MATH2211 SPRING 2022
PROBLEM SET 10

DUE FRIDAY, APRIL 29, 2022 AT 11:59 PM

Relevant reading: Axler Chapter 6 and Section 7.A.

Problem 1. Let V be a finite-dimensional inner product space and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis. Prove *Parseval's Identity*: for all $x, y \in V$,

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, y \rangle .$$

Solution

Write $x = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$ and $y = \sum_{i=1}^{n} \langle y, e_i \rangle e_i$. Then

$$\begin{aligned}
\langle x, y \rangle &= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \langle x, e_i \rangle e_i, \langle y, e_j \rangle e_j \rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, e_i \rangle \overline{\langle y, e_j \rangle} \langle e_i, e_j \rangle \\
&= \sum_{i=1}^{n} \langle x, e_i \rangle \langle e_i, y \rangle .
\end{aligned}$$

Problem 2.

(a) Let V be a real vector space and let $\langle \cdot, \cdot \rangle$ be a bilinear form on V. Prove that for every $v \in V$, the map φ_v defined by $\varphi_v(x) = \langle v, x \rangle$ is a linear functional on V.

Remark: This shows that, given a bilinear form on V, we get a natural map φ from V to V^* sending each $v \in V$ to φ_v. The Riesz representation theorem says that if the bilinear form $\langle \cdot, \cdot \rangle$ is an inner product, then $\varphi \colon V \to V^*$ is an isomorphism. Moreover, in $\mathbb{R}^n$ with the standard Euclidean inner product, φ is exactly the map that turns a column vector into a row vector, which is classically denoted $\cdot^T$.

Solution

We check that φ_v is a linear map from V to $\mathbb{R}$. This follows from the bilinarity of the real inner product: for all $a, b \in \mathbb{R}$ and $x_1, x_2 \in V$,

$$\langle v, ax_1 + bx_2 \rangle = a \langle v, x_1 \rangle + b \langle v, x_2 \rangle .$$

(b) Let's do a concrete example. Let $V = P_2(\mathbb{R})$ with inner product given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

Let $[x^2]$ denote the linear functional which when given a polynomial, returns its coefficient of the x^2 term. By the Riesz representation theorem, $[x^2]$ can be represented as $\langle f, \cdot \rangle$ for a unique polynomial $f \in P_2(\mathbb{R})$. Find f.

Solution

We need to find the unique polynomial $f \in P_2(\mathbb{R})$ such that

$$\int_0^1 f(x) \cdot 1\, dx = 0,$$
$$\int_0^1 f(x) \cdot x\, dx = 0,$$
$$\int_0^1 f(x) \cdot x^2\, dx = 1.$$

If we write $f(x) = a + bx + cx^2$, this gives the equations

$$a + \frac{1}{2}b + \frac{1}{3}c = 0,$$
$$\frac{1}{2}a + \frac{1}{3}b + \frac{1}{4}c = 0,$$
$$\frac{1}{3}a + \frac{1}{4}b + \frac{1}{5}c = 1.$$

This can be solved using whatever method we like to obtain $f(x) = 30 - 180x + 180x^2$.

Problem 3. Let V be an inner product space and let U be a subspace of V. In this problem we investigate the orthogonal projection operator P_U.

(a) Prove that $P_U^2 = P_U$ and that P_U is self-adjoint (that is, $P_U = P_U^*$). Show that the identity $P_U^2 = P_U$ is equivalent to saying that $P_U|_{\operatorname{im} P_U} = I_{\operatorname{im} P_U}$.

Solution

For any $v \in V$, write $v = u + w$ for $u \in U$ and $w \in U^\perp$. Then $P_U^2 v = P_U(P_U v) = P_U u = u$, which is the same as $P_U v$.
Now let us prove self-adjointness of P_U. Let $v = u + w$ and $v' = u' + w'$ be decompositions of two arbitrary vectors v and v' into components in U and $U^\perp$. Then

$$\left\langle Pv, v' \right\rangle = \left\langle u, v' \right\rangle = \left\langle u, u' + w' \right\rangle = \left\langle u, u' \right\rangle$$

and

$$\left\langle v, Pv' \right\rangle = \left\langle v, u' \right\rangle = \left\langle u + w, u' \right\rangle = \left\langle u, u' \right\rangle$$

as well. Therefore, $\langle Pv, v' \rangle = \langle v, Pv' \rangle$ which shows that P is self-adjoint.

(b) In this problem, suppose that $U = \operatorname{span}(u)^\perp$ for some nonzero vector $u \in V$. Prove that P_U can be expressed as $I - u\varphi_u$ (more classically denoted $I - uu^T$).

Solution

Oops, this problem needs u to be a unit vector, and I didn't state that. The correct expression in the general case is $I - \frac{u\varphi_u}{\|u\|^2}$.
We saw in class that the projection onto the line spanned by u can be given by the formula

$$P_u v = \frac{\langle v, u \rangle}{\langle u, u \rangle} u.$$

For simplicity let us assume $\langle u, u \rangle = 1$, i.e. u is a unit vector. So then $P_u v = \langle v, u \rangle u = \varphi_u(v) \cdot u = u\varphi_u(v)$. (If u is not a unit vector, $P_u v$ is equal to $u\varphi_u(v)/\|u\|^2$.) The residual vector $v - P_u v$ is precisely the projection of v onto $\operatorname{span}(u)^\perp$. Therefore, the projection onto $\operatorname{span}(u)^\perp$ is given by $I - u\varphi_u$.

(c) Prove that the eigenvalues of P_U are 0 and 1, with the multiplicity of 1 being the dimension of U and the multiplicity of 0 being the dimension of $U^\perp$.

Solution

Let $u_1, \ldots, u_r$ be any basis of U where $r = \dim U$, and let $w_1, \ldots, w_m$ be any basis of $U^\perp$, where $m = \dim U^\perp$. Together these vectors form a basis of V. We have $Pu_i = u_i$ for each $1 \le i \le r$ and $Pw_i = 0$ for each $1 \le i \le m$, giving a basis of eigenvectors of V where r of them have eigenvalue 1 and m of them have eigenvalue 0. This finishes the proof.

(d) Prove that the eigenspace of 1 of P_U is U and that the eigenspace of 0 is $U^\perp$, and that P_U is diagonalizable.

Solution

The eigenspace of 1 is the kernel of $P_U - I$ or equivalently the kernel of $I - P_U$, which is just $P_{U^\perp}$. In any of these formulations it is clear the kernel is U itself. Similarly, the eigenspace of 0 is the kernel of P_U, which is by definition $U^\perp$. Finally, since the two eigenspaces span V, P_U is diagonalizable.

Problem 4. Prove that if $T\colon V \to V$ is self-adjoint, then its eigenspaces corresponding to two different eigenvalues are orthogonal to each other.

Solution

Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues and v_1, v_2 be two corresponding eigenvectors. Then

$$\lambda_1 \langle v_1, v_2 \rangle = \langle T v_1, v_2 \rangle = \langle v_1, T v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle ,$$

showing that $(\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0$, which shows that $\langle v_1, v_2 \rangle = 0$.

Problem 5. Let $A \in M_{m\times n}(\mathbb{R})$, and let A^* denote the adjoint of A, namely the unique matrix in $M_{n\times m}(\mathbb{R})$ such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

(a) Prove that $\ker(A) = \ker(A^*A)$, where A^* denotes the adjoint of A.

Solution

First, it is clear that if $Av = 0$ then $A^*Av = 0$ as well, so $\ker A \subseteq \ker(A^*A)$. Now let us suppose that $v \in \ker(A^*A)$, so $A^*Av = 0$. Then

$$0 = \langle v, A^*Av \rangle = \langle Av, Av \rangle ,$$

which implies that $Av = 0$. Therefore, $\ker A \supseteq \ker(A^*A)$ as well.

(b) For a general matrix equation $Ax = b$, recall that there may be no solutions. Multiplying both sides on the left by A^*, we get the equation $A^*Ax = A^*b$. This is called the least squares normal equation for the matrix equation. It turns out that $A^*Ax = A^*b$ always has a solution, and that any solution x to the normal equation minimizes the value of $\|Ax - b\|$.

In this exercise, we prove this last part. Prove that if x is a solution to the equation $A^*Ax = A^*b$, then the projection of b onto the image of A is equal to Ax. Show therefore that $\|Ax - b\|$ is minimized at those x which solve $A^*Ax = A^*b$.

Solution

Showing that the projection of b onto the image of A is equal to Ax is equivalent to showing that $Ax - b \perp (\operatorname{im} A)$, which is equivalent to

$$\langle Ay, Ax - b\rangle = 0$$

for all $y \in V$. This is in turn equivalent to

$$\langle y, A^*Ax - A^*b\rangle = 0$$

for all $y \in V$, which is equivalent to $A^*Ax - A^*b = 0$. Therefore, if x is a solution to $A^*Ax = A^*b$, then $Ax - b \perp (\operatorname{im} A)$, which is what we wanted to prove.

Since we have proven that $P_{\operatorname{im} A} b = Ax$, let us write $b = Ax + w$ for some $w \in (Ax)^{\perp}$. Now let x' be any other vector in V. Then

$$\|Ax' - b\|^2 = \|Ax' - Ax - w\|^2 = \|A(x' - x)\|^2 + \|w\|^2$$

by the Pythagorean theorem, using the fact that $w \perp A(x' - x)$ since $A(x' - x) \in \operatorname{im} A$. Since w does not depend on x', the value of $\|A(x' - x)\|^2 + \|w\|^2$ is minimized when $A(x' - x) = 0$, which is equivalent to saying that $Ax' = Ax = b$.