MATH1103 FALL 2022
PROBLEM SET 4

This problem set is due on Wednesday, September 28 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.6.4) What is the average value of the function $\sqrt{x}$ between 0 and 4?

(b) (Strang 8.1.14) Find the area bounded by $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$.

(c) (Strang 5.6.17) What number $\overline{v}$ gives

$$\int_a^b (v(x) - \overline{v})\, dx = 0?$$

Justify your answer.

(d) (Strang 8.1.53) If a roll of paper with inner radius 2 cm and outer radius 10 cm has about 10 thicknesses per centimeter, approximately how long is the paper when unrolled?

Hint: Computing the volume of the roll would be a good place to start.

(e) (Adapted from Strang 5.6.32) I choose a number at random between 0 and 1, and you choose a number at random between 0 and 1 as well. What is the probability that the square of my number is less than your number? (For example, if I chose 0.4 and you chose 0.2, then the square of my number, 0.16, would be less than your number.)

Problem 2 (Adapted from Strang 5.6.27)**.** On the curved portion of a semicircle with radius 1 centered at the origin lying above the x-axis, a point P is chosen at random. What is the average height (i.e. y-coordinate) of P? In fact, this problem has multiple different answers depending on how exactly the random point is chosen.

(a) Find the answer assuming P is chosen by choosing a random number a between -1 and 1, and then taking the point P on the semicircle with x-coordinate equal to a.

(b) Now find the answer assuming P is chosen by choosing a random angle θ between 0 and π, and then taking the point P to be $(\cos\theta, \sin\theta)$.

Problem 3 (Adapted from Strang 5.6.24)**.** Let $v_1, v_2, \ldots$ be positive numbers such that $v_{n+1} < v_n$ for all n, in other words, the sequence v_n is decreasing. (For example, we could have $v_1 = 1, v_2 = 0.5, v_3 = 0.2, v_4 = 0.1, v_5 = 0.05$, and so on.) For each n, let $a_n = (v_1 + \cdots + v_n)/n$, i.e. the average of the first n terms. Prove that $a_{n+1} < a_n$ for all n, in other words, the sequence a_n is decreasing.

Hint: Equivalently, you have to prove that $a_{n+1} - a_n < 0$ for all n. Can you write $a_{n+1} - a_n$ in terms of the v_i in a helpful way?

Problem 4. Find the hyper-volume of the unit sphere in 4 dimensions, which has equation $x^2 + y^2 + z^2 + w^2 \leq 1$. (This problem is an advertisement for how powerful math is when dealing with objects we cannot visualize.)

Hint: Slice it. What are the cross sections?

Hint 2: To see if you made any mistakes, here is the answer: $\frac{1}{2}\pi^2$. Of course you must still show a derivation.

Note: In solving this problem it may become necessary to find the antiderivative of $\cos^4 \theta$. To do this, read Chapter 7.2 of Strang, pages 288–290, and use reduction formula (7).

Note 2: Here is a real life interpretation of this seemingly abstract 4-dimensional volume. It says that the probability that 4 numbers x, y, z, w, each chosen randomly from -1 to 1, will satisfy $x^2 + y^2 + z^2 + w^2 \leq 1$, is $\pi^2/32 \approx 31\%$. Contrast that with the 2 dimensional case, where the probability is $\pi/4 \approx 78.5\%$!

Optional challenge: Continue to higher dimensional spheres. Can you find a pattern? Or a recursion?