MATH2211 SPRING 2022
PROBLEM SET 2

DUE WEDNESDAY, FEBRUARY 9 2022 AT 11:59 PM

Problem 1. Compute the real and imaginary parts of $\dfrac{\pi + i}{5 - i}$.

Problem 2.

(a) Use power series expansions to prove Euler's formula[1]
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

(b) Use Euler's formula to prove the identity
$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

(c) Use the same technique to derive a formula for $\cos(3\theta)$ in terms of $\cos\theta$.[2]

Problem 3. Let $z = e^{\frac{2\pi i}{n}}$, where $n \in \mathbb{Z}^+$. Prove that $1 + z + z^2 + \cdots + z^{n-1} = 0$.[3]

Problem 4. Read up about Fermat's little theorem by looking it up on the internet. Using Fermat's little theorem, find the roots of $x^{10} - 1$ over $\mathbb{F}_{11}$.

Problem 5.

(a) Is $U = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ a subspace of $\mathbb{C}^3$?

(b) Is $U = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 : x_1x_2x_3 = 0\}$ a subspace of $\mathbb{Q}^3$?

(c) Let P be the $\mathbb{R}$-vector space of all polynomials with real coefficients. is
$$U = \{f \in P : f'(-1) = 3f(2)\}$$
a subspace of P? Here, f' means the derivative of f.

Problem 6.

[1] If you don't remember what the power series of $\exp, \sin$, and $\cos$ are, you can look them up on the internet.

[2] This can be generalized to $\cos(n\theta)$: look up *Chebyshev polynomials of the first kind* on the internet.

[3] Hint: Factor the polynomial $x^n - 1$.

(a) Is $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{C}^3$ a linear combination of $\begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$?

(b) In the real vector space consisting of all polynomials with real coefficients, is

$$x + 1 \in \operatorname{span}\{x^2 + 1, x^3 + x, 2x^2 + x, x + 3\}?$$

Problem 7. Show that a subset W of a vector space is a subspace if and only if $\operatorname{span}(W) = W$.