MATH2211 SPRING 2022
PROBLEM SET 3

DUE WEDNESDAY, FEBRUARY 16 2022 AT 11:59 PM

Let F be a field and let V be an F-vector space.

Problem 1. Suppose we are given a list $v_1, \dots, v_n \in V$.

(a) Show that $v_1, \dots, v_n$ are linearly dependent if and only if there is some $1 \leq i \leq n$ such that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ (i.e. the span of the list with v_i taken out).

(b) Show that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ if and only if

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

(That is, the span doesn't change when v_i is taken out.)

Problem 2. Suppose $v_1, v_2, v_3, v_4 \in V$ and set

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_3 - v_4, \quad w_4 = v_4.$$

(a) Show that $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(w_1, w_2, w_3, w_4)$.

(b) Show that v_1, v_2, v_3, v_4 are linearly independent if and only if w_1, w_2, w_3, w_4 are linearly independent.

Problem 3. Suppose that $\{v_1, v_2, \dots, v_n\}$ is linearly independent in V. Show that $\{v_1, v_1 + v_2, \dots, v_1 + v_2 + \cdots + v_n\}$ is linearly independent as well.

Problem 4.

(a) Show that V is infinite dimensional if and only if it satisfies the following property: for every integer $k > 0$, one can find k linearly independent vectors $v_1, \dots, v_k \in V$.

(b) Show that the vector space $\mathbb{R}^\infty := \{$ all sequences $(a_1, a_2, a_3, \dots)$ of real numbers$\}$ is infinite dimensional.

(c) Give an example of a subspace of $\mathbb{R}^\infty$ which is strictly contained in $\mathbb{R}^\infty$ but is still infinite dimensional.

Problem 5. For each positive integer n, let

$$B_n = \{(-1,1,\dots,1),(1,-1,1,\dots,1),\dots,(1,1,\dots,-1)\} \subseteq F^n.$$

That is, B_n is the set of vectors in F^n with one component equal to -1 and $n-1$ components equal to 1.

(a) Let $F = \mathbb{R}$. For which n is B_n a basis of $\mathbb{R}^n$?[1]

(b) Let $F = \mathbb{F}_3$. Show that if n is of the form $3k+2$ for some $k \in \mathbb{Z}_{\geq 0}$ (i.e. $n \in \{2,5,8,\dots\}$), then B_n is not a basis of $\mathbb{F}_3^n$.[2]

[1]If you need a hint for where to start, try to check whether the vectors in B_n are linearly independent.

[2]Hint: Try to show that B_n is contained in a proper subspace.