}
& \frac{10^{k+1}}{7+5^{k+1}} > 2^k \\
\iff\quad & 10^{k+1} > 2^k(7+5^{k+1}) \\
\iff\quad & 10^{k+1} > 7\cdot 2^k + 2^k\cdot 5^{k+1} \\
\iff\quad & 10\cdot 10^k > 7\cdot 2^k + 5\cdot 10^k \\
\iff\quad & (10-5)\cdot 10^k > 7\cdot 2^k \\
\iff\quad & 5\cdot 10^k > 7\cdot 2^k \\
\iff\quad & \frac{10^k}{2^k} > \frac{7}{5} \\
\iff\quad & 5^k > \frac{7}{5}.
\end{aligned}$$

The last statement is true for all $k \geq 1$ hence the first statement is also true for all $k \geq 1$.

Solution

Method 2: Another way to do the comparison. Notice that $7+5^k < 5^{k+1}$ for all $k \geq 1$ because $5^{k+1} - 5^k = (5-1)5^k = 4\cdot 5^k \geq 4\cdot 5 = 20 > 7$ for $k \geq 1$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{10^k}{7+5^k} &< \sum_{k=1}^{\infty} \frac{10^k}{5^{k+1}} \\
&< \sum_{k=1}^{\infty} \frac{1}{5}\frac{10^k}{5^k} \\
&< \sum_{k=1}^{\infty} \frac{1}{5}2^k,
\end{aligned}$$

and this latter series diverges because it is a geometric series with common ratio greater than 1.

Solution

Method 3: Ratio test. If a_k denotes the kth term of the series, then the ratio between a_{k+1} and a_k (absolute values ignored because everything is positive)

$$\frac{10^{k+1}}{7+5^{k+1}} \cdot \frac{7+5^k}{10^k} = 10 \cdot \frac{7+5^k}{7+5^{k+1}} = 10 \cdot \frac{7/5^k+1}{7/5^k+5} \xrightarrow{k\to\infty} 10 \cdot \frac{1}{5} = 2.$$

This limit is greater than 1, therefore the series diverges.

(b) $\displaystyle\sum_{k=1}^{\infty} k \cdot 3^{-k}$.

Solution

The series converges.
Method 1: Comparison. For any $k \geq 1$, we have $k < 2^k$. Therefore,

$$\sum_{k=1}^{\infty} k \cdot 3^{-k} < \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k,$$

and this series converges because it is a geometric series with common ratio less than 1.

Solution

Method 2: Ratio test. Ratio test. If a_k denotes the kth term of the series, then the ratio between a_{k+1} and a_k (absolute values ignored because everything is positive)

$$\frac{k+1}{3^{k+1}} \cdot \frac{3^k}{k} = \frac{1}{3} \cdot \frac{k+1}{k} \xrightarrow{k\to\infty} \frac{1}{3} < 1,$$

so the series converges.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k}$.

Solution

If you tried the ratio test, you would have found that it is inconclusive as the ratio is 1. Now let's do comparison. Since the log in the problem was accidentally ambiguous, some of you chose the base of the log to be 10 while others chose it to be e. In either case, $\log k$ becomes greater than 1 as soon as k is greater than 10 (resp. e). From that point onward the series becomes greater than the corresponding tail of the harmonic series, so the series diverges.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.

Solution

Notice that $\sqrt{n^2-1} < \sqrt{n^2} = n$, therefore $\frac{1}{\sqrt{n^2-1}} > \frac{1}{n}$. By comparison with the tail of the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$, we conclude the series diverges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$.

Solution

We can do a comparison between this and the series $\frac{1}{2^n}$, at least starting from $n = 2$. Indeed, for all $n \geq 2$, $n^n \geq 2^n$, so $\frac{1}{n^n} \leq \frac{1}{2^n}$. Thus,

$$\sum_{n=2}^{\infty} \frac{1}{n^n} \leq \sum_{n=2}^{\infty} \frac{1}{2^n}$$

which converges since it is a geometric series with common ratio less than 1. Therefore, the original sum starting at 1 converges too.
The ratio test is also plausible for this but requires knowing that $\left(\frac{n}{n+1}\right)^n$ converges to $\frac{1}{e}$ as $n \to \infty$, which is techincally something you should be able to deduce from the definition of e as $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, but might be difficult to do on your own. It takes several steps of logical deduction.

Problem 2. Partial fractions.

(a) What is $\frac{1}{2} - \frac{1}{3}$? What is $\frac{1}{3} - \frac{1}{4}$? What is $\frac{1}{4} - \frac{1}{5}$? Make a conjecture based on your findings, then prove it.

Solution

$$\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}.$$
$$\frac{1}{3} - \frac{1}{4} = \frac{4-3}{12} = \frac{1}{12}.$$
$$\frac{1}{4} - \frac{1}{5} = \frac{5-4}{20} = \frac{1}{20}.$$

It looks like $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$.

Proof: $\frac{1}{n} - \frac{1}{n+1} = \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$.

(b) Using what you proved in part (a), find the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{99\cdot 100}$$

and the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots.$$

Solution

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right).$$

In this sum, every term from $\frac{1}{2}$ to $\frac{1}{99}$ gets canceled out. Fun fact: Sums/series like this are called telescoping sums/series. Anyway, what remains is $1 - \frac{1}{100} = \frac{99}{100}$, so this is the answer.
To get the value of the infinite series, one must take the limit of $1 - \frac{1}{n}$ as $n \to \infty$, which is evidently 1. We got $1 - \frac{1}{n}$ by generalizing our reasoning from the first part.

(c) Challenge! Use your thinking skills, reflecting on how you solved the previous part, to find the sum

$$\frac{1}{1\cdot 4} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 6} + \cdots.$$

Solution

It looks like the general term of this series is $\frac{1}{n(n+3)}$. Let's see what $\frac{1}{n} - \frac{1}{n+3}$ equals:

$$\frac{1}{n} - \frac{1}{n+3} = \frac{(n+3)-n}{n(n+3)} = \frac{3}{n(n+3)}.$$

It looks like this is 3 times too big, so we divide everything by 3 to get what we want:

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right).$$

Therefore,

$$\begin{aligned}
&\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\cdots\\
&\quad=\frac{1}{3}\left(1-\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)+\cdots\\
&\quad=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)\\
&\quad=\frac{1}{3}\cdot\frac{11}{6}\\
&\quad=\frac{11}{18},
\end{aligned}$$

where the third line follows from the second because 1, $\frac{1}{2}$, and $\frac{1}{3}$ are the only terms that survive the telescoping series.

(d) A slight change can make a problem much much harder. Let's now look at the following sum:

$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$$

This sum is similar in form to the one in part (b) but the limit is now irrational! What does the internet (e.g. Wolfram Alpha) say this sum equals? (You might want to figure out how to express it as a summation so you can input it into the service.)

Then give a guess as to how one might prove it. (Hint: the sum in the Zax problem of 2 psets ago converged to $1-\ln 2$. Maybe there's a connection...)

Solution

WA gives a value of $\ln 2$. To input it into WA I had to turn the sum into the form

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k)}$$

and type in something like

```
sum of 1/((2k-1)(2k)) from k=1 to infinity.
```

As to the how the proof might go: no spoilers! (Spoiler in the solution to the next problem set.) I'll just say that maybe the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ might be involved in some way. Any reasonable and logical guess is acceptable for this part.

(e) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, using comparison and the result of part (b).[1]

Solution

The naive comparison between $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ has the wrong direction, so we need to do some shifting. Instead, let's compare $\sum_{n=2}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ with $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We have for every positive integer n that $(n+1)^2 = (n+1)(n+1) > n(n+1)$, therefore $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges as proven in part (b), so does the series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$. But this one is a tail of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, so the latter series converges too.

Problem 3. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered that

$$\frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}} = \frac{1}{\pi}.$$

Prove the more modest assertion, that the series converges at all.

Solution

Let's ignore the $\frac{2\sqrt{2}}{9801}$ factor as it does not affect convergence. (It will also get cancelled in the ratio test.) Let's apply the ratio test: the ratio between the $(n+1)$st term and the nth term will be

$$\begin{aligned}
&\frac{(4(n+1))!(1103 + 26390(n+1))}{((n+1)!)^4 396^{4(n+1)}} \cdot \frac{(n!)^4 396^{4n}}{(4n)!(1103 + 26390n)} \\
&= \frac{(4n+4)!}{(4n)!} \cdot \frac{(n!)^4}{(n+1)!^4} \cdot \frac{396^{4n}}{396^{4(n+1)}} \cdot \frac{1103 + 26390(n+1)}{1103 + 26390n} \\
&= \frac{(4n+4)!}{(4n)!} \cdot \frac{1}{(n+1)^4} \cdot \frac{1}{396^4} \cdot \frac{1103 + 26390(n+1)}{1103 + 26390n}.
\end{aligned}$$

We seek the limit of this as $n \to \infty$. Now, $(4n+4)! = (4n)!(4n+1)(4n+2)(4n+3)(4n+4)$, so that the first term in the last line simplifies to $(4n+1)(4n+2)(4n+3)(4n+4)$. The last term in the last line converges to 1 as $n \to \infty$ since both the numerator and

[1] Finding the sum was a famous problem, called the Basel Problem because the Bernoulli family and Euler (all from Basel, Switzerland) worked on it. It was Euler who found the sum in 1734. We may see later how he did it.

denominator are linear polynomials in n with leading coefficient 26390. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(4n+4)!}{(4n)!} \cdot \frac{1}{(n+1)^4} \cdot \frac{1}{396^4} \cdot \frac{1103+26390(n+1)}{1103+26390n} \\
&= \frac{1}{396^4} \lim_{n\to\infty} \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)(n+1)(n+1)(n+1)} \\
&= \frac{1}{396^4} \lim_{n\to\infty} \frac{4^4 n^4 + O(n^3)}{n^4 + O(n^3)} \\
&= \frac{1}{396^4} \cdot 4^4 \\
&= \frac{4^4}{396^4} \\
&= \left(\frac{1}{99}\right)^4 \\
&< 1.
\end{aligned}$$

Therefore, the series converges.

Problem 4. Rate the difficulty of each problem (1a, 1b, 1c, 1d, 1e, 2a, 2b, 2c, 2d, 2e, 3) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!
- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.
- 6 – I could not start this problem without help, but after getting help I was able to solve the problem. However, I still don't feel like I understand what is going on in this problem.
- 7 – I could not solve the problem, even with help.

Solution

:)