MATH2211 SPRING 2022
PROBLEM SET 5

DUE WEDNESDAY, MARCH 16, 2022 AT 11:59 PM

Problem 1. Find a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^2$ with kernel

$$\{(x_1, x_2, x_3, x_4, x_5) : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$$

or prove that no such T exists.

Solution

The desired kernel is the set $\{(3x, x, y, y, y) : x, y \in \mathbb{R}\}$, which is manifestly is a subspace of $\mathbb{R}^5$ of dimension 2. However, T has rank at most 2, implying that $\dim \ker T \geq 3$ by the rank-nullity theorem. This is incompatible with the desired kernel, so no such T exists.

Problem 2. Show that there is a unique linear map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ satisfying

$$T\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}, \quad T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix},$$

and find the corresponding 3×3 matrix. Is this linear map an isomorphism?

Solution

Adding the latter two equations, then dividing by 2, says that $T\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$. Subtracting the latter two equations, then dividing by 2, says that $T\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$. Finally, we use the fact that $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right) - \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) - 2\left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$ to obtain that $T\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right) - \left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) - 2\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}-2\\-1\\0\end{smallmatrix}\right)$. Therefore we have deduced that T is equal to

$$\begin{pmatrix}1 & 1 & -2\\ 1 & 0 & -1\\ 1 & 0 & 0.\end{pmatrix}$$

This proves existence and uniqueness of T. This also proves that $\left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right)$ is a basis of $\mathbb{R}^3$. Since the vectors $\left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2\\1\\1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0\\1\\1\end{smallmatrix}\right)$ are also a basis (by computation), this shows that the map T is an isomorphism because T sends a basis to a basis.

Solution

Let $U = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, the matrix formed from the input vectors. Let $S = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, the matrix formed from the desired output vectors. One sees that both matrices are invertible (their determinant is nonzero). Moreover, SU^{-1} satisfies the 3 desired equations. Therefore $T = SU^{-1} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 0 & 0. \end{pmatrix}$ and T is an isomorphism since both S and U are isomorphisms.

Problem 3.

(a) Is there a linear map $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with $\operatorname{im}(T) = \ker(T)$?

Solution

Yes, an example is
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
In this case, $\ker(T) = (*, *, 0, 0)$ and $\operatorname{im}(T)$ is also $(*, *, 0, 0)$. (Here, $(*, *, 0, 0)$ is defined to mean $\{(a, b, 0, 0) : a, b \in \mathbb{R}\}$.)

(b) Is there a linear map $T\colon \mathbb{R}^5 \to \mathbb{R}^5$ with $\operatorname{im}(T) = \ker(T)$?

Solution

No. By the rank-nullity theorem, $\dim \operatorname{im} T + \dim \ker T = 5$. On the other hand, if $\operatorname{im} T = \ker T$, then $\dim \operatorname{im} T + \dim \ker T$ must be an even number.

Problem 4. A *linear functional* on an F-vector space V means a linear map from V to F (the one-dimensional F-vector space). For example, $(x, y) \mapsto 2x + 3y$ is a linear functional on $\mathbb{R}^2$.

(a) Suppose that T is a linear functional on a vector space V of dimension n. Prove that the kernel of T has dimension either n or $n-1$. When does the kernel have dimension $n-1$?

Solution

We know that the image of T is either F or 0. By the rank-nullity theorem, in the first case, the kernel of T has dimension $n-1$, and in the second case, the kernel of T has dimension n. We also see that the kernel has dimension $n-1$ precisely when T is not the zero map.

(b) Suppose S and T are two linear functionals on a vector space V with the same kernel. Prove that there exists a scalar $c \in F$ such that $T = cS$.

Solution

First, if $\ker S = \ker T = V$, then both S and T are the zero map, so any scalar works.
Now assume $K = \ker S = \ker T$ is a proper subspace of V. Pick some $w \notin K$. Then we have $V = K \oplus \operatorname{span}(w)$, and furthermore $Sw \neq 0$ and $Tw \neq 0$.
Now, S sends an arbitrary vector $v = k + aw$ $(k \in K, c \in F)$ to $Sk + Saw = 0 + Scw = a(Sw) \in F$, and similarly T sends the same vector $v = k + aw$ to $a(Tw) \in F$. Therefore, we can set $c = (Tw)/(Sw)$ and we have $T = cS$.

Problem 5.

(a) Let $T\colon \mathbb{R}^5 \to \mathbb{R}^3$ be the linear map corresponding to

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Find bases for the kernel and image of T and find all solutions to $Tx = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$.

Solution

The kernel is the set of all $x \in \mathbb{R}^5$ such that $Ax = 0$, so doing row operations (left multiplication by elementary matrices) will preserve the kernel. The rref form of A is

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & \frac{3}{2} & 3 \end{pmatrix}.$$

Suppose that $x = (a, b, c, d, e)$ is in the kernel of this matrix. This gives the equations $a = -\frac{1}{2}d - 2e$, $b = d + 2e$, and $c = -\frac{3}{2}d - 3e$. Thus $\ker A$ is completely parametrized by the pair (d, e), so setting $d = 1, e = 0$ and $d = 0, e = 1$ should give a basis of $\ker A$. This yields

$$\left\{ \left(-\frac{1}{2}, 1, -\frac{3}{2}, 1, 0 \right), (-2, 2, -3, 0, 1) \right\}$$

for a basis of $\ker A$.

The rref of A shows that the rank of A is 3. Therefore, A is surjective, so a basis of the image of T is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

To find all solutions to $Tx = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$, we first find that a particular solution is $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ (since we notice that $\left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$ is the third column of A). The set of all solutions is therefore

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1/2 \\ 1 \\ -3/2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} : c_1, c_2 \in \mathbb{R} \right\}.$$

(b) Find bases for the kernel and image of the linear map $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x, y) = (x + y, 0, 2x - y)$.

Solution

The same method as part (a) works here too. The following is a quick solution using some nice observations. The map $S \colon (x, y) \mapsto (x + y, 2x - y)$ is an isomorphism. One can view T as the composition $U \cdot S$ where $U(x, y) = (x, 0, y)$. Since both U and S are injective, this shows that $\ker T = 0$, so $\ker T$ has basis the empty set. Moreover, the image of T is the image of U which one easily sees is 2-dimensional with basis $\{(1, 0, 0), (0, 0, 1)\}$.

Problem 6.

(a) Let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Prove that the matrix of T^{-1} is precisely the $n \times n$ matrix such that for $i = 1, 2, \ldots, n$, its ith column is the unique vector v such that $Tv = e_i$.

Solution

The ith column of the matrix of T^{-1} is $T^{-1}e_i$, where e_i denotes the ith standard basis vector. By definition $T^{-1}e_i$ is the unique vector v such that $Tv = e_i$.

(b) Using part (a), find the inverse of

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Try not to use any previously learned matrix inversion methods from high school or such.

Solution

Let A denote the given matrix. Part (a) suggests that we solve the three systems

$$\begin{aligned} x + y + z &= 1 \\ 3x - y &= 0 \\ x + 2z &= 0, \end{aligned}$$

$$\begin{aligned} x + y + z &= 0 \\ 3x - y &= 1 \\ x + 2z &= 0, \end{aligned}$$

$$\begin{aligned} x + y + z &= 0 \\ 3x - y &= 0 \\ x + 2z &= 1. \end{aligned}$$

For the first system, we have $x = -2z$ and $y = 3x = -6z$, so that the first equation becomes $-2z - 6z + z = 1$, or $z = -\frac{1}{7}$, giving $x = \frac{2}{7}$ and $y = \frac{6}{7}$. Hence the first column of A^{-1} is $\begin{pmatrix} 2/7 \\ 6/7 \\ -1/7 \end{pmatrix}$.

For the second system, we have $x = -2z$ and $y = 3x - 1 = -6z - 1$, so the first equation becomes $-2z - 6z - 1 + z = 0$, or $z = -\frac{1}{7}$, giving $x = \frac{2}{7}$ and $y = -\frac{1}{7}$. Hence the second column of A^{-1} is $\begin{pmatrix} 2/7 \\ -1/7 \\ -1/7 \end{pmatrix}$.

For the third system, we have $x = 1 - 2z$ and $y = 3x = 3 - 6z$, so the first equation becomes $1 - 2z + 3 - 6z + z = 0$, or $z = \frac{4}{7}$, giving $x = -\frac{1}{7}$ and $y = -\frac{3}{7}$. Hence the third column of A^{-1} is $\begin{pmatrix} -1/7 \\ -3/7 \\ 4/7 \end{pmatrix}$.

Therefore,

$$A^{-1} = \begin{pmatrix} \frac{2}{7} & \frac{2}{7} & -\frac{1}{7} \\ \frac{6}{7} & -\frac{1}{7} & -\frac{3}{7} \\ -\frac{1}{7} & -\frac{1}{7} & \frac{4}{7} \end{pmatrix}.$$