MATH2211 SPRING 2022
FINAL EXAM

FRIDAY, MAY 13 2022

Name: ______________________________

This exam is open notes, and the time limit is 3 hours. There are 100 points total in this exam.

High 87, low 39, mean 70.15, median 71. Good exam. The intended structure was that the first 5 problems were standard linear algebra problems, while the last 5 required at least one step of original thought. The scores reflected this quite well; average score across problems 1 to 5 was 85.5%, while average score across problems 6 to 10 was 54.8%. The hardest problems were 7(a), 7(b), 8(b), 9, and 10(b), with average points obtained being 43%, 44%, 39%, 43%, and 50%. It also turned out that 8(a) and 10(a) from the second half of the exam were quite easy; average score on those was 80%.

The top score on problem 9 was 6/10, which means that the problem should have a substantial hint in the future.

Nobody got full points on problem 8(b), despite the extensive hint. Two people came close, saying that I and C^{-1} are both in V_C. The issue is of course that C^{-1} does not always exist, and they should have used C instead of C^{-1}.

Problem 1. Let $M = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 2 & -1 & 0 \end{pmatrix}$.

(a) (5 points) Solve the system

$$M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$$

in terms of a.

(b) (5 points) Find $\operatorname{tr} M$, $\det M$, and the characteristic polynomial of M.

Problem 2. (10 points) For which $t \in \mathbb{R}$ do the vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ t \end{pmatrix}, \quad v_2 = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$?

Problem 3. (10 points) Show that every complex solution to $z^4 + 1 = 0$ also satisfies $z^{1200} = 1$.

Problem 4. (10 points) Find the inverse of the linear operator $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$T(x, y, z) = (x + y, x + z, y + z),$$

or prove that no such inverse exists.

Problem 5. (10 points) Find a basis of eigenvectors for the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hint: It will be helpful to let $\zeta = e^{2\pi i/3}$. (Make use of the zeta drawing skills you learned in class!)

Problem 6. (10 points) Let X be a 2×2 real matrix with trace 0 and rank 1. Prove that X only has 0 as an eigenvalue.

Problem 7. Let $V = C([0,1], \mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$. The integral operator I defined by

$$(I(f))(x) = \int_0^x f(t)\,dt$$

is a linear operator on V.

(a) (5 points) Prove that I is not surjective.

Hint: The non-surjectivity comes from a simple observation; no real analysis knowledge is required.

(b) (5 points) What is $({}^tI)(\delta)$, where δ is the Dirac delta functional? Show that your answer gives another proof that I is not surjective.

Problem 8. Let $n \geq 2$ be a positive integer. For any matrix $C \in M_n(\mathbb{R})$, let V_C be the set of all $n \times n$ matrices $A \in M_n(\mathbb{R})$ such that $AC = CA$.

(a) (5 points) Prove that V_C is a subspace of $M_n(\mathbb{R})$.

(b) (5 points) Prove that $\dim V_C \geq 2$ for all $C \in M_n(\mathbb{R})$.

Hint: Consider the case when C is a scalar matrix and the case when C is not a scalar matrix separately. In the latter case, try to come up with two linearly independent matrices in V_C.

Problem 9. (10 points) Find a counterexample to the following (reasonable-sounding) claim: If P and Q are orthogonal projection operators, then PQ is also an orthogonal projection operator.

Problem 10.

(a) (5 points) Let $e_1 \ldots e_n$ be an orthonormal basis of an inner product space V. Prove that the functionals $\varepsilon_i \in V^*$ defined by

$$\varepsilon_i(x) = \langle x, e_i \rangle, \quad 1 \leq i \leq n,$$

are precisely the dual basis of $e_1, \ldots, e_n$.

(b) (5 points) Suppose now that $e_1, \ldots, e_n$ is only an orthogonal basis, meaning that $\langle e_i, e_j \rangle = 0$ for $i \neq j$, but the $\|e_i\|$ are not necessarily equal to 1. For each i, let $\ell_i = \|e_i\|$, the length of e_i. Find the dual basis to $e_1, \ldots, e_n$, in terms of the ε_i defined in part (a) as well as the ℓ_i.