MATH2211 SPRING 2022
PROBLEM SET 4

DUE WEDNESDAY, FEBRUARY 23 2022 AT 11:59 PM

Useful reading for this problem set: Axler pages 20-23 (Sums of subspaces), Axler page 47 (Dimension of a sum and proof)

Problem 1. For each $c \in \mathbb{R}$, determine the dimension of

$$U = \operatorname{span}\left(\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \\ c \end{pmatrix} \right).$$

Problem 2. Suppose that U and W are 4-dimensional $\mathbb{C}$-subspaces of $\mathbb{C}^6$. Show that one can find two vectors in $U \cap W$, neither of which is a scalar multiple of the other.[1]

Problem 3.

(a) Find a basis for the subspace

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0 \text{ and } x_3 - x_4 = 0\}$$

and extend it to a basis of $\mathbb{R}^4$.

(b) Let $P_5(\mathbb{R})$ be the vector space of all polynomials of degree at most 5 with real coefficients. Find a basis for the subspace

$$U = \{f(x) \in P_5(\mathbb{R}) : f(1) = 0 \text{ and } f(2) = 0\}$$

and extend it to a basis of $P_5(\mathbb{R})$.

Problem 4. Given subspaces $U_1, U_2 \subseteq V$, prove that the following properties are equivalent.

- $V = U_1 + U_2$ and $\{0\} = U_1 \cap U_2$.
- Every vector $v \in V$ can be written in a **unique** way as $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$.

When these hold, we say that V is the *direct sum* of U_1 and U_2, and write $V = U_1 \oplus U_2$.

[1]Hint: the existence of such two vectors is precisely equivalent to a simple dimension condition on $U \cap W$. Can you see it?

Problem 5. Prove or give a counterexample: if $f\colon A \to B$ and $g\colon B \to C$ are invertible functions, then $g \circ f\colon A \to C$ is also invertible.

Problem 6. Suppose $T\colon V \to W$ is a linear map, and $v_1, \ldots, v_n \in V$. For each statement, give a proof or a counterexample.

(a) If T is injective and $v_1, \ldots, v_n$ are linearly independent, then $T(v_1), \ldots, T(v_n)$ are linearly independent.

(b) If $T(v_1), \ldots, T(v_n)$ are linearly independent, then $v_1, \ldots, v_n$ are linearly independent.

(c) If T is surjective and $v_1, \ldots, v_n$ span V, then $T(v_1), \ldots, T(v_n)$ span W.

(d) If $T(v_1), \ldots, T(v_n)$ span W, then $v_1, \ldots, v_n$ span V.