MATH2211 SPRING 2022
EXAM 2 REVIEW PROBLEMS

1. Major topics

(1) Matrix equations and elementary row operations
(2) Exterior algebra and determinants
(3) Duals, transpose, four fundamental subspaces, rank
(4) Eigenvalues

2. Problems

Problem 1. State and prove the rank-nullity theorem.

Problem 2. Factor the matrix
$$A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$$
as $A = CDC^{-1}$ with D diagonal, or explain why this cannot be done.

Problem 3. Prove that every 2×2 matrix with negative determinant is diagonalizable.

Problem 4. Prove that the geometric multiplicity of an eigenvalue λ of a matrix A is less than or equal to the algebraic multiplicity of λ for A.

Problem 5. Let $e_1, \ldots, e_n$ be the standard basis of F^n and let $\varepsilon_1, \ldots, \varepsilon_n$ be the standard dual basis. Let $A = (a_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix. Let i and j be some integers between 1 and n. What is $\varepsilon_i A e_j$?

Problem 6. Let $A \in M_2(F)$. Prove that there exist linear functionals $\lambda_1, \lambda_2 \in (F^2)^*$ and vectors $v_1, v_2 \in F^2$ such that
$$Aw = \lambda_1(w)v_1 + \lambda_2(w)v_2$$
for all $w \in F^2$. (This is the rank 2 version of problem set 7 problem 4).

Problem 7. Let $E \in M_n(F)$ be an elementary matrix and let $A \in M_n(F)$ be a matrix.

(a) Does EA have the same kernel as A? Does AE have the same kernel as A?
(b) Does EA have the same nullity as A? Does AE have the same nullity as A?
(c) Does EA have the same rank as A? Does AE have the same rank as A?

(d) Does EA have the same image as A? Does AE have the same image as A?

(e) Does EA have the same determinant as A? Does AE have the same determinant as A?

(f) Does EA have the same eigenvalues as A? Does AE have the same eigenvalues as A?

Problem 8. The fact that $v \wedge v = 0$ for all vectors $v \in V$ is an axiom of the exterior algebra. Does it follow that for all $\alpha \in \bigwedge^2 V$, we have $\alpha \wedge \alpha = 0$?

Problem 9. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) What is the matrix of $\bigwedge^2 A$ with respect to the ordered basis $(e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3)$?

(b) What is the matrix of $\bigwedge^3 A$ with respect to the ordered basis $e_1 \wedge e_2 \wedge e_3$?

Problem 10.

(a) The space V of real sequences with finitely many nonzero terms has dual $V^* = \mathbb{R}^\infty$, the space of real sequences without any conditions. Let S be the left shift operator on V, which sends $(a_0, a_1, \dots)$ to $(a_1, a_2, \dots)$. Describe ${}^tS$ as an operator on $\mathbb{R}^\infty$.

(b) Is S injective? Is S surjective? Is ${}^tS$ injective? Is ${}^tS$ surjective?

See the next page for remarks/spoilers to these problems! Stop scrolling if you don't want to be spoiled.

3. Hints and remarks

Remark 1. If you need to, look it up in the course notes and/or Axler!

Remark 2. Diagonalization, find eigenvalues and eigenvectors. Eigenvalues will turn out to be 3 and -2.

Remark 3. Hint: Eigenvalues have to be distinct or else the determinant is positive or zero!

Remark 4. Let $v_1, \dots, v_k$ be a basis of the eigenspace of λ, so that the geometric multiplicity of λ is equal to k. Extend this to a basis of V. In this basis, the matrix of A has the block form

$$\begin{pmatrix} \lambda I_k & B \\ 0 & C \end{pmatrix}.$$

The matrix $tI - A$ therefore has the matrix form

$$\begin{pmatrix} tI_k - \lambda I_k & B \\ 0 & tI_{n-k} - C \end{pmatrix}.$$

The determinant of this is $\det(tI_k - \lambda I_k)\det(tI_{n-k} - C) = (t-\lambda)^k \det(tI_{n-k} - C)$, so the algebraic multiplicity of λ is at least k.

Remark 5. It's a_{ij}.

Remark 6. Let $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$. Think of $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ as taking x times the first column of A plus y times the second column of A. Now x is just $\varepsilon_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and y is just $\varepsilon_2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, so we see that $A\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\varepsilon_1\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right) + \left(\varepsilon_2\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right)$! In tensor notation we would write

$$A = v_1 \otimes \varepsilon_1 + v_2 \otimes \varepsilon_2 \in F^2 \otimes (F^2)^*$$

where $v_1 = \left(\begin{smallmatrix} a_{11} \\ a_{21} \end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix} a_{12} \\ a_{22} \end{smallmatrix}\right)$.

This isn't the only way. If we think of Av as being the vector composed of (A's first row) times v and (A's second row) times v, then we could also write A as

$$A = e_1 \otimes \lambda_1 + e_2 \otimes \lambda_2$$

where λ_1 and λ_2 are the rows of A!

Remark 7. (a) Yes, no. (What's the kernel of AE?)

(b) Yes, yes.

(c) Yes, yes.

(d) No, yes.

(e) Yes, yes.

(f) No, no.

Remark 8. Nope! Let $V = F^4$ and let $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 V$. Then

$$\alpha \wedge \alpha = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0.$$

Interestingly, 4 is the minimum dimension in which this is possible.

Remark 9.

(a) We compute

$$Ae_1 \wedge Ae_2 = (e_1 + 4e_2 + 7e_3) \wedge (2e_1 + 5e_2 + 8e_3) = -3e_1 \wedge e_2 - 6e_1 \wedge e_3 - 3e_2 \wedge e_3$$
$$Ae_1 \wedge Ae_3 = (e_1 + 4e_2 + 7e_3) \wedge (3e_1 + 6e_2 + 9e_3) = -6e_1 \wedge e_2 - 12e_1 \wedge e_3 - 6e_2 \wedge e_3$$
$$Ae_2 \wedge Ae_3 = (2e_1 + 5e_2 + 8e_3) \wedge (3e_1 + 6e_2 + 9e_3) = -3e_1 \wedge e_2 - 6e_1 \wedge e_3 - 3e_2 \wedge e_3.$$

Thus in the given ordered basis,

$$\bigwedge^2 A = \begin{pmatrix} -3 & -6 & -3 \\ -6 & -12 & -6 \\ -3 & -6 & -3 \end{pmatrix}.$$

(b) This is just the determinant of A, which is 0 because A is not invertible (one way to see this is that the second column is the average of the first and third columns).

Remark 10. (a) Let $(b_0, b_1, b_2, \dots) \in \mathbb{R}^\infty$. As a linear functional on V, it sends $(a_0, a_1, a_2, \dots)$ to

$$a_0 b_0 + a_1 b_1 + a_2 b_2 + \cdots$$

(which is well defined because finitely many a_i are nonzero). Now $(b_0, b_1, b_2, \dots)S$ sends $(a_0, a_1, a_2, \dots)$ to

$$\begin{aligned}(b_0, b_1, b_2, \dots)S(a_0, a_1, a_2, \dots) &= (b_0, b_1, b_2, \dots)(a_1, a_2, a_3, \dots) \\ &= b_0 a_1 + b_1 a_2 + b_2 a_3 + \cdots \\ &= (0, b_0, b_1, \dots)(a_0, a_1, a_2, \dots).\end{aligned}$$

In other words, $(b_0, b_1, b_2, \dots)S = (0, b_0, b_1, b_2, \dots)$. Thus ${}^tS$ is the right shift operator on $\mathbb{R}^\infty$ which inserts a 0 into the beginning of the sequence and shifts everything else to the right by one.

(b) S is not injective but it is surjective. ${}^tS$ is injective but it is not surjective.