MATH2211 SPRING 2022
PROBLEM SET 10

DUE FRIDAY, APRIL 29, 2022 AT 11:59 PM

Relevant reading: Axler Chapter 6 and Section 7.A.

Problem 1. Let V be a finite-dimensional inner product space and let $e_1, e_2, \ldots, e_n$ be an orthonormal basis. Prove *Parseval's Identity*: for all $x, y \in V$,

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, e_i\rangle\langle e_i, y\rangle.$$

Problem 2.

(a) Let V be a real vector space and let $\langle\cdot,\cdot\rangle$ be a bilinear form on V. Prove that for every $v \in V$, the map φ_v defined by $\varphi_v(x) = \langle v, x\rangle$ is a linear functional on V.

Remark: This shows that, given a bilinear form on V, we get a natural map φ from V to V^* sending each $v \in V$ to φ_v. The Riesz representation theorem says that if the bilinear form $\langle\cdot,\cdot\rangle$ is an inner product, then $\varphi\colon V \to V^*$ is an isomorphism. Moreover, in $\mathbb{R}^n$ with the standard Euclidean inner product, φ is exactly the map that turns a column vector into a row vector, which is classically denoted $\cdot^T$.

(b) Let's do a concrete example. Let $V = P_2(\mathbb{R})$ with inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

Let $[x^2]$ denote the linear functional which when given a polynomial, returns its coefficient of the x^2 term. By the Riesz representation theorem, $[x^2]$ can be represented as $\langle f, \cdot\rangle$ for a unique polynomial $f \in P_2(\mathbb{R})$. Find f.

Problem 3. Let V be an inner product space and let U be a subspace of V. In this problem we investigate the orthogonal projection operator P_U.

(a) Prove that $P_U^2 = P_U$ and that P_U is self-adjoint (that is, $P_U = P_U^*$). Show that the identity $P_U^2 = P_U$ is equivalent to saying that $P_U|_{\operatorname{im} P_U} = I_{\operatorname{im} P_U}$.

(b) In this problem, suppose that $U = \operatorname{span}(u)^\perp$ for some nonzero unit vector $u \in V$. Prove that P_U can be expressed as $I - u\varphi_u$ (more classically denoted $I - uu^T$).

(c) Prove that the eigenvalues of P_U are 0 and 1, with the multiplicity of 1 being the dimension of U and the multiplicity of 0 being the dimension of $U^\perp$.

(d) Prove that the eigenspace of 1 of P_U is U and that the eigenspace of 0 is $U^\perp$, and that P_U is diagonalizable.

Problem 4. Prove that if $T\colon V \to V$ is self-adjoint, then its eigenspaces corresponding to two different eigenvalues are orthogonal to each other.

Problem 5. Let $A \in M_{m\times n}(\mathbb{R})$, and let A^* denote the adjoint of A, namely the unique matrix in $M_{n\times m}(\mathbb{R})$ such that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

(a) Prove that $\ker(A) = \ker(A^*A)$, where A^* denotes the adjoint of A.

(b) For a general matrix equation $Ax = b$, recall that there may be no solutions. Multiplying both sides on the left by A^*, we get the equation $A^*Ax = A^*b$. This is called the least squares normal equation for the matrix equation. It turns out that $A^*Ax = A^*b$ always has a solution, and that any solution x to the normal equation minimizes the value of $\|Ax - b\|$.

In this exercise, we prove this last part. Prove that if x is a solution to the equation $A^*Ax = A^*b$, then the projection of b onto the image of A is equal to Ax. Show therefore that $\|Ax - b\|$ is minimized at those x which solve $A^*Ax = A^*b$.