MATH2211 SPRING 2022
PROBLEM SET 3 SOLUTIONS

Let F be a field and let V be an F-vector space.

Problem 1. Suppose we are given a list $v_1, \dots, v_n \in V$.

(a) Show that $v_1, \dots, v_n$ are linearly dependent if and only if there is some $1 \leq i \leq n$ such that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ (i.e. the span of the list with v_i taken out).

Solution

First suppose that $v_1, \dots, v_n$ are linearly dependent. One characterization of linear dependence is that some vector v_i is a linear combination of the other vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. Therefore, for this choice of i, $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.
Now suppose that some v_i is in the span of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. This says that the vector v_i is a linear combination of the other vectors in the list. That implies that $v_1, \dots, v_n$ is a linearly dependent set.

(b) Show that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ if and only if

$$\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n).$$

(That is, the span doesn't change when v_i is taken out.)

Solution

First note that for two subsets $A \subseteq B$ of V, $\operatorname{span} A \leq \operatorname{span} B$ automatically. So to say that $\operatorname{span}(v_1, \dots, v_n) = \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ is equivalent to saying that $\operatorname{span}(v_1, \dots, v_n) \subseteq \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$, because the other containment always holds. This in turn is equivalent to saying that every linear combination of $v_1, \dots, v_n$ is also a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$.
For one direction, suppose that $v_i \in \operatorname{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ for some i. This says that there exist scalars $a_j \in F$ for $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$ such that $v_i = \sum_{j \in \{1,2,\dots,i-1,i+1,\dots,n\}} a_j v_j$. Now let

$$v = b_1 v_1 + \cdots + b_n v_n$$

be an arbitrary element in $\text{span}(v_1, \dots, v_n)$. By replacing the $b_i v_i$ term with the sum $\sum_{j \in \{1,2,\dots,i-1,i+1,\dots,n\}} b_i a_j v_j$, we can rewrite v as

$$v = (b_i a_1 + b_1)v_1 + \cdots + (b_i a_{i-1} + b_{i-1})v_{i-1} + (b_i a_{i+1} + b_{i+1})v_{i+1} + (b_i a_n + b_n)v_n,$$

showing that v is in the span of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$.
For the other direction, suppose that every linear combination of $v_1, \dots, v_n$ is also a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. Now it is certainly true that v_i is a linear combination of $v_1, \dots, v_n$, so this says that v_i is also able to be written as a linear combination of $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. This shows that $v_i \in \text{span}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ and concludes the proof.

Problem 2. Suppose $v_1, v_2, v_3, v_4 \in V$ and set

$$w_1 = v_1 - v_2, \quad w_2 = v_2 - v_3, \quad w_3 = v_3 - v_4, \quad w_4 = v_4.$$

(a) Show that $\text{span}(v_1, v_2, v_3, v_4) = \text{span}(w_1, w_2, w_3, w_4)$.

Solution

As was given, each w_i is a linear combination of the v_i. In other words, $w_i \in \text{span}(v_1, v_2, v_3, v_4)$ for all i. Therefore,

$$\text{span}(w_1, w_2, w_3, w_4) \leq \text{span}(v_1, v_2, v_3, v_4)$$

since $\text{span}(w_1, w_2, w_3, w_4)$ is the smallest subspace of V containing the vectors w_1, w_2, w_3, w_4.
Now we can also write each v_i as a linear combination of the w_i as follows:

$$v_1 = w_1 + w_2 + w_3 + w_4, \quad v_2 = w_2 + w_3 + w_4, \quad v_3 = w_3 + w_4, \quad v_4 = w_4.$$

By the same argument as above, this shows that

$$\text{span}(w_1, w_2, w_3, w_4) \geq \text{span}(v_1, v_2, v_3, v_4).$$

Thus we are done.

(b) Show that v_1, v_2, v_3, v_4 are linearly independent if and only if w_1, w_2, w_3, w_4 are linearly independent.

Solution

Here is a very neat proof for this problem. We first have the following lemma: A finite set of vectors $S \subseteq V$ is linearly independent if and only if the dimension of $\text{span}(S)$ is equal to $|S|$. Proof: If S is linearly independent, then S is a basis of $\text{span}(S)$ since S is linearly independent and S spans $\text{span}(S)$. Conversely, if $\dim \text{span}\, S = |S|$, Corollary 4.26 from the course notes along

with the fact that S spans $\mathrm{span}(S)$ shows that S is a basis of $\mathrm{span}(S)$, and therefore linearly independent.
Using this lemma, we have that v_1, v_2, v_3, v_4 are linearly independent iff $\mathrm{span}(v_1, v_2, v_3, v_4)$ has dimension 4, and the same for w_1, w_2, w_3, w_4. Since part (a) proved that

$$\mathrm{span}(v_1, v_2, v_3, v_4) = \mathrm{span}(w_1, w_2, w_3, w_4),$$

it follows that one side has dimension 4 iff the other side has dimension 4. By the lemma, this is the same as saying that v_1, v_2, v_3, v_4 is linearly independent iff w_1, w_2, w_3, w_4 is linarly independent. This concludes the proof.

Problem 3. Suppose that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V. Show that $\{v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n\}$ is linearly independent as well.

Solution

We know that linear independence of $v_1, \ldots, v_n$ says that the only solution to $\sum_i a_i v_i = 0$ is $a_i = 0$ for all i. Now let us suppose that $a_i \in \mathbb{R}$ satisfies

$$a_1 v_1 + a_2(v_1 + v_2) + \cdots + a_n(v_1 + \cdots + v_n) = 0.$$

Let us rewrite this as

$$(a_1 + \cdots + a_n)v_1 + (a_2 + \cdots + a_n)v_2 + \cdots + a_n v_n = 0.$$

By linear independence of $v_1, \ldots, v_n$, we see that

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= 0, \\ a_2 + \cdots + a_n &= 0, \\ &\vdots \\ a_n &= 0. \end{aligned}$$

Using back-substitution we find that the only solution to this system is $a_i = 0$ for all i. Thus, $v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n$ are linearly independent.

Problem 4.

(a) Show that V is infinite dimensional if and only if it satisfies the following property: for every integer $k > 0$, one can find k linearly independent vectors $v_1, \ldots, v_k \in V$.

Solution

Recall that the definition of V being infinite dimensional is that there does not exist a finite basis for V.

For one direction, suppose that V is infinite dimensional. Thus we know that there does not exist a finite basis for V. We prove that for every $k > 0$ we can find k linearly independent vectors $v_1, \dots, v_k \in V$ by induction. For the base case $k = 1$, we can certainly find a nonzero vector in V since V cannot be the zero vector space. Now suppose we have found $k - 1$ linearly independent vectors $v_1, \dots, v_{k-1} \in V$. Because $v_1, \dots, v_{k-1}$ is not a basis (because V is not supposed to have a finite basis), but is linearly independent, it must be that they do not span V. Hence we can let v_k be any element of $V - \operatorname{span}(v_1, \dots, v_{k-1})$ to complete the induction.
Now suppose that V is finite dimensional and has dimension n. Then any set of $n + 1$ vectors must be linearly dependent. This completes the proof.

(b) Show that the vector space $\mathbb{R}^\infty := \{$ all sequences $(a_1, a_2, a_3, \dots)$ of real numbers$\}$ is infinite dimensional.

Solution

The sequences $(1, 0, 0, 0, \dots,), (0, 1, 0, 0, \dots), (0, 0, 1, 0, \dots), \dots$ gives a sequence of linearly independent sequences in V. For any integer $k > 0$ we can take the first k terms of this sequence to get k linearly independent vectors in $\mathbb{R}^\infty$. By part (a) it follows that $\mathbb{R}^\infty$ is infinite dimensional.

(c) Give an example of a subspace of $\mathbb{R}^\infty$ which is strictly contained in $\mathbb{R}^\infty$ but is still infinite dimensional.

Solution

Some examples:

- All sequences whose first term is 0.
- All sequences whose second term is 0.
- All sequences whose first and 42th term are zero.
- All sequences whose first through 42nd term are zero.
- All sequences with finitely many nonzero terms (fun fact: $(1, 0, 0, 0, \dots,), (0, 1, 0, 0, \dots), (0, 0, 1, 0, \dots), \dots$ is a basis of this space but **not** a basis of $\mathbb{R}^\infty$!)

Can you come up with a cool example that is not like these?
A non-example is the set S of sequences whose terms are integers. This set fails to be closed under scalar multiplication: $0.5 \cdot (1, 0, \dots, 0) = (0.5, 0, \dots, 0)$ is not in S.

Problem 5. For each positive integer n, let

$$B_n = \{(-1, 1, \dots, 1), (1, -1, 1, \dots, 1), \dots, (1, 1, \dots, -1)\} \subseteq F^n.$$

That is, B_n is the set of vectors in F^n with one component equal to -1 and $n-1$ components equal to 1.

(a) Let $F = \mathbb{R}$. For which n is B_n a basis of $\mathbb{R}^n$?[1]

Solution

I claim that the set of n for which B_n is a basis of $\mathbb{R}^n$ is the set of positive integers except 2.
For $n = 1$, B_n consists of $\{-1\}$ which is clearly linearly independent. For $n = 2$, $B_n = \{(-1, 1), (1, -1)\}$. These vectors are multiples of each other, so cannot be a basis of $\mathbb{R}^2$ as they are not linearly independent.
For $n \geq 3$, we will prove that B_n is linearly independent, which proves that B_n is a basis of $\mathbb{R}^n$ as $|B_n| = n$. To prove that B_n is linearly independent, let us suppose that $a_1, \dots, a_n \in \mathbb{R}$ such that

$$a_1(-1, 1, \dots, 1) + a_2(1, -1, 1, \dots, 1) + \cdots + a_n(1, 1, \dots, -1) = 0.$$

This is equivalent to the system of equations

$$\begin{aligned} -a_1 + a_2 + \cdots + a_n &= 0 \\ a_1 - a_2 + \cdots + a_n &= 0 \\ &\vdots \\ a_1 + a_2 \cdots - a_n &= 0. \end{aligned}$$

To solve this system, let's note that if we add these equations together we get

$$(n-2)(a_1 + \cdots + a_n) = 0,$$

which implies that $a_1 + \cdots + a_n = 0$, since $n - 2 \neq 0$ for $n \geq 3$. Now subtracting this last equation by each of the equations in the original system, we see that $2a_i = 0$ for all i. Hence $a_i = 0$ for all i is the only solution to the system, therefore B_n is linearly independent.

(b) Let $F = \mathbb{F}_3$. Show that if n is of the form $3k + 2$ for some $k \in \mathbb{Z}_{\geq 0}$ (i.e. $n \in \{2, 5, 8, \dots\}$), then B_n is not a basis of $\mathbb{F}_3^n$.[2]

Solution

(Side note: If $n \not\equiv 2 \pmod 3$, then $n - 2 \neq 0$ in $\mathbb{F}_3$, so everything in the solution to part (a) continues to hold. So in this case, B_n is a basis of $\mathbb{F}_3^n$. But this argument is not needed because the problem did not ask to show that B_n is a basis when $n \notin \{2, 5, 8, \dots\}$.)

[1] If you need a hint for where to start, try to check whether the vectors in B_n are linearly independent.
[2] Hint: Try to show that B_n is contained in a proper subspace.

Let $n \equiv 2 \pmod 3$. The argument from (a) fails because $n - 2 = 0$, so the equation

$$(n-2)(a_1 + \cdots + a_n) = 0$$

says nothing. However, this alone is not enough to conclude that B_n is not a basis, because maybe some other argument might show that B_n is a basis. To conclude that B_n is not a basis, it suffices to make the following observation: when $n \equiv 2 \pmod 3$, every vector in B_n has the property that the sum of its components is 0. Indeed, each vector has one component equal to -1 and $n-1$ components equal to 1, which makes the sum of the components equal to $-1 + (n-1)1 = n - 2 = 0 \in \mathbb{F}_3$ when $n \equiv 2 \pmod 3$. Therefore,

$$B_n \subseteq U \coloneqq \{(a_1, \ldots, a_n) \in \mathbb{F}_3^n : a_1 + \cdots + a_n = 0\}.$$

Note that U is a vector space, in fact a proper subspace of $\mathbb{F}_3^n$. It follows that $\operatorname{span} B_n \subseteq U$ as well. Hence, B_n does not span $\mathbb{F}_3^n$, so B_n is not a basis of $\mathbb{F}_3^n$. (By the basis theorem, it follows that this set of vectors must be linearly dependent when $n \equiv 2 \pmod 3$. Can anyone find an explicit linear relation?)