MATH1103 FALL 2022
EXAM 2 REVIEW

Many of these review problems are similar to homework problems. For best learning, please try to work them out at first **without** referring to notes or your homework.

Problem 1. Determine whether the following series converges or not. You may use comparison test/ratio test/any other logical reasoning steps that make perfect sense.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n.$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+3}.$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n\cdot\sin^2 n}.$

Note: You may use the fact that $\sin n \neq 0$, for any positive integer n.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}.$

(e) $\displaystyle\sum_{k=1}^{\infty}\ln k.$

(f) $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k}.$

(g) $\displaystyle\frac{1}{20000}+\frac{1}{20001}+\frac{1}{20002}+\frac{1}{20003}+\cdots.$

(h) $\displaystyle\frac{1}{10000}+\frac{1}{40000}+\frac{1}{90000}+\frac{1}{160000}+\cdots.$

(i) $\displaystyle\frac{1}{10000}+\frac{1}{30000}+\frac{1}{90000}+\frac{1}{270000}+\cdots.$

(j) $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$

(k) $\displaystyle\sum_{k=1}^{\infty} a_k$ where $a_k = e^{-k}$ if $k < 10^{100}$ but $a_k = \frac{1}{k}$ if $k \geq 10^{100}$.

We did not have a question exactly like this in class or on homework, so please work this problem out very carefully!

(l) $\displaystyle\sum_{k=1}^{\infty} a_k$ where $a_k = 2^{-k}$ if k is not a power of 2, but $a_k = \frac{1}{1000}$ if k is a power of 2.
The first few terms of (a_k) look like: $\frac{1}{1000}, \frac{1}{1000}, \frac{1}{8}, \frac{1}{1000}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{1000}$, because 3, 5, 6, and 7 are not powers of 2 while 1, 2, 4, and 8 are.

Problem 2. If (x_n) and (y_n) both diverge, then answer the following questions. Notice that we didn't discuss these exact questions in class nor did you have them on your homework (so don't bother looking in your notes!), but nevertheless these can be answered completely convincingly using only your thinking and some logic. Exercise your creative mind to try to come up with counterexamples!

(a) Must $(x_n + y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample

(b) Must $(x_n \cdot y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample.

(c) Must $(x_n - y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample.

(d) Must $\left(\dfrac{x_n}{y_n}\right)$ be divergent (say $y_n \neq 0$ for all n in this case)? If so, give a proof. Otherwise, give a counterexample.

Problem 3.

(a) Prove that $\left(\frac{999}{1000}\right)^n$ converges to 0, using the ε definition of convergence and logarithms.

How to start your proof, if you're stuck: "Let $\varepsilon > 0$ be arbitrary. [Now you must find an integer N such that $\left|\left(\frac{999}{1000}\right)^n - 0\right| < \varepsilon$ for all $n \geq N$.]" Also check the front page of the Canvas site for a guide.

(b) Let $a_0 = 1000$ and $a_n = a_{n-1} - \frac{1}{1000}a_{n-1}$ for $n \geq 1$. Prove that a_n converges to 0.

Hint: Make observations first! You should find something that reduces this to a problem you already solved.... Hence avoiding having to write an ε proof for this.

(c) Prove that $\left(\frac{1000}{999}\right)^n$ does **not** converge to 0, using the ε definition of convergence.

Hint: First prove that $\left(\frac{1000}{999}\right)^{n+1} > \left(\frac{1000}{999}\right)^n$ for all n.

Problem 4. Deriving the formula for geometric series.

(a) if
$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$
for some numbers a, r, then what kind of formula can you come up with to calculate s_n?

(b) Now we assume that $|r| < 1$. Then the (geometric series)
$$s = a + ar + ar^2 + ar^3 + \cdots$$
converges and we are very familiar with a formula that we've been using again and again that $s = \dfrac{a}{1-r}$. First, prove this formula with the result you calculated in part(a).

(c) Besides proving part(b) with the result from part(a), we can also prove the formula $s = \dfrac{a}{1-r}$ directly. Finish the proof yourself.

(d) Use geometric series to show that $1 = .999\cdots$

Problem 5. Recall that the decimal expression of a number $x = 0.a_1a_2a_3\cdots$, where $a_i \in \{0, 1, \cdots, 9\}$ means
$$x = a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + \cdots.$$
While in the homework, we explored that a binary expression of a number $x = 0.b_1b_2b_3\cdots$, where $b_i \in \{0, 1\}$ means
$$x = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + b_3 \cdot 2^{-3} + \cdots.$$
Similarly, we can also define a ternary expression for a number $x = 0.c_1c_2c_3\cdots$, where $c_i \in \{0, 1, 2\}$ means
$$x = c_1 \cdot 3^{-1} + c_2 \cdot 3^{-2} + c_3 \cdot 3^{-3} + \cdots.$$
Then determine what the number $x = 0.1111\ldots$ really stands for in decimal, binary, ternary expressions respectively.

Problem 6. What is $0.99989998\ldots$, where the 9998 is repeating?

Problem 7. What does $\frac{\cos n}{n}$ approach as $n \to \infty$? Prove it.

Problem 8. Knowing that $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$ is a true identity that you have already proved in homework, deduce a closed form for $2 + 3x + 4x^2 + 5x^3 + \ldots.$

Problem 9. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Problem 10. Prove that there are infinitely many odd numbers.

Problem 11. Find the exact values of:

(a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$

(b) $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots$

(c) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$

(d) $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots$ (I think there's a π in the answer to this one.)

(e) $\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$

(f) $\sum_{k=1}^{100}((k+1)^2-k^2)$.

(g) $2+\frac{3}{5}+\frac{4}{25}+\frac{5}{125}+\cdots$. (Note: the denominators are powers of 5. If you are stuck and need inspiration, first solve Problem 8.)

Problem 12. What is the power series (around 0) for:

(a) e^{2x}?

(b) e^x-1?

(c) $\arctan x$?

(d) $\arctan(-x)$?

(e) Prove that $\arctan x$ is an odd function using its power series, applying the previous 2 parts.

(f) $\ln(1+x)$?

(g) $400\ln(1+x)$?

(h) $\frac{x^2}{1-x}$?

(i) $\frac{1}{(1+x)(1-x)}$? (Hint: Multiply out the denominator.)

(j) $\frac{x^2}{(1+x)(1-x)}$?

Problem 13. Prove that

$$\frac{1}{0!}+\frac{1}{2!}+\frac{1}{4!}+\cdots=\frac{e+\frac{1}{e}}{2}.$$

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

Remark 1. Please let me know if I made any mistakes in the below!

(a) Diverges, because the limit of terms is not 0.

(b) Converges, by comparison with $\frac{1}{n^2}$.

(c) Diverges, by comparison with the harmonic series.

(d) Converges, by comparison with $\frac{1}{n^2}$.

(e) Diverges, because the limit of the terms is not 0.

(f) Diverges, by comparison with the harmonic series (after the 3rd term).

(g) Diverges, because it is a tail of the harmonic series.

(h) Converges, because it is a constant multiple of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(i) Converges, because it is a geometric series with common ratio between 0 and 1.

(j) Converges, because by partial fractions this becomes a decreasing alternating series whose terms converge to 0.

(k) The tail is what matters, and the tail is the tail of a harmonic series, so this series diverges.

(l) The sequence of terms was constructed in such a way that it does not converge to 0. Indeed, put $\varepsilon = \frac{1}{2000}$, then no matter what N is, there exists $n \geq N$ such that $a_n \geq \varepsilon$. (Just pick the next power of 2 after N.) This proves that $a_n \not\to 0$, so the series does not converge.

Or you can say that the series includes infinitely many $\frac{1}{1000}$ terms, and all other terms are positive, so it diverges by comparison with the sum $\frac{1}{1000} + \frac{1}{1000} + \cdots$.

Remark 2.

(a) Counterexample: Let $x_n = 2^n$ and $y_n = -2^n$, then $x_n + y_n$ is the constant zero sequence. Thus (x_n) and (y_n) both diverge but $(x_n + y_n)$ converges!

(b) Counterexample: Let x_n be any sequence whose odd subsequence (meaning $x_1, x_3, x_5, \ldots$) is divergent and whose even terms are all 0, and let y_n be the other way around. Then $x_n y_n$ is always 0.

(c) Counterexample: Let x_n and y_n be the same divergent sequence. $x_n - y_n$ is the constant sequence 0.

(d) Exact same idea as in part (c) applies here too! Haha. x_n/y_n will be the constant sequence 1.

Remark 3.

(a) This was in problem set 8 or so.

(b) This looks like a completely new problem on first sight, but actually, the equation $a_n = a_{n-1} - \frac{1}{1000}a_{n-1}$ simplifies to
$$a_n = \left(1 - \frac{1}{1000}\right) a_{n-1} = \frac{999}{1000}a_{n-1}.$$
So each term is 999/1000 times as large as the previous one. This is the geometric series again! So this series and the one in part (a) are identical.

(c) Since $\frac{1000}{999} > 1$, we can multiply this inequality on both sides by $\left(\frac{1000}{999}\right)^n$ to deduce that $\left(\frac{1000}{999}\right)^{n+1} > \left(\frac{1000}{999}\right)^n$ for all n. It follows that the sequence $\left(\frac{1000}{999}\right)^n$ is an increasing sequence of positive real numbers, so cannot converge to 0. (More detail: the first term is 1, and the sequence is increasing, so every term is at least a distance 1 away from 0.)

Remark 4. Check out one of the discussions. Also the internet has plenty of derivations for the geometric series formula. Part (d) was a homework problem.

Remark 5. In base n, $0.111\ldots$ is the series $\sum_{k=1}^{\infty} \frac{1}{n^k}$. So in binary ($n = 2$), the series converges to 1, which leads us to say that $0.111_2 \ldots = 1$. (The subscript 2 after a string of digits is standard notation which indicates the number is to be read in binary.) This should be very reminiscent of the $0.999\ldots = 1$ equality in base 10. In fact, if $\triangle$ represents the digit $(n-1)$ in base n, then $0.\triangle\triangle\triangle\ldots$ always equals 1! Maybe you can try to prove this. . .

Similarly, one can plug in $n = 3$ into the series to find that $0.111\ldots_3$ is the ternary expansion for the very familiar number 1/2!

And of course, $0.111\ldots_{10}$ is 1/9. (Plug in $n = 10$ into the series to verify this.)

Remark 6. Answer for your checking purposes:
$$\frac{9998}{9999}.$$

Remark 7. $\cos n$ is bounded while $\frac{1}{n}$ converges to 0. Therefore $\frac{\cos n}{n}$, being the product of a bounded sequence and a sequence that converges to 0, converges to 0. (You can also use the squeeze theorem, which could be one of our favorite theorems ■)

Remark 8. You have to notice that $2 + 3x + 4x^2 + 5x^3 + \ldots$ is the sum of $1 + 2x + 3x^2 + 4x^3 + \ldots$ and $1 + x + x^2 + x^3 + \ldots$, both of which you already know. The rest is some algebra!

Remark 9. We compare the series with the series of Problem 1(j)! Just as in problem set 10 problem 2(e).

Remark 10. This is supposed to remind you of Euclid's proof of the infinitude of primes. A very short proof that there are infinitely many odd numbers might be as follows:

> Suppose there are finitely many odd numbers. Then there is a largest odd number, let's call it N. But then $N+2$ is also an odd number, and $N+2$ is greater than N, contradicting the fact that N was the largest odd number.

If you want to also justify why $N+2$ is odd if N is odd, you can use the following: The definition of N being odd is that $N = 2k+1$ for some positive integer k. To prove $N+2$ is odd, one has to show that $N+2 = 2m+1$ for some positive integer m. But $N+2 = (2k+1)+2 = 2k+3 = 2(k+1)+1$. So we have found our m, it is $k+1$ which is an integer!

Remark 11. Answers:

(a) 1

(b) $\ln 2$

(c) $\dfrac{1}{2}$

(d) $\dfrac{\pi}{8}$

(e) $e - \dfrac{1}{0!} - \dfrac{1}{1!} - \dfrac{1}{2!} = e - 1 - 1 - \dfrac{1}{2} = e - \dfrac{5}{2}$

(f) $101^2 - 1^2 = 10200$ (keyword: telescope) And if you remember the idea from Problem Set 1 Problem 1, you could compute the value as follows: $101^2 - 1^2 = (101+1)(101-1) = 102 \cdot 100 = 10200$. Neat!

(g) Take what you got for problem 8 and plug in $x = \frac{1}{5}$.

Remark 12. Answers (in non-sigma form. Either sigma or non-sigma form is fine):

(a) $1 + 2x + \dfrac{4x^2}{2!} + \dfrac{8x^3}{3!} + \cdots$

(b) $x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

(c) $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots$

(d) $(-x) + \dfrac{x^3}{3} - \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots$

(e) The answers to the two previous parts are exact negatives of each other. This shows that $\arctan(-x)$ and $-\arctan(x)$ represent the same function, which shows that $\arctan(-x) = -\arctan(x)$ for all real numbers x, which is exactly what proves that arctan is an odd function.

(f) We know that

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$$

so if we plug in $-x$ in place for x and negate everything, we'll get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

(g)

$$400\ln(1+x) = 400x - 200x^2 + \frac{400x^3}{3} - 100x^4 + \cdots.$$

(h) It's x^2 times the geometric series $1 + x + x^2 + \cdots$, which is $x^2 + x^3 + x^4 + \cdots$.

(i)

$$\frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots.$$

(j) It's x^2 times the above, so it's $x^2 + x^4 + x^6 + \cdots$.

Remark 13. The power series for e^x is $1/0! + x/1! + x^2/2! + x^3/3! + \cdots$. The number e can be obtained by plugging in $x = 1$ to this power series, so we get $1/0! + 1/1! + 1/2! + 1/3! + 1/4! + \cdots$. The number $1/e$, which is also equal to e^{-1}, can be obtained by plugging in $x = -1$ to the power series for e^x, so we get $1/0! - 1/1! + 1/2! - 1/3! + 1/4! - \cdots$. If we add the power series of e to that of $1/e$, the even terms double while the odd terms cancel out. Finally dividing the result by 2 gives us

$$\frac{e + \frac{1}{e}}{2} = \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots.$$