MATH2211 SPRING 2022
PROBLEM SET 9

DUE FRIDAY, APRIL 15, 2022 AT 11:59 PM

Useful reading for Problems 1 and 2: Section 8.A of Axler.

Problem 1. A *nilpotent* linear operator is defined to be a linear operator $T\colon V \to V$ such that some power of T is equal to zero. In parts (b) to (d), let $N\colon V \to V$ be a nilpotent operator on a finite dimensional vector space V.

(a) Give an example of a 2×2 real nilpotent matrix (i.e. a nilpotent linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$) none of whose entries are 0 (or prove they don't exist).

(b) Now let N be a nilpotent operator on a finite dimensional vector space V. Prove that 0 is the only eigenvalue of N.

(c) Let U be any nonzero subspace of V and suppose that $NU \subseteq U$. Prove that NU is strictly contained in U.

Hint: Use contradiction. Comment: A useful notation for strict containment is $\subsetneq$.

(d) Prove that $N^{\dim V} = 0$. Hint: Use the previous part iteratively starting with $U = V$.

Problem 2. Given a polynomial $p \in F[t]$ (note: $F[t]$ just means the set of polynomials in the variable t with coefficients in F) and a linear operator $T\colon V \to V$ on a vector space V over F, the expression $p(T)$ makes sense if we use powers and addition of linear operators, to give a linear operator $p(T)\colon V \to V$.

(a) Let $V = F^2$ and $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+y \\ y \end{smallmatrix}\right)$. For the polynomial $p(t) = t^2 - 2t$, find $p(T)$.

(b) Prove that for every linear operator $T\colon V \to V$ with $\dim V = n$, there exists a polynomial $p \in F[t]$, of degree at most n^2, such that $p(T) = 0$.

Note: Do not use the theory of minimal polynomials or any theorems named after multiple people.

Hint: Use linear dependence ideas.

(c) Let $J_{n,\lambda}$ be the $n \times n$ Jordan block (this is the standard name for what I called an "atomic Jordan matrix" in class) with λ's on the diagonal and 1's above the diagonal. Prove that $p(J_{n,\lambda}) = 0$ for the polynomial $p(t) = (t-\lambda)^n$, but $q(J_{n,\lambda}) \neq 0$ for the polynomial $q(t) = (t-\lambda)^{n-1}$.

Comment (not a hint): This problem essentially says that $J_{n,\lambda} - \lambda I_n$ is a nilpotent operator and that n is the least power k that makes $(J_{n,\lambda} - \lambda I_n)^k = 0$.

Problem 3. A *permutation matrix* is a square matrix where in each column and each row, there is exactly one nonzero entry and that nonzero entry is a 1. The name is because an $n \times n$ permutation matrix times an $n \times 1$ vector is the same vector but with entries permuted. Let P be an $n \times n$ permutation matrix.

(a) Prove that the product of two $n \times n$ permutation matrices is another permutation matrix.

(b) Using part (a), prove that some positive power of P is equal to the identity.

Hint: Are there infinitely many $n \times n$ permutation matrices?

(c) Provide a counterexample to the claim that for all $n \in \mathbb{Z}^+$ and all $n \times n$ permutaion matrices P, there is some $1 \leq k \leq n$ such that P^k is the identity.

Hint: You will not find a counterexample by looking at $n \leq 4$.

Problem 4. Let P_3 be the space of real polynomials of degree at most 3, and define an inner product on P_3 by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

Set $U = \text{span}\{x, x^2\}$.

(a) Find an orthonormal basis for U.

(b) Write $x^3 = p(x) + q(x)$ with $p(x) \in U$ and $q(x) \in U^\perp$.