MATH2211 SPRING 2022
PROBLEM SET 5

DUE WEDNESDAY, MARCH 16, 2022 AT 11:59 PM

Problem 1. Find a linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with kernel
$$\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$$
or prove that no such T exists.

Problem 2. Show that there is a unique linear map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying
$$T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix},$$
and find the corresponding 3×3 matrix. Is this linear map an isomorphism?

Problem 3.

(a) Is there a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with $\mathrm{im}(T) = \ker(T)$?

(b) Is there a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ with $\mathrm{im}(T) = \ker(T)$?

Problem 4. A *linear functional* on an F-vector space V means a linear map from V to F (the one-dimensional F-vector space). For example, $(x, y) \mapsto 2x + 3y$ is a linear functional on $\mathbb{R}^2$.

(a) Suppose that T is a linear functional on a vector space V of dimension n. Prove that the kernel of T has dimension either n or $n - 1$. When does the kernel have dimension $n - 1$?

(b) Suppose S and T are two linear functionals on a vector space V with the same kernel. Prove that there exists a scalar $c \in F$ such that $T = cS$.

Problem 5.

(a) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map corresponding to
$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$
Find bases for the kernel and image of T and find all solutions to $Tx = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$.

(b) Find bases for the kernel and image of the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (x + y, 0, 2x - y)$.

Problem 6.

(a) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the matrix of T^{-1} is precisely the $n \times n$ matrix such that for $i = 1, 2, \ldots, n$, its ith column is the unique vector v such that $Tv = e_i$.

(b) Using part (a), find the inverse of

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Try not to use any previously learned matrix inversion methods from high school or such.