MATH2211 SPRING 2022
PROBLEM SET 9

DUE FRIDAY, APRIL 15, 2022 AT 11:59 PM

Useful reading for Problems 1 and 2: Section 8.A of Axler.

Problem 1. A *nilpotent* linear operator is defined to be a linear operator $T\colon V \to V$ such that some power of T is equal to zero. In parts (b) to (d), let $N\colon V \to V$ be a nilpotent operator on a finite dimensional vector space V.

(a) Give an example of a 2×2 real nilpotent matrix (i.e. a nilpotent linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$) none of whose entries are 0 (or prove they don't exist).

Solution

There are lots of examples. One is
$$N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

A quick way to see that this operator squares to zero (hence is nilpotent) is to notice that the kernel of N equals the image of N in this example. In general $N^2 = 0$ iff $\ker N \supseteq \operatorname{im} N$ as one can easily show.

(b) Prove that 0 is the only eigenvalue of N.

Solution

Suppose $N^k = 0$. Let $Nv = \lambda v$. Then $0 = N^k v = \lambda^k v$, so $\lambda^k = 0$, so $\lambda = 0$.

(c) Let U be any nonzero subspace of V and suppose that $NU \subseteq U$. Prove that NU is strictly contained in U.

Hint: Use contradiction. Comment: A useful notation for strict containment is $\subsetneq$.

Solution

If $NU = U$, then $N^k U = U$ for all positive integers k, so N cannot be nilpotent.

(d) Prove that $N^{\dim V} = 0$. Hint: Use the previous part iteratively starting with $U = V$.

Solution

We have a strict descending chain

$$V \supsetneq NV \supsetneq N^2V \cdots .$$

In such a descending chain, the dimensions must decrease by at least 1. Therefore the $(\dim V)$-th step after V has dimension 0, in other words, $N^{\dim V}V$ is the 0 vector space. This says $N^{\dim V}$ is the 0 operator.

Problem 2. Given a polynomial $p \in F[t]$ (note: $F[t]$ just means the set of polynomials in the variable t with coefficients in F) and a linear operator $T\colon V \to V$ on a vector space V over F, the expression $p(T)$ makes sense if we use powers and addition of linear operators, to give a linear operator $p(T)\colon V \to V$.

(a) Let $V = F^2$ and $T\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+y \\ y \end{smallmatrix}\right)$. For the polynomial $p(t) = t^2 - 2t$, find $p(T)$.

Solution

T^2 sends (x, y) to $(x + 2y, y)$. Thus $T^2 - 2T$ sends (x, y) to $(x + 2y, y) - 2(x + y, y) = (-x, -y)$. It is -1 times the identity matrix.

(b) Prove that for every linear operator $T\colon V \to V$ with $\dim V = n$, there exists a polynomial $p \in F[t]$, of degree at most n^2, such that $p(T) = 0$.

Note: Do not use the theory of minimal polynomials or any theorems named after multiple people.

Hint: Use linear dependence ideas.

Solution

The space of $n \times n$ matrices has dimension n^2. Therefore the space of linear operators on V, where $\dim V = n$, also has dimension n^2. Now the list $1, T, T^2, \ldots, T^n$ is a list of $n + 1$ elements of the space of linear operators on V, so they must be linearly dependent. The linear dependence between these gives our polynomial p.

(c) Let $J_{n,\lambda}$ be the $n \times n$ Jordan block (this is the standard name for what I called an "atomic Jordan matrix" in class) with λ's on the diagonal and 1's above the diagonal. Prove that $p(J_{n,\lambda}) = 0$ for the polynomial $p(t) = (t - \lambda)^n$, but $q(J_{n,\lambda}) \neq 0$ for the polynomial $q(t) = (t - \lambda)^{n-1}$.

Comment (not a hint): This problem essentially says that $J_{n,\lambda} - \lambda I_n$ is a nilpotent operator and that n is the least power k that makes $(J_{n,\lambda} - \lambda I_n)^k = 0$.

Solution

Notice that $J_{n,\lambda} - \lambda I_n$ is a matrix with 1's one step above the diagonal and 0's everywhere else. One proves by induction that $(J_{n,\lambda} - \lambda I_n)^k$ is the matrix with 1's k steps above the diagonal and 0's everywhere else. Thus $(J_{n,\lambda} - \lambda I_n)^{n-1}$ is the matrix with a sole 1 in the upper right corner, and $(J_{n,\lambda} - \lambda I_n)^n = 0$.

Problem 3. A *permutation matrix* is a square matrix where in each column and each row, there is exactly one nonzero entry and that nonzero entry is a 1. The name is because an $n \times n$ permutation matrix times an $n \times 1$ vector is the same vector but with entries permuted. Let P be an $n \times n$ permutation matrix.

(a) Prove that the product of two $n \times n$ permutation matrices is another permutation matrix.

Solution

Any permutation matrix restricts to a set-theoretic bijection from $\{e_1, \ldots, e_n\}$ to itself. In other words, if P is a permutation matrix, then for all i, Pe_i is of the form e_j for some j, and each e_j is represented exactly once. Thus the composition of two permutation matrices also restricts to a set-theoretic bijection from $\{e_1, \ldots, e_n\}$ to itself.

(b) Using part (a), prove that some positive power of P is equal to the identity.

Hint: Are there infinitely many $n \times n$ permutation matrices?

Solution

There are finitely many $n \times n$ permutation matrices, in fact $n!$ of them. Thus by the pigeonhole principle, in the list $1, P, P^2, \ldots, P^{n!}$ there must be a repeat, namely there are some $0 \leq \ell < k \leq n!$ such that $P^k = P^\ell$. We also observe that permutation matrices are invertible. Therefore $P^{k-\ell} = I_n$.

(c) Provide a counterexample to the claim that for all $n \in \mathbb{Z}^+$ and all $n \times n$ permutaion matrices P, there is some $1 \leq k \leq n$ such that P^k is the identity.

Hint: You will not find a counterexample by looking at $n \leq 4$.

Solution

One example is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6 is the smallest positive power of P equal to the identity. Remark: In cycle notation (which you will most likely learn in abstract algebra) this matrix corresponds to the permutation $(12)(345)$, a product of a 2-cycle and a 3-cycle. This permutation has order $2 \times 3 = 6$.

Problem 4. Let P_3 be the space of real polynomials of degree at most 3, and define an inner product on P_3 by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

Set $U = \operatorname{span}\{x, x^2\}$.

(a) Find an orthonormal basis for U.

Solution

First let's find an orthogonal basis. We keep x and make a replacement for x^2. Let's project x^2 onto the line spanned by x:

$$P_x(x^2) = \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = \frac{1/4}{1/3} x = \frac{3}{4} x.$$

What we want is the orthogonal part $x^2 - \frac{3}{4}x$. This vector is indeed orthogonal to x so $\{x, x^2 - \frac{3}{4}x\}$ is an orthogonal basis. Now we just need to divide them by their norms to make it an orthonormal basis. The norm of x is $\frac{1}{\sqrt{3}}$ and the norm of $x^2 - \frac{3}{4}x$ is $\frac{1}{\sqrt{80}} = \frac{1}{4\sqrt{5}}$. Thus an orthonormal basis for U is

$$\left\{ \sqrt{3}x, 4\sqrt{5}(x^2 - \frac{3}{4}x) \right\} = \{\sqrt{3}x, 4\sqrt{5}x^2 - 3\sqrt{5}x\}.$$

We can double-check our answer by taking inner products of every pair in the list and checking that they are 1 if we take the same vector and 0 if we take different vectors.

(b) Write $x^3 = p(x) + q(x)$ with $p(x) \in U$ and $q(x) \in U^{\perp}$.

Solution

We can find $p(x)$ by taking the projections of x^3 onto each vector in the orthonormal basis we just found, and adding them up. Then the difference between x^3 and that will lie in $U^\perp$. Note that $\langle v, v\rangle = 1$ when v is a vector in an orthonormal basis, so the projection formulas are simpler:

$$P_{\sqrt{3}x}(x^3) = \langle x^3, \sqrt{3}x\rangle\sqrt{3}x = \frac{3}{5}x,$$

$$P_{4\sqrt{5}x^2-3\sqrt{5}x}(x^3) = \langle x^3, 4\sqrt{5}x^2 - 3\sqrt{5}x\rangle(4\sqrt{5}x^2 - 3\sqrt{5}x) = \frac{4}{3}x^2 - x.$$

So we have $p(x) = \frac{3}{5}x + \frac{4}{3}x^2 - x = \frac{4}{3}x^2 - \frac{2}{5}x$, and so $q(x) = x^3 - \frac{4}{3}x^2 + \frac{2}{5}x$. To double check our work we can calculate the inner product of $q(x)$ with both x and x^2 and check that they both give 0, verifying that $q(x) \in U^\perp$.