MATH2211 SPRING 2022
PROBLEM SET 6

DUE WEDNESDAY, MARCH 23, 2022 AT 11:59 PM

Problem 1. Let $T\colon V \to V$ be a linear map from a vector space V to itself. Let $\mathcal{B} = (v_1, \ldots, v_n)$ be an ordered basis of V and let $A \in M_n(F)$ be the matrix of V with respect to $\mathcal{B}$. Let $\mathcal{C} = (v_1', \ldots, v_n')$ be another ordered basis of V and let $A' \in M_n(F)$ be the matrix of T with respect to $\mathcal{C}$. Prove that there exists an invertible matrix $B \in M_n(F)$ such that $A' = BAB^{-1}$.

Problem 2. Factor $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ as a product of elementary matrices.

Hint: One way to approach this is to try to turn A into the identity matrix via elementary matrices, then see how what you've done is useful.

Problem 3.

(a) In this problem we show that a one-sided inverse of a square matrix is a two-sided inverse. More precisely, let $A, B \in M_n(F)$. Show that

$$AB = I_n \iff BA = I_n.$$

Hint: For the forward direction, assume $AB = I_n$ and first show that $A\colon F^n \to F^n$ is surjective. Similar hint applies to the other direction.

(b) In this problem we show that inverses are unique. More precisely, suppose $A, B, C \in M_n(F)$ such that $AB = I_n$ and $AC = I_n$. Prove that $B = C$.

Hint: In this proof, you are not allowed to multiply both sides by A^{-1}, because doing that presupposes that A^{-1} is unique! Instead, start by proving that A is surjective.

Problem 4. Let $A \in M_n(\mathbb{R})$ be an invertible matrix with integer entries, such that A^{-1} also has integer entries. Prove that $\det A = \pm 1$.

Problem 5. Compute:

(a) $\det \begin{pmatrix} 0 & 1 & 2 \\ 2 & 6 & -1 \\ 3 & 0 & 4 \end{pmatrix}$.

(b) $\det \begin{pmatrix} 2 & 0 & 2 & -4 \\ 12 & 6 & 6 & 1 \\ 0 & -1 & 4 & 5 \\ 3 & 2 & 3 & 2 \end{pmatrix}$.

Problem 6. Directly from the definition of $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ for the 2×2 determinant, prove the following:

(a) $\det(AB) = \det(A)\det(B)$ for all $A, B \in M_2(F)$.

(b) A is invertible if and only if $\det(A) \neq 0$, in which case $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(c) The function $\det\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}; (v, w) \mapsto \det \begin{pmatrix} v & w \end{pmatrix}$ (the determinant of the matrix whose columns are v and w) is *bilinear*, meaning that for all $\alpha, \beta \in F$ and $v_1, v_2, v, w_1, w_2, w \in \mathbb{R}^2$, we have

(i) $\det(\alpha v_1 + \beta v_2, w) = \alpha \det(v_1, w) + \beta \det(v_2, w)$,

(ii) $\det(v, \alpha w_1 + \beta w_2) = \alpha \det(v, w_1) + \beta \det(v, w_2)$.