MATH2211 SPRING 2022
PROBLEM SET 2

DUE WEDNESDAY, FEBRUARY 9 2022 AT 11:59 PM

Problem 1. Compute the real and imaginary parts of $\dfrac{\pi+i}{5-i}$.

Solution

$$\begin{aligned}\frac{\pi+i}{5-i} &= \frac{(\pi+i)(5+i)}{5^2+1} \\ &= \frac{5\pi-1+(5+\pi)i}{26}.\end{aligned}$$

The real part of this number is $\frac{5\pi-1}{26}$ and the imaginary part of this number is $\frac{5+\pi}{26}$.

Problem 2.

(a) Use power series expansions to prove Euler's formula[1]

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Solution

The power series for e^z is $\sum_{n=0}^{\infty}\frac{z^n}{n!}$, and the function e^z is equal to its power series for all complex numbers z. Therefore, when $z = i\theta$, we have

$$\begin{aligned}e^{i\theta} &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!} \\ &= \sum_{n=0}^{\infty} i^n\frac{\theta^n}{n!}.\end{aligned}$$

When $n = 2k$ for some $k \in \mathbb{Z}$, we have $i^n = (-1)^k$, and when $n = 2k+1$ for some $k \in \mathbb{Z}$, we have $i^n = i(-1)^k$. Splitting the above sum into even and odd

[1] If you don't remember what the power series of $\exp, \sin$, and $\cos$ are, you can look them up on the internet.

terms yields

$$e^{i\theta} = \sum_{k=0}^{\infty}(-1)^k \frac{\theta^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}(-1)^k \frac{\theta^{2k+1}}{(k+1)!}$$
$$= \cos\theta + i\sin\theta,$$

since two terms in the right hand side are the power series for $\cos\theta$ and $i\sin\theta$ respectively.

(b) Use Euler's formula to prove the identity

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2).$$

Solution

Noticing that $\sin(\theta_1 + \theta_2) = \operatorname{Im} e^{i(\theta_1+\theta_2)}$, we compute

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= e^{i\theta_1}e^{i\theta_2} \\ &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2). \end{aligned}$$

The imaginary part of the last expression is $\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2$, which gives the result.

(c) Use the same technique to derive a formula for $\cos(3\theta)$ in terms of $\cos\theta$.[2]

Solution

Noticing that $\cos(3\theta) = \operatorname{Re} e^{3i\theta}$, we compute

$$\begin{aligned} e^{3i\theta} &= (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \end{aligned}$$

The real part of this last expression is $\cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = 4\cos^3\theta - 3\cos\theta$.

Problem 3. Let $z = e^{\frac{2\pi i}{n}}$, where $n \in \mathbb{Z}^+$. Prove that $1 + z + z^2 + \cdots + z^{n-1} = 0$.[3]

[2] This can be generalized to $\cos(n\theta)$: look up *Chebyshev polynomials of the first kind* on the internet.
[3] Hint: Factor the polynomial $x^n - 1$.

Solution

Raising z to the nth power yields $e^{2\pi i}$, which is 1. Therefore, $z^n - 1 = 0$. On the other hand, $z^n - 1$ can be factored as $(z-1)(z^{n-1} + z^{n-2} + \cdots + z^2 + z + 1)$. Since $z - 1 \neq 0$ (as $z \neq 1$), we must have $z^{n-1} + z^{n-2} + \cdots + z^2 + z + 1 = 0$.

Problem 4. Read up about Fermat's little theorem by looking it up on the internet. Using Fermat's little theorem, find the roots of $x^{10} - 1$ over $\mathbb{F}_{11}$.

Solution

Fermat's little theorem says that for all integers n and all primes p, we have $n^p \equiv n \pmod{p}$. In other words, $n^p - n$ is divisible by p for all integers n and all primes p. Furthermore, when n is not divisible by p, the equation $n^p - n \equiv 0 \pmod{p}$ implies that $n^{p-1} - 1 \equiv 0 \pmod{p}$. (We used the primality of p here, which is actually equivalent to the zero product property of $\mathbb{F}_p$!)
Letting $p = 11$, and translating everything from integers to $\mathbb{F}_{11}$, this says that $x^{10} - 1 = 0$ for all $x \in \mathbb{F}_{11} \setminus \{0\}$. Moreover, $0^{10} - 1 = -1 \neq 0$. Hence, the polynomial $x^{10} - 1$ has all elements of $\mathbb{F}_{11}$ except 0 as its roots.

Problem 5.

(a) Is $U = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 + 2x_2 + 3x_3 = 0\}$ a subspace of $\mathbb{C}^3$?

Solution

Yes, U is a subspace of $\mathbb{C}^3$. To check this, we check that the zero vector is in U (yes). Next we check that if $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in U$, then $v + w \in U$. The hypothesis that $v, w \in U$ is equivalent to the following two hypotheses:

$$v_1 + 2v_2 + 3v_3 = 0,$$
$$w_1 + 2w_2 + 3w_3 = 0.$$

Therefore $(v_1 + w_1) + 2(v_2 + w_2) + 3(v_3 + w_3) = (v_1 + 2v_2 + 3v_3) + (w_1 + 2w_2 + 3w_3) = 0$ as well. This shows that $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$ is in U. Finally, we check that for $v = (v_1, v_2, v_3) \in U$ and $a \in \mathbb{C}$ that $av \in \mathbb{C}$. The hypothesis that $v \in U$ is equivalent to the statement that $v_1 + 2v_2 + 3v_3 = 0$. Therefore, $av_1 + 2av_2 + 3av_3 = a(v_1 + 2v_2 + 3v_3) = 0$ as well, which shows that $av = (av_1, av_2, av_3)$ is in U.

(b) Is $U = \{(x_1, x_2, x_3) \in \mathbb{Q}^3 : x_1 x_2 x_3 = 0\}$ a subspace of $\mathbb{Q}^3$?

Solution

No, U is not a subspace of $\mathbb{Q}^3$. The elements $(1, 1, 0)$ and $(0, 0, 1)$ belong to U but their sum, $(1, 1, 1)$ does not.

(c) Let P be the $\mathbb{R}$-vector space of all polynomials with real coefficients. is

$$U = \{f \in P : f'(-1) = 3f(2)\}$$

a subspace of P? Here, f' means the derivative of f.

Solution

Yes, U is a subspace of P. First, the zero function, which we shall name zero, is in U, because $\mathrm{zero}'(-1) = 0 = 3\,\mathrm{zero}(2)$. Next, let $f, g \in U$ and $a \in \mathbb{R}$. The hypothesis that $f, g \in U$ says that $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. These two equations imply that $f'(-1) + g'(-1) = 3f(2) + 3g(2)$. This is the same as saying that $(f + g)'(-1) = 3(f + g)(2)$, which is precisely the condition for $f + g$ to be in U. Finally, $(af)'(-1) = af'(-1) = 3af(2) = 3(af)(2)$, so $af \in U$. This verifies all the conditions for U to be a subspace of P.
(We freely used the definition of sum of two functions, as well as the product of a function with a scalar, in this solution.)

Problem 6.

(a) Is $w = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{C}^3$ a linear combination of $\begin{pmatrix} 1 \\ 1 \\ -i \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$?

Solution

Suppose that $\left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right) = a\left(\begin{smallmatrix} 1 \\ 1 \\ -i \end{smallmatrix}\right) + b\left(\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}\right) + c\left(\begin{smallmatrix} 1 \\ 0 \\ i \end{smallmatrix}\right)$ for some $a, b, c \in \mathbb{C}$. In other words, a, b, c solve the system of equations

$$\begin{aligned} a + 2b + c &= 1 \\ a + b &= -1 \\ -ia + ic &= 0. \end{aligned}$$

The third equation implies that $c = a$. So the first two equations can now be written in terms of a and b only:

$$\begin{aligned} 2a + 2b &= 1 \\ a + b &= -1. \end{aligned}$$

This system has no solution because $a + b = -1$ implies $2a + 2b = -2$. Therefore, $(1, -1, 0)$ is not a complex linear combination of the three given vectors.

(b) In the real vector space consisting of all polynomials with real coefficients, is
$$x + 1 \in \operatorname{span}\{x^2 + 1, x^3 + x, 2x^2 + x, x + 3\}?$$

Solution

Suppose that $x + 1 = a(x^2 + 1) + b(x^3 + x) + c(2x^2 + x) + d(x + 3)$ for some real numbers a, b, c, d. Using the fact that two polynomials are equal iff their coefficients are equal, it follows that a, b, c, d solve the system of equations
$$\begin{aligned} b &= 0 \\ a + 2c &= 0 \\ b + c + d &= 1 \\ a + 3d &= 1. \end{aligned}$$
Therefore, $b = 0$ and $a = -2c$. Substituting this into the last two equations, we find that c, d must solve the system of equations
$$\begin{aligned} c + d &= 1 \\ -2c + 3d &= 1. \end{aligned}$$
Using any favorite 2×2 system solving method, we obtain $c = \frac{2}{5}$ and $d = \frac{3}{5}$ as the unique solution (and therefore $a = -2c = -\frac{4}{5}$). This set of values for a, b, c, d is indeed a solution, so $x + 1$ is in the span of the four given polynomials.

Problem 7. Show that a subset W of a vector space is a subspace if and only if $\operatorname{span}(W) = W$.

Solution

Let V be the unnamed vector space in the problem and first suppose that W is a subspace of V. We know that $\operatorname{span}(W)$ is the smallest subspace of W containing every element of W (we proved this characterization in class). But W is such a subspace, so $\operatorname{span}(W) \subseteq W$. We also have $\operatorname{span}(W) \supseteq W$ since $\operatorname{span}(W)$ is supposed to contain every element of W. Therefore, $\operatorname{span}(W) = W$.
Now suppose that $\operatorname{span}(W) = W$. In class we proved that the span of any subset of V is a subspace of V. Therefore, by the equation $\operatorname{span}(W) = W$, it follows that W is a subspace of V.