MATH1103 FALL 2022
PROBLEM SET 5

This problem set is due on Wednesday, October 5 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 8.1.50) Rotate $y = x^3$ around the y-axis from $y = 0$ to $y = 8$. Write down the volume integral by shells and disks and compute both ways.

(b) (Strang 8.2.12) What integral gives the length of Archimedes' spiral $x = t \cos t,\ y = t \sin t$?

(c) (Strang 8.3.1) Find the surface area when the curve $y = \sqrt{x},\ 2 \le x \le 6$ is revolved around the x-axis.

(d) (Strang 8.3.19 and 8.3.20) A lamp shade is constructed by rotating $y = 1/x$ around the y-axis, and keeping the part from $y = 1$ to $y = 2$. What is the surface area of the lamp shade?

Hint: The integral computation is surprisingly tricky. See the third page for a guide.

(e) (Strang 8.4.7) If you choose x completely at random between 0 and π, what is the density $p(x)$ and the cumulative density $F(x)$?

Problem 2 (Strang 8.2.33)**.**

(a) Write down the integral for the length L of $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(b) Show (without computing the integrals) that $y = \frac{1}{2}x^2$ from $(0, 0)$ to $(2, 2)$ is exactly twice as long.

Problem 3. You toss a coin repeatedly and stop when you get heads. Let X be a random variable representing the number of coins tossed. (So the minimum X can be is 1, which happens if your first flip lands heads. But there is no maximum.)

(a) For each n let $p_n = \Pr[X = n]$, the probability that X equals n. Show that $p_n = 1/2^n$ for every positive integer n.

(b) (Strang 8.4.2) What is the probability that X is odd?

(c) (Strang 8.4.4) Show that the probability P that X is a prime number satisfies

$$\frac{6}{16} \leq P \leq \frac{7}{16}.$$

(d) (Strang 8.4.20) Find the average number μ of coin tosses by writing $p_1 + 2p_2 + 3p_3 + \dots$ as $(p_1 + p_2 + p_3 + \dots) + (p_2 + p_3 + p_4 + \dots) + (p_3 + p_4 + p_5 + \dots) + \dots$.

Guide on how to compute the integral for 1(d)

For this guide I will walk through a computation of the indefinite integral

$$\int \frac{\sqrt{y^4+1}}{y^3}\,dy.$$

First make the substitution $u = y^2$, $du = 2y\,dy$. Then $\sqrt{y^4+1}$ can be written as $\sqrt{u^2+1}$, while $1/y^3\,dy$ can be written as $1/(2u^2)\,du$. So our integral is now

$$\int \frac{\sqrt{u^2+1}}{2u^2}\,du = \frac{1}{2}\int \frac{\sqrt{u^2+1}}{u^2}\,du.$$

There are two choices now and they both work.

Choice 1. Make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$. This takes advantage of the fact that $\sqrt{\tan^2\theta+1} = \sec\theta$. We now have

$$\frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\sec^2\theta\,d\theta.$$

The integrand simplifies (using $\tan\theta = \sin\theta/\cos\theta$) to $\csc^2\theta\sec\theta$. Finally, using $\csc^2\theta = 1+\cot^2\theta$ the integrand simplifies to $\sec\theta + \cot\theta\csc\theta$. The rest, as they say, is history. You can integrate both terms easily by a lookup table. Finally don't forget that a definite integral awaits after you find your antiderivative.

Choice 2. If you like hyperbolic functions, you can make the substitution $u = \sinh\theta$, $du = \cosh\theta\,d\theta$. I won't walk through this beyond this point because I haven't talked about hyperbolic functions at all.