INSTRUCTOR'S REPORT, FALL 2022
January 20, 2023

Course number, name: MATH1103, Calculus II (Math/Science Majors)

Instructor: Yongyi Chen

1. Report

1.1. **Texts.**

- Gilbert Strang's *Calclus* for integration
- Mark Reeder's in-house MATH1103 notes for integration and series.

1.2. **Topics (include sections of text).** Each bullet point at level 2 represents one week of class.

- Integration and applications
  - History, areas, definition of definite integral, properties of definite integral, indefinite integrals (Strang 5.1, 5.2, 5.3, 5.5, 5.6)
  - Fundamental theorem of calculus I and II, u-substitution (Reeder 9.5, 9.7, 9.9; Strang 5.4, 5.7)
  - Integration by parts, trig substitution (Strang 5.4, 7.1, 7.2; Reeder 9.8, 9.9)
  - Expected value, basic probability, areas, volumes (Strang 5.6, 8.1; Reeder 9.15)
  - Arc length, surface area, advanced probability (Strang 8.2, 8.3, 8.4; Reeder 9.13, 9.12)
  - More advanced probability (Strang 8.4, Reeder 9.12)
  - Improper integrals and proof that π is irrational (Strang 7.5, Niven)
- Sequences and series
  - Intro to sequences, the ε-lemma, the ε definition of convergence (Reeder 1)
  - More work on the ε definition, intro to series (Reeder 1, 3)
  - Geometric series (Reeder 3)
  - Comparison theorem and other convergence tests (Reeder 3)
  - Power series (Reeder 4)

- Additional topics
  - Proof that e is irrational, combining power series and integration, for example representing erf values as an infinite series that converges very fast.

1.3. **Some comments about the text.**

1.3.1. *Strang's textbook.* The textbook is very well organized. In my opinion it has excellent exposition, motivation, and examples, although I find that most students do not attempt to read the exposition and only prefer to read their handwritten notes (a learned habit from their pre-college math education no doubt). It may be that school does not adequately teach them the level of mathematical maturity required to learn from the exposition. The exercises and problems at the end of each section had excellent variety and challenge, and I drew from them a lot when designing my problem sets.

I also very much liked the way Strang presented the probability section.

1.3.2. *Reeder's notes.* Mark Reeder's MATH1103 notes were excellent too. The flow of topics was non-traditional but, in my opinion, much better motivated than, say, Stewart's. However, I found that the students' schooling did not teach the students enough mathematical maturity to fully absorb the rigorous mathematical language in the notes, so I frequently had to adapt the content to their level of mathematical rigor. This is not a drawback of the notes themselves by any means; one could say the students are not adequately prepared to fully absorb any heavy logical content (such as proofs) that would appear in any calculus textbook.

The notes themselves do not contain many exercises or problems but Mark Reeder graciously provided me with his old homework which I could draw problems from. The problems were excellent, as well.

It's important to note that Reeder's notes are intended for a series-first approach to calculus, and since I was asked to take the integration-first approach, I had to select a different text for the integration part of the course due to the heavy integration of power series in Reeder's integration chapters. As I already mentioned, Strang was my choice.

1.4. **Course format.** In-person, one-hour lectures 3 times per week. Weekly problem sets, with one multi-part (6-8 parts) problem for skill practice and then 2 to 4 challenging problems. In these problem sets, full solutions were expected to be written up. No online homework. According to course evaluations, students still spent on average around 6–9 hours per week on homework, which is higher than the school average.

1.5. **Number of exams.** Two midterm exams (50 minutes) and one final exam (3 hours). All were done in person, and all were open notes.

1.6. **Enrollment.**

- Section 1: 19 (plus 1 listener).
- Section 2: 12.
- Total: 31 (plus 1 listener).

(This semester saw 0 late drops or withdrawals.)

1.7. **Grading policy.**

- 20% homework,
- 20% first exam,
- 20% second exam,
- 40% final.

Each exam was designed so that half of it is skill-based and the other half is problem solving-based. I designed the difficulty of the problems so that scoring 50% of the possible points indicates B level performance and scoring 75% of the possible points indicates A level performance. This may seem like an extreme curve but this is because my exam problems were designed to push the students' thinking.

On the second midterm and final, I gave the instruction to write down partial progress and failed solution attempts for partial credit. I now think this is a very good idea.

Each homework problem was graded on a scale of 0–3 points. I think this is also a very good idea. It may seem like the coarse scale gives students permission to be sloppy with their work but, counterintuitively, that was not the case at all! They still put their best work into the problems. So in some sense, this is a small example of "ungrading."

1.8. **General comments about this course.** There was a serious challenge I had to tackle in this class, but by the end it was a big success. First, the successes:

- The lectures, homework, and exams in the course ended up being very cohesive and there was a clear sense of mathematical story throughout the semester. Moreover, students were unanimously happy (as far as I can tell, at least) about the the grading of the homework and exams.
- Students enjoyed the challenging problems and even called them fun, although some initially complained that they had trouble starting them.
- Students liked the historical approach in Reeder's notes, especially seeing how mathematicans such as Euler contributed to calculus. It really showed them that math is a lively human endeavor, not a cold subject. In fact, after showing how Euler proved that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ using an infinite product expansion of $\sin x$, two students were extremely eager and wondered what would happen if you look at $\cos x$ instead. They worked it out in my office hours all by themselves and got

$\sum_{n=0}^{\infty} 1/(2n+1)^2 = \pi^2/8$. Amazing! And this was just one of many examples of students being excited about mathematics.

- Also thanks to Reeder's notes, students were very quick to understand the nature of a mathematical proof and what a proof is. I have not heard a single expression of fear of the word "proof" in this class, and the proof questions on exams had high success rates.

- After the interventions I made mid-semester (described below), performance on exams rose drastically, from 60% average on the first exam to 80% average on the second exam and 73% average on the final. I think the performance on the final is especially impressive given the problems I put on there – the last four problems on the final consisted entirely of problems that were new to them to some extent.

Now onto the challenges. The main challenge was finding out after the first exam that about half of the students were not picking up on the habits of mathematical thinking even though I demonstrated them during lectures and had them practice them by doing challenging homework problems. By mathematical thinking here, I do not mean something as advanced as proof skills required of math majors, but something like the following:

- The ability to apply knowledge to a new context.

- The ability to start a problem for which the solution is not known in advance.

- The ability to make logical deductions from previous work and knowledge. (A Chinese phrase that Yaoying taught me that is about this is the Confucian saying "ju yi fan san," meaning "learn one, deduce three.")

These three skills are absolutely crucial in every walk of life. They are also required if you want students to have a true understanding of the material in any math class (especially the skill of making logical deductions, aka "ju yi fan san").

After a lot of thinking, I came up with the following three reasons why students did not gain mathematical thinking habits by mid-semester:

- Many students complained that they could not start the challenging problems and needed to get help. After getting help, they were able to proceed. But maybe the help they got was too guided and they failed to generalize their lessons.

- Most students thought lectures were meant to be pure informational material (to be copied down).

- Lecture is ineffective in changing habits, namely the (non)-problem solving habits they learned in school. Even doubly so if they are just writing things down during lectures without thinking.

I made several interventions in response to this. They are as follows:

- On each homework, I started asking them to rate the difficulty of each homework problem on a 1–7 scale (you can see in the attached homeworks), with 6–7 representing that had to seek help and still don't understand the problem afterwards. This benefitted me in that I could see which homework problems they struggled on, and also benefitted them in knowing which homework problems to redo when it came time to study for exams.
- In order to teach them how to start a novel problem, I spent significant class time calling groups of 4 at a time to the board to start problems by writing down observations and logical deductions. It is important to note that this is not the same as "groupwork," in fact I did not ask them to finish the problem (although many groups did manage to finish the problem).

It is interesting to note that the second item not only teaches how to start a problem but also has the effect of slowing down the pace of the lectures (a good thing for students who are struggling to keep up) as well as allowing students to show different approaches for a problem to the whole class.

1.8.1. *Yaoying's comments.* Yaoying has told me that often in discussions, students would sit and stare at the discussion questions until Yaoying comes to help. When this is the case, a compromise must be struck between helping students individually and helping the class as a whole. This is another case where it's clearly better if students were taught how to start on problems before entering college!

1.9. **Final grade distribution.**

- 16 A
- 4 A−
- 4 B+
- 3 B
- 2 B−
- 2 C+

Attached are the syllabus, all tests (including final), list of HW assignments, and any other pertinent material.

COURSE INFORMATION FOR MATH1103 (FALL 2022)
CALCULUS II (MATH/SCIENCE MAJORS)

Calculus has a rich history, interesting mathematics, and many applications. In this class we will explore two of the main themes: integration and applications in the first half, and sequences and series in the second half. Time permitting, we may explore polar coordinates and parametric equations.

For questions about correct course placement, talk to me or see advice at `https://www.bc.edu/bc-web/schools/mcas/departments/math/undergraduate/about-calculus.html`.

Instructor: Yongyi Chen
Email: `yongyi.chen@bc.edu`

Teaching assistant: Yaoying Fu
Email: `fugh@bc.edu`

Lectures:

- Section 1: MWF 1:00 pm–1:50 pm in Gasson Hall 302
- Section 2: MWF 2:00 pm–2:50 pm in Gasson Hall 302

Homework: Weekly, due on Wednesdays at 11:59 pm.

Office: Maloney 532
Office hours:

- Yongyi: Fridays 3–4 pm, Wednesdays 3–5 pm in Maloney 532, both in person.
- Yaoying: Wednesdays 12–1 pm, Thursadys 12–2 pm in Maloney 537.

1. Course information

Course website. On Canvas. There you will find homework assignments, homework solutions, and supplemental course materials.

Course format. In person. Both office hours are in person. I may change hours or add more office hours based on demand.

Textbooks. We will be loosely following selected sections from the following two course notes:

- Mark Reeder's MATH1103 notes.
- Gilbert Strang's calculus textbook, freely available online.

Both of these two are linked on Canvas.

As a supplementary resource for light and fun reading, I also suggest reading *Hitchhiker's Guide to Calculus* by Spivak.

Homework. There will be weekly homework, due on Wednesdays at 11:59 pm. Because homework solutions will be posted on Canvas, late homework will not be accepted. To submit your homework, upload a single PDF file to Gradescope (accessible from within the Canvas assignment page as well).

You are encouraged to collaborate on homework with your classmates, but the work that you turn in must be your own and must be written in your own words. Working together is good; copying somebody else's work is plagiarism.

Typesetting your homework using LaTeX is strongly encouraged, but not required.

Discussion. Yaoying Fu will be running our discussion sections, on Thursdays at 9 am, 10 am, and 11 am. Attendance is strongly encouraged, as you will be able to practice on additional problems, work with classmates, and ask the TA any questions.

Exams and grading. There will be two in-class exams (50 minutes each) and a final (120 minutes). Final grades will be determined by a weighted average of homework and exam scores. Homework counts for 20%, each in-class exam counts for 20%, and the final counts for 40%.

All exams will be given in class, in the same classroom as lectures are held in. Dates are as follows:

- Midterm 1: Wednesday, October 19
- Midterm 2: Wednesday, November 30
- Final exam:
  - Section 1: Wednesday, December 14 at 12:30 pm
  - Section 2: Friday, December 16 at 12:30 pm

There will be no homework due on the same weeks as exams are held.

Academic integrity. Cheating of any kind will result in a failing grade for the course and referral to the Dean's office for disciplinary action. For more information on academic integrity see `https://www.bc.edu/integrity`.

Resources. Here are some Resources to take advantage of:

(1) Come to class!

(2) I have office hours, listed above.

(3) The Connors Family Learning Center provides peer tutoring for all Boston College Students. See `www.bc.edu/libraries/help/tutoring.html` or call 617-552-0611 to schedule an appoint- ment, after add/drop.

(4) Math Department Tutoring: This is a drop-in tutoring staffed by math majors.

(5) The Math Department office maintains a list of tutors-for-hire who have indicated their availability for the term. Contact me if you are interested in being put in touch with a personal, paid tutor.

If you are a student with a documented disability seeking reasonable accommodations in this course, please contact the Connors Family Learning Center regarding learning disabilities and ADHD, or the Disability Services Office (617) 552-3470 regarding all other types of disabilities, including temporary disabilities. Advance notice and appropriate documentation are required for accommodations.

2. List of topics

(1) Integration and applications
- Areas and distances
- The definite integral
- The fundamental theorem of calculus
- Integration techniques: substitution, parts, trigonometric integrals, partial fractions, improper integrals
- Applications: area, volume, probability

(2) Sequences and series
- Definition of sequences, definition of series, definition of convergence
- Convergence theorems: comparison theorem, integral test, alternating series test, ratio test, root test
- Power series and Taylor series

(3) Additional topics
- Combining power series and integration

MATH1103 FALL 2022
EXAM 1

WEDNESDAY, OCTOBER 19, 2022

Name: ______________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam.

Problem 1. Integrals. If you are using a result from class/homework/discussion, make sure you state it clearly.

(a) (2 points) Calculate $\displaystyle\int_{-\pi}^{\pi} \sin x \, dx$.

(b) (2 points) Calculate the indefinite integral $\displaystyle\int \frac{e^x}{e^x+1} \, dx$.

(c) (2 points) Calculate $\displaystyle\int_1^3 \ln x \, dx$.

(d) (2 points) Calculate $\displaystyle\int_{-1}^{1} (x^5 + \sin(x^3)) \, dx$.

(e) (2 points) For which values of x is $\displaystyle\int_1^x \left(\frac{1}{|t|} + e^{t^2} \right) dt$ a well-defined number?

Problem 2. Let P be the paraboloid formed by rotating the region bounded by $y = \frac{1}{2}x^2$, $x = 0$, and $y = 2$ around the y-axis.

(a) (5 points) What is the volume of P?

Hint: I found the disk method the easiest here.

(b) (5 points) Show that the lateral surface area of P is $\frac{2}{3}\pi(5\sqrt{5} - 1)$. (Lateral just means not including the top lid portion.)

Problem 3. A dartboard has the shape of a circle of radius 1. A dart hits a random point in the circle, where by random we mean that the probability that the dart lands in any region R inside the circle is equal to the area of R divided by the area of the circle.

(a) (5 points) Let X be the random variable representing the dart's distance from the center. For any r between 0 and 1, what is the probability that $X \leq r$?

(b) (5 points) What is the expected distance of the dart from the center?

Problem 4 (10 points)**.** Find the variance of the exponential distribution given by $p(x) = ae^{-ax}$ for $x \geq 0$ (and 0 for $x < 0$). You may use the fact that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ for any random variable X, and the fact the mean of this exponential distribution is $1/a$.

Problem 5 (10 points)**.** Let f be a continuous function and suppose that $f(x) = f(-x)$ for all x. Prove algebraically (not graphically) that

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

for all a.

MATH1103 FALL 2022
EXAM 2

WEDNESDAY, NOVEMBER 30, 2022

Name: ______________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1. Convergence tests. Any proof will do. You may use results from class and homework.

(a) (5 points) Find, with proof, whether $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{314159^k}$ converges or diverges.

(b) (5 points) Find, with proof, whether $\displaystyle\sum_{n=314159}^{\infty} \frac{1}{n-1}$ converges or diverges.

Problem 2. Power series. You may freely use without proof the power series for e^x we established in class:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

You do not need to write any series in sigma notation.

(a) (5 points) What is the power series of e^{-5x}?

(b) (5 points) The below series converges. (You do not need to prove it.)

$$\frac{1}{0!} - \frac{10}{1!} + \frac{100}{2!} - \frac{1000}{3!} + \cdots .$$

What is its exact value? This problem does not depend on part (a).

Problem 3 (10 points)**.** You are freely given (thanks to Euler) that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

(a) (5 points) Prove, from the above equation, that

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\frac{\pi^2}{24}.$$

(b) (5 points) Using the result of part (a), find the value of

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots.$$

Problem 4 (10 points)**.** What are the first 10 digits of $\dfrac{1}{98}$?

Note: A solution using long division will be awarded at most 5 points.

Hint: First prove that

$$\frac{1}{98} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{100^n},$$

then use that result to figure out the digits.

Problem 5 (10 points)**.** For $n \geq 1$, let

$$b_n = \begin{cases} 4^n & \text{if } n < 314159 \\ 271828 & \text{if } n \geq 314159. \end{cases}$$

What is $\lim_{n\to\infty} b_n$? Prove it using the ε definition of limit.

MATH1103 FALL 2022
FINAL EXAM

WEDNESDAY, DECEMBER 14, 2022
FRIDAY, DECEMBER 16, 2022

Name: ________________________________

This exam is open notes, but calculators are not allowed. There are 100 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1.

(a) (3 points) Calculate $\displaystyle\int_{-\pi}^{\pi}(\sin x + \cos x)\, dx$.

(b) (3 points) Calculate $\displaystyle\int \frac{\ln x}{x}\, dx$.

(c) (4 points) Calculate $\displaystyle\int_0^1 |e^x - 2|\, dx$.

Problem 2.

(a) (5 points) Show that $\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$.

(b) (5 points) Calculate $\int x\cos^2 x\,dx$.

Problem 3.

(a) (5 points) Find the area bounded by the parabola $y = x^2$ and the line through the points $(-1, 1)$ and $(2, 4)$.

(b) (5 points) Let R be the part of the region described in (a) to the right of the y-axis, and form a solid of revolution by revolving R around the y-axis. What is the volume of this solid?

Problem 4 (10 points)**.** For each of the following sums, find its exact value or prove that it diverges.

(a) (3 points) $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \cdots$.

(b) (3 points) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$.

(c) (4 points) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$.

Problem 5 (10 points)**.** For which values of x does the power series $\displaystyle\sum_{n=1}^{\infty} \frac{nx^{n-1}}{5^{n-1}}$ converge?

Problem 6.

(a) (5 points) Show that $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$.

Hint: Start by writing $\tan x = \dfrac{\sin x}{\cos x}$.

(b) (5 points) An angle θ is chosen uniformly at random between 0 and $\pi/4$. What is the average slope of the line joining $(0, 0)$ and $(\cos\theta, \sin\theta)$?

Problem 7 (10 points)**.** Let (a_n) be a sequence such that $0 \leq a_n \leq 1$ for all positive integers n and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Problem 8 (10 points)**.** Prove that

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}.$$

Hint: Power series may be useful.

Problem 9 (10 points)**.** The standard normal distribution is given by the probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Let $A = \int_{-1}^{1} p(x)\, dx$. Prove that

$$A > \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{1}{2}} + e^{-\frac{1}{8}} \right).$$

Hint: Draw a picture and think about Riemann sums! (Maybe try looking at a lower Riemann sum with 4 subdivisions. . .)

Problem 10 (10 points)**.** You push a button to call the elevator. The instant you push the button, a music cycle starts which consists of 10 seconds of music, then 10 seconds of silence, then 10 seconds of music, then 10 seconds of silence, and so on, until the elevator arrives.

Of course, the waiting time for the elevator is random. The probability distribution for the amount of time, in seconds, it takes for the elevator to arrive is modeled by the exponential density function $p(t) = \frac{1}{10}e^{-\frac{1}{10}t}$, $t \geq 0$. (So for example, the average waiting time is 10 seconds according to the model.)

What is the probability that there is music playing at the moment the elevator arrives?

MATH1103 FALL 2022
PROBLEM SET 1

This problem set is due on Wednesday, September 7 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Algebra, functions, and differential calculus review and practice.

(a) Find a clever and simple way to evaluate $1002 \cdot 998$.

(b) Derive a formula for $1 + 2 + \cdots + n$, the sum of the first n positive integers. Try to think of as short a derivation as possible.

(c) What is $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \cdots}}}$?

You do not have to prove this continued fraction converges; just use algebra to find its value.

(d) What is the coefficient of xyz in $(x + y + z)^3$?

(e) Find three different real-valued functions f such that $(f(x))^2 = x^2$ for all $x \in \mathbb{R}$.

(f) State the intermediate value theorem as precisely as possible, but in your own words.

(g) Find a formula, in terms of the positive integer n, for the nth derivative of $\ln(x)$.

Problem 2. Using only Euclidean geometry (no trigonometry), determine the area of a regular 12-sided polygon inscribed in a unit circle (i.e. a circle of radius 1). You should be able to get an exact answer. How close is the area to π?

Problem 3. Use the method of Riemann sums with 20 equal divisions to approximate π, using the function $f(x) = \sqrt{1 - x^2}$. Use a calculator or computer to help you with all the additions! (Excel or Google Sheets should be helpful here.)

Problem 4. Find a general (possibly piecewise) expression for

$$\int_a^b |x| \, dx$$

in terms of the two real numbers a, b (which can each be positive, negative, or zero!).

Problem 5. For any sequence $\underline{a} = (a_1, a_2, a_3, \dots)$ of real numbers, define the *finite difference operator* Δ by

$$\Delta(\underline{a}) = (a_2 - a_1, a_3 - a_2, a_4 - a_3, \dots).$$

For example, $\Delta(1, 2, 3, 4, \dots) = (1, 1, 1, 1, \dots)$. Also define the *cumulative sum operator* Σ by

$$\Sigma(\underline{a}) = (a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots).$$

For example, $\Sigma(1, 2, 3, 4, \dots) = (1, 3, 6, 10, \dots)$. In general, what is $\Delta(\Sigma(\underline{a}))$? (If you aren't sure, try on a few examples of your own.) Once you find a result, give a proof that your result holds.

Remark: This is the discrete analog of the fundamental theorem of calculus! Finite differences are the discrete version of derivatives, and cumulative sums are the discrete version of integrals.

MATH1103 FALL 2022
PROBLEM SET 2

This problem set is due on Wednesday, September 14 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) Find $\int_a^b 1\,dx$ (in terms of a and b).

(b) Find $\int_a^b x^{100}\,dx$.

(c) (Strang 5.7.7) Find
$$\frac{d}{dx}\int_x^{x+1} v(t)\,dt.$$
(The integral here is sometimes called a *running average* of v.)

(d) (Strang 5.7.8) Find
$$\frac{d}{dx}\left(\frac{1}{x}\int_0^x v(t)\,dt\right).$$
(The integral here represents the average value of v from 0 to x. A hint is to use the product rule for derivatives.)

(e) Let a, b be real numbers. Explain why
$$\frac{d^2}{dx^2}\int_0^x \int_a^b \sin(u^3)\,du\,dt = 0.$$

Hint: This is actually not a deep problem. The lesson here is to not be scared of the "double" integral; it's just an integral of an expression which happens to itself be an integral.

Problem 2 (Strang 5.4.43)**.** If $f(t)$ is an antiderivative of $v(t)$, find an antiderivative of

(a) $v(t+3)$.

(b) $v(t)+3$.

Problem 3. Find
$$\int_0^{76.5} \lfloor x \rfloor\,dx.$$

Recall that for any real number x, the notation $\lfloor x \rfloor$, read *floor* of x, means the greatest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$, $\lfloor -\pi \rfloor = -4$, and $\lfloor 15 \rfloor = 15$.

Problem 4. Define

$$f(x) = \begin{cases} 1 & x \text{ is rational,} \\ 0 & x \text{ is irrational.} \end{cases}$$

It turns out that the definite integral

$$\int_0^1 f(x)\,dx$$

does not exist! In this problem you will find out why.

You may take for granted the following fact about the real number line:

Between any two distinct real numbers, no matter how close they are to each other, one can find both rational numbers and irrational numbers.

Using this fact, show that no matter how large n is, a Riemann sum for the integral $\int_0^1 f(x)\,dx$ with n equal partitions can be made equal to 1 or 0 depending on how you pick the sample points.

Remark: Since the Riemann sums do not approach a limit, the aforementioned definite integral therefore does not exist, and such a function is said to be *non-integrable.*

Remark 2: Of course, the same reasoning applies to any bounds on the integral, not just from 0 to 1.

Problem 5. Let f be an odd function, meaning that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Give two different proofs that

$$\int_{-a}^{a} f(x)\,dx = 0$$

for any real number a:

(a) Graphically using the definition of definite integral as a signed area.

(b) Algebraically using u-substitution.

MATH1103 FALL 2022
PROBLEM SET 3

This problem set is due on Wednesday, September 21 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.4.1) Find
$$\int \sqrt{2+x}\,dx.$$
(Don't forget to add $+C$ to your final answer.)

(b) (Strang 5.4.14) Find
$$\int t^3\sqrt{1-t^2}\,dt.$$
Hint: Starting with $u = 1 - t^2$ worked for me.

(c) (Strang 5.4.20) Find
$$\int \sin^3 x\,dx.$$
Hint: The identity $\sin^2 x = 1 - \cos^2 x$ may be useful.

(d) (Strang 7.1.2) Find
$$\int xe^{4x}\,dx.$$

(e) (Strang 7.1.27) Find
$$\int_0^1 \ln(x)\,dx.$$

(f) (Strang 7.3.3) Using trigonometric substitution, find
$$\int \sqrt{4-x^2}\,dx$$
and use this to calculate
$$\int_{-2}^{2} \sqrt{4-x^2}\,dx.$$
Could you have found this definite integral using geometry instead?

Problem 2.

(a) Use software such as Desmos or WolframAlpha to find an approximation to the following definite integral:

$$\int_0^4 e^{(x-2)^4}\,dx.$$

Do the same for the following definite integral:

$$\int_0^4 xe^{(x-2)^4}\,dx.$$

If you divide the bigger number by the smaller, what do you get?

Hint: If you didn't get a positive integer, you probably inputted one or more integrals wrong somehow.

(b) Prove that your observation is in fact exactly true. You may find it useful to use u-substitution along with a hefty dose of symmetry.

Hint: The values of the definite integrals in part (a) have no known closed forms! This suggests that trying to get exact values for the integrals will be a dead end.

Hint 2: If you are still stuck, see the footnote.[1] Try plotting the first term in the right hand side of the footnote and see if you notice anything.

Problem 3. Let

$$f(x) = \int_1^x \frac{1}{t}\,dt.$$

For example, this means that $f(3) = \int_1^3 \frac{1}{t}\,dt$, $f(s) = \int_1^s \frac{1}{t}\,dt$, and $f(xy) = \int_1^{xy} \frac{1}{t}\,dt$.

Put yourself in the mind of someone who does not know yet that f is the natural logarithm function and wants to prove that f is the natural logarithm function. One way to start proving this is to show that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y. That is, f turns multiplication into addition. This is one of the defining properties of the natural logarithm.

Using properties of integrals and u-substitution, prove that $f(xy) = f(x) + f(y)$ for all positive real numbers x and y.

Optional challenge: The other ingredient needed to prove that f is the natural logarithm is to prove that f is continuous and that $f(e) = 1$. Continuity (in fact, differentiability) follows from the fact that f is an area function whose derivative is $1/x$. Can you prove that $f(e) = 1$ from first principles? Of course, you'll need a working definition of e. Here is one:

$$e \coloneqq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

[1] $xe^{(x-2)^4} = (x-2)e^{(x-2)^4} + 2e^{(x-2)^4}$.

Problem 4. Recall that I mentioned in passing that areas are also invariant under rotations and reflections. In particular, areas are invariant under a reflection across the line $y = x$. This is interesting because the graph of the inverse of a function f is also the reflection across the line $y = x$ of the graph of f. We will apply this to show a surprising relationship between the definite integrals of a function and its inverse, using the example of the inverse pair e^x and $\ln x$.

(a) First, show using the integration by parts technique that, for any $b > 1$,

$$\int_1^b \ln t \, dt = b \ln b - b + 1.$$

(b) Now we will show the same identity without using any integration techniques! Do the following:

(i) Draw on coordinate axes the graph of e^x and shade the region under this curve from $x = 0$ to $x = \ln b$. Also find the area of this region.

(ii) Reflect everything across the line $y = x$ and show the result on a new set of coordinate axes. Also draw the rectangle with corners $(0, 0)$, $(b, 0)$, $(b, \ln b)$, and $(0, \ln b)$.

(iii) If you drew everything right, the unshaded region within the rectangle should correspond exactly to the definite integral

$$\int_1^b \ln t \, dt.$$

Deduce the formula from part (a) by combining this observation with previous observations.

MATH1103 FALL 2022
PROBLEM SET 4

This problem set is due on Wednesday, September 28 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 5.6.4) What is the average value of the function $\sqrt{x}$ between 0 and 4?

(b) (Strang 8.1.14) Find the area bounded by $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$.

(c) (Strang 5.6.17) What number $\overline{v}$ gives

$$\int_a^b (v(x) - \overline{v})\, dx = 0?$$

Justify your answer.

(d) (Strang 8.1.53) If a roll of paper with inner radius 2 cm and outer radius 10 cm has about 10 thicknesses per centimeter, approximately how long is the paper when unrolled?

Hint: Computing the volume of the roll would be a good place to start.

(e) (Adapted from Strang 5.6.32) I choose a number at random between 0 and 1, and you choose a number at random between 0 and 1 as well. What is the probability that the square of my number is less than your number? (For example, if I chose 0.4 and you chose 0.2, then the square of my number, 0.16, would be less than your number.)

Problem 2 (Adapted from Strang 5.6.27)**.** On the curved portion of a semicircle with radius 1 centered at the origin lying above the x-axis, a point P is chosen at random. What is the average height (i.e. y-coordinate) of P? In fact, this problem has multiple different answers depending on how exactly the random point is chosen.

(a) Find the answer assuming P is chosen by choosing a random number a between -1 and 1, and then taking the point P on the semicircle with x-coordinate equal to a.

(b) Now find the answer assuming P is chosen by choosing a random angle θ between 0 and π, and then taking the point P to be $(\cos\theta, \sin\theta)$.

Problem 3 (Adapted from Strang 5.6.24)**.** Let $v_1, v_2, \dots$ be positive numbers such that $v_{n+1} < v_n$ for all n, in other words, the sequence v_n is decreasing. (For example, we could have $v_1 = 1, v_2 = 0.5, v_3 = 0.2, v_4 = 0.1, v_5 = 0.05$, and so on.) For each n, let $a_n = (v_1 + \cdots + v_n)/n$, i.e. the average of the first n terms. Prove that $a_{n+1} < a_n$ for all n, in other words, the sequence a_n is decreasing.

Hint: Equivalently, you have to prove that $a_{n+1} - a_n < 0$ for all n. Can you write $a_{n+1} - a_n$ in terms of the v_i in a helpful way?

Problem 4. Find the hyper-volume of the unit sphere in 4 dimensions, which has equation $x^2 + y^2 + z^2 + w^2 \leq 1$. (This problem is an advertisement for how powerful math is when dealing with objects we cannot visualize.)

Hint: Slice it. What are the cross sections?

Hint 2: To see if you made any mistakes, here is the answer: $\frac{1}{2}\pi^2$. Of course you must still show a derivation.

Note: In solving this problem it may become necessary to find the antiderivative of $\cos^4\theta$. To do this, read Chapter 7.2 of Strang, pages 288–290, and use reduction formula (7).

Note 2: Here is a real life interpretation of this seemingly abstract 4-dimensional volume. It says that the probability that 4 numbers x, y, z, w, each chosen randomly from -1 to 1, will satisfy $x^2 + y^2 + z^2 + w^2 \leq 1$, is $\pi^2/32 \approx 31\%$. Contrast that with the 2 dimensional case, where the probability is $\pi/4 \approx 78.5\%$!

Optional challenge: Continue to higher dimensional spheres. Can you find a pattern? Or a recursion?

MATH1103 FALL 2022
PROBLEM SET 5

This problem set is due on Wednesday, October 5 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 8.1.50) Rotate $y = x^3$ around the y-axis from $y = 0$ to $y = 8$. Write down the volume integral by shells and disks and compute both ways.

(b) (Strang 8.2.12) What integral gives the length of Archimedes' spiral $x = t\cos t,\ y = t\sin t$?

(c) (Strang 8.3.1) Find the surface area when the curve $y = \sqrt{x},\ 2 \leq x \leq 6$ is revolved around the x-axis.

(d) (Strang 8.3.19 and 8.3.20) A lamp shade is constructed by rotating $y = 1/x$ around the y-axis, and keeping the part from $y = 1$ to $y = 2$. What is the surface area of the lamp shade?

Hint: The integral computation is surprisingly tricky. See the third page for a guide.

(e) (Strang 8.4.7) If you choose x completely at random between 0 and π, what is the density $p(x)$ and the cumulative density $F(x)$?

Problem 2 (Strang 8.2.33)**.**

(a) Write down the integral for the length L of $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(b) Show (without computing the integrals) that $y = \frac{1}{2}x^2$ from $(0, 0)$ to $(2, 2)$ is exactly twice as long.

Problem 3. You toss a coin repeatedly and stop when you get heads. Let X be a random variable representing the number of coins tossed. (So the minimum X can be is 1, which happens if your first flip lands heads. But there is no maximum.)

(a) For each n let $p_n = \Pr[X = n]$, the probability that X equals n. Show that $p_n = 1/2^n$ for every positive integer n.

(b) (Strang 8.4.2) What is the probability that X is odd?

(c) (Strang 8.4.4) Show that the probability P that X is a prime number satisfies

$$\frac{6}{16} \leq P \leq \frac{7}{16}.$$

(d) (Strang 8.4.20) Find the average number μ of coin tosses by writing $p_1 + 2p_2 + 3p_3 + \ldots$ as $(p_1 + p_2 + p_3 + \ldots) + (p_2 + p_3 + p_4 + \ldots) + (p_3 + p_4 + p_5 + \ldots) + \ldots$.

Guide on how to compute the integral for 1(d)

For this guide I will walk through a computation of the indefinite integral

$$\int \frac{\sqrt{y^4+1}}{y^3}\,dy.$$

First make the substitution $u = y^2$, $du = 2y\,dy$. Then $\sqrt{y^4+1}$ can be written as $\sqrt{u^2+1}$, while $1/y^3\,dy$ can be written as $1/(2u^2)\,du$. So our integral is now

$$\int \frac{\sqrt{u^2+1}}{2u^2}\,du = \frac{1}{2}\int \frac{\sqrt{u^2+1}}{u^2}\,du.$$

There are two choices now and they both work.

Choice 1. Make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$. This takes advantage of the fact that $\sqrt{\tan^2\theta+1} = \sec\theta$. We now have

$$\frac{1}{2}\int \frac{\sec\theta}{\tan^2\theta}\sec^2\theta\,d\theta.$$

The integrand simplifies (using $\tan\theta = \sin\theta/\cos\theta$) to $\csc^2\theta\sec\theta$. Finally, using $\csc^2\theta = 1+\cot^2\theta$ the integrand simplifies to $\sec\theta + \cot\theta\csc\theta$. The rest, as they say, is history. You can integrate both terms easily by a lookup table. Finally don't forget that a definite integral awaits after you find your antiderivative.

Choice 2. If you like hyperbolic functions, you can make the substitution $u = \sinh\theta$, $du = \cosh\theta\,d\theta$. I won't walk through this beyond this point because I haven't talked about hyperbolic functions at all.

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PROBLEM SET 6

This problem set is due on Wednesday, October 12 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Some exercises.

(a) (Strang 8.4.3) Why is $p(x) = e^{-2x}$ not an acceptable probability density for $x \geq 0$? Why is $p(x) = 4e^{-2x} - e^{-x}$ (also for $x \geq 0$) not acceptable?

(b) (Strang 8.4.5) If $p(x) = e^{-x}$ for $x \geq 0$, find the probability that $X \geq 2$ and the approximate probability that $1 \leq X \leq 1.01$.

(c) (Strang 8.4.12) Find the mean of the distribution given by the PDF $p(x) = e^{-x}$ for $x \geq 0$. (Integrate by parts.)

(d) (Strang 8.4.19) Supernovas are expected about every 100 years. What is the probability that you will be alive for the next one?

Hint: You can use a Poisson model with $\lambda = 0.01 \times$ (your lifetime) and estimate your lifetime. You can also use an exponential distribution.

(Supernovas actually occurred in 1054 (Crab nebula), 1572, 1604, and 1987. But the future distribution doesn't depend on the date of the last one.)

Remark: Strang discusses the Poisson distribution on pages 331–332, including in the table at the end of page 332.

(e) (Strang 8.4.24) What is the variance of the uniform distribution on $[0, 1]$? What is the standard deviation?

(f) (Strang 8.4.33) Suppose grades have a normal distribution with mean 70 and standard deviation 10. If 300 students take the test and passing is 55, how many are expected to fail? Write your answer as a definite integral then give an approximation. What passing grade will fail 1/10 of the class?

Problem 2. You know from class that $\mathrm{Var}(X)$ is defined as $\mathbb{E}[(X - \mu)^2]$, where $\mu = \mathbb{E}[X]$. Use calculus to prove the following famous identity for variance:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Hint: If you are stuck, Exercise 8.4.23 in Strang has some spoilers to get you started, but you still need to explain the step Strang asks to explain.

Problem 3. Two random numbers X and Y are drawn (independently) from a uniform distribution on $[0, 1]$. Find $\mathbb{E}[\max(X, Y)]$.

Hint: This is a pretty fun problem. Start by finding the CDF of $\max(X, Y)$. Use the fact that $\max(X, Y) \leq z$ if and only if both $X \leq z$ and $Y \leq z$.

Optional challenge: Generalize to the $\max(X_1, \ldots, X_n)$ where each X_i is drawn from a uniform distribution on $[0, 1]$.

Problem 4 (Optional)**.** This video talks about the surprisingly non-intuitive problem of generating a random point inside a circle. After watching the video, think about the following.

(a) What do you think is the fairest way to pick a random point on a circle?

(b) The video mentioned something called inverse transform sampling. To test if you understood it, what is the probability density function of X^2 if X is drawn from a uniform distribution on $[0, 1]$? (Recall we saw in class a while ago that the average value of X^2 was $1/3$.)

Hint: As usual, start by finding the CDF.

Spoiler: the PDF of X^2 is $\frac{1}{2\sqrt{x}}$ for $0 \leq x \leq 1$.

Problem 5 (Optional)**.**

(a) Let X and Y be two independent random variables with PDFs $p(x)$ and $q(x)$ respectively. Prove that the PDF of $X + Y$ is given by

$$\mathrm{PDF}_{X+Y}(x) = \int_{-\infty}^{\infty} p(t)q(x - t)\, dt.$$

Hint: Find the CDF of $X + Y$, then differentiate under the integral sign.

(b) We write $X \sim N(\mu, \sigma)$ to mean that X follows a normal distribution with mean μ and standard deviation σ. Using part (a), prove that if $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, then $X + Y \sim N(0, \sqrt{2})$.

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PROBLEM SET 7

This problem set is due on Wednesday, October 26 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Euclid's proof of the infinitude of the sequence of primes, shown in full on the last page, is one of the oldest and most classic proofs in mathematics. Read it 15 times (I am serious!). On each reading, ask yourself why each statement follows from previous statements or already known facts, and try to answer it.

(a) Once you have finished, rewrite the proof in your own words.

(b) Give two examples of finite lists of primes and verify that the proof does indeed produce a prime not in the list for both examples.

(c) To show you understand the proof, say what happens if you try to set $P = p_1 p_2 \cdots p_r + 2$ instead of $p_1 p_2 \cdots p_r + 1$. The proof won't work anymore, but where and why does the proof break down?

Problem 2. Example 3 on page 26 of Reeder's notes proves that r^n converges to 0 for any $0 < r < 1$. Use this result to extend it to the result that $r^n \to 0$ for any $-1 < r < 1$, but without needing to write a long proof using the Binomial Theorem again. Do it by the following steps.

(a) Handling the $r = 0$ case first. Let's take for granted that we know that all constant sequences, that is sequences of the form $x_n = a$ for some number a independent of n, converge to a. (We might prove this result in class or in a future homework.) How can you use this result to handle the $r = 0$ case?

(b) Complete the following statement (fill in the blanks, but write everything on your paper):

The theorem that r^n converges to 0 for all $0 < r < 1$ says that _____(for all/there exists) $0 < r < 1$, _____(for all/there exists) $\varepsilon > 0$, _____(for every/there exists an) integer N such that, _____(for all/there exists) $n \geq N$, $|r^n - 0| < \varepsilon$. The inequality $|r^n - 0| < \varepsilon$ can be written more simply as __________.

Let's call this theorem "Theorem A." Read this theorem 15 times.

(c) Complete the following statement (fill in the blanks, but write everything on your paper):

We wish to prove that r^n converges to 0 when $-1 < r < 0$. To this end, let $\varepsilon > 0$ be arbitrary. Also define $r' = -r$. Since r satisfies $-1 < r < 0$, we deduce that r' satisfies ______, so we can use Theroem A on r', plugging into it our ε as the ε in Theorem A. When we do this, we obtain an integer N such that, ______(for all/there exists) $n \geq N$, ____________(same as the last blank of part (b), but with r' in place of r). Let us pick the same N for our proof. Then, for all $n \geq N$, (the number of blanks is not required to be 2, but these will take some thinking)

$$|r^n - 0| = |r^n| = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_ < \varepsilon,$$

so we win.

After you've written your proof, read it 15 times so you understand it by heart. (This part is mandatory.)

Problem 3. We can use the ε-lemma to prove that $0.999\ldots = 1$. Do it by the following steps.

(a) Prove that for any $\varepsilon > 0$, there exists a positive integer n such that $10^{-n} < \varepsilon$.

Big hint: Use the result mentioned in the beginning of Problem 2 (what is a suitable choice of r that might be useful here?), then you barely have to write anything here.

(b) If we define $a_n = 0.99\ldots9$ with n 9s, then explain why $0.999\ldots > a_n$.

Hint: What's the decimal expansion of $0.999\ldots - a_n$?

(c) Also explain why $1 - a_n = 10^{-n}$.

(d) Also show why part (b) implies that $1 - 0.999\ldots < 1 - a_n$.

(e) You can finally write the proof that $0.999\ldots = 1$ as follows (fill in the blanks; write out the whole proof in your submission, of course):

Let $\varepsilon > 0$ be arbitrary. Pick n large enough so that $10^{-n} < \varepsilon$. Then $|1-0.999\ldots| = 1 - 0.999\ldots < 1 -$ ______ $= 10^{-n} <$ ______. Therefore, $|1 - 0.999\ldots|$ is less than every positive number, so by the ε-lemma, $1 - 0.999\ldots =$ ______, meaning that $0.999\ldots = 1$.

After you've written your proof, read it 15 times so you understand it by heart. (This part is mandatory.)

Problem 4. Prove that $\frac{1}{\sqrt{n}}$ converges to 0 using the ε definition of convergence.

Problem 5. Rate the difficulty of each problem (1a, 1b, 1c, 2a, 2b, 2c, 3a, 3b, 3c, 3d, 3e, 4) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!

- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.
- 6 – I could not start this problem without help, but after getting help I was able to solve the problem. However, I still don't feel like I understand what is going on in this problem.
- 7 – I could not solve the problem, even with help.

1. Euclid's proof of the infinitude or primes (c. 300 BC)

Theorem 1. *There are more primes than can be found in any finite list of primes.*

Proof. Call the primes in our finite list $p_1, p_2, \ldots, p_r$. Let

$$P = p_1 p_2 \cdots p_r + 1.$$

Now P is either prime or it is not. If it is prime, then P is a prime that was not in our list. If P is not prime, then it is divisible by some prime, call it p. Notice p cannot be any of $p_1, p_2, \ldots, p_r$, otherwise p would divide $P - p_1 p_2 \cdots p_r = 1$, which is impossible. So this prime p is some prime that was not in our original list. Either way, the original list was incomplete. □

Some background knowledge necessary for the understanding of the proof.

- A *prime number* is a positive integer, greater than 1, which is not divisible by any other positive integer except 1 and itself.
- Every positive integer has a unique decomposition as a product of prime numbers. For example, $60 = 2 \cdot 2 \cdot 3 \cdot 5$. A seemingly weaker, but actually equivalent, statement is that every positive integer has at least one prime factor.
- When we say "a divides b," this just means that b is a multiple of a, or in other words, a is a factor of b. Or more formally, there exists an integer k such that $b = ka$.
- If a divides b and a divides c, then a divides $b + c$ as well as $b - c$. (This is actually a theorem that can be proved from the previous bullet.)

MATH1103 FALL 2022
PROBLEM SET 8

This problem set is due on Wednesday, November 2 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Reeder proves that $r^n \to 0$ for $0 < r < 1$ using the Binomial theorem and it is understandably hard to understand. In fact we can just use logarithms to prove it, provided we are willing to accept facts about the logarithm from algebra. This problem walks you through the proof.

(a) For concreteness's sake let's take $r = 0.9 = 9/10$, so we want to prove that $0.9^n \to 0$ as $n \to \infty$. Suppose your opponent hands you $\varepsilon = 0.1$. What is the minimum value of N needed so that 0.9^n is less than 0.1 away from 0, for all $n \geq N$?

(b) Suppose your opponent hands you $\varepsilon = 0.01$. What is the minimum value of N needed now?

(c) Now beat your opponent even before he hands you any ε. Give a formula/rule that can produce a sufficient value of N, as a function of whatever ε your opponent may give you.

(d) With this, write your proof in full, in complete sentences. Before you write your proof, write down the answer to the following: What are you proving?

(e) Generalize your proof to any $0 < r < 1$.

Problem 2. Let a be a real number and let (x_n) be the constant sequence $x_n = a$ for all n. Prove that $x_n \to a$, using the ε definition.

Problem 3. Prove that $\dfrac{\sin n}{n} \to 0$ as $n \to \infty$.

Hint: You can use the squeeze theorem. Because you use an existing theorem, you won't need to do any ε reasoning here.

Problem 4. Use the Zax Theorem to prove that the sequence (z_n) given by

$$z_n = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{(-1)^n}{n}$$

converges. (You don't have to find the limit. Note that z_n only makes sense for $n \geq 2$, so we're starting the sequence at $n = 2$.)

Problem 5. Last week you proved that $0.999\ldots = 1$ using the ε-lemma. This time, prove that $0.999\ldots = 1$ using geometric series.

Problem 6. Rate the difficulty of each problem (1a, 1b, 1c, 1d, 1e, 2, 3, 4, 5) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!
- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.
- 6 – I could not start this problem without help, but after getting help I was able to solve the problem. However, I still don't feel like I understand what is going on in this problem.
- 7 – I could not solve the problem, even with help.

MATH1103 FALL 2022
PROBLEM SET 9

This problem set is due on Wednesday, November 9 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Binary numbers are the bread and butter of how computers calculate things. What is the exact value of this binary number?

$$x = 0.01010101\dots.$$

If you are unfamiliar with binary, the kth position after the decimal point is the $\frac{1}{2^k}$-place value. So for example, a finite binary expansion such as 0.01101 would be equal to $\frac{1}{4}+\frac{1}{8}+\frac{1}{32}$.

Problem 2. A king once lost in a game of chess to a traveller, and offered the traveller a prize of his choice. The traveller said:

> I am a modest man, so I will only request this: on this chessboard in front of us, put one grain of rice on the first square, two grains of rice on the second square, four on the third square, and so on, doubling each time until the 64th square.

The king laughed and said, "That's all? You are too modest."

Do you agree with the king that the traveller is too modest? How many grains of rice did the traveller request? What fraction of the total rice belongs to the last square?

Problem 3. Determine whether the following series converge or diverge, and find the sum of those that converge.

(a)
$$\frac{1}{10000}+\frac{1}{10001}+\frac{1}{10002}+\frac{1}{10003}+\cdots$$

(b)
$$\frac{1}{10000}+\frac{1}{20000}+\frac{1}{30000}+\frac{1}{40000}+\cdots$$

(c)
$$\frac{1}{10000}+\frac{1}{20000}+\frac{1}{40000}+\frac{1}{80000}+\cdots$$

Problem 4. In this problem you will uncover many different ways to sum the series

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n.$$

Spoiler alert: we'll find that the series sums to $1/(1-x)^2$ (whenever it converges).

(a) For now, assume x is any number that makes the series converge. Recall the method mentioned in Problem Set 5 Problem 3(d) (Strang 8.4.20)! If necessary, relearning the method is part of this problem! Then work out the sum using that method.

(b) Here is a completely different approach. First notice that the kth term above is kx^{k-1} which is precisely the derivative of x^k. So

$$1 + 2x + 3x^2 + 4x^3 + \cdots$$

is precisely the derivative of

$$1 + x + x^2 + x^3 + \cdots .$$

This is, of course, the geometric series we all know and love. You know what the geometric series sums to... perhaps you can take the derivative of both sides of the geometric series formula? See where this leads.

(c) Notice that $1/(1-x)^2$ is the square of $1/(1-x)$. Therefore, it somehow must be true that

$$(1 + x + x^2 + \dots)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots .$$

See if you can argue why this equation is true by going through the expanding process on the left hand side. For example, I can see that the coefficient of x in $(1+x+x^2+\dots)^2$ should be 2 because when we multiply $(1+x+x^2+\dots)(1+x+x^2\dots)$, the only two ways to get an x^1 term are to pick 1 from the first group and x from the second group, or to pick x from the first group and 1 from the second group.

(d) What series do you think equals $1/(1-x)^3$? (Any reasonable guess with some explanation of why you think it's true suffices.)

(e) (Optional) Let's address convergence. Find a formula for the *finite* sum

$$1 + 2x + 3x^2 + 4x^3 + \cdots + (n+1)x^n$$

(notice I'm stopping at n here), and use what you know about growth rates to find the values of x for which the series converges.

Problem 5. Rate the difficulty of each problem (1, 2, 3a, 3b, 3c, 4a, 4b, 4c, 4d) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!

- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.
- 6 – I could not start this problem without help, but after getting help I was able to solve the problem. However, I still don't feel like I understand what is going on in this problem.
- 7 – I could not solve the problem, even with help.

MATH1103 FALL 2022
PROBLEM SET 10

This problem set is due on Wednesday, November 16 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Comparison and/or ratio test practice. Determine whether each of the following series converges. General tip: Think about comparison before thinking about ratio test. Of course, think about both strategies in case one of them doesn't seem to be leading anywhere useful.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{10^k}{7+5^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} k \cdot 3^{-k}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{\log k}{k}$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$.

Problem 2. Partial fractions.

(a) What is $\frac{1}{2}-\frac{1}{3}$? What is $\frac{1}{3}-\frac{1}{4}$? What is $\frac{1}{4}-\frac{1}{5}$? Make a conjecture based on your findings, then prove it.

(b) Using what you proved in part (a), find the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{99\cdot 100}$$

and the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots.$$

(c) Challenge! Use your thinking skills, reflecting on how you solved the previous part, to find the sum
$$\frac{1}{1\cdot 4}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 6}+\cdots.$$

(d) A slight change can make a problem much much harder. Let's now look at the following sum:
$$\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$$
This sum is similar in form to the one in part (b) but the limit is now irrational! What does the internet (e.g. Wolfram Alpha) say this sum equals? (You might want to figure out how to express it as a summation so you can input it into the service.)

Then give a guess as to how one might prove it. (Hint: the sum in the Zax problem of 2 psets ago converged to $1-\ln 2$. Maybe there's a connection. . .)

(e) Prove that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, using comparison and the result of part (b).[1]

Problem 3. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered that
$$\frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}=\frac{1}{\pi}.$$
Prove the more modest assertion, that the series converges at all.

Problem 4. Rate the difficulty of each problem (1a, 1b, 1c, 1d, 1e, 2a, 2b, 2c, 2d, 2e, 3) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!
- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.

[1] Finding the sum was a famous problem, called the Basel Problem because the Bernoulli family and Euler (all from Basel, Switzerland) worked on it. It was Euler who found the sum in 1734. We may see later how he did it.

- 6 – I could not start this problem without help, but after getting help I was able to solve the problem. However, I still don't feel like I understand what is going on in this problem.
- 7 – I could not solve the problem, even with help.

MATH1103 FALL 2022
PROBLEM SET $10\frac{1}{2}$

This half-problem set is due on **Monday**, November 21 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Euler's identity! For this problem, let i denote the square root of -1.

(a) Derive the power series of $\sin x$ (around 0) using the fact that the coefficient of x^n in the power series of $f(x)$ should be $\frac{1}{n!}f^{(n)}(0)$.

(b) Do the same for e^x as well as $\cos x$.

(c) If you take the power series for e^x and plug in $x = i\theta$, you get a power series in θ. What is the coefficient of θ^n in this power series? Recall that $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. So the answer will depend on the remainder of n mod 4.

(d) Using part (c), prove the #1 most beautiful math equation (Euler's identity)

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(e) To see why it's so beautiful, let's derive the double angle identities for $\sin(2\theta)$ and $\cos(2\theta)$ without breaking a sweat, using the result of part (d): start with

$$\cos(2\theta) + i\sin(2\theta) = e^{2i\theta} = e^{i\theta}e^{i\theta} = \cdots$$

and finish this line of thinking to get the proof. You will need to recall that $a + bi = c + di$ for real numbers a, b, c, d if and only if $a = c$ and $b = d$.

Problem 2. You now have the tools to prove something hinted at from last week's problem set:

$$\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \ln 2.$$

The task for this problem: prove it. I won't tell you exactly how to do it. I'll just say you can freely use the following result we proved in class:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Also, try to do this problem without looking at your notes, even if you feel tempted to because you don't know how to start! Remember the good problem solving habits you learned in class.

Problem 3. Rate the difficulty of each problem (1a, 1b, 1c, 1d, 1e, 2) according to the following scale. Your ratings will collectively let me know which areas are difficult in this class. Thanks for your feedback!

- 1 – Super easy, barely an inconvenience!
- 2 – Not easy, but I was able to solve the problem on my own by comparing it with an example from class or the textbook.
- 3 – Not easy, but I was able to solve the problem on my own through observations, analysis, and/or creative reasoning.
- 4 – I made some progress but got stuck, and with help, I was able to solve the problem. I feel like I understand it now.
- 5 – I could not start this problem without help, but after getting help I was able to solve the problem. I feel like I understand it now.
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- 7 – I could not solve the problem, even with help.

MATH1103 FALL 2022
EXAM 1 REVIEW

Problem 1. Some integration problems.

(a) Find $\displaystyle\int_{-2}^{2} |x-1|\,dx$.

(b) Find $\int \sqrt{x}\ln x\,dx$.

(c) Find $\int \ln(3x+2)\,dx$.

(d) Find $\displaystyle\int x^5 e^{x^3}\,dx$.

(e) Find $\displaystyle\int_0^4 \sqrt{9-\sqrt{x}}\,dx$.

(f) Find $\displaystyle\int_1^e \frac{1}{x+x\ln(x)}\,dx$.

(g) If $\displaystyle\int_0^4 f(x)\,dx = 50$, find $\displaystyle\int_0^2 xf(x^2)\,dx$.

(h) What is $\displaystyle\int_{-4}^{10} f'(t)\,dt$?

Problem 2. You are given that $\frac{d}{dx}(\ln(x-1)-\ln(x+1)) = \frac{2}{x^2-1}$ on the interval $(1,\infty)$. Compute

$$\int_3^5 \frac{1}{1-x^2}\,dx.$$

(Note: don't use partial fractions; we haven't touched that in class yet.)

Problem 3. Prove or disprove whether the limit $\lim_{a\to\infty}\int_{-a}^{a} x^3\,dx$ converges. What about $\int_{-\infty}^{\infty} x^3\,dx$?

Problem 4. Suppose the acceleration due to gravity is constant at 10 meters per second squared. I throw an object upwards such that it takes 2 seconds to land on the floor 2 meters below where it started. What was the initial upward velocity of this object?

If you already happen to know relevant physics formulas, derive them using calculus.

Problem 5. Show that the infamous Gabriel's Horn, which is the solid of revolution obtained by revolving $y = \frac{1}{x}$ around the x-axis and taking the part from $x = 1$ all the way to $x = \infty$, has infinite surface area but finite volume.

A popular slogan that one gets out of this is that you can fill Gabriel's Horn with a finite amount of paint, even though you can't paint the outside!

Problem 6. Recall that Buffon's needle experiment produced a probability of $1/\pi$ in the case that the needle's length was half the spacing between the lines. What if the needle's length is the same as the spacing between the lines? What if the needle's length is arbitrary?

Problem 7. (Exercise from class) Prove, from the definitions, that the variance of the normal distribution $N(\mu, \sigma)$ is σ^2, which shows that the parameter σ really is permitted to be called the standard deviation.

Recall that the PDF of the normal distribution $N(\mu, \sigma)$ is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2/2}.$$

New problems!

Problem 8. An ice-cream cone can be constructed by rotating $y = x$ from $y = 0$ to $y = \frac{\sqrt{2}}{2}$, and rotating $y = \sqrt{1 - x^2}$ around the y-axis, and from $y = \frac{\sqrt{2}}{2}$ to $y = 1$ also around the y-axis. Can you draw a picture to illustrate this? Find the volume of this ice-cream cone.

Problem 9. A cafe is interested in studying the number of people who come to order coffee in the early morning, i.e., 6:00 am. 2 customers, on average, come to order coffee every 15 minutes between 6:00 am and 7:30 am. What is the probability that there are fewer than or equal to 3 people coming to order the coffee between 7:00 am and 7:15 am? *Hint*: Recall that for Poisson distribution, we have $E(X) = \lambda$, where λ is the parameter of the (discrete!) probability distribution function $p_n = \frac{\lambda^n e^{-\lambda}}{n!}$.

Problem 10. (Exercise from discussion) Prove that for any exponential distribution with probability density function $p(x) = \alpha e^{-\alpha x}$, the expected value $E(X) = 1/\alpha$.

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

Remark 1. Each of these is solvable by either u-substitution or integration by parts, or neither. Answers can be verified by computer!

Remark 2. Use FTC 2. Final answer is also verifiable by computer.

Remark 3. The first limit converges. The expression inside the limit is 0 for all values of a! The second expression diverges because because it involves $\infty - \infty$, the two infinities being $\int_0^\infty x^3\,dx$ and $\int_{-\infty}^0 x^3\,dx$.

Remark 4. I got 9 m/s. Integrate acceleration (and use initial conditions to determine the constant of integration) to find the velocity as a function of time, then integrate velocity to find the position as a function of time (with initial velocity as a parameter, which can then be solved for). Keep your signs straight! If you choose up to be positive, then gravity's acceleration should be denoted as -10.

Remark 5. The integral for the area is something you already computed in a homework, but you can use comparison to save yourself a lot of algebra. (See the Wikipedia article on Gabriel's horn to see what comparison the article uses.)

Remark 6. If the needle's length is the same as the spacing between the lines, I believe the probability that a needle hits a line is $2/\pi$.

Remark 7. First, use the u-substitution $u = (x-\mu)/\sigma$ to get rid of the μ term and σ term (leaving a σ^2 factor outside the integral); effectively reducing the problem to a normal distribution with mean 0 and standard deviation 1.

We reduce the problem to showing that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx = 1.$$

Use integration by parts to finish. Obviously a non-elementary integral will be involved so you have to use the following result, that the total area under the bell curve is 1:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1.$$

Remark 8. Using disks for both parts leads to the easiest computation. For the cone part you could just use the formula we proved in class for the volume of a cone, which is $V = \frac{1}{3}\pi r^2 h$, where $r = h = 1/\sqrt{2}$. The integral for the cone part should be

$$\int_{\frac{1}{\sqrt{2}}}^{1} \pi(1-y^2)\,dy.$$

The final volume should end up being $\pi\left(\frac{2-\sqrt{2}}{3}\right) \approx 0.613434$.

Remark 9. According to the problem, we have $\lambda = 2$, so the required probability is

$$p_0 + p_1 + p_2 + p_3 = \sum_{n=0}^{3} \frac{2^n e^2}{n!} = \frac{19}{3e^2} \approx 0.857.$$

Remark 10. Very similar to pset 6 problem 1(c)!

MATH1103 FALL 2022
EXAM 2 REVIEW

Many of these review problems are similar to homework problems. For best learning, please try to work them out at first **without** referring to notes or your homework.

Problem 1. Determine whether the following series converges or not. You may use comparison test/ratio test/any other logical reasoning steps that make perfect sense.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n.$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3+3}.$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n\cdot\sin^2 n}.$

Note: You may use the fact that $\sin n \neq 0$, for any positive integer n.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}.$

(e) $\displaystyle\sum_{k=1}^{\infty}\ln k.$

(f) $\displaystyle\sum_{k=1}^{\infty}\frac{\ln k}{k}.$

(g) $\displaystyle\frac{1}{20000}+\frac{1}{20001}+\frac{1}{20002}+\frac{1}{20003}+\cdots.$

(h) $\displaystyle\frac{1}{10000}+\frac{1}{40000}+\frac{1}{90000}+\frac{1}{160000}+\cdots.$

(i) $\displaystyle\frac{1}{10000}+\frac{1}{30000}+\frac{1}{90000}+\frac{1}{270000}+\cdots.$

(j) $\displaystyle\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots.$

(k) $\sum_{k=1}^{\infty} a_k$ where $a_k = e^{-k}$ if $k < 10^{100}$ but $a_k = \frac{1}{k}$ if $k \geq 10^{100}$.

We did not have a question exactly like this in class or on homework, so please work this problem out very carefully!

(l) $\sum_{k=1}^{\infty} a_k$ where $a_k = 2^{-k}$ if k is not a power of 2, but $a_k = \frac{1}{1000}$ if k is a power of 2. The first few terms of (a_k) look like: $\frac{1}{1000}, \frac{1}{1000}, \frac{1}{8}, \frac{1}{1000}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{1000}$, because 3, 5, 6, and 7 are not powers of 2 while 1, 2, 4, and 8 are.

Problem 2. If (x_n) and (y_n) both diverge, then answer the following questions. Notice that we didn't discuss these exact questions in class nor did you have them on your homework (so don't bother looking in your notes!), but nevertheless these can be answered completely convincingly using only your thinking and some logic. Exercise your creative mind to try to come up with counterexamples!

(a) Must $(x_n + y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample

(b) Must $(x_n \cdot y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample.

(c) Must $(x_n - y_n)$ be divergent? If so, give a proof. Otherwise, give a counterexample.

(d) Must $\left(\frac{x_n}{y_n}\right)$ be divergent (say $y_n \neq 0$ for all n in this case)? If so, give a proof. Otherwise, give a counterexample.

Problem 3.

(a) Prove that $\left(\frac{999}{1000}\right)^n$ converges to 0, using the ε definition of convergence and logarithms.

How to start your proof, if you're stuck: "Let $\varepsilon > 0$ be arbitrary. [Now you must find an integer N such that $\left|\left(\frac{999}{1000}\right)^n - 0\right| < \varepsilon$ for all $n \geq N$.]" Also check the front page of the Canvas site for a guide.

(b) Let $a_0 = 1000$ and $a_n = a_{n-1} - \frac{1}{1000}a_{n-1}$ for $n \geq 1$. Prove that a_n converges to 0.

Hint: Make observations first! You should find something that reduces this to a problem you already solved.... Hence avoiding having to write an ε proof for this.

(c) Prove that $\left(\frac{1000}{999}\right)^n$ does **not** converge to 0, using the ε definition of convergence.

Hint: First prove that $\left(\frac{1000}{999}\right)^{n+1} > \left(\frac{1000}{999}\right)^n$ for all n.

Problem 4. Deriving the formula for geometric series.

(a) if

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

for some numbers a, r, then what kind of formula can you come up with to calculate s_n?

(b) Now we assume that $|r| < 1$. Then the (geometric series)

$$s = a + ar + ar^2 + ar^3 + \cdots$$

converges and we are very familiar with a formula that we've been using again and again that $s = \dfrac{a}{1-r}$. First, prove this formula with the result you calculated in part(a).

(c) Besides proving part(b) with the result from part(a), we can also prove the formula $s = \dfrac{a}{1-r}$ directly. Finish the proof yourself.

(d) Use geometric series to show that $1 = .999\cdots$

Problem 5. Recall that the decimal expression of a number $x = 0.a_1a_2a_3\cdots$, where $a_i \in \{0, 1, \cdots, 9\}$ means

$$x = a_1 \cdot 10^{-1} + a_2 \cdot 10^{-2} + a_3 \cdot 10^{-3} + \cdots .$$

While in the homework, we explored that a binary expression of a number $x = 0.b_1b_2b_3\cdots$, where $b_i \in \{0, 1\}$ means

$$x = b_1 \cdot 2^{-1} + b_2 \cdot 2^{-2} + b_3 \cdot 2^{-3} + \cdots .$$

Similarly, we can also define a ternary expression for a number $x = 0.c_1c_2c_3\cdots$, where $c_i \in \{0, 1, 2\}$ means

$$x = c_1 \cdot 3^{-1} + c_2 \cdot 3^{-2} + c_3 \cdot 3^{-3} + \cdots .$$

Then determine what the number $x = 0.1111\ldots$ really stands for in decimal, binary, ternary expressions respectively.

Problem 6. What is $0.99989998\ldots$, where the 9998 is repeating?

Problem 7. What does $\frac{\cos n}{n}$ approach as $n \to \infty$? Prove it.

Problem 8. Knowing that $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$ is a true identity that you have already proved in homework, deduce a closed form for $2 + 3x + 4x^2 + 5x^3 + \ldots$.

Problem 9. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Problem 10. Prove that there are infinitely many odd numbers.

Problem 11. Find the exact values of:

(a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots$

(b) $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots$

(c) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots$

(d) $\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots$ (I think there's a π in the answer to this one.)

(e) $\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$

(f) $\sum_{k=1}^{100}((k+1)^2-k^2)$.

(g) $2+\frac{3}{5}+\frac{4}{25}+\frac{5}{125}+\cdots$. (Note: the denominators are powers of 5. If you are stuck and need inspiration, first solve Problem 8.)

Problem 12. What is the power series (around 0) for:

(a) e^{2x}?

(b) e^x-1?

(c) $\arctan x$?

(d) $\arctan(-x)$?

(e) Prove that $\arctan x$ is an odd function using its power series, applying the previous 2 parts.

(f) $\ln(1+x)$?

(g) $400\ln(1+x)$?

(h) $\frac{x^2}{1-x}$?

(i) $\frac{1}{(1+x)(1-x)}$? (Hint: Multiply out the denominator.)

(j) $\frac{x^2}{(1+x)(1-x)}$?

Problem 13. Prove that

$$\frac{1}{0!}+\frac{1}{2!}+\frac{1}{4!}+\cdots=\frac{e+\frac{1}{e}}{2}.$$

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

Remark 1. Please let me know if I made any mistakes in the below!

(a) Diverges, because the limit of terms is not 0.

(b) Converges, by comparison with $\frac{1}{n^2}$.

(c) Diverges, by comparison with the harmonic series.

(d) Converges, by comparison with $\frac{1}{n^2}$.

(e) Diverges, because the limit of the terms is not 0.

(f) Diverges, by comparison with the harmonic series (after the 3rd term).

(g) Diverges, because it is a tail of the harmonic series.

(h) Converges, because it is a constant multiple of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(i) Converges, because it is a geometric series with common ratio between 0 and 1.

(j) Converges, because by partial fractions this becomes a decreasing alternating series whose terms converge to 0.

(k) The tail is what matters, and the tail is the tail of a harmonic series, so this series diverges.

(l) The sequence of terms was constructed in such a way that it does not converge to 0. Indeed, put $\varepsilon = \frac{1}{2000}$, then no matter what N is, there exists $n \geq N$ such that $a_n \geq \varepsilon$. (Just pick the next power of 2 after N.) This proves that $a_n \not\to 0$, so the series does not converge.

Or you can say that the series includes infinitely many $\frac{1}{1000}$ terms, and all other terms are positive, so it diverges by comparison with the sum $\frac{1}{1000} + \frac{1}{1000} + \cdots$.

Remark 2.

(a) Counterexample: Let $x_n = 2^n$ and $y_n = -2^n$, then $x_n + y_n$ is the constant zero sequence. Thus (x_n) and (y_n) both diverge but $(x_n + y_n)$ converges!

(b) Counterexample: Let x_n be any sequence whose odd subsequence (meaning $x_1, x_3, x_5, \ldots$) is divergent and whose even terms are all 0, and let y_n be the other way around. Then $x_n y_n$ is always 0.

(c) Counterexample: Let x_n and y_n be the same divergent sequence. $x_n - y_n$ is the constant sequence 0.

(d) Exact same idea as in part (c) applies here too! Haha. x_n/y_n will be the constant sequence 1.

Remark 3.

(a) This was in problem set 8 or so.

(b) This looks like a completely new problem on first sight, but actually, the equation $a_n = a_{n-1} - \frac{1}{1000}a_{n-1}$ simplifies to

$$a_n = \left(1 - \frac{1}{1000}\right) a_{n-1} = \frac{999}{1000}a_{n-1}.$$

So each term is 999/1000 times as large as the previous one. This is the geometric series again! So this series and the one in part (a) are identical.

(c) Since $\frac{1000}{999} > 1$, we can multiply this inequality on both sides by $\left(\frac{1000}{999}\right)^n$ to deduce that $\left(\frac{1000}{999}\right)^{n+1} > \left(\frac{1000}{999}\right)^n$ for all n. It follows that the sequence $\left(\frac{1000}{999}\right)^n$ is an increasing sequence of positive real numbers, so cannot converge to 0. (More detail: the first term is 1, and the sequence is increasing, so every term is at least a distance 1 away from 0.)

Remark 4. Check out one of the discussions. Also the internet has plenty of derivations for the geometric series formula. Part (d) was a homework problem.

Remark 5. In base n, 0.111... is the series $\sum_{k=1}^{\infty} \frac{1}{n^k}$. So in binary ($n = 2$), the series converges to 1, which leads us to say that $0.111_2 \ldots = 1$. (The subscript 2 after a string of digits is standard notation which indicates the number is to be read in binary.) This should be very reminiscent of the $0.999\ldots = 1$ equality in base 10. In fact, if $\triangle$ represents the digit $(n-1)$ in base n, then $0.\triangle\triangle\triangle\ldots$ always equals 1! Maybe you can try to prove this...

Similarly, one can plug in $n = 3$ into the series to find that $0.111\ldots_3$ is the ternary expansion for the very familiar number 1/2!

And of course, $0.111\ldots_{10}$ is 1/9. (Plug in $n = 10$ into the series to verify this.)

Remark 6. Answer for your checking purposes:

$$\frac{9998}{9999}.$$

Remark 7. $\cos n$ is bounded while $\frac{1}{n}$ converges to 0. Therefore $\frac{\cos n}{n}$, being the product of a bounded sequence and a sequence that converges to 0, converges to 0. (You can also use the squeeze theorem, which could be one of our favorite theorems ■)

Remark 8. You have to notice that $2 + 3x + 4x^2 + 5x^3 + \ldots$ is the sum of $1 + 2x + 3x^2 + 4x^3 + \ldots$ and $1 + x + x^2 + x^3 + \ldots$, both of which you already know. The rest is some algebra!

Remark 9. We compare the series with the series of Problem 1(j)! Just as in problem set 10 problem 2(e).

Remark 10. This is supposed to remind you of Euclid's proof of the infinitude of primes. A very short proof that there are infinitely many odd numbers might be as follows:

> Suppose there are finitely many odd numbers. Then there is a largest odd number, let's call it N. But then $N+2$ is also an odd number, and $N+2$ is greater than N, contradicting the fact that N was the largest odd number.

If you want to also justify why $N+2$ is odd if N is odd, you can use the following: The definition of N being odd is that $N = 2k+1$ for some positive integer k. To prove $N+2$ is odd, one has to show that $N+2 = 2m+1$ for some positive integer m. But $N+2 = (2k+1)+2 = 2k+3 = 2(k+1)+1$. So we have found our m, it is $k+1$ which is an integer!

Remark 11. Answers:

(a) 1

(b) $\ln 2$

(c) $\frac{1}{2}$

(d) $\frac{\pi}{8}$

(e) $e - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} = e - 1 - 1 - \frac{1}{2} = e - \frac{5}{2}$

(f) $101^2 - 1^2 = 10200$ (keyword: telescope) And if you remember the idea from Problem Set 1 Problem 1, you could compute the value as follows: $101^2 - 1^2 = (101+1)(101-1) = 102 \cdot 100 = 10200$. Neat!

(g) Take what you got for problem 8 and plug in $x = \frac{1}{5}$.

Remark 12. Answers (in non-sigma form. Either sigma or non-sigma form is fine):

(a) $1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$

(b) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(c) $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(d) $(-x) + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} + \cdots$

(e) The answers to the two previous parts are exact negatives of each other. This shows that $\arctan(-x)$ and $-\arctan(x)$ represent the same function, which shows that $\arctan(-x) = -\arctan(x)$ for all real numbers x, which is exactly what proves that arctan is an odd function.

(f) We know that

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots,$$

so if we plug in $-x$ in place for x and negate everything, we'll get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

(g)

$$400\ln(1+x) = 400x - 200x^2 + \frac{400x^3}{3} - 100x^4 + \cdots.$$

(h) It's x^2 times the geometric series $1 + x + x^2 + \cdots$, which is $x^2 + x^3 + x^4 + \cdots$.

(i)

$$\frac{1}{(1+x)(1-x)} = \frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots.$$

(j) It's x^2 times the above, so it's $x^2 + x^4 + x^6 + \cdots$.

Remark 13. The power series for e^x is $1/0! + x/1! + x^2/2! + x^3/3! + \cdots$. The number e can be obtained by plugging in $x = 1$ to this power series, so we get $1/0! + 1/1! + 1/2! + 1/3! + 1/4! + \cdots$. The number $1/e$, which is also equal to e^{-1}, can be obtained by plugging in $x = -1$ to the power series for e^x, so we get $1/0! - 1/1! + 1/2! - 1/3! + 1/4! - \cdots$. If we add the power series of e to that of $1/e$, the even terms double while the odd terms cancel out. Finally dividing the result by 2 gives us

$$\frac{e + \frac{1}{e}}{2} = \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots.$$

MATH1103 FALL 2022
FINAL EXAM REVIEW

Many of these review problems are similar to homework problems. For best learning, please try to work them out at first **without** referring to notes or your homework.

Problem 1 (Warm-up)**.** What is $1+2+\cdots+200$?

Problem 2.

(a) Given that we worked out in class that
$$x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$$
is the power series for $-\ln(1-x)$, what is the power series for $\ln(1+x)$?

Coincidentally, Problem Set 1, Problem 1(g) is very related to this power series if we use the Taylor series method for $\ln(1+x)$ instead of substitution. See if you can find what the relationship is.

(b) Prove that the power series for $-\ln(1-x)$ converges if $-1\le x<1$, and diverges otherwise. Prove that the power series for $\ln(1+x)$ converges if $-1<x\le 1$ and diverges otherwise.

Hint: You only need to do significant work for one of these, then deduce the other one by logical reasoning.

(c) What function has
$$x+\frac{x^2}{4}+\frac{x^3}{9}+\frac{x^4}{16}+\cdots$$
as its power series? In fact, the answer is an integral expression that cannot be simplified.

Hint: If you're stuck, here's the answer, and try to work out how you might have gotten it.
$$\int_0^x -\frac{\ln(1-t)}{t}\,dt.$$

Fun fact: This integral expression is called the *dilogarithm* function, "di" meaning "two" owing to the fact that its power series has inverse squares as its coefficients.

(d) For which values of x does the series of part (c) converge?

Hint: The answer is $-1\le x\le 1$.

(e) Let $f(x) = \int_0^x e^{-t^2}\,dt$. What is the power series of f?

Hint: You could try the Taylor series method (evaluating nth derivatives at 0, then dividing by $n!$) but it will get messy pretty fast. Can you think of another way?

Problem 3. Revisiting Problem Set 1, Problem 4, which asked you to find a general expression for

$$\int_a^b |x|\,dx$$

in terms of a and b. Try this problem again. Does this problem make more sense to you now than at the beginning of the semester?

Problem 4. Let $v(t)$ represent some velocity function, and let $f(x) = \displaystyle\int_x^{x+1} v(t)\,dt$. You saw this function in Problem Set 2 where the function f was called the *running average* function of v. In other words, the value of f at a number x is the average of v from x to $x+1$.

(a) Prove that

$$f(x) + f(x+1) = \int_x^{x+2} v(t)\,dt.$$

(b) Suppose that $\int_{-\infty}^{\infty} v(t)\,dt$ converges. Prove that

$$\sum_{n=0}^{\infty} f(x+n) = \int_x^{\infty} v(t)\,dt.$$

(c) Imagine you were to take the derivative of both sides of part (b). Explain why the derivative of the left hand side is equal to

$$\sum_{n=0}^{\infty}(v(x+n+1) - v(x+n)),$$

and why the derivative of the right hand side is equal to

$$-v(x).$$

(d) Of course, the two expressions in part (c) are equal by virtue of part (b). But pretending you forgot that you did part (b), can you see algebraically how to show that the two expressions in part (c) are equal using telescoping series ideas?

Problem 5. Suppose $\int_{-x}^{0} v(t)\,dt = \int_0^x v(t)\,dt$ for all $x \in \mathbb{R}$. Alice claims this proves that v is an odd function, while Bob claims that this proves that v is an even function, while Steve claims that this proves neither. Who is right? Prove it. (Hint: You can very easily prove Alice or Bob *wrong* by demonstrating a single counterexample. This won't help with Steve though.)

Problem 6.

(a) Why are $\int_1^x v(t)\,dt$, $\int_3^x v(t)\,dt$, and $\int_\pi^x v(t)\,dt$ all valid antiderivatives of $v(x)$?

(b) Part (a) shows that the 3 integrals all differ by a constant, because the antiderivatives of a function differ from each other by a constant. Can you show more directly why the integrals differ by constant?

(c) Why is $\int_x^\infty v(t)\,dt$ an antiderivative of $-v(x)$ rather than an antiderivative of $v(x)$?

Problem 7. Integration practice, old and new.

(a) Integrate the constant function $f(x) = c$ from $x = 0$ to $x = 100$.

(b) Find $\int_0^\pi \sin(2x+\pi)\,dx$.

(c) Find $\int_0^x (u^3 - e^{u/2})\,du$.

(d) You can find the power series for the integrand in part (c), namely you can find the power series for $u^3 - e^{u/2}$, then integrate term-by-term to get a power series. You can also take the answer of part (c) and find its power series. See if they're equal.

(e) Find $e^{x/2} \cdot \int_0^x \frac{t^2}{2} e^{-t/2}\,dt$. This integral came up in the recent 2022 Putnam A4 probability problem! You will need to do integration by parts twice and it will be messy, but not too messy.

Fun fact: This integral expression (including the factor of $e^{x/2}$ in the front) is the solution to the differential equation $f'(x) = \frac{1}{2}f(x) + \frac{x^2}{2}$ with initial value $f(0) = 0$.

(f) Find $\int \frac{e^x}{e^x+1}\,dx$.

(g) Find $\int h(x)h'(x)\,dx$.

(h) Find $\int (g(x)h'(x) + g'(x)h(x))\,dx$.

(i) Use what you found in the previous part to prove the integration by parts rule!

(j) Find $\int \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}\,dx$.

(k) Find $\int_0^x \arctan(t)\,dt$.

(l) Using the power series for $\arctan(x)$, verify your result for the previous part through power series. It should be quite satisfying to see everything match up!

Problem 8.

(a) Prove that $\int_3^{15} \frac{1}{t}\,dt = \int_1^5 \frac{1}{t}\,dt$ without using anything about logarithms.

(b) The same does not hold with $1/t^2$ instead of $1/t$. In fact, suppose that
$$\int_3^{15} \frac{1}{t^2}\,dt = \int_1^x \frac{1}{t^2}\,dt.$$
Can you find x? You are free to take antiderivatives this time to help solve this problem.

Problem 9. Is the length of the curve $y = \frac{1}{3}x^2$ from $(0,0)$ to $(3,3)$ exactly 3 times the length of the curve $y = x^2$ from $(0,0)$ to $(1,1)$?

Problem 10. Work out the volume of a unit sphere (unit meaning radius 1) from first principles using both the washer and shell method. Do not look at your notes. To check your answer, compare what you got with
$$\frac{4}{3}\pi.$$

Problem 11.

(a) If I cut a stick of length 1 at a random point along the stick, what is the expected size of the piece to the left of the cut?

(b) If I cut a stick of length 1 at a random point along the stick, what is the expected size of the smaller piece? Expected size of the larger piece?

(c) If I cut a stick of length 1 at a random point along the stick, what is the expected absolute length difference between the two pieces?

Problem 12.

(a) Take a look at the formula for the expected value of a random variable X:
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\,\mathrm{PDF}_X(x)\,dx$$
and try to understand why it makes sense to define the expected value that way. (Recall the expected value formula in the discrete case if you're stuck.)

Hint: Here's a motto that may be helpful: The expected value is the weighted average of all the possible values of the outcome where the weights are the probabilities of the outcomes. In the integrand of the integral expression above, which part represents the outcome and which part represents the probability?

(b) Suppose that you want to measure the expected value of X^2 instead. The notation X^2 means that you generate a random number according to the distribution of X, then square it. Try to understand why
$$\int_{-\infty}^{\infty} x^2\,\mathrm{PDF}_X(x)\,dx$$

makes sense as the expected value of X^2.

(c) Let $p(x)$ stand for $\mathrm{PDF}_X(x)$. Let μ represent the mean of X. You are given that $\mathrm{Var}(X)$ is shorthand for $\mathbb{E}[(X-\mu)^2]$. Write down the integral that will compute $\mathrm{Var}(X)$. Don't look at your notes.

Problem 13. Why are the following three quantities equal according to the normal distribution model? You can appeal to something you know about the normal distribution, but you should also demonstrate the equality by writing down each probability as an integral, and then showing that all three integral expressions have the same value.

- The probability that the IQ of a randomly selected person is 125 or higher. IQ is modeled as a normal distribution with mean 100 and standard deviation 15.
- The probability that the IQ of a randomly selected person is 75 or lower.
- The probability that the height of a randomly selected person is 73 inches or greater. Height is modeled as a normal distribution with mean 68 inches and standard deviation 3 inches.

Problem 14. A dartboard has the shape of a circle of radius 1. A dart hits a random point in the circle, where by random we mean that the probability that the dart lands in any region R inside the circle is equal to the area of R divided by the area of the circle.

(a) Let X be the random variable representing the dart's distance from the center. For any r between 0 and 1, what is the probability that $X \leq r$?

(b) What is the expected distance of the dart from the center?

Problem 15. Buffon's needle problem. Try to derive the fact that the probability of a needle of length L landing on a line when dropped on a grid with uniformly spaced vertical lines $2L$ apart, is $1/\pi$, without looking at your notes.

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

Remark 1. If you use Gauss's trick, twice the sum is 201 added to itself 200 times, which is 40200. Divide by 2, you get 20100.

Remark 2.

(a) Plug in $-x$ in the place of x, and then negate the entire result. Answer:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

The relationship to Problem 1(g) in Problem Set 1 is: the nth derivative of $\ln x$ was found to be $(-1)^{n-1}(n-1)!/x^n$. Therefore, the nth derivative of $\ln(1+x)$ will be $(-1)^{n-1}(n-1)!/(1+x)^n$, because the chain rule for the function $1+x$ does nothing. When that is evaluated at $x = 0$ and divided by $n!$, you get

$$(-1)^{n-1}\frac{(n-1)!}{n!} = \frac{(-1)^{n-1}}{n},$$

as the nth coefficient of the series (for $n \geq 1$). This agrees with the answer!

(b) Everything here can be solved by using the ratio test, then investigating endpoints separately. The nth term of the power series for $-\ln(1-x)$ is, according to part (a), $\frac{x^n}{n}$. Using the ratio test, we get the relevant ratio being:

$$\left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = \left|x \cdot \frac{n}{n+1}\right| \xrightarrow{n\to\infty} |x|.$$

This limit is strictly less than 1 iff $-1 < x < 1$, and strictly greater than 1 if $|x| > 1$. So we can already conclude convergence for $-1 < x < 1$ and divergence for $x < -1$ and $x > 1$. It remains to check $x = -1$ and $x = 1$. The $x = -1$ case is the alternating harmonic series which converges, and the $x = 1$ case is the harmonic series which diverges.

(c) The goal is to do a combination of multiplications or divisions by x, along with differentiation or integration, to turn the coefficients from reciprocals to reciprocals of squares. I expect you to try a lot of things here.

Soon enough you will make the observation that integration should be involved in some way. If you integrate right off the bat though, the power series becomes

$$\frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots ,$$

which is close but no cigar. What if you divide by x first, then integrate? Dividing by x gets you

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots ,$$

then integration gives you

$$x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \cdots .$$

Winner winner chicken dinner!

(d) Let's do the ratio test again! The $(n+1)^2/n^2$ part of the ratio test still converges to 1 as $n \to \infty$ (dominant term idea!), so we still get convergence for $-1 < x < 1$ and divergence for $|x| > 1$. But now, at $x = -1$ we have an alternating sum of reciprocal squares, which converges since the series is an alternating decreasing series whose terms converge to 0. At $x = 1$, we get the famous $\pi^2/6$ sum, so this also converges.

(e) We did this on Monday! The answer is (if I recall, you should check this for yourself in case I have bad memory)

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

Remark 3. Copied from pset 1 solutions:

The definition of the definite integral of $|x|$ with respect to x from a to b is the signed area under the graph of $|x|$ from a to b. For example, if $a = 7$ and $b = 9$, the definite integral would evaluate to the signed area under the graph of $|x|$ from 7 to 9. This can be found by taking a difference of the areas of two isosceles right triangles of leg lengths 9 and 7, respectively. The task at hand is to generalize this computation to any values of a and b.

If $a, b \geq 0$, then the area under the absolute value curve from a to b is equal to $\frac{1}{2}(b^2 - a^2)$.

If $b \geq 0$ and $a \leq 0$, then the area is equal to $\frac{1}{2}(b^2 + a^2)$.

If $b \leq 0$ and $a \geq 0$, then the area is equal to $-\frac{1}{2}(b^2 + a^2)$.

If $a, b \leq 0$, then the area is equal to $\frac{1}{2}(a^2 - b^2)$.

Summing up,

$$\int_a^b |x|\, dx = \begin{cases} \frac{1}{2}(b^2 - a^2) & a, b \geq 0 \\ \frac{1}{2}(b^2 + a^2) & b \geq 0, a \leq 0, \\ -\frac{1}{2}(b^2 + a^2) & b \leq 0, a \geq 0, \\ \frac{1}{2}(a^2 - b^2) & a, b \leq 0. \end{cases}$$

Remark 4. (a) Addition property of integrals!

(b) Addition property of integrals, infinite series version!

(c) Using FTC 1 we have $f'(x) = v(x+1) - v(x)$, therefore $f'(x+n) = v(x+n+1) - v(x+n)$.

FTC 1 on the right hand side gives $-v(x)$ for the derivative.

Advanced note: There is actually a subtle real analysis issue with part (c) but I won't say anything about it here.

(d) The sum

$$\sum_{n=0}^{\infty}(v(x+n+1)-v(x+n)),$$

telescopes!

Advanced note: There is also a subtle real analysis issue Here but I won't say anything about it here.

Remark 5. The easiest way to see what's going on with this question is to try to draw a graph of a function v satisfying the equation in the first sentence. For example, if areas from 0 to anywhere on the right half of the graph of v are positive, then areas from anywhere on the left half of the graph of v to 0 must also be positive. This rules out the possibility that v is an odd function. Maybe we can now prove that v must be an even function. How? Let's take the derivative of both sides of the integral equation! We get

$$v(-x) = v(x) \qquad \text{for all } x \in \mathbb{R}.$$

Getting $v(-x)$ for the derivative of the left hand side is quite tricky, so I'll lay it out in more detail here. First the idea: the fact that the dependence on x is in the lower bound of the integral introduces a negative sign on the outside. But also, the chain rule on $-x$ introduces a second negative sign canceling out the first negative sign. Here's an algebraic way to say the above: Define an "intermediate" integral

$$V(x) \coloneqq \int_x^0 v(t)\,dt.$$

We can also write $V(x) = -\int_0^x v(t)\,dt$. Then the derivative of $V(x)$ is $-v(x)$. The actual LHS is $V(-x)$ and the derivative of this with respect to x is, by the chain rule, $V'(-x)\cdot(-1) = -(-v(-x)) = v(-x)$.

Remark 6.

(a) Their derivatives all give the same function $v(x)$ back.

(b) For example, $\int_3^x v(t)\,dt - \int_1^x v(t)\,dt = \int_1^3 v(t)\,dt$ which is a constant!

(c) You notice that the x is in the lower bound this time, so roughly speaking, increasing x removes area instead of adding it. That's the intuitive reason, now the mathematically rigorous explanation:

$$\frac{d}{dx}\int_x^{\infty} v(t)\,dt = \frac{d}{dx}\left(-\int_{\infty}^{x} v(t)\,dt\right) = -v(x).$$

Remark 7. (a) Rectangle of width 100 and height c.

(b) One thing you can notice before you do the integration is to observe that $\sin(2x+\pi) = -\sin(2x)$ for all x. Or, you can just perform the u-substitution $u = 2x+\pi$. Either way the answer is 0.

(c) Term-by-term integration should do the trick here.

(d) The power series of u^3 is itself. The power series of $e^{u/2}$ is $1 + u/2 + (u/2)^2/2! + (u/2)^3/3! + \cdots$. The integrand is the subtraction of the former by the latter. Integrating we get

$$\frac{x^4}{4} - x - \frac{x^2}{2 \cdot 2} - \frac{x^3}{3 \cdot 2^2 \cdot 2!} - \frac{x^4}{4 \cdot 2^3 \cdot 3!} - \cdots .$$

Noticing that $n \cdot (n-1)! = n!$ for all positive integers n, this simplifies to

$$\frac{x^4}{4} - x - \frac{x^2}{2 \cdot 2!} - \frac{x^3}{2^2 \cdot 3!} - \frac{x^4}{2^3 \cdot 4!} - \cdots .$$

On the other hand, the answer to part (c) is $\frac{x^4}{4} - 2(e^{x/2} - 1)$. The power series of $\frac{x^4}{4}$ is itself. The power series of $e^{x/2} - 1$ is $x/2 + (x/2)^2/2! + (x/2)^3/3! + \cdots$. We subtract the former by twice the latter to get

$$\frac{x^4}{4} - x - \frac{x^2}{2 \cdot 2!} - \frac{x^3}{2^2 \cdot 3!} - \cdots$$

Lo and behold, the power series match.

(Maybe it was easier to see this by writing everything as a summation. I'd recommend that actually.)

(e) Double integration by parts. The answer is $-x^2 - 4x + 8e^{x/2} - 8$.

(f) This was on Exam 1! Do u-substitution $u = e^x + 1$, with $du = e^x\,dx$. This takes care of everything and we have

$$\int \frac{du}{u} = \ln|u| + C = \ln|e^x + 1| + C.$$

For these types of integrations, it's natural to not think of the right method off the bat (many tried integration by parts), but you should be mindful enough to try the other method if you're not getting anywhere with your first choice.

(g) Let $u = h(x)$, then $du = h'(x)\,dx$. We get that the integral equals $\frac{1}{2}h(x)^2 + C$.

(h) Notice that $g(x)h'(x) + g'(x)h(x) = \frac{d}{dx}(g(x)h(x))$. Since antiderivative and derivative are inverse processes (up to a constant), the desired antiderivative is just $g(x)h(x) + C$. As mentioned in class, the fact that derivative and antiderivative are inverse processes is a matter of definition chasing and is not the content of FTC 1 or 2! The FTCs deal with the link between **definite** integrals and derivatives which is a much deeper statement.

(i) We know that $\int (gh' + g'h)\,dx = gh + C$. Therefore, $\int (gh')\,dx = gh - \int (g'h)\,dx + C$. That's integration by parts!

(j) We did this in class. This is the antiderivative of the quotient rule so the answer is $f(x)/g(x) + C$.

(k) Integration by parts with $u = \arctan t$ and $dv = dt$ ($v = t$). The answer is $t\arctan t - \frac{1}{2}\ln(t^2+1)\Big|_0^x = x\arctan x - \frac{1}{2}\ln(x^2+1)$.

(l) So the power series for $\arctan x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. (i.e. coefficients are odd reciprocals with alternating signs.) Doing term-by-term integration we get

$$\int_0^x \arctan(t)\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)(2n+2)}.$$

Well this is nice, we can do partial fractions on this! Noticing that $1/((2n+1)(2n+2)) = 1/(2n+1) - 1/(2n+2)$, this allows us to rewrite the above as

$$\begin{aligned}
\int_0^x \arctan(t)\,dt &= \sum_{n=0}^{\infty}(-1)^n x^{2n+2}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x\cdot x^{2n+1}}{2n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+2}\\
&= x\arctan x - \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{n+1}}{n+1}\\
&= x\arctan x - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x^2)^{n}}{n}\\
&= x\arctan x - \frac{1}{2}\ln(1+x^2).
\end{aligned}$$

Hard work but neat!

Remark 8.

(a) We did this in class! A u-substitution looks good because we can notice that $3 = 3\cdot 1$ and $15 = 3\cdot 5$, suggesting the substitution $u = 3t$ for the right hand integral. Magically the u-substitution does not change the function. This is what is special about the $1/t$ function.

(b) Let's see: $\int_3^{15} \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_3^{15} = \frac{1}{3} - \frac{1}{15} = \frac{4}{15}$. So we need to solve

$$\frac{4}{15} = \int_1^x \frac{1}{t^2}\,dt = -\frac{1}{t}\Big|_1^x = 1 - \frac{1}{x}.$$

Using algebra, we rewrite this equation as $\frac{11}{15} = \frac{1}{x}$, so $x = \frac{15}{11}$. So we can see that $x = 5$ does not work here anymore.

Remark 9. Here's a tip as to how I re-construct the arc length formula (because it's natural to forget!) The key idea is based on the Pythagorean theorem, just a differential version of it. Instead of $\sqrt{a^2+b^2}$, it's $\sqrt{dx^2+dy^2}$. And we are integrating this infinitesimal hypotenuse length over the interval of interest, and that is the arc length. So here, our curve has equation $y = \frac{1}{3}x^2$, we have $dx = dx$ and $dy = \frac{2}{3}x\,dx$. So the hypotenuse element is $\sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{4}{9}x^2 dx^2} = dx\sqrt{1 + \frac{4}{9}x^2}$. And we integrate this from $x = 0$ to $x = 3$, giving our length

$$L_3 = \int_0^3 \sqrt{1 + \frac{4}{9}x^2}\,dx.$$

Similarly,

$$L_1 = \int_0^1 \sqrt{1 + 4x^2}\,dx.$$

In order to compare these two integrals, in the integral L_3 let's make the substitution $u = \frac{x}{3}$ so $x = 3u$ and $dx = 3\,du$. This transforms our integral to

$$L_3 = \int_0^1 \sqrt{1 + 4u^2}\,3du = 3\int_0^1 \sqrt{1 + 4u^2}\,du.$$

So indeed, L_3 is exactly 3 times L_1.

Remark 10. I think this is in the textbook (both Strang and Reeder). This document is long enough as is!

Remark 11. For all these parts, the key thing that makes this problem manageable is to let x stand for the position of the cut (0 for leftmost and 1 for rightmost), and express our desired value in terms of x, then average over $x \in [0, 1]$ (as x is uniformly distributed). Remember the "averaging" method is only appropriate when the parameter (x in this problem) is uniformly distributed.

There's also a CDF/PDF approach that works here too. It is a nice exercise to try to switch perspectives back and forth.

(a) The size of the piece to the left of the cut, if the cut was at x, is just x. Averaging x over 0 to 1 we get

$$\frac{1}{1-0}\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Another approach is to notice that the size of the piece to the left of the cut follows a uniform distribution on $[0, 1]$, and the expected value of this uniform distribution is

$$\int_0^1 x \cdot 1\,dx$$

which is the same integral (here 1 is the PDF of a uniform distribution on $[0, 1]$).

(b) Here, the length of the smaller piece as a function of x is $\min(x, 1-x)$. Integrating this gives us

$$\frac{1}{1-0}\int_0^1 \min(x,1-x)\,dx = \int_0^{\frac{1}{2}} x\,dx + \int_{\frac{1}{2}}^1 (1-x)\,dx = \frac{1}{8}+\frac{1}{8} = \frac{1}{4}.$$

We can also approach this with distributions. The key method of this approach is to introduce a new variable z (the letter name isn't important, it's just important that it doesn't clash with already used variable names like x) and ask: "What is the probability that the length of the smaller piece is less than z?"

Well in order for the smallest piece have length less than z we must have cut the stick anywhere from 0 to z, or anywhere from $1-z$ to 1. The length of the set of valid places to cut is $2z$. And the relevant values of z range from 0 to $1/2$. So this tells us that $\mathrm{CDF}(z) = 2z$ for $z \in [0, \frac{1}{2}]$. For $z < 0$, $\mathrm{CDF}(z) = 0$ and for $z > \frac{1}{2}$, $\mathrm{CDF}(z) = 1$. The PDF is then $\mathrm{PDF}(z) = 2$ on $[0, \frac{1}{2}]$ and 0 outside that range. The expected value of the length of the smaller piece is therefore

$$\int_0^{\frac{1}{2}} z\,\mathrm{PDF}(z)\,dz = \int_0^{\frac{1}{2}} 2z\,dz = \frac{1}{4}.$$

(c) Here, the length of the larger piece as a function of x is $\max(x, 1-x)$. Integrating this gives us

$$\frac{1}{1-0}\int_0^1 \max(x,1-x)\,dx = \int_0^{\frac{1}{2}} (1-x)\,dx + \int_{\frac{1}{2}}^1 x\,dx = \frac{3}{8}+\frac{3}{8} = \frac{3}{4}.$$

A CDF/PDF approach works similar to above.

Remark 12.

(a) The outcomes are represented by x and the weights (probabilities) are represented by $\mathrm{PDF}_X(x)\,dx$.

(b) For each outcome x the measured outcome is x^2, and the weight (probability) is $\mathrm{PDF}_X(x)\,dx$.

(c)

$$\int_{-\infty}^{\infty} (x-\mu)^2\,\mathrm{PDF}_X(x)\,dx.$$

Basically, we are taking the expected value of "our outcome, minus μ, then squared." Turning this into our integral, our outcome gets represented by x (the variable of integration). And what we measure given x is $(x-\mu)^2$. Then $\mathrm{PDF}_X(x)\,dx$ gives us the weights.

Remark 13. The core explanation is that all of these IQ/height values are exactly 5/3 standard deviations away from the mean. The calculus side of this is that if you write each probability as an integral involving the normal distribution function, a u-substitution transforms any of these integrals into any of the other integrals, proving they have the same value.

Remark 14.

(a) The event $X \leq r$ means that the dart landed within a smaller circle of radius r around the center. The region of such points inside this circle has area πr^2, and the total region has area $\pi \cdot 1^2 = \pi$. So the probability that $X \leq r$ is r^2.

(b) The purpose of part (a) is that it gives us the CDF of X as r^2, $0 \leq r \leq 1$. The PDF is therefore $\mathrm{PDF}_X(r) = 2r$, $0 \leq r \leq 1$. The expected distance is therefore

$$\int_0^1 r \,\mathrm{PDF}_X(r)\, dr = \int_0^1 r \cdot 2r\, dr = \int_0^1 2r^2\, dr = \frac{2}{3} r^3 \Big|_0^1 = \frac{2}{3}.$$

Remark 15. Here's a link to a webpage that gives an explanation of Buffon's needle problem. I think it's a nice explanation. Although one difference is that the author chooses to parametrize the position with the center of the needle rather than the edge. However, this changes pretty much nothing in the analysis.

Appendix A. Supplementary materials

EARLY MATH1103 FEEDBACK

The following is a quick survey of how the course is going so far for you. Responses are anonymous.

1. Questions about the class

For each of the scale questions below, mark your opinion with a $\times$ on the scale.

(1) Pace?

Too slow | Too fast

(2) Volume/speech clarity?

Can't hear/understand | Easy to hear/understand

(3) Handwriting?

Hard to read | Easy to read

(4) Difficulty of lectures?

I get lost | Can follow in real time

(5) Problem set difficulty?

Too easy | Too hard

(6) Additional comments?

Flip to the back for questions about office hours!

2. Office hours

(1) Check all the times where you would likely be able to attend office hours if it was offered at that time. If you want, you can put a star in preferred time slots.

Time	Monday	Tuesday	Wednesday	Thursday	Friday
11 am					
1 pm	(class)		(class)		(class)
2 pm	(class)		(class)		(class)
3 pm					
4 pm					

(2) What's something you're looking forward to learning about (not necessarily something in the syllabus)?