Examples of proofs that $\sqrt{2}$ is irrational

MATH2211 Spring 2022

January 26, 2022

1. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Also assume $\gcd(a, b) = 1$. Then we have
$$\frac{a^2}{b^2} = 2,$$
so $a^2 = 2b^2$. This equation implies that a^2 is even because $2b^2$ is even. Therefore, a is even. Let $a = 2k$ for some positive integer k. Then $a^2 = 2b^2$ can be rewritten as
$$(2k)^2 = 2b^2$$
or
$$4k^2 = 2b^2$$
or
$$2k^2 = b^2.$$
Therefore, b is even. Therefore, a and b are both even, which contradicts $\gcd(a, b) = 1$.

2. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Let S be the set of all pairs of positive integers (a, b) such that $a/b = \sqrt{2}$. If S is empty, then we're done. Assume S is nonempty. Pick some $(a, b) \in S$. Then we have
$$\frac{a^2}{b^2} = 2,$$
so $a^2 = 2b^2$. This equation implies that a^2 is even because $2b^2$ is even. Therefore, a is even. Let $a = 2k$ for some positive integer k. Then $a^2 = 2b^2$ can be rewritten as
$$(2k)^2 = 2b^2$$

or

$$4k^2 = 2b^2$$

or

$$2k^2 = b^2.$$

Therefore, b is even. Therefore, a and b are both even. Therefore, $a/2$ and $b/2$ are both positive integers and $(a/2, b/2) \in S$. Repeat this argument on $(a/2, b/2)$ to conclude that $a/2$ and $b/2$ are themselves even, so $(a/4, b/4) \in S$. This can be repeated forever, producing an infinite decreasing sequence of positive integers $a, a/2, a/4, a/8, \ldots$. This is impossible, so $\sqrt{2}$ is irrational. (This is called proof by infinite descent.)

3. We know that $x^2 - 2$ has $\pm\sqrt{2}$ as roots. Let's apply the rational root theorem to $x^2 - 2$. The theorem says that if a/b (in lowest terms) is a root of $x^2 - 2$, then a divides 2 and b divides 1. In other words, the only possible rational roots of $x^2 - 2$ are

$$\{-2, -1, 1, 2\}.$$

Now, $(-2)^2 - 2 = 2 \neq 0$. $(-1)^2 - 2 = -1 \neq 0$. $1^2 - 2 = -1 \neq 0$. $2^2 - 2 = 2 \neq 0$. Therefore, $x^2 - 2$ has no rational roots. So $\sqrt{2}$ is irrational.

4. Assume that $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$. Let S be the set of all pairs of positive integers (a, b) such that $a/b = \sqrt{2}$. If S is empty, then we're done. Assume S is nonempty. Pick some $(a, b) \in S$. Then we have

$$\frac{a^2}{b^2} = 2,$$

so $a^2 = 2b^2$. Now we compute

$$\begin{aligned}(2b - a)^2 &= 4b^2 - 4ab + a^2 \\ &= 6b^2 - 4ab.\end{aligned}$$

We also compute

$$\begin{aligned}(a - b)^2 &= a^2 - 2ab + b^2 \\ &= 3b^2 - 2ab.\end{aligned}$$

Therefore, $(2b - a)^2 = 2(a - b)^2$. Therefore, $(2b - a, a - b) \in S$. Moreover, $a - b < b$ because $1 < \sqrt{2} < 2$, which implies $b < \sqrt{2}b < 2b$, which implies $b < a < 2b$, which implies $0 < a - b < b$. So again contradiction by infinite descent.