# MATH1103 FALL 2022
# PROBLEM SET 2

This problem set is due on Wednesday, September 14 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

**Problem 1.** Some exercises.

(a) Find $\int_a^b 1\,dx$ (in terms of $a$ and $b$).

(b) Find $\int_a^b x^{100}\,dx$.

(c) (Strang 5.7.7) Find
$$\frac{d}{dx}\int_x^{x+1} v(t)\,dt.$$
(The integral here is sometimes called a *running average* of $v$.)

(d) (Strang 5.7.8) Find
$$\frac{d}{dx}\left(\frac{1}{x}\int_0^x v(t)\,dt\right).$$
(The integral here represents the average value of $v$ from 0 to $x$. A hint is to use the product rule for derivatives.)

(e) Let $a, b$ be real numbers. Explain why
$$\frac{d^2}{dx^2}\int_0^x \int_a^b \sin(u^3)\,du\,dt = 0.$$

*Hint*: This is actually not a deep problem. The lesson here is to not be scared of the "double" integral; it's just an integral of an expression which happens to itself be an integral.

**Problem 2** (Strang 5.4.43)**.** If $f(t)$ is an antiderivative of $v(t)$, find an antiderivative of

(a) $v(t+3)$.

(b) $v(t)+3$.

**Problem 3.** Find
$$\int_0^{76.5} \lfloor x \rfloor\,dx.$$

Recall that for any real number $x$, the notation $\lfloor x \rfloor$, read *floor* of $x$, means the greatest integer less than or equal to $x$. For example, $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$, and $\lfloor 15 \rfloor = 15$.

**Problem 4.** Define

$$f(x) = \begin{cases} 1 & x \text{ is rational,} \\ 0 & x \text{ is irrational.} \end{cases}$$

It turns out that the definite integral

$$\int_0^1 f(x)\,dx$$

does not exist! In this problem you will find out why.

You may take for granted the following fact about the real number line:

*Between any two distinct real numbers, no matter how close they are to each other, one can find both rational numbers and irrational numbers.*

Using this fact, show that no matter how large $n$ is, a Riemann sum for the integral $\int_0^1 f(x)\,dx$ with $n$ equal partitions can be made equal to 1 or 0 depending on how you pick the sample points.

*Remark*: Since the Riemann sums do not approach a limit, the aforementioned definite integral therefore does not exist, and such a function is said to be *non-integrable*.

*Remark 2*: Of course, the same reasoning applies to any bounds on the integral, not just from 0 to 1.

**Problem 5.** Let $f$ be an odd function, meaning that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Give two different proofs that

$$\int_{-a}^{a} f(x)\,dx = 0$$

for any real number $a$:

(a) Graphically using the definition of definite integral as a signed area.

(b) Algebraically using $u$-substitution.