# MATH2211 SPRING 2022
# PROBLEM SET 6 SOLUTIONS

DUE WEDNESDAY, MARCH 23, 2022 AT 11:59 PM

**Problem 1.** Let $T: V \to V$ be a linear map from a vector space $V$ to itself. Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis of $V$ and let $A \in M_n(F)$ be the matrix of $V$ with respect to $\mathcal{B}$. Let $\mathcal{C} = (v'_1, \ldots, v'_n)$ be another ordered basis of $V$ and let $A' \in M_n(F)$ be the matrix of $T$ with respect to $\mathcal{C}$. Prove that there exists an invertible matrix $B \in M_n(F)$ such that $A' = BAB^{-1}$.

**Solution**

Let $C: F^n \to V$ be defined by $Ce_i = v_i$, and let $C': F^n \to V$ be defined by $C'e_i = v'_i$. Note that $(C')^{-1}C$ goes from $F^n$ to $F^n$, so it is an $n \times n$ matrix (with respect to the standard basis of $F^n$). I claim that we can take $B = (C')^{-1}C$.
To see why this is the case, recall that the matrix $A$, when thought of as a linear transformation $F^n \to F^n$, is the composition $C^{-1}TC$. The matrix $A'$ is likewise the composition $C'^{-1}TC'$. Finally,

$$BAB^{-1} = (C')^{-1}CC^{-1}TCC^{-1}C' = (C')^{-1}TC,$$

as desired.

**Problem 2.** Factor $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ as a product of elementary matrices.

Hint: One way to approach this is to try to turn $A$ into the identity matrix via elementary matrices, then see how what you've done is useful.

**Solution**

Let's try to turn $A$ into the identity matrix via elementary matrices:

$$\begin{pmatrix}2&1&2\\0&1&0\\2&0&1\end{pmatrix} \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\\-1&0&1\end{smallmatrix}\right)} \begin{pmatrix}2&1&2\\0&1&0\\0&-1&-1\end{pmatrix}$$
$$\xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&0\\0&1&1\end{smallmatrix}\right)} \begin{pmatrix}2&1&2\\0&1&0\\0&0&-1\end{pmatrix}$$
$$\xrightarrow{\left(\begin{smallmatrix}1&-1&0\\0&1&0\\0&0&1\end{smallmatrix}\right)} \begin{pmatrix}2&0&2\\0&1&0\\0&0&-1\end{pmatrix}$$
$$\xrightarrow{\left(\begin{smallmatrix}1&0&2\\0&1&0\\0&0&1\end{smallmatrix}\right)} \begin{pmatrix}2&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$
$$\xrightarrow{\left(\begin{smallmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&-1\end{smallmatrix}\right)} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Therefore, it follows that

$$A = \begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&-2\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

The decomposition is not unique, of course.

**Problem 3.**

(a) In this problem we show that a one-sided inverse of a square matrix is a two-sided inverse. More precisely, let $A, B \in M_n(F)$. Show that

$$AB = I_n \iff BA = I_n.$$

Hint: For the forward direction, assume $AB = I_n$ and first show that $A\colon F^n \to F^n$ is surjective. Similar hint applies to the other direction.

**Solution**

$(\implies)$ Suppose that $AB = I_n$. This implies that $ABv = v$ for all $v \in F^n$, which shows that $A$ is surjective. Because the dimension of the source and target of $A$ are equal, $A$ is therefore injective too. Now let $v \in F^n$ be arbitrary.

We have

$$ABAv = I_n Av = Av,$$

and by injectivity of $A$ this implies that $BAv = v$. This shows that $BA = I_n$.
($\Leftarrow$) Swap the role of $A$ and $B$ and use the above argument verbatim again.

(b) In this problem we show that inverses are unique. More precisely, suppose $A, B, C \in M_n(F)$ such that $AB = I_n$ and $AC = I_n$. Prove that $B = C$.

Hint: In this proof, you are not allowed to multiply both sides by $A^{-1}$, because doing that presupposes that $A^{-1}$ is unique! Instead, start by proving that $A$ is surjective.

Solution

In fact the hint was not needed. Suppose $AB = AC = I_n$. Then $BA = I_n$ by part (a). Therefore, $B = B(AC) = (BA)C = C$.

**Problem 4.** Let $A \in M_n(\mathbb{R})$ be an invertible matrix with integer entries, such that $A^{-1}$ also has integer entries. Prove that $\det A = \pm 1$.

Solution

We know that $(\det A^{-1}) = (\det A)^{-1}$, so $\det A$ is an integer whose reciprocal is also an integer. The only integers with this property are $\pm 1$.

**Problem 5.** Compute:

(a) $\det \begin{pmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{pmatrix}$.

**Solution**

Let's compute it with wedge products. (You could also use the 6-term formula or cofactor expansion if you wanted.)

$$\begin{aligned}(2e_2 + 3e_3) \wedge (e_1 + 6e_2) \wedge (2e_1 - e_2 + 4e_3) \\ &= 2e_2 \wedge e_1 \wedge 4e_3 + 3e_3 \wedge e_1 \wedge -e_2 + 3e_3 \wedge 6e_2 \wedge 2e_1 \\ &= (-8 - 3 - 36) e_1 \wedge e_2 \wedge e_3 \\ &= -47 e_1 \wedge e_2 \wedge e_3.\end{aligned}$$

Hence the determinant is $-47$.

(b) $\det \begin{pmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{pmatrix}$.

**Solution**

The answer is $-232$. This is a heavy slog of a computation (no matter which method) and this is the one problem I think you have least need for a written out solution for :)

The most efficient method uses the method whereby the matrix is turned into an upper triangular matrix by determinant-1 elementary matrices. This is by far a faster determinant algorithm for larger (dense) matrices. Unfortunately this method was not discussed in class yet!

**Problem 6.** Directly from the definition of $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ for the $2 \times 2$ determinant, prove the following:

(a) $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_2(F)$.

**Solution**

First let's write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix},$$

and

$$\begin{aligned} \det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg \\ &= adeh + bcfg - adfg - bceh \\ &= (ad-bc)(eh-fg) \\ &= \det A \cdot \det B. \end{aligned}$$

(b) $A$ is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

**Solution**

We have $A$ invertible iff $\ker A = 0$. But $\ker A$ is the set of solutions to the system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

By high school algebra, this system has a unique solution iff the two lines (the graphs of the two equations) are not parallel, iff $ad \neq bc$.
Now suppose $A$ is invertible. To verify the identity for $A^{-1}$ we multiply:

$$\frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = I_2.$$

(c) The function $\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}; (v, w) \mapsto \det \begin{pmatrix} v & w \end{pmatrix}$ (the determinant of the matrix whose columns are $v$ and $w$) is *bilinear*, meaning that for all $\alpha, \beta \in F$ and $v_1, v_2, v, w_1, w_2, w \in \mathbb{R}^2$, we have

(i) $\det(\alpha v_1 + \beta v_2, w) = \alpha \det(v_1, w) + \beta \det(v_2, w)$,

(ii) $\det(v, \alpha w_1 + \beta w_2) = \alpha \det(v, w_1) + \beta \det(v, w_2)$.

**Solution**

Let $v_1 = \binom{a_1}{b_1}$, $v_2 = \binom{a_2}{b_2}$, $w_1 = \binom{c_1}{d_1}$, and $w_2 = \binom{c_2}{d_2}$. Let us verify (i):

$$\begin{aligned}
\det(\alpha v_1 + \beta v_2, w_1) &= \det \begin{pmatrix} \alpha a_1 + \beta a_2 & c_1 \\ \alpha b_1 + \beta b_2 & d_1 \end{pmatrix} \\
&= (\alpha a_1 + \beta a_2) d_1 - (\alpha b_1 + \beta b_2) c_1 \\
&= \alpha(a_1 d_1 - b_1 c_1) + \beta(a_2 d_1 - b_2 c_1) \\
&= \alpha \det(v_1, w_1) + \beta \det(v_2, w_1).
\end{aligned}$$

Now let us verify (ii):

$$\begin{aligned}
\det(v_1, \alpha w_1 + \beta w_2) &= \det \begin{pmatrix} a_1 & \alpha c_1 + \beta c_2 \\ b_1 & \alpha d_1 + \beta d_2 \end{pmatrix} \\
&= a_1(\alpha d_1 + \beta d_2) - b_1(\alpha c_1 + \beta c_2) \\
&= \alpha(a_1 d_1 - b_1 c_1) + \beta(a_1 d_2 - b_1 c_2).
\end{aligned}$$

This concludes the proof of bilinearity.