MATH2211 SPRING 2022
PROBLEM SET 8

DUE FRIDAY, APRIL 8, 2022 AT 11:59 PM

Problem 1. For each of the following matrices A, find the eigenvalues of A and a basis for each eigenspace, and say whether A is diagonalizable.

(a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) $A = \begin{pmatrix} 7 & 4 & 4 \\ 0 & -1 & 0 \\ -8 & -4 & -5 \end{pmatrix}$.

Problem 2. Compute the characteristic polynomial of the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

Problem 3. A *stochastic matrix* is a square matrix of real numbers, all of whose entries are between 0 and 1 (inclusive), and with the property that the numbers in each column add to 1.

Prove that 1 is an eigenvalue of any stochastic matrix.

Problem 4. Let $T\colon V \to V$ be a linear transformation and $B\colon V \to V$ be an invertible linear transformation. Prove that

$$\chi_T(t) = \chi_{B^{-1}TB}(t).$$

Problem 5. Prove that the eigenvalues of an upper triangular square matrix are the numbers on the diagonal.

Problem 6. In this problem you will be working out the derivation of the closed form of the Fibonacci sequence that I showed at the beginning of the semester.

Let $\mathbb{R}^\infty$ be the vector space of infinite sequences of real numbers. Define the left shift operator $S\colon \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$S(a_0, a_1, a_2, \dots) = (a_1, a_2, \dots).$$

(a) Because $\mathbb{R}^\infty$ is infinite-dimensional (in fact, uncountably-dimensional), weird things can happen. Prove that *every real number* is an eigenvalue of S. Also find the corresponding eigenvector for an arbitrary eigenvalue $\lambda \in \mathbb{R}$.

(b) Let $V \subseteq \mathbb{R}^\infty$ be the subspace consisting of Fibonacci-like sequences; that is, consisting of sequences $(a_0, a_1, a_2, \dots)$ satisfying $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. Prove that if $v \in V$, then $Sv \in V$ as well.

(c) Part (b) showed that S restricts to a linear map $S|_V\colon V \to V$. What is the dimension of V? What are the eigenvalues and eigenvectors of $S|_V$?

(d) Derive the closed form for the Fibonacci sequence $a_0 = 0, a_1 = 1; a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, using part (c).

(e) To convince you that this "abstract" proof is not actually that abstract, here is a final exercise. Let $\varphi\colon \mathbb{R}^2 \to V$ be the isomorphism sending (x, y) to the Fibonacci-like sequence whose first two terms are $a_0 = x$ and $a_1 = y$. Then $\varphi^{-1}\colon V \to \mathbb{R}$ is the projection $(a_0, a_1, \dots,) \mapsto (a_0, a_1)$. Thus, the composition

$$\mathbb{R}^2 \xrightarrow{\varphi} V \xrightarrow{S} V \xrightarrow{\varphi^{-1}} \mathbb{R}^2$$

is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$, i.e. a 2×2 matrix. What is this matrix?