MATH2211 SPRING 2022
EXAM 1 REVIEW PROBLEMS

Problem 1. Find three different complex solutions to

$$z^3 = -1.$$

Express your answers in polar form $z = re^{i\theta}$.

Problem 2. Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Delete one or more vectors from the list v_1, v_2, v_3, v_4 to produce a basis of $\mathbb{R}^3$, or explain why this can't be done.

Problem 3. Suppose that $U \subseteq \mathbb{C}^n$ is a subspace. Show that there exists another subspace $V \subseteq \mathbb{C}^n$ such that

$$U + V = \mathbb{C}^n \quad \text{and} \quad U \cap V = \{0\}.$$

Problem 4. Show that $T \colon \mathbb{R}^2 \to \mathbb{R}^3$, defined by $T(x, y) = (x + y, 0, 2x - y)$, is a linear transformation.

Problem 5. Let W_1 and W_2 be subspaces of V.

(a) Show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(b) Show that $W_1 \cap W_2$ is a subspace of V.

Problem 6.

(a) Let $V \subseteq \mathbb{F}_3^3$ be the space

$$\{(x, y, z) \in \mathbb{F}_3^3 : x + y + 2z = 0\}.$$

How many elements does V contain? (Hint: you can always just do this by exhaustion. There are only 27 vectors in $\mathbb{F}_3^3$.)

(b) Generalize this: It turns out that **any** r-dimensional subspace of $\mathbb{F}_p^n$ will always have p^r elements. Can you prove why?

Problem 7.

(a) Show expilcitly that the solutions in $\mathbb{C}$ to $z^3 = -1$ (the answer to Problem 1) are linearly dependent over $\mathbb{R}$.

(b) What is the dimension of their $\mathbb{R}$-span over $\mathbb{R}$?

(c) What is the dimension of their $\mathbb{C}$-span over $\mathbb{C}$?

Problem 8. Let $\mathbb{R}^\infty$ be the space of infinite sequences $(a_1, a_2, \dots)$ of real numbers. Let $S\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the shift operator

$$(a_1, a_2, a_3, \dots) \mapsto (a_2, a_3, \dots),$$

in other words, S deletes the first term of its input and shifts all other terms to the left by one.

(a) Show that S is a linear transformation.

(b) What is the kernel of S?

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

Remark 1. My thinking would go as follows: find one solution ($-1 = e^{\pi i}$ would be a solution) and the multiply it by the three cube roots of unity $1, e^{2\pi i/3}, e^{4\pi i/3}$.

Remark 2. By inspection it looks like v_1, v_2, v_3 are linearly independent which is enough. To check linear independence just write out the system of equations you get from the equation

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

and check its only solution is $a_1 = a_2 = a_3 = 0$.

If you do this, you'll find out there are nontrivial solutions! So this first try doesn't work. If you have time, try the other 3 combinations. None of them work! So maybe the span of this list of 4 vectors might actually have dimension 2....

Let's take the two vectors with zeros in them, v_3 and v_4 (to make things easier; strictly speaking, you don't need to pick vectors with zeros in them), and see if v_1 and v_2 are expressible as linear combinations of them. That is enough to show that $\dim_{\mathbb{R}} \operatorname{span}(v_1, v_2, v_3, v_4) = 2$. Try this yourself; it shouldn't be too hard.

Remark 3. My thinking would go as follows: Pick a basis of U and extend it to a basis of the entire space $\mathbb{C}^n$. Let V be the span of the vectors you added. This should work. Note that V isn't uniquely determined by U; it depends very much on how what vectors were added to the basis. After figuring this out I would write a proof that $U + V = \mathbb{C}^n$ and $U \cap V = \{0\}$.

Remark 4. Just definitions and algebra.

Remark 5.

(a) The backward direction is immediate from definitions. For the forward direction, start by supposing that $W_1 \cup W_2$ is a subspace and letting $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1 + w_2 \in W_1 \cup W_2$. Two cases: either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. In the first case, we obtain $w_2 = (w_1 + w_2) - w_1 \in W_1$, and because w_2 was arbitrary, this says that $W_2 \subseteq W_1$. The other case is similar.

(b) Just definitions and algebra.

Remark 6.

(a) Following the hint is a very good idea if you need a feel for what's going on.

(b) Pick a basis. Every vector is a unique linear combination of the basis. There are p^r combinations of coefficients you can chose in this linear combination. It's amazing this stuff works over finite fields with no modifications!

Remark 7. The solutions add to 0! (A linear relation with coefficients in $\mathbb{R}$!) Their $\mathbb{R}$-span has dimension 2 since any two of the complex numbers are linearly independent over $\mathbb{R}$.

Their $\mathbb{C}$-span has dimension 1 over $\mathbb{C}$—it can't have dimension more than 1 because $\mathbb{C}$ has dimension 1 over $\mathbb{C}$. And the only way for the span to have dimension 0 is for all the elements in the set to be 0, which is clearly not the case here.

Remark 8.

(a) Just definitions and algebra!

(b) The kernel of S is all sequences whose 2nd term onward are zero, i.e. $\{(a, 0, 0, \dots) : a \in \mathbb{R}\}$. You can fill in the proof yourself.