INSTRUCTOR'S REPORT, SPRING 2022
May 31, 2022

Course number, name: MATH2211, Honors Linear Algebra

Instructor: Yongyi Chen

1. Report

1.1. **Texts.** A combination of Bill Keane's 2019S linear algebra notes and Axler's *Linear Algebra Done Right*.

1.2. **Topics (include sections of text).**

- Introduction, set theory, intro to proofs, proofs by induction
- Fields, complex numbers, finite fields
- Vector spaces over a field, subspaces, spans
- Linear independence, bases, dimension,
- Functions, linear transformations, matrices
- Kernel, image, rank, nullity
- Isomorphisms, change of basis
- Solving linear equations
- Intro to determinants
- Determinants via the exterior algebra
- Dual vector spaces, annihilator, the four fundamental subspaces
- Eigenvalues, characteristic polynomial, eigenspaces, trace
- Inner products, orthonormal bases, Riesz representation theorem
- Projections, adjoints, self-adjoint operators
- (Not on final exam) Sneak peek of spectral graph theory and Markov chains

1.3. **Some comments about the text.** Between Keane's notes and Axler's book, the students had no problems with the text and using it to solve their homework problems. Both resources were precisely and concisely written. The exercises in Keane's notes are on the easy side, but I had no trouble creating harder problems on my own.

Notably missing from both texts is a unified approach to determinants. Keane's notes takes the ad-hoc computational approach, defining determinants by a cofactor expansion algorithm, while Axler avoids the discussion of determinants until eigenvalues are developed. I took the approach of defining determinants via the exterior algebra and wrote up some notes for this. The exterior algebra approach makes the proofs of most properties of determinants self-evident. However, the caveat is that some students found the exterior algebra content somewhat difficult—perhaps the hardest topic of the course.

Also missing from Keane's notes are quotients, duals, and products. Of these three topics I only discussed duals. It would be nice, although possibly too ambitious, to try to cover more than that.

1.4. **Course format.** In-person, one-hour lectures 3 times per week.

1.5. **Number of exams.** Two midterm exams (50 minutes) and one final exam (3 hours). All were done in person, and all were open notes.

1.6. **Enrollment.** 10.

1.7. **Grading policy.**

- 20% homework,
- 20% first exam,
- 20% second exam,
- 40% final.

1.8. **General comments about this course.** The course was a very big success. I was able to cover a strict superset of what has been covered in the past. The new topics included exterior algebra, dual spaces, the four fundamental subspaces, projections, self-adjoint operators, and the spectral theorem. The students overall handled all of these topics just fine.

1.9. **Final grade distribution.**

- 5 A
- 1 A−
- 1 B+
- 2 B
- 1 C+

Attached are the syllabus, all tests (including final), list of HW assignments, and any other pertinent material.

COURSE INFORMATION FOR MATH2211 (SPRING 2022)
HONORS LINEAR ALGEBRA

Instructor: Yongyi Chen
Email: `yongyi.chen@bc.edu`

Lectures: MWF 12:00 pm–12:50 pm in Campion Hall 9
Homework: Weekly, due on Wednesdays at 11:59 pm.

Office: Maloney 532
Office hours: (tentative) Mondays 2-3 pm in Maloney 532, Tuesdays 4-6 pm over Zoom

1. Course information

Course website. On Canvas. There you will find homework assignments, homework solutions, and supplemental course materials.

Course format. In person. 1 office hour is provided in person in my office and 2 office hours will be over Zoom.

Textbooks. There is no official textbook for the course. We will follow the notes written by Prof. Keane for the Spring 2019 version of the course. The notes can be found on Canvas. For a few topics we will follow *Linear Algebra Done Right* by Sheldon Axler, available on SpringerLink.

In addition to these course notes, you may find the following textbooks useful:

- *Linear Algebra Done Wrong* by Sergei Treil. Available free online.
- An experimental linear algebra zyBook (more information on Canvas).

Homework. There will be weekly homework, due on Wednesdays at 11:59 pm. Because homework solutions will be posted on Canvas, late homework will not be accepted. To submit your homework, upload a single PDF file to Gradescope (accessible from within the Canvas assignment page as well).

You are encouraged to collaborate on homework with your classmates, but the work that you turn in must be your own and must be written in your own words. Working together is good; copying somebody else's work is plagiarism.

One of the primary differences between this course and its non-honors variant is the emphasis on careful mathematical reasoning and proof. As such, writing style counts as much as having the right answer (often you will be told the answer and asked to justify it). Homework solutions must be written in complete sentences, and must be clear, concise, and readable. A correct but poorly expressed solution will not receive full credit.

Typesetting your homework using LaTeX is strongly encouraged, but not required.

Exams and grading. There will be two in-class exams (50 minutes each) and a final (120 minutes). Final grades will be determined by a weighted average of homework and exam scores. Homework counts for 20%, each in-class exam counts for 20%, and the final counts for 40%.

All exams will be given in class.

Academic integrity. Cheating of any kind will result in a failing grade for the course and referral to the Dean's office for disciplinary action. For more information on academic integrity see `https://www.bc.edu/integrity`.

2. List of topics

(1) Warm-up
- Sets, fields, functions, induction, complex numbers

(2) Vector spaces
- Subspaces, span, and linear independence
- Bases and dimension

(3) Linear transformations
- Linear transformations and matrices
- Kernels, images, and invertibility
- Products, quotients, and duals

(4) Gaussian elimination
- Systems of linear equations
- Row reduction and elementary matrices
- Computing inverses

(5) Determinants
- Determinants and invertibility
- Expansion by minors
- Cramer's rule

(6) Spectral theory
- Polynomials
- Eigenvectors and eigenvalues

- The characteristic polynomial
- Diagonalizing matrices

(7) Other topics, as time permits:
- Inner product spaces
- The Cayley-Hamilton theorem and Jordan normal form

MATH2211 SPRING 2022
EXAM 1

WEDNESDAY, MARCH 2 2022

Name: ______________________________

This exam is open notes. There are 50 points total in this exam.

High 49.5/50, low 20/50, mean 35.2/50, median 36.5/50. Good exam. Problems 2 and 5 were the most difficult, with average points obtained on them being 59% and 58% respectively. Every problem was approachable, and everyone had enough time to attempt every problem.

Problem 1.

(a) (5 points) Find, with proof, the dimension of

$$V \coloneqq \operatorname{span}\{(1,1,1),(2,2,2),(3,3,3)\} \subseteq \mathbb{R}^3.$$

(b) (5 points) Let $P_2(\mathbb{R})$ be the space of polynomials of degree at most 2. Write down the matrix of the derivative operator $\frac{d}{dx}$ from $P_2(\mathbb{R})$ to $P_2(\mathbb{R})$ with respect to the ordered basis $(1, x, x^2)$ for both the source and target.

Problem 2. (10 points) Find, with proof, all complex numbers $z \in \mathbb{C}$ such that z and z^2 are linearly independent over $\mathbb{R}$.

Problem 3. (10 points) Prove or disprove: Let v_1, v_2, v_3 be a basis of a real vector space V. Then the set $\{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$ is also a basis of V.

Problem 4. (10 points) Let $T \colon V \to W$ be a linear transformation between two finite-dimensional $\mathbb{R}$-vector spaces. Prove that if T is injective, then $\dim V \leq \dim W$.

Problem 5. (10 points) Let $\mathbb{R}^\infty$ denote the space of infinite sequences of real numbers. For each $n \in \mathbb{Z}^+$, let v_n be the element $(n, n+1, n+2, n+3, \dots) \in \mathbb{R}^\infty$. Prove that $\operatorname{span}(v_1, v_2, v_3, \dots)$ is finite dimensional and determine its dimension.

MATH2211 SPRING 2022
EXAM 2

WEDNESDAY, APRIL 20 2022

Name: ______________________________

This exam is open notes. There are 50 points total in this exam.

High 46/50, low 15.5/50, mean 33.3/50, median 34.5/50. Good exam. This exam was tougher than the first exam. Every problem was fully solved by at least one student, however. The hardest problem was problem 5, with average points obtained being 41%. The next hardest was 4(a), with average points obtained being 54%.

It is also worth noting that Problem 5 was essentially done in the intro to eigenvalues lecture, albeit with a different matrix. However, the presentation of the result here as an explicit limit (which wasn't done in lecture) may have thrown most students off.

Problem 1. Let $A = \begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$.

(a) (5 points) Find an elementary matrix E such that EA is upper triangular.

(b) (5 points) Give a basis of $\bigwedge^2 \mathbb{R}^2$ and write the matrix of the linear operator
$$\textstyle\bigwedge^2 A \colon \bigwedge^2 \mathbb{R}^2 \to \bigwedge^2 \mathbb{R}^2$$
in this basis.

Problem 2. (10 points) The *right shift* operator $T \colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined by
$$T(a_0, a_1, a_2, \dots) = (0, a_0, a_1, a_2, \dots).$$
What are the eigenvalues and eigenvectors of T?

Problem 3. (10 points) Prove that if $T \colon V \to V$ is diagonalizable, then $T^2 + T + I_V$ is also diagonalizable.

Hint: Eigenvectors.

Problem 4. Let A be a 50×100 real matrix whose kernel is 74-dimensional.

(a) (5 points) What is the dimension of the space of row vectors y such that $yA = 0$?

(b) (5 points) Suppose you know that the vector $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{100}$ is in the kernel of A. This implies that every row vector $(y_1 \; y_2 \; \cdots \; y_{100})$ in the image of ${}^tA$ satisfies some condition. What is this condition? Prove your answer.

Problem 5. (10 points) Let $M = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$. For $n \geq 0$, let a_n be the top left entry of M^n. Prove that

$$\lim_{n\to\infty} \frac{a_n}{(1+\sqrt{3})^n} = \frac{1}{2}.$$

MATH2211 SPRING 2022
FINAL EXAM

FRIDAY, MAY 13 2022

Name: ______________________________

This exam is open notes, and the time limit is 3 hours. There are 100 points total in this exam.

High 87, low 39, mean 70.15, median 71. Good exam. The intended structure was that the first 5 problems were standard linear algebra problems, while the last 5 required at least one step of original thought. The scores reflected this quite well; average score across problems 1 to 5 was 85.5%, while average score across problems 6 to 10 was 54.8%. The hardest problems were 7(a), 7(b), 8(b), 9, and 10(b), with average points obtained being 43%, 44%, 39%, 43%, and 50%. It also turned out that 8(a) and 10(a) from the second half of the exam were quite easy; average score on those was 80%.

The top score on problem 9 was 6/10, which means that the problem should have a substantial hint in the future.

Nobody got full points on problem 8(b), despite the extensive hint. Two people came close, saying that I and C^{-1} are both in V_C. The issue is of course that C^{-1} does not always exist, and they should have used C instead of C^{-1}.

Problem 1. Let $M = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$.

(a) (5 points) Solve the system

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$$

in terms of a.

(b) (5 points) Find $\operatorname{tr} M$, $\det M$, and the characteristic polynomial of M.

Problem 2. (10 points) For which $t \in \mathbb{R}$ do the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}, \quad v_2 = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$?

Problem 3. (10 points) Show that every complex solution to $z^4 + 1 = 0$ also satisfies $z^{1200} = 1$.

Problem 4. (10 points) Find the inverse of the linear operator $T \colon \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$T(x, y, z) = (x + y, x + z, y + z),$$

or prove that no such inverse exists.

Problem 5. (10 points) Find a basis of eigenvectors for the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hint: It will be helpful to let $\zeta = e^{2\pi i/3}$. (Make use of the zeta drawing skills you learned in class!)

Problem 6. (10 points) Let X be a 2×2 real matrix with trace 0 and rank 1. Prove that X only has 0 as an eigenvalue.

Problem 7. Let $V = C([0,1], \mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. The integral operator I defined by

$$(I(f))(x) = \int_0^x f(t)\, dt$$

is a linear operator on V.

(a) (5 points) Prove that I is not surjective.

Hint: The non-surjectivity comes from a simple observation; no real analysis knowledge is required.

(b) (5 points) What is $({}^tI)(\delta)$, where δ is the Dirac delta functional? Show that your answer gives another proof that I is not surjective.

Problem 8. Let $n \geq 2$ be a positive integer. For any matrix $C \in M_n(\mathbb{R})$, let V_C be the set of all $n \times n$ matrices $A \in M_n(\mathbb{R})$ such that $AC = CA$.

(a) (5 points) Prove that V_C is a subspace of $M_n(\mathbb{R})$.

(b) (5 points) Prove that $\dim V_C \geq 2$ for all $C \in M_n(\mathbb{R})$.

Hint: Consider the case when C is a scalar matrix and the case when C is not a scalar matrix separately. In the latter case, try to come up with two linearly independent matrices in V_C.

Problem 9. (10 points) Find a counterexample to the following (reasonable-sounding) claim: If P and Q are orthogonal projection operators, then PQ is also an orthogonal projection operator.

Problem 10.

(a) (5 points) Let $e_1 \ldots e_n$ be an orthonormal basis of an inner product space V. Prove that the functionals $\varepsilon_i \in V^*$ defined by

$$\varepsilon_i(x) = \langle x, e_i \rangle, \quad 1 \leq i \leq n,$$

are precisely the dual basis of $e_1, \ldots, e_n$.

(b) (5 points) Suppose now that $e_1, \ldots, e_n$ is only an orthogonal basis, meaning that $\langle e_i, e_j \rangle = 0$ for $i \neq j$, but the $\|e_i\|$ are not necessarily equal to 1. For each i, let $\ell_i = \|e_i\|$, the length of e_i. Find the dual basis to $e_1, \ldots, e_n$, in terms of the ε_i defined in part (a) as well as the ℓ_i.

MATH2211 SPRING 2022
PROBLEM SET 1

This problem set is due on Wednesday, February 2 at 11:59 pm. Each problem part is worth 3 points.

Problem 1. Let S be a set and let $A, B, C \subseteq S$. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Problem 2. Show that the set of real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$, with addition and multiplication as in $\mathbb{R}$, is a field.

Problem 3. Define the Fibonacci numbers $F_0, F_1, F_2, \ldots$ by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Use induction to prove that $F_1 + \cdots + F_n = F_{n+2} - 1$.

(b) Use induction to prove that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

Problem 4. Prove that for every $n \in \mathbb{Z}^+$

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Problem 5. Suppose that $z_1, z_1 \in \mathbb{C}$. Prove that

(a) $|z_1 z_2| = |z_1| \cdot |z_2|$.

(b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

Problem 6.

(a) Find all complex solutions to $z^4 = -1$.

(b) Find all complex solutions to $z^3 = i$.

Express your answers both in the form $re^{i\theta}$ and by giving the real and imaginary parts.

Problem 7. Suppose $z_1, z_2 \in \mathbb{C}$. Prove the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

MATH2211 SPRING 2022
PROBLEM SET 2

DUE WEDNESDAY, FEBRUARY 9 2022 AT 11:59 PM

Problem 1. Compute the real and imaginary parts of $\dfrac{\pi + i}{5 - i}$.

Problem 2.

(a) Use power series expansions to prove Euler's formula[1]
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(b) Use Euler's formula to prove the identity
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

(c) Use the same technique to derive a formula for $\cos(3\theta)$ in terms of $\cos\theta$.[2]

Problem 3. Let $z = e^{\frac{2\pi i}{n}}$, where $n \in \mathbb{Z}^+$. Prove that $1 + z + z^2 + \cdots + z^{n-1} = 0$.[3]

Problem 4. Read up about Fermat's little theorem by looking it up on the internet. Using Fermat's little theorem, find the roots of $x^{10} - 1$ over $\mathbb{F}_{11}$.

Problem 5.

(a) Is $U = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ a subspace of $\mathbb{C}^3$?

(b) Is $U = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 : x_1 x_2 x_3 = 0\}$ a subspace of $\mathbb{Q}^3$?

(c) Let P be the $\mathbb{R}$-vector space of all polynomials with real coefficients. is
$$U = \{f \in P : f'(-1) = 3f(2)\}$$
a subspace of P? Here, f' means the derivative of f.

Problem 6.

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[1] If you don't remember what the power series of exp, sin, and cos are, you can look them up on the internet.

[2] This can be generalized to $\cos(n\theta)$: look up *Chebyshev polynomials of the first kind* on the internet.

[3] Hint: Factor the polynomial $x^n - 1$.

(a) Is $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{C}^3$ a linear combination of $\begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$?

(b) In the real vector space consisting of all polynomials with real coefficients, is

$$x + 1 \in \operatorname{span}\{x^2 + 1, x^3 + x, 2x^2 + x, x + 3\}?$$

Problem 7. Show that a subset W of a vector space is a subspace if and only if $\operatorname{span}(W) = W$.

MATH2211 SPRING 2022
PROBLEM SET 3

DUE WEDNESDAY, FEBRUARY 16 2022 AT 11:59 PM

Let F be a field and let V be an F-vector space.

Problem 1. Suppose we are given a list $v_1, \dots, v_n \in V$.

(a) Show that $v_1, \dots, v_n$ are linearly dependent if and only if there is some $1 \leq i \leq n$ such that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ (i.e. the span of the list with v_i taken out).

(b) Show that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ if and only if

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

(That is, the span doesn't change when v_i is taken out.)

Problem 2. Suppose $v_1, v_2, v_3, v_4 \in V$ and set

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_3 - v_4, \quad w_4 = v_4.$$

(a) Show that $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(w_1, w_2, w_3, w_4)$.

(b) Show that v_1, v_2, v_3, v_4 are linearly independent if and only if w_1, w_2, w_3, w_4 are linearly independent.

Problem 3. Suppose that $\{v_1, v_2, \dots, v_n\}$ is linearly independent in V. Show that $\{v_1, v_1 + v_2, \dots, v_1 + v_2 + \cdots + v_n\}$ is linearly independent as well.

Problem 4.

(a) Show that V is infinite dimensional if and only if it satisfies the following property: for every integer $k > 0$, one can find k linearly independent vectors $v_1, \dots, v_k \in V$.

(b) Show that the vector space $\mathbb{R}^\infty := \{$ all sequences $(a_1, a_2, a_3, \dots)$ of real numbers$\}$ is infinite dimensional.

(c) Give an example of a subspace of $\mathbb{R}^\infty$ which is strictly contained in $\mathbb{R}^\infty$ but is still infinite dimensional.

Problem 5. For each positive integer n, let

$$B_n = \{(-1, 1, \dots, 1), (1, -1, 1, \dots, 1), \dots, (1, 1, \dots, -1)\} \subseteq F^n.$$

That is, B_n is the set of vectors in F^n with one component equal to -1 and $n-1$ components equal to 1.

(a) Let $F = \mathbb{R}$. For which n is B_n a basis of $\mathbb{R}^n$?[1]

(b) Let $F = \mathbb{F}_3$. Show that if n is of the form $3k + 2$ for some $k \in \mathbb{Z}_{\geq 0}$ (i.e. $n \in \{2, 5, 8, \dots\}$), then B_n is not a basis of $\mathbb{F}_3^n$.[2]

[1] If you need a hint for where to start, try to check whether the vectors in B_n are linearly independent.

[2] Hint: Try to show that B_n is contained in a proper subspace.

MATH2211 SPRING 2022
PROBLEM SET 4

DUE WEDNESDAY, FEBRUARY 23 2022 AT 11:59 PM

Useful reading for this problem set: Axler pages 20-23 (Sums of subspaces), Axler page 47 (Dimension of a sum and proof)

Problem 1. For each $c \in \mathbb{R}$, determine the dimension of

$$U = \operatorname{span}\left(\begin{pmatrix}2\\3\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}7\\3\\c\end{pmatrix}\right).$$

Problem 2. Suppose that U and W are 4-dimensional $\mathbb{C}$-subspaces of $\mathbb{C}^6$. Show that one can find two vectors in $U \cap W$, neither of which is a scalar multiple of the other.[1]

Problem 3.

(a) Find a basis for the subspace

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0 \text{ and } x_3 - x_4 = 0\}$$

and extend it to a basis of $\mathbb{R}^4$.

(b) Let $P_5(\mathbb{R})$ be the vector space of all polynomials of degree at most 5 with real coefficients. Find a basis for the subspace

$$U = \{f(x) \in P_5(\mathbb{R}) : f(1) = 0 \text{ and } f(2) = 0\}$$

and extend it to a basis of $P_5(\mathbb{R})$.

Problem 4. Given subspaces $U_1, U_2 \subseteq V$, prove that the following properties are equivalent.

- $V = U_1 + U_2$ and $\{0\} = U_1 \cap U_2$.
- Every vector $v \in V$ can be written in a **unique** way as $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$.

When these hold, we say that V is the *direct sum* of U_1 and U_2, and write $V = U_1 \oplus U_2$.

[1] Hint: the existence of such two vectors is precisely equivalent to a simple dimension condition on $U \cap W$. Can you see it?

Problem 5. Prove or give a counterexample: if $f\colon A \to B$ and $g\colon B \to C$ are invertible functions, then $g \circ f\colon A \to C$ is also invertible.

Problem 6. Suppose $T\colon V \to W$ is a linear map, and $v_1, \dots, v_n \in V$. For each statement, give a proof or a counterexample.

(a) If T is injective and $v_1, \dots, v_n$ are linearly independent, then $T(v_1), \dots, T(v_n)$ are linearly independent.

(b) If $T(v_1), \dots, T(v_n)$ are linearly independent, then $v_1, \dots, v_n$ are linearly independent.

(c) If T is surjective and $v_1, \dots, v_n$ span V, then $T(v_1), \dots, T(v_n)$ span W.

(d) If $T(v_1), \dots, T(v_n)$ span W, then $v_1, \dots, v_n$ span V.

MATH2211 SPRING 2022
PROBLEM SET 5

DUE WEDNESDAY, MARCH 16, 2022 AT 11:59 PM

Problem 1. Find a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with kernel

$$\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$$

or prove that no such T exists.

Problem 2. Show that there is a unique linear map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix},$$

and find the corresponding 3×3 matrix. Is this linear map an isomorphism?

Problem 3.

(a) Is there a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with $\operatorname{im}(T) = \ker(T)$?

(b) Is there a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ with $\operatorname{im}(T) = \ker(T)$?

Problem 4. A *linear functional* on an F-vector space V means a linear map from V to F (the one-dimensional F-vector space). For example, $(x, y) \mapsto 2x + 3y$ is a linear functional on $\mathbb{R}^2$.

(a) Suppose that T is a linear functional on a vector space V of dimension n. Prove that the kernel of T has dimension either n or $n - 1$. When does the kernel have dimension $n - 1$?

(b) Suppose S and T are two linear functionals on a vector space V with the same kernel. Prove that there exists a scalar $c \in F$ such that $T = cS$.

Problem 5.

(a) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map corresponding to

$$A = \begin{pmatrix}1 & 2 & 1 & 0 & 1\\ 2 & 1 & 0 & 0 & 2\\ -1 & 0 & 1 & 1 & 1\end{pmatrix}.$$

Find bases for the kernel and image of T and find all solutions to $Tx = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$.

(b) Find bases for the kernel and image of the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x,y) = (x+y, 0, 2x-y)$.

Problem 6.

(a) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the matrix of T^{-1} is precisely the $n \times n$ matrix such that for $i = 1, 2, \ldots, n$, its ith column is the unique vector v such that $Tv = e_i$.

(b) Using part (a), find the inverse of

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Try not to use any previously learned matrix inversion methods from high school or such.

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PROBLEM SET 6

DUE WEDNESDAY, MARCH 23, 2022 AT 11:59 PM

Problem 1. Let $T\colon V \to V$ be a linear map from a vector space V to itself. Let $\mathcal{B} = (v_1, \dots, v_n)$ be an ordered basis of V and let $A \in M_n(F)$ be the matrix of V with respect to $\mathcal{B}$. Let $\mathcal{C} = (v_1', \dots, v_n')$ be another ordered basis of V and let $A' \in M_n(F)$ be the matrix of T with respect to $\mathcal{C}$. Prove that there exists an invertible matrix $B \in M_n(F)$ such that $A' = BAB^{-1}$.

Problem 2. Factor $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ as a product of elementary matrices.

Hint: One way to approach this is to try to turn A into the identity matrix via elementary matrices, then see how what you've done is useful.

Problem 3.

(a) In this problem we show that a one-sided inverse of a square matrix is a two-sided inverse. More precisely, let $A, B \in M_n(F)$. Show that

$$AB = I_n \iff BA = I_n.$$

Hint: For the forward direction, assume $AB = I_n$ and first show that $A\colon F^n \to F^n$ is surjective. Similar hint applies to the other direction.

(b) In this problem we show that inverses are unique. More precisely, suppose $A, B, C \in M_n(F)$ such that $AB = I_n$ and $AC = I_n$. Prove that $B = C$.

Hint: In this proof, you are not allowed to multiply both sides by A^{-1}, because doing that presupposes that A^{-1} is unique! Instead, start by proving that A is surjective.

Problem 4. Let $A \in M_n(\mathbb{R})$ be an invertible matrix with integer entries, such that A^{-1} also has integer entries. Prove that $\det A = \pm 1$.

Problem 5. Compute:

(a) $\det \begin{pmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{pmatrix}$.

(b) $\det \begin{pmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{pmatrix}$.

Problem 6. Directly from the definition of $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ for the 2×2 determinant, prove the following:

(a) $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_2(F)$.

(b) A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(c) The function $\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}; (v, w) \mapsto \det \begin{pmatrix} v & w \end{pmatrix}$ (the determinant of the matrix whose columns are v and w) is *bilinear*, meaning that for all $\alpha, \beta \in F$ and $v_1, v_2, v, w_1, w_2, w \in \mathbb{R}^2$, we have

(i) $\det(\alpha v_1 + \beta v_2, w) = \alpha \det(v_1, w) + \beta \det(v_2, w)$,

(ii) $\det(v, \alpha w_1 + \beta w_2) = \alpha \det(v, w_1) + \beta \det(v, w_2)$.

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PROBLEM SET 7

DUE WEDNESDAY, MARCH 30, 2022 AT 11:59 PM

Problem 1. Suppose $A \in M_{m\times n}(F)$. Prove or find counterexamples:

(a) For any $B \in M_{n\times p}(F)$, $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

(b) For any $B \in M_{p\times m}(F)$, $\operatorname{rank}(BA) \le \operatorname{rank}(A)$.

Problem 2.

(a) Prove, using the exterior algebra definition of the determinant, that the determinant of an upper-triangular square matrix is the product of the diagonal entries.

(b) You may assume that a similar proof as in part (a) shows that the same result holds for lower triangular square matrices as well.

Prove that that the elementary matrix E representing the row operation "add a multiple of row j to row i" ($i \neq j$) has determinant 1.

Problem 3.

(a) Prove that if $T\colon V \to W$ is surjective, then ${}^tT\colon W^* \to V^*$ is injective.

(b) Prove that if $T\colon V \to W$ is injective, then ${}^tT\colon W^* \to V^*$ is surjective.

(c) Prove that every linear transformation $T\colon V \to W$ can be factored as a composition

$$V \overset{T'}{\twoheadrightarrow} U \overset{i}{\hookrightarrow} W,$$

where T' is surjective and i is injective. (Hint: Set $U = \operatorname{im} T$.)

(d) Recall that the rank of a linear transformation $T\colon V \to W$ is defined to be $\dim \operatorname{im} T$. Use parts (a), (b), and (c) to prove that $\operatorname{rank}(T) = \operatorname{rank}({}^tT)$.

Problem 4. Let V be a vector space. Given a nonzero element $v \in V$ and a nonzero linear functional $\lambda\colon V \to \mathbb{R}$, we can make a linear transformation $T_{v,\lambda}\colon V \to V$ by sending $x \in V$ to $\lambda(x)\cdot v$. Prove that T has rank 1, and prove that every rank 1 linear transformation from V to V is equal to $T_{v,\lambda}$ for some $0 \neq v \in V$ and $0 \neq \lambda\colon V \to \mathbb{R}$.

Hint for the second part: Problem 3(c) is very helpful.

Problem 5. Let $\frac{d}{dx}\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the derivative operator on the space of polynomials of degree at most 2 over $\mathbb{R}$.

(a) Prove that ${}^t(\frac{d}{dx})\colon P_2(\mathbb{R})^* \to P_2(\mathbb{R})^*$ has a 1-dimensional kernel.

(b) Prove that $\ker({}^t(\frac{d}{dx}))$ is spanned by the functional taking a polynomial to its x^2 coefficient. Can you provide a plain English interpretation of what this is saying?

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PROBLEM SET 8

DUE FRIDAY, APRIL 8, 2022 AT 11:59 PM

Problem 1. For each of the following matrices A, find the eigenvalues of A and a basis for each eigenspace, and say whether A is diagonalizable.

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{pmatrix}$.

Problem 2. Compute the characteristic polynomial of the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

Problem 3. A *stochastic matrix* is a square matrix of real numbers, all of whose entries are between 0 and 1 (inclusive), and with the property that the numbers in each column add to 1.

Prove that 1 is an eigenvalue of any stochastic matrix.

Problem 4. Let $T\colon V \to V$ be a linear transformation and $B\colon V \to V$ be an invertible linear transformation. Prove that

$$\chi_T(t) = \chi_{B^{-1}TB}(t).$$

Problem 5. Prove that the eigenvalues of an upper triangular square matrix are the numbers on the diagonal.

Problem 6. In this problem you will be working out the derivation of the closed form of the Fibonacci sequence that I showed at the beginning of the semester.

Let $\mathbb{R}^\infty$ be the vector space of infinite sequences of real numbers. Define the left shift operator $S\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$S(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots).$$

(a) Because $\mathbb{R}^\infty$ is infinite-dimensional (in fact, uncountably-dimensional), weird things can happen. Prove that *every real number* is an eigenvalue of S. Also find the corresponding eigenvector for an arbitrary eigenvalue $\lambda \in \mathbb{R}$.

(b) Let $V \subseteq \mathbb{R}^\infty$ be the subspace consisting of Fibonacci-like sequences; that is, consisting of sequences $(a_0, a_1, a_2, \dots)$ satisfying $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that if $v \in V$, then $Sv \in V$ as well.

(c) Part (b) showed that S restricts to a linear map $S|_V\colon V \to V$. What is the dimension of V? What are the eigenvalues and eigenvectors of $S|_V$?

(d) Derive the closed form for the Fibonacci sequence $a_0 = 0, a_1 = 1; a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, using part (c).

(e) To convince you that this "abstract" proof is not actually that abstract, here is a final exercise. Let $\varphi\colon \mathbb{R}^2 \to V$ be the isomorphism sending (x, y) to the Fibonacci-like sequence whose first two terms are $a_0 = x$ and $a_1 = y$. Then $\varphi^{-1}\colon V \to \mathbb{R}$ is the projection $(a_0, a_1, \dots,) \mapsto (a_0, a_1)$. Thus, the composition

$$\mathbb{R}^2 \xrightarrow{\varphi} V \xrightarrow{S} V \xrightarrow{\varphi^{-1}} \mathbb{R}^2$$

is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, i.e. a 2×2 matrix. What is this matrix?

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PROBLEM SET 9

DUE FRIDAY, APRIL 15, 2022 AT 11:59 PM

Useful reading for Problems 1 and 2: Section 8.A of Axler.

Problem 1. A *nilpotent* linear operator is defined to be a linear operator $T\colon V \to V$ such that some power of T is equal to zero. In parts (b) to (d), let $N\colon V \to V$ be a nilpotent operator on a finite dimensional vector space V.

(a) Give an example of a 2×2 real nilpotent matrix (i.e. a nilpotent linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$) none of whose entries are 0 (or prove they don't exist).

(b) Now let N be a nilpotent operator on a finite dimensional vector space V. Prove that 0 is the only eigenvalue of N.

(c) Let U be any nonzero subspace of V and suppose that $NU \subseteq U$. Prove that NU is strictly contained in U.

Hint: Use contradiction. Comment: A useful notation for strict containment is $\subsetneq$.

(d) Prove that $N^{\dim V} = 0$. Hint: Use the previous part iteratively starting with $U = V$.

Problem 2. Given a polynomial $p \in F[t]$ (note: $F[t]$ just means the set of polynomials in the variable t with coefficients in F) and a linear operator $T\colon V \to V$ on a vector space V over F, the expression $p(T)$ makes sense if we use powers and addition of linear operators, to give a linear operator $p(T)\colon V \to V$.

(a) Let $V = F^2$ and $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+y \\ y \end{smallmatrix}\right)$. For the polynomial $p(t) = t^2 - 2t$, find $p(T)$.

(b) Prove that for every linear operator $T\colon V \to V$ with $\dim V = n$, there exists a polynomial $p \in F[t]$, of degree at most n^2, such that $p(T) = 0$.

Note: Do not use the theory of minimal polynomials or any theorems named after multiple people.

Hint: Use linear dependence ideas.

(c) Let $J_{n,\lambda}$ be the $n \times n$ Jordan block (this is the standard name for what I called an "atomic Jordan matrix" in class) with λ's on the diagonal and 1's above the diagonal. Prove that $p(J_{n,\lambda}) = 0$ for the polynomial $p(t) = (t - \lambda)^n$, but $q(J_{n,\lambda}) \neq 0$ for the polynomial $q(t) = (t - \lambda)^{n-1}$.

Comment (not a hint): This problem essentially says that $J_{n,\lambda} - \lambda I_n$ is a nilpotent operator and that n is the least power k that makes $(J_{n,\lambda} - \lambda I_n)^k = 0$.

Problem 3. A *permutation matrix* is a square matrix where in each column and each row, there is exactly one nonzero entry and that nonzero entry is a 1. The name is because an $n \times n$ permutation matrix times an $n \times 1$ vector is the same vector but with entries permuted. Let P be an $n \times n$ permutation matrix.

(a) Prove that the product of two $n \times n$ permutation matrices is another permutation matrix.

(b) Using part (a), prove that some positive power of P is equal to the identity.

Hint: Are there infinitely many $n \times n$ permutation matrices?

(c) Provide a counterexample to the claim that for all $n \in \mathbb{Z}^+$ and all $n \times n$ permutaion matrices P, there is some $1 \leq k \leq n$ such that P^k is the identity.

Hint: You will not find a counterexample by looking at $n \leq 4$.

Problem 4. Let P_3 be the space of real polynomials of degree at most 3, and define an inner product on P_3 by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

Set $U = \operatorname{span}\{x, x^2\}$.

(a) Find an orthonormal basis for U.

(b) Write $x^3 = p(x) + q(x)$ with $p(x) \in U$ and $q(x) \in U^\perp$.

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PROBLEM SET 10

DUE FRIDAY, APRIL 29, 2022 AT 11:59 PM

Relevant reading: Axler Chapter 6 and Section 7.A.

Problem 1. Let V be a finite-dimensional inner product space and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis. Prove *Parseval's Identity*: for all $x, y \in V$,

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, e_i\rangle\langle e_i, y\rangle.$$

Problem 2.

(a) Let V be a real vector space and let $\langle\cdot,\cdot\rangle$ be a bilinear form on V. Prove that for every $v \in V$, the map φ_v defined by $\varphi_v(x) = \langle v, x\rangle$ is a linear functional on V.

Remark: This shows that, given a bilinear form on V, we get a natural map φ from V to V^* sending each $v \in V$ to φ_v. The Riesz representation theorem says that if the bilinear form $\langle\cdot,\cdot\rangle$ is an inner product, then $\varphi\colon V \to V^*$ is an isomorphism. Moreover, in $\mathbb{R}^n$ with the standard Euclidean inner product, φ is exactly the map that turns a column vector into a row vector, which is classically denoted $\cdot^T$.

(b) Let's do a concrete example. Let $V = P_2(\mathbb{R})$ with inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Let $[x^2]$ denote the linear functional which when given a polynomial, returns its coefficient of the x^2 term. By the Riesz representation theorem, $[x^2]$ can be represented as $\langle f, \cdot\rangle$ for a unique polynomial $f \in P_2(\mathbb{R})$. Find f.

Problem 3. Let V be an inner product space and let U be a subspace of V. In this problem we investigate the orthogonal projection operator P_U.

(a) Prove that $P_U^2 = P_U$ and that P_U is self-adjoint (that is, $P_U = P_U^*$). Show that the identity $P_U^2 = P_U$ is equivalent to saying that $P_U|_{\operatorname{im} P_U} = I_{\operatorname{im} P_U}$.

(b) In this problem, suppose that $U = \operatorname{span}(u)^\perp$ for some nonzero unit vector $u \in V$. Prove that P_U can be expressed as $I - u\varphi_u$ (more classically denoted $I - uu^T$).

(c) Prove that the eigenvalues of P_U are 0 and 1, with the multiplicity of 1 being the dimension of U and the multiplicity of 0 being the dimension of $U^\perp$.

(d) Prove that the eigenspace of 1 of P_U is U and that the eigenspace of 0 is $U^\perp$, and that P_U is diagonalizable.

Problem 4. Prove that if $T\colon V \to V$ is self-adjoint, then its eigenspaces corresponding to two different eigenvalues are orthogonal to each other.

Problem 5. Let $A \in M_{m\times n}(\mathbb{R})$, and let A^* denote the adjoint of A, namely the unique matrix in $M_{n\times m}(\mathbb{R})$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

(a) Prove that $\ker(A) = \ker(A^*A)$, where A^* denotes the adjoint of A.

(b) For a general matrix equation $Ax = b$, recall that there may be no solutions. Multiplying both sides on the left by A^*, we get the equation $A^*Ax = A^*b$. This is called the least squares normal equation for the matrix equation. It turns out that $A^*Ax = A^*b$ always has a solution, and that any solution x to the normal equation minimizes the value of $\|Ax - b\|$.

In this exercise, we prove this last part. Prove that if x is a solution to the equation $A^*Ax = A^*b$, then the projection of b onto the image of A is equal to Ax. Show therefore that $\|Ax - b\|$ is minimized at those x which solve $A^*Ax = A^*b$.

Appendix A. Supplementary materials

Examples of proofs that $\sqrt{2}$ is irrational

MATH2211 Spring 2022

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1. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Also assume $\gcd(a, b) = 1$. Then we have
$$\frac{a^2}{b^2} = 2,$$
so $a^2 = 2b^2$. This equation implies that a^2 is even because $2b^2$ is even. Therefore, a is even. Let $a = 2k$ for some positive integer k. Then $a^2 = 2b^2$ can be rewritten as
$$(2k)^2 = 2b^2$$
or
$$4k^2 = 2b^2$$
or
$$2k^2 = b^2.$$
Therefore, b is even. Therefore, a and b are both even, which contradicts $\gcd(a, b) = 1$.

2. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Let S be the set of all pairs of positive integers (a, b) such that $a/b = \sqrt{2}$. If S is empty, then we're done. Assume S is nonempty. Pick some $(a, b) \in S$. Then we have
$$\frac{a^2}{b^2} = 2,$$
so $a^2 = 2b^2$. This equation implies that a^2 is even because $2b^2$ is even. Therefore, a is even. Let $a = 2k$ for some positive integer k. Then $a^2 = 2b^2$ can be rewritten as
$$(2k)^2 = 2b^2$$

or

$$4k^2 = 2b^2$$

or

$$2k^2 = b^2.$$

Therefore, b is even. Therefore, a and b are both even. Therefore, $a/2$ and $b/2$ are both positive integers and $(a/2, b/2) \in S$. Repeat this argument on $(a/2, b/2)$ to conclude that $a/2$ and $b/2$ are themselves even, so $(a/4, b/4) \in S$. This can be repeated forever, producing an infinite decreasing sequence of positive integers $a, a/2, a/4, a/8, \ldots$. This is impossible, so $\sqrt{2}$ is irrational. (This is called proof by infinite descent.)

3. We know that $x^2 - 2$ has $\pm\sqrt{2}$ as roots. Let's apply the rational root theorem to $x^2 - 2$. The theorem says that if a/b (in lowest terms) is a root of $x^2 - 2$, then a divides 2 and b divides 1. In other words, the only possible rational roots of $x^2 - 2$ are

$$\{-2, -1, 1, 2\}.$$

Now, $(-2)^2 - 2 = 2 \neq 0$. $(-1)^2 - 2 = -1 \neq 0$. $1^2 - 2 = -1 \neq 0$. $2^2 - 2 = 2 \neq 0$. Therefore, $x^2 - 2$ has no rational roots. So $\sqrt{2}$ is irrational.

4. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Let S be the set of all pairs of positive integers (a, b) such that $a/b = \sqrt{2}$. If S is empty, then we're done. Assume S is nonempty. Pick some $(a, b) \in S$. Then we have

$$\frac{a^2}{b^2} = 2,$$

so $a^2 = 2b^2$. Now we compute

$$\begin{aligned}(2b - a)^2 &= 4b^2 - 4ab + a^2 \\ &= 6b^2 - 4ab.\end{aligned}$$

We also compute

$$\begin{aligned}(a - b)^2 &= a^2 - 2ab + b^2 \\ &= 3b^2 - 2ab.\end{aligned}$$

Therefore, $(2b - a)^2 = 2(a - b)^2$. Therefore, $(2b - a, a - b) \in S$. Moreover, $a - b < b$ because $1 < \sqrt{2} < 2$, which implies $b < \sqrt{2}b < 2b$, which implies $b < a < 2b$, which implies $0 < a - b < b$. So again contradiction by infinite descent.

DETERMINANTS VIA THE EXTERIOR ALGEBRA

YONGYI CHEN

1. Introduction

I have not found on the internet any notes on determinants via the exterior algebra which are friendly to first year undergraduates, so I shall attempt to create such a note here.

2. Motivation for the exterior algebra

Recall the *cross product* of vectors in $\mathbb{R}^3$. This notion is used, for example, to define torque and angular momentum in physics. These are commonly called *pseudovectors* because they are not real vectors in a physical sense in an important way: If the whole space is transformed by some linear transformation, the cross product of the transformed vectors is not the transformation applied to the cross product!

As a simple example, suppose space was stretched by a factor of 2. This would make any vector twice as long, and so would make the cross product of two vectors four times as long! So as we can see, the cross product of two vectors behaves more like an *area* than a *length*.

You may also have heard that cross product is only well defined in 3 dimensions (and 7 dimensions, but that's a different kind of cross product). The reason for this turns out to be the following: In 3 dimensions, the number of basis planes you can make out of basis vectors is 3 (namely, the xy, yz, and xz planes), which is the same as the number of basis vectors. Furthermore, 3 is only the number of dimensions with this property.

As we will see, the cross product of two vectors in V should therefore belong not to the vector space V itself, but to some vector space of "areas" in V. This leads to the notion of bivectors, which we will define as elements of the second exterior power of V. As a bonus, this gives us a definition of cross product that works in any dimension. This leads us to the next section.

3. The exterior algebra

Let V be a finite dimensional vector space over some field F. Let $n = \dim V$.

Definition 1. Let k be a nonnegative integer. A *k-blade* on V is a formal concatenation of k vectors (with each vector belonging to V) with the $\wedge$ symbol inserted between them. A *k-vector* on V is a linear combination of k-blades.

Remark 1. A 1-blade is just a vector. The definition does not make clear what a 0-blade is, so we will define a 0-blade to be a scalar.

Example 1. Let $V = F^2$ with standard basis e_1, e_2. Some examples of 2-blades are

$$e_1 \wedge e_2, \quad e_2 \wedge e_1, \quad (e_1 + e_2) \wedge e_2.$$

An example of a 3-blade on V is $e_1 \wedge e_1 \wedge e_2$. An example of a 3-vector on V is $e_1 \wedge e_1 \wedge e_2 + e_2 \wedge (e_1 + 2e_2) \wedge e_1$ (a sum of two 3-blades).

Definition 2. The *kth exterior power* of V, denoted $\bigwedge^k V$, is the vector space of k-vectors on V, subject to the following axiomatic relations:

(a) Multilinarity: For any $v_1, \ldots, v_k \in V$ and any $w \in V$, we have

$$v_1 \wedge \cdots \wedge (v_i + w) \wedge \cdots \wedge v_k = (v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k) + (v_1 \wedge \cdots \wedge w \wedge \cdots \wedge v_k),$$

and moreover for any $c \in F$ we have

$$v_1 \wedge \cdots \wedge cv_i \wedge \cdots \wedge v_k = c \cdot (v_1 \wedge \cdots \wedge v_k).$$

(b) Skew-symmetry: For any $v_1, \ldots, v_k \in V$ and any $1 \leq i \leq k-1$, we have

$$v_1 \wedge \cdots \wedge v_i \wedge v_{i+1} \wedge \cdots \wedge v_k = -v_1 \wedge \cdots \wedge v_{i+1} \wedge v_i \wedge \cdots \wedge v_k.$$

(c) Alternating property: For any $v_1, \ldots, v_k \in V$, if some v_i is equal to some v_j, then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = 0.$$

(d) Nonzero for linearly independent vectors: If $v_1, \ldots, v_k \in V$ are linearly independent, then

$$v_1 \wedge \cdots \wedge v_k \neq 0.$$

Furthermore, from now on, k-blades and k-vectors will be thought of as their corresponding elements of $\bigwedge^k V$ (in other words, some k-vectors will end up being equal to 0 according to these relations, or to other k-vectors which don't look the same at first glance).

Remark 2. In fact, for fields of characteristic other than 2, property (b) is equivalent to property (c). One can also check that a generalization of property (b) holds: any swap of the components of a blade (not necessarily adjacent components) induces multiplication by -1.

Remark 3. We have by definition $\bigwedge^0 V = F$ and $\bigwedge^1 V = V$.

A 2-blade in $\bigwedge^2 V$ should be thought of as an "oriented area"—the orientation being given by the span of the vectors in the blade and the order of the vectors. In a similar way, a 3-blade should be thought of as an "oriented volume." For example, $e_1 \wedge e_2$ can be thought of the oriented area of $+1$ in the plane spanned by e_1 and e_2, and $e_2 \wedge e_1 = -e_1 \wedge e_2$ can be thought of as the oriented area of -1 in the same plane.

Example 2. Let us look at Example 1 more closely. In $V = F^2$, we have $e_2 \wedge e_1 = -e_1 \wedge e_2$, and $(e_1 + e_2) \wedge e_2 = e_1 \wedge e_2 + e_2 \wedge e_2 = e_1 \wedge e_2$. Furthermore, $e_1 \wedge e_1 \wedge e_2 = 0$. In fact, as we will see soon, $\bigwedge^3 F^2$ is the zero vector space.

Example 3. Sometimes a sum of k-blades can be written as a single k-blade. For example, in F^3, $e_1 \wedge e_2 + e_2 \wedge e_3$ can be rewritten using the axioms as $(e_1 - e_3) \wedge e_2$. However, not all k-vectors can be so written. For example, in F^4,

$$e_1 \wedge e_2 + e_3 \wedge e_4$$

cannot be expressed as a single 2-blade. Such k-vectors are said to be *indecomposable.*

Proposition 1. *Let $e_1, \ldots, e_n$ be a ordered basis of V. Then $\bigwedge^k V$ has, as a basis, the set of k-blades $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for all tuples $(i_1, \ldots, i_k)$ satisfying $1 \le i_1 < \cdots < i_k \le n$.*

Proof. First, let us show that the set of k-blades $e_{i_1} \wedge \cdots \wedge e_{i_k}$ without the condition $1 \le i_1 < \cdots < i_k \le n$ form a spanning set of $\bigwedge^k V$. Let $v_1 \wedge \cdots \wedge v_k$ be a k-blade. Writing each v_i as a linear combination of the e_i and using multilinearity, we see that $v_1 \wedge \cdots \wedge v_k$ is a linear combination of the $e_{i_1} \wedge \cdots \wedge e_{i_k}$ for various tuples $(i_1, \ldots, i_k)$. Therefore, the $e_{i_1} \wedge \cdots \wedge e_{i_k}$ span $\bigwedge^k V$.

Moreover, for any $e_{i_1} \wedge \cdots \wedge e_{i_k}$ which violates the ordering condition on $(i_1, \ldots, i_k)$, the blade is just zero if some two integers among $i_1, \ldots, i_k$ are equal. If no two $i_1, \ldots, i_k$ are equal, we can reorder the factors by swaps using the skew-symmetry axiom until $i_1 < i_2 < \cdots < i_k$, picking up a factor of ± 1 in the process. Hence we have proved the proposition. $\square$

Corollary 1. *The dimension of $\bigwedge^k V$ is $\binom{n}{k}$.*

Proof. This is because $\binom{n}{k}$ is the number of ways to choose a k-element set of an n-element set, and counting the number of tuples of integers $(i_1, \ldots, i_k)$ satisfying $1 \le i_1 < \cdots < i_k \le n$ is exactly the same counting problem. $\square$

Corollary 2. $\dim \bigwedge^n V = 1$, *and for $k > n$, $\bigwedge^k V = 0$.*

Here is a very important proposition showing that the exterior algebra detects linear dependence.

Proposition 2. *If $v_1, \ldots, v_k$ are linearly dependent, then*

$$v_1 \wedge \cdots \wedge v_k = 0.$$

Proof. Without loss of generality suppose that v_k is a linear combination of the $v_1, \ldots, v_{k-1}$:

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}.$$

Then

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &= v_1 \wedge \cdots \wedge (a_1 v_1 + \cdots + a_{k-1} v_{k-1}) \\ &= \sum_{i=1}^{k-1} a_i (v_1 \wedge \cdots \wedge v_{k-1} \wedge v_i) \\ &= 0, \end{aligned}$$

where in the second equality we used multilinarity and in the last equality we used the alternating property, because in each of the $k-1$ terms in the summation, some vector appears twice. $\square$

The preceding proposition therefore hints at what follows: because the determinant of a matrix should have something to do with linear independence of the column vectors, it turns out that we should define the determinant in terms of the top exterior power, $\bigwedge^n V$.

4. The determinant

First, recall the following very nifty fact:

Proposition 3. *Let W be a 1-dimensional vector space over a field F. Denote by* $\mathrm{End}(W)$ *the set of set of linear transformations from W to W. Then*

$$\mathrm{End}(W) \cong F,$$

in other words, every endomorphism of W is just multiplication by some scalar.

Proof. Let $\{e\}$ be a basis of W. Let $T\colon W \to W$ be a linear transformation. By definition, T is determined by where it sends e. Since W is 1-dimensional, Te must be a scalar multiple of e. Let $Te = \lambda e$ for some $\lambda \in F$. Then associating T to the number λ gives the desired one-to-one correspondence bewteen $\mathrm{End}(W)$ and F. One easily checks this correspondence is linear, and as a bonus, does not depend on the choice of basis vector e. □

Now for a vector space V of dimension n, and pick an oriented basis $e_1, e_2, \ldots, e_n$ of V. Consider the nth exterior power (the top exterior power) $\bigwedge^n V$. It is one dimensional and is spanned by the single basis element

$$e_1 \wedge e_2 \cdots \wedge e_n.$$

Let us discuss some intuition about what this space represents. For example, let $V = F^n$ and let $e_1, \ldots, e_n$ be the standard basis. We can think of $e_1 \wedge e_2 \cdots \wedge e_n$ as the standard oriented unit cube in F^n, with signed volume 1 according to, say, the right hand rule. If we swap any two of these basis elements, the resulting signed volume becomes -1, indicating that the orientation now follows the left hand rule. If we do two swaps, we get signed volume 1 again, since we have returned to the right hand rule. Also, if any basis vector is scaled by some factor c, then the resulting volume is also scaled by c. Finally, if a basis vector is replaced by a sum of that basis vector and another basis vector, the volume does not change, corresponding to the fact that the resulting parallelpiped is just a shear of the original cube.

Now let $v_1, \ldots, v_n$ be some arbitrary vectors in V. Since $\bigwedge^n V$ is 1-dimensional, the n-blade $v_1 \wedge \cdots \wedge v_n$ must be a scalar multiple of $e_1 \wedge \cdots \wedge e_n$. That is,

$$v_1 \wedge \cdots \wedge v_n = c \cdot (e_1 \wedge \cdots \wedge e_n)$$

for some $c \in F$. Because the signed volume satisfies the same axioms as those we specified for $\bigwedge^n V$, the number c turns out to be precisely the signed volume of the oriented parallelepiped spanned by $v_1, \ldots, v_n$.

With this intuition in mind, let us define the determinant. First, let us define the map of a linear transformation on a vector space's top exterior power.

Definition 3. Let $T\colon V \to V$ be a linear map. Define $\bigwedge^n T\colon \bigwedge^n V \to \bigwedge^n V$ by the equation

$$(\textstyle\bigwedge^n T)(v_1 \wedge \cdots \wedge v_n) = (Tv_1 \wedge \cdots \wedge Tv_n).$$

Now $\bigwedge^n T \in \operatorname{End}(\bigwedge^n V)$ and $\bigwedge^n V$ is a 1-dimensional vector space. Therefore, by Proposition 3, $\bigwedge^n T$ correponds to a scalar in F.

Definition 4 (The determinant)**.** Let $T\colon V \to V$ be a linear map. We define $\det T \in F$ to be the element of F corresponding to $\bigwedge^n T$ under the correspondence of Proposition 3.

This definition may be slightly abstract. With the help of a basis, a down-to-earth equivalent definition can be given as follows: Let $e_1, \ldots, e_n$ be an oriented basis of V, and let $T\colon V \to V$ be a linear transformation. Then $\det T$ is defined to be the unique scalar $c \in F$ such that

$$Te_1 \wedge \cdots \wedge Te_n = c \cdot (e_1 \wedge \cdots \wedge e_n).$$

By this definition we get immediately the most important interpretation of determinant:

> *The determinant of a linear transformation is the factor by which the linear transformation scales signed volumes.*

We also get for free the second most important interpretation of determinant, which we will state as a proposition:

Proposition 4. *Let $T\colon V \to V$ be a linear transformation. Then T is an isomorphism iff $\det T \neq 0$.*

Proof. T is an isomorphism iff $Te_1, \ldots, Te_n$ is a linearly independent set of vectors iff $Te_1 \wedge \cdots \wedge Te_n \neq 0$ (by the nonzero axiom and Proposition 2) iff $\det T \neq 0$. ☐

Example 4. Now let us compute the determinant of the general 3×3 matrix using the machinery of the exterior power. Let

$$T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Thus

$$\begin{aligned} Te_1 &= ae_1 + de_2 + ge_3 \\ Te_2 &= be_1 + ee_2 + he_3 \\ Te_3 &= ce_1 + fe_2 + ie_3. \end{aligned}$$

Therefore,

$$\begin{aligned} Te_1 \wedge Te_2 \wedge Te_3 &= (ae_1 + de_2 + ge_3) \wedge (be_1 + ee_2 + he_3) \wedge (ce_1 + fe_2 + ie_3) \\ &= aei \cdot e_1 \wedge e_2 \wedge e_3 + ahf \cdot e_1 \wedge e_3 \wedge e_2 + dbi \cdot e_2 \wedge e_1 \wedge e_3 + \\ &\qquad + dhc \cdot e_2 \wedge e_3 \wedge e_1 + gbf \cdot e_3 \wedge e_1 \wedge e_2 + gec \cdot e_3 \wedge e_2 \wedge e_1 \\ &= (aei - ahf - dbi + dhc + gbf - gec) \cdot e_1 \wedge e_2 \wedge e_3. \end{aligned}$$

Note the wedge product of 3 vectors each with 3 terms has 27 terms, but 21 of those 27 terms are zero because they have a repeated basis vector. One can verify that the formula we got is the same as the usual formula for the 3×3 determinant.

The cofactor expansion inductive calculation of determinant (at least, using a column for the expansion) can also be easily proven, but is nasty to write down in LaTeX. To get the cofactor expansion along a row, we need to use $\det(A) = \det(A^T)$, which is a theorem to be proven later.

Another thing we get for free is the following properties of the determinant:

(a) Multilinearity in the columns of the matrix: Let M, N, N' be square $n \times n$ matrices such that, for some integer $1 \leq i \leq n$, the ith column of M is equal to the sum of the ith columns of N and N', and the jth columns of M, N, N' are equal for $j \neq i$. Then $\det M = \det N + \det N'$.

Moreover, if M' is the matrix made out of M by scaling some column by a factor of c, then $\det M' = c \det M$.

(b) Skew-symmetry: If M' is obtained from the square matrix M by swapping two columns, then $\det M' = -\det M$.

(c) Alternating property: The determinant of a matrix with two equal columns is zero.

(d) Nonzero for linearly independent columns: If the columns of a matrix are linearly independent, then determinant of the matrix is not zero.

These properties follow directly from the corresponding axioms for $\bigwedge^n V$.

Finally, we show that determinant is multiplicative:

Proposition 5. *Let $S, T \colon V \to V$ be linear transformations. Then*

$$\det(ST) = \det(S)\det(T).$$

Proof. This follows because

$$(\textstyle\bigwedge^n(ST)) = (\bigwedge^n S) \circ (\bigwedge^n T) \colon \bigwedge^n V \to \bigwedge^n V,$$

and composition of two endomorphisms of a 1-dimensional vector space corresponds to multiplication of their corresponding scaling factors. □

5. Bonus material: cross product

Since I promised a better viewpoint on the cross product, I will provide it here.

Let V be a vector space of dimension n. The proper generalization of cross product is simply the wedge product:

Definition 5. The wedge product $\wedge$ is the bilinear map $V \times V \to \bigwedge^2 V$ sending a pair of vectors v and w to the 2-blade $v \wedge w \in \bigwedge^2 V$.

Note that the wedge product works in any dimension. It maps two vectors from an n-dimensional space to an $\binom{n}{2}$-dimensional space.

Let us now compare this to the traditional cross product in 3 dimensions. Let $V = F^3$ with standard basis e_1, e_2, e_3. Then, the dimension of $\bigwedge^2 V$ is equal to $\binom{3}{2} = 3$, with a basis given by $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.

We now describe how to recover the cross product from the wedge product. We simply use the fact that $\bigwedge^2 V$ and V have the same dimension, which means that there is some isomorphism between them. In fact, there are many isomorphisms between them. Let us pick this isomorphism:

$$\begin{aligned}\varphi\colon \textstyle\bigwedge^2 V &\xrightarrow{\sim} V \\ e_1 \wedge e_2 &\mapsto e_3 \\ e_1 \wedge e_3 &\mapsto -e_2 \\ e_2 \wedge e_3 &\mapsto e_1.\end{aligned}$$

Then $v \times w$ can be recovered as $\varphi(v \wedge w)$. Of course, the isomorphism φ does not respect linear transformations, and only exists because $\bigwedge^2 V \cong V$, a feat that only happens for $\dim V = 3$.

The notion of torque and angular momentum in physics works much better as bivectors in $\bigwedge^2 V$ than as vectors in V. To see why, recall that the angular momentum of an object is some quantity that measures rotational momentum along some plane of rotation. In our 3-dimensional world, we can define the angular momentum vector as the vector normal to that plane and whose length is the magnitude of the angular momentum, and orientation given by the right hand rule. The assignment of a normal vector to a plane is possible because the orthogonal complement of a plane in $\mathbb{R}^3$ is a line.

What if our universe was not 3-dimensional, say our universe was 9-dimensional? In this space, the orthogonal complement of a plane is a 7-dimensional subspace. Therefore, the whole idea of using a normal vector to represent angular momentum breaks down violently. This is why in a general universe, one must represent angular momentum as a bivector. The notion of bivector for measuring a quantity oriented along a plane is well behaved in any number of dimensions, unlike the notion of normal vector to a plane.

EARLY MATH2211 FEEDBACK

The following is a quick survey of how the course is going so far for you. Responses are anonymous.

1. Questions about the class

For each of the scale questions below, mark your opinion with a $\times$ on the scale.

(1) Pace?

Too slow — Too fast

(2) Volume/speech clarity?

Can't hear/understand — Easy to hear/understand

(3) Handwriting?

Hard to read — Easy to read

(4) Difficulty of lectures?

I get lost — Can follow in real time

(5) Problem set difficulty?

Too easy — Too hard

(6) Additional comments?

Flip to the back for questions about office hours!

2. Office hours

(1) Do you prefer Zoom or in-person office hours (or no preference)?

(2) Check all the times where you would likely be able to attend office hours if it was offered at that time. If you want, you can put a star in preferred time slots.

Time	Monday	Tuesday	Wednesday	Thursday	Friday
11 am					
1 pm					
2 pm					
3 pm					
4 pm					
5 pm (Zoom)					
6 pm or later (Zoom)					