MATH1103 FALL 2022
PROBLEM SET 4

This problem set is due on Wednesday, September 28 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.6.4) What is the average value of the function $\sqrt{x}$ between 0 and 4?

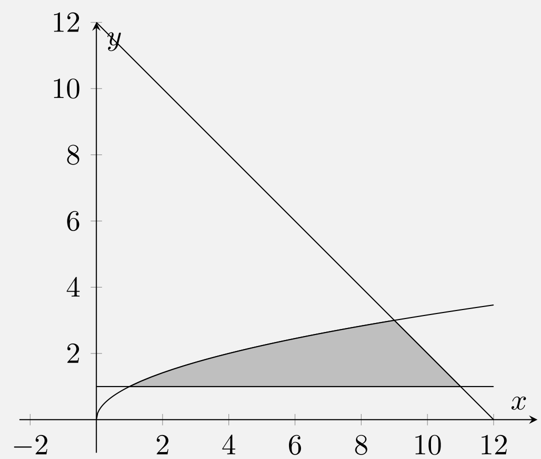
Solution

$\frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{1}{6}(4^{\frac{3}{2}} - 0^{\frac{3}{2}}) = \frac{8}{6} = \frac{4}{3}$.

(b) (Strang 8.1.14) Find the area bounded by $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$.

Solution

The diagram is as follows:



The equations $y = \sqrt{x}$ and $y = 1$ intersect at $(1, 1)$. The equations $y = \sqrt{x}$ and $y = 12-x$ intersect at $(9, 3)$. The equations $y = 12-x$ and $y = 1$ intersect at $(11, 1)$. So between $x = 1$ and $x = 9$, the vertical cross-section of the region at position x has length $\sqrt{x} - 1$, and between $x = 9$ and $x = 11$, the vertical cross-section of the region at position x has length $(12 - x) - 1 = 11 - x$.

Therefore, the total area is

$$\begin{aligned}
A &= \int_1^9 (\sqrt{x}-1)\,dx + \int_9^{11} (11-x)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - x\right]_1^9 + \left[11x - \frac{1}{2}x^2\right]_9^{11} \\
&= \left((18-9) - \left(\frac{2}{3}-1\right)\right) + \left(\left(11\cdot 11 - \frac{1}{2}\cdot 121\right) - \left(11\cdot 9 - \frac{1}{2}\cdot 81\right)\right) \\
&= \frac{28}{3} + 2 \\
&= \frac{34}{3}.
\end{aligned}$$

(c) (Strang 5.6.17) What number $\overline{v}$ gives

$$\int_a^b (v(x) - \overline{v})\,dx = 0?$$

Justify your answer.

Solution

If $\int_a^b (v(x) - \overline{v})\,dx = 0$, then

$$\begin{aligned}
& \int_a^b v(x)\,dx - \int_a^b \overline{v}\,dx = 0 \\
\iff\quad & \int_a^b v(x)\,dx - (b-a)\overline{v} = 0 \\
\iff\quad & \overline{v} = \frac{1}{b-a}\int_a^b v(x)\,dx.
\end{aligned}$$

(d) (Strang 8.1.53) If a roll of paper with inner radius 2 cm and outer radius 10 cm has about 10 thicknesses per centimeter, approximately how long is the paper when unrolled?

Hint: Computing the volume of the roll would be a good place to start.

Solution

The key insight: the rolled paper and the unrolled paper must have the same volume, because no paper is gained or lost when unrolling! Let h be the height of the roll (unspecified, but it will turn out to not matter). Let ℓ be the length of the roll, i.e. the quantity we are trying to find. The thickness of the paper is given to be $1/10$ cm. The volume of the unrolled paper is therefore $\frac{1}{10}\ell h$.

To compute the volume of the rolled paper, notice that it has a constant cross-sectional area where each cross-section is a washer with outer radius 10 and inner radius 2, which has area $\pi(100-4) = 96\pi$. Therefore, the volume of the roll is $96\pi h$. Equating the two volumes gives the equation

$$\frac{1}{10}\ell h = 96\pi h$$

which implies that $\ell = 960\pi$. No integrals were harmed in the making of this solution. :)

(e) (Adapted from Strang 5.6.32) I choose a number at random between 0 and 1, and you choose a number at random between 0 and 1 as well. What is the probability that the square of my number is less than your number? (For example, if I chose 0.4 and you chose 0.2, then the square of my number, 0.16, would be less than your number.)

Solution

The space of possibilities of my number x and your number y can be represented as the square $\{(x,y) \in \mathbb{R}^2 \colon 0 \le x \le 1,\ 0 \le y \le 1\}$, and picking our numbers corresponds to picking a random point in this square. The choices satisfy $x^2 < y$ if and only if the point (x,y) is to the left of (or equivalently, above) the parabola $y = x^2$. Since the area under the parabola is $1/3$, the area above the parabola is $2/3$. So the probability that the square of my number is less than your number is $2/3$.

Problem 2 (Adapted from Strang 5.6.27)**.** On the curved portion of a semicircle with radius 1 centered at the origin lying above the x-axis, a point P is chosen at random. What is the average height (i.e. y-coordinate) of P? In fact, this problem has multiple different answers depending on how exactly the random point is chosen.

(a) Find the answer assuming P is chosen by choosing a random number a between -1 and 1, and then taking the point P on the semicircle with x-coordinate equal to a.

Solution

We are taking the average value of $\sqrt{1-x^2}$ from -1 to 1. This is

$$\frac{1}{1-(-1)}\int_{-1}^{1}\sqrt{1-x^2}\,dx = \frac{1}{2}(\text{area of the unit semicircle}) = \frac{\pi}{4}.$$

In this solution we recognized the definite integral $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ as the area of the unit semicircle, which is $\frac{\pi}{2}$.

(b) Now find the answer assuming P is chosen by choosing a random angle θ between 0 and π, and then taking the point P to be $(\cos\theta, \sin\theta)$.

Solution

We are taking the average value of $\sin\theta$ from 0 to π. This is
$$\frac{1}{\pi}\int_0^\pi \sin\theta\, d\theta = \frac{2}{\pi}.$$
In this solution we calculated $\int_0^\pi \sin\theta\, d\theta = -\cos\theta\Big|_0^\pi = -(-1-1) = 2$.

Problem 3 (Adapted from Strang 5.6.24)**.** Let $v_1, v_2, \ldots$ be positive numbers such that $v_{n+1} < v_n$ for all n, in other words, the sequence v_n is decreasing. (For example, we could have $v_1 = 1, v_2 = 0.5, v_3 = 0.2, v_4 = 0.1, v_5 = 0.05$, and so on.) For each n, let $a_n = (v_1 + \cdots + v_n)/n$, i.e. the average of the first n terms. Prove that $a_{n+1} < a_n$ for all n, in other words, the sequence a_n is decreasing.

Hint: Equivalently, you have to prove that $a_{n+1} - a_n < 0$ for all n. Can you write $a_{n+1} - a_n$ in terms of the v_i in a helpful way?

Solution

First let's algebraically simplify our goal. We want to show
$$\frac{v_1 + \cdots + v_{n+1}}{n+1} - \frac{v_1 + \cdots + v_n}{n} < 0.$$
By multiplying both sides by $n(n+1)$, it is equivalent to show that
$$n(v_1 + \cdots + v_{n+1}) - (n+1)(v_1 + \cdots + v_n) < 0.$$
This equation simplifies to
$$-v_1 - v_2 - \cdots - v_n + nv_{n+1} < 0$$
or
$$nv_{n+1} < v_1 + v_2 + \cdots + v_n.$$
So it suffices to show that $nv_{n+1} < v_1 + v_2 + \cdots + v_n$. But this follows from the fact that $v_{n+1} < v_1$, $v_{n+1} < v_2$, $\ldots$, $v_{n+1} < v_n$, and adding these n inequalities together.

Problem 4. Find the hyper-volume of the unit sphere in 4 dimensions, which has equation $x^2 + y^2 + z^2 + w^2 \leq 1$. (This problem is an advertisement for how powerful math is when dealing with objects we cannot visualize.)

Hint: Slice it. What are the cross sections?

Hint 2: To see if you made any mistakes, here is the answer: $\frac{1}{2}\pi^2$. Of course you must still show a derivation.

Note: In solving this problem it may become necessary to find the antiderivative of $\cos^4\theta$. To do this, read Chapter 7.2 of Strang, pages 288–290, and use reduction formula (7).

Note 2: Here is a real life interpretation of this seemingly abstract 4-dimensional volume. It says that the probability that 4 numbers x, y, z, w, each chosen randomly from -1 to 1, will satisfy $x^2 + y^2 + z^2 + w^2 \leq 1$, is $\pi^2/32 \approx 31\%$. Contrast that with the 2 dimensional case, where the probability is $\pi/4 \approx 78.5\%$!

Optional challenge: Continue to higher dimensional spheres. Can you find a pattern? Or a recursion?

Solution

If we slice the 4D sphere by the planes $y = a$ for $a \in [-1, 1]$, we get 3D spheres of radii $\sqrt{1-a^2}$. In class I showed this in one lower dimension by appealing to geometry. In 4D it's not easy to do geometry by visualization, so here is a nice way to see this in the 4D case. The 4D sphere has equation $x^2 + y^2 + z^2 + w^2 \leq 1$. Its intersection with the hyperplane $y = a$ is by definition the set of points $(x, y, z, w) \in \mathbb{R}^4$ satisfying simultaneously $x^2 + y^2 + z^2 + w^2 \leq 1$ and $y = a$, i.e. the set

$$\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 \leq 1 \text{ and } y = a\}.$$

This set can be written another way as

$$\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + z^2 + w^2 \leq 1 - a^2 \text{ and } y = a\}.$$

Now we see the first equation specifies a 3D sphere of radius $\sqrt{1-a^2}$, as expected. So in other words, the slice at y is a 3D sphere of radius $\sqrt{1-y^2}$, which has volume $\frac{4}{3}\pi(1-y^2)^{\frac{3}{2}}$. The hyper-volume of the 4D sphere is then

$$\int_{-1}^{1} \frac{4}{3}\pi(1-y^2)^{\frac{3}{2}}\, dy.$$

Let us make the inverse substitution $y = \sin\theta$, $dy = \cos\theta\, d\theta$. The integral then becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{4}{3}\pi(1-\sin^2\theta)^{\frac{3}{2}} \cos\theta\, d\theta = \frac{4}{3}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta \cos\theta\, d\theta = \frac{4}{3}\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

Now we appeal to the reduction formula (7). For $n = 4$ it says

$$4\int \cos^4\theta\, d\theta = \cos^3\theta \sin\theta + 3\int \cos^2\theta\, d\theta.$$

We know the integral of $\cos^2\theta$ from class, but just for fun let us apply (7) again for the $n = 2$ case:

$$2\int \cos^2\theta \, d\theta = \cos\theta\sin\theta + \int 1 \, d\theta = \cos\theta\sin\theta + \theta + C.$$

So

$$\int \cos^4\theta \, d\theta = \frac{1}{4}\left(\cos^3\theta\sin\theta + \frac{3}{2}(\cos\theta\sin\theta + \theta + C)\right).$$

Therefore,

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta \, d\theta &= \frac{1}{4}\left(\cos^3\theta\sin\theta + \frac{3}{2}(\cos\theta\sin\theta + \theta + C)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{4}\left(0 + \frac{3}{2}\left(0 + \frac{\pi}{2}\right)\right) - \frac{1}{4}\left(0 + \frac{3}{2}\left(0 - \frac{\pi}{2}\right)\right) \\
&= \frac{3}{8}\pi.
\end{aligned}$$

Finally we multiply this by $\frac{4}{3}\pi$ to get a hyper-volume of $\frac{1}{2}\pi^2$, as desired.
Let me know if you want to see how this generalizes to higher dimensions! Fun fact: the power of π that appears in the volume for $n = 1, 2, 3, 4, \ldots$ goes $0, 1, 1, 2, 2, 3, 3, \ldots$.