MATH2211 SPRING 2022
PROBLEM SET 7

DUE WEDNESDAY, MARCH 30, 2022 AT 11:59 PM

Problem 1. Suppose $A \in M_{m\times n}(F)$. Prove or find counterexamples:

(a) For any $B \in M_{n\times p}(F)$, $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

(b) For any $B \in M_{p\times m}(F)$, $\operatorname{rank}(BA) \leq \operatorname{rank}(A)$.

Problem 2.

(a) Prove, using the exterior algebra definition of the determinant, that the determinant of an upper-triangular square matrix is the product of the diagonal entries.

(b) You may assume that a similar proof as in part (a) shows that the same result holds for lower triangular square matrices as well.

Prove that that the elementary matrix E representing the row operation "add a multiple of row j to row i" ($i \neq j$) has determinant 1.

Problem 3.

(a) Prove that if $T\colon V \to W$ is surjective, then ${}^tT\colon W^* \to V^*$ is injective.

(b) Prove that if $T\colon V \to W$ is injective, then ${}^tT\colon W^* \to V^*$ is surjective.

(c) Prove that every linear transformation $T\colon V \to W$ can be factored as a composition

$$V \xrightarrow{T'} \!\!\!\!\rightarrow U \overset{i}{\hookrightarrow} W,$$

where T' is surjective and i is injective. (Hint: Set $U = \operatorname{im} T$.)

(d) Recall that the rank of a linear transformation $T\colon V \to W$ is defined to be $\dim \operatorname{im} T$. Use parts (a), (b), and (c) to prove that $\operatorname{rank}(T) = \operatorname{rank}({}^tT)$.

Problem 4. Let V be a vector space. Given a nonzero element $v \in V$ and a nonzero linear functional $\lambda\colon V \to \mathbb{R}$, we can make a linear transformation $T_{v,\lambda}\colon V \to V$ by sending $x \in V$ to $\lambda(x) \cdot v$. Prove that T has rank 1, and prove that every rank 1 linear transformation from V to V is equal to $T_{v,\lambda}$ for some $0 \neq v \in V$ and $0 \neq \lambda\colon V \to \mathbb{R}$.

Hint for the second part: Problem 3(c) is very helpful.

Problem 5. Let $\frac{d}{dx}\colon P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the derivative operator on the space of polynomials of degree at most 2 over $\mathbb{R}$.

(a) Prove that ${}^t(\frac{d}{dx})\colon P_2(\mathbb{R})^* \to P_2(\mathbb{R})^*$ has a 1-dimensional kernel.

(b) Prove that $\ker({}^t(\frac{d}{dx}))$ is spanned by the functional taking a polynomial to its x^2 coefficient. Can you provide a plain English interpretation of what this is saying?