MATH1103 FALL 2022
EXAM 2 REMARKS

WEDNESDAY, NOVEMBER 30, 2022

Name: ______________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1. Convergence tests. Any proof will do. You may use results from class and homework.

(a) (5 points) Find, with proof, whether $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{314159^k}$ converges or diverges.

Remark. Using the ratio test, we have

$$\left| \frac{(k+1)!}{314159^{k+1}} \cdot \frac{314159^k}{k!} \right| = \left| \frac{k+1}{314159} \right| \xrightarrow{k\to\infty} \infty,$$

from which we conclude the series diverges.

(b) (5 points) Find, with proof, whether $\displaystyle\sum_{n=314159}^{\infty} \frac{1}{n-1}$ converges or diverges.

Remark. The series is a tail of the harmonic series, so it diverges.

Problem 2. Power series. You may freely use without proof the power series for e^x we established in class:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

You do not need to write any series in sigma notation.

(a) (5 points) What is the power series of e^{-5x}?

Remark. You can substitute $-5x$ for x in the power series of e^x, or you can calculate $\frac{1}{n!}\frac{d^n}{dx^n}\Big|_{x=0} e^{-5x}$ for each n. Either way works.

(b) (5 points) The below series converges. (You do not need to prove it.)

$$\frac{1}{0!} - \frac{10}{1!} + \frac{100}{2!} - \frac{1000}{3!} + \cdots.$$

What is its exact value? This problem does not depend on part (a).

Remark. This series equals the power series of e^x at $x = -10$, so the exact value is e^{-10}.

Problem 3 (10 points)**.** You are freely given (thanks to Euler) that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

(a) (5 points) Prove, from the above equation, that

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{24}.$$

Remark. Divide the given equation by 4, and use the fact that $2^2=4$.

(b) (5 points) Using the result of part (a), find the value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots .$$

Remark. If you subtract the first equation by the second equation, what remains is $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}$.

Problem 4 (10 points)**.** What are the first 10 digits of $\frac{1}{98}$?

Note: A solution using long division will be awarded at most 5 points.

Hint: First prove that

$$\frac{1}{98} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n}{100^n},$$

then use that result to figure out the digits.

Remark. To prove that

$$\frac{1}{98} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n}{100^n},$$

we notice that the right hand side is a geometric series with common ratio $1/50$ (because $2^n/100^n = (2/100)^n = (1/50)^n$) and with first term $1/2 \cdot 1/50 = 1/100$, so the sum of the geometric series is

$$\frac{1/100}{1-1/50} = \frac{1/100}{49/50} = \frac{1}{100} \cdot \frac{50}{49} = \frac{1}{98}.$$

This proves the hint. Now to use it to figure out the first 10 digits, notice

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{\infty} \frac{2^n}{100^n} &= \frac{1}{2}\left(\frac{2}{10^2} + \frac{4}{10^4} + \frac{8}{10^6} + \frac{16}{10^8} + \frac{32}{10^{10}} + \cdots\right) \\ &= \frac{1}{10^2} + \frac{2}{10^4} + \frac{4}{10^6} + \frac{8}{10^8} + \frac{16}{10^{10}} + \cdots \\ &= 0.0102040816\ldots.\end{aligned}$$

For fun, here is 1/98 to more digits.

$$0.010204081632653061224489795918367346938775510204081632 65\ldots$$

Let me know if you notice any other patterns! (For example, does it repeat?)

Problem 5 (10 points)**.** For $n \geq 1$, let

$$b_n = \begin{cases} 4^n & \text{if } n < 314159 \\ 271828 & \text{if } n \geq 314159. \end{cases}$$

What is $\lim_{n\to\infty} b_n$? Prove it using the ε definition of limit.

Remark. The limit is 271828. Proof: Let $\varepsilon > 0$ be arbitrarily chosen. Choose $N = 314159$. (You could also have chosen N to be any number greater than or equal to 314159 if you wanted.) Then for all $n \geq N$,

$$|b_n - 271828| = |271828 - 271828| = 0 < \varepsilon.$$