

MATH2211 SPRING 2022
PROBLEM SET 1 SOLUTIONS

Problem 1. Let S be a set and let $A, B, C \subseteq S$. Prove that $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Solution

Let $x \in (A \cup B) \cap C$. This means that $x \in A \cup B$ and $x \in C$, i.e. $(x \in A \text{ or } x \in B)$ and $x \in C$. If $x \in A$, then $x \in A \cap C$. If $x \in B$, then $x \in B \cap C$. Therefore, $(A \cup B) \cap C \subseteq (A \cap C) \cup (B \cap C)$.

In the other direction, suppose $x \in (A \cap C) \cup (B \cap C)$. In other words, $x \in A \cap C$ or $x \in B \cap C$. In either case, $x \in A \cup B$ and $x \in C$, so $x \in (A \cup B) \cap C$. Hence $(A \cup B) \cap C \supseteq (A \cap C) \cup (B \cap C)$.

Problem 2. Show that the set of real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$, with addition and multiplication as in \mathbb{R} , is a field.

Solution

Let's call this set K . Since we are using the addition and multiplication from \mathbb{R} , many of the properties are inherited. More specifically, commutativity, associativity, and distributivity are automatically true. We just need to make sure that the sum and product of any two elements in K are in K , that 0 and 1 are in K , and that additive and multiplicative inverses of elements in K (besides 0) are in K .

Let us check that the sum and product of any two elements of K are still in K . For $a + b\sqrt{2} \in K$ and $c + d\sqrt{2} \in K$, we have

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in K.$$

Similarly, for products, for any $a + b\sqrt{2} \in K$ and $c + d\sqrt{2} \in K$, we have

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (bc + ad)\sqrt{2} \in K.$$

The numbers 0 and 1 are $0 + 0\sqrt{2}$ and $1 + 0\sqrt{2}$, which are both in K . The additive inverse of $a + b\sqrt{2} \in K$ is $-a - b\sqrt{2}$, which is in K . The multiplicative inverse is the most nontrivial part of this problem. Let $0 \neq a + b\sqrt{2} \in K$. Over \mathbb{R} , the multiplicative inverse of $a + b\sqrt{2}$ can be worked out to be

$$\frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} + \frac{-b}{a^2 - 2b^2}\sqrt{2},$$

and the denominators are nonzero because $a^2 - 2b^2$ is never 0. (Otherwise, $\sqrt{2}$ would be rational!) Therefore $(a + b\sqrt{2})^{-1} \in K$. This concludes the proof.

Problem 3. Define the Fibonacci numbers F_0, F_1, F_2, \dots by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

- (a) Use induction to prove that $F_1 + \dots + F_n = F_{n+2} - 1$.

Solution

The base case is $n = 1$, for which the statement says $F_1 = F_3 - 1$, which is true because $F_1 = 1$ and $F_3 = 2$.

Now let $n \geq 2$. We may assume we've proved that $F_1 + \dots + F_k = F_{k+2} - 1$ for all $1 \leq k \leq n - 1$. (In fact as we will see, we only need the $n - 1$ case.) We have

$$F_1 + \dots + F_n = (F_1 + \dots + F_{n-1}) + F_n = (F_{n+1} - 1) + F_n = F_{n+2} - 1,$$

completing the induction.

- (b) Use induction to prove that $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$.

Solution

Here is a solution that uses the previous result and avoids use of induction. Write F_3 as $F_1 + F_2$, F_5 as $F_3 + F_4$, and so on. The left hand side of the equation is now

$$F_1 + F_1 + F_2 + F_3 + \dots + F_{2n-2},$$

which is equal to $F_1 + F_{2n} - 1$ by part (a). This is equal to F_{2n} , so we're done.

Solution

We can also use induction here if we want. The base case is $n = 1$, which reads $F_1 = F_2$, which is true since $F_1 = F_2 = 1$.

Now let $n \geq 2$ and assume we've proved that $F_1 + F_3 + \dots + F_{2k-1} = F_{2k}$ for all $1 \leq k \leq n - 1$. We can use this to write $F_1 + F_3 + \dots + F_{2n-1}$ as $(F_1 + F_3 + \dots + F_{2n-3}) + F_{2n-1} = F_{2n-2} + F_{2n-1} = F_{2n}$, which completes the induction.

(Again, we actually only made use of the $k = n - 1$ case of the induction hypothesis.)

Problem 4. Prove that for every $n \in \mathbb{Z}^+$

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Solution

We use induction. The base case is $n = 1$, which reads $\frac{1}{1^2} \leq 2 - \frac{1}{1}$, which is true as both sides are equal to 1. Now let $n \geq 2$ and suppose we've proved that $\frac{1}{1^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ for all $1 \leq k \leq n-1$. Then

$$\frac{1}{1^2} + \cdots + \frac{1}{n^2} = \frac{1}{1^2} + \cdots + \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq 2 - \frac{1}{n-1} + \frac{1}{n^2}.$$

Now,

$$\begin{aligned} 2 - \frac{1}{n-1} + \frac{1}{n^2} &= 2 - \left(\frac{1}{n-1} - \frac{1}{n^2} \right) = 2 - \frac{n^2 - n + 1}{n^2(n-1)} \\ &< 2 - \frac{n^2 - n}{n^2(n-1)} \\ &= 2 - \frac{1}{n}, \end{aligned}$$

which completes the induction.

Problem 5. Suppose that $z_1, z_2 \in \mathbb{C}$. Prove that

(a) $|z_1 z_2| = |z_1| \cdot |z_2|$.

Solution

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$\begin{aligned} |z_1 z_2| &= |(a + bi)(c + di)| = |(ac - bd) + (ad + bc)i| \\ &= \sqrt{(ac - bd)^2 + (bc + ad)^2} \\ &= \sqrt{(ac)^2 + (bd)^2 + (bc)^2 + (ad)^2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |z_1| \cdot |z_2| &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} = \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{(ac)^2 + (ad)^2 + (bc)^2 + (bd)^2}. \end{aligned}$$

Both expressions are equal, so we are done.

(b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$.

Solution

Use the same notation as in (a). We have

$$\overline{z_1 z_2} = \overline{(a + bi)(c + di)} = \overline{(ac - bd) + (ad + bc)i} = (ac - bd) - (ad + bc)i.$$

On the other hand,

$$\overline{z_1} \cdot \overline{z_2} = (a - bi)(c - di) = (ac - bd) + (-ad - bc)i.$$

Both expressions are equal, so we are done.

Problem 6.

- (a) Find all complex solutions to $z^4 = -1$.

Solution

Suppose $z = re^{i\theta}$ is a solution. Then $e^{\pi i} = -1 = z^4 = r^4 e^{4i\theta}$, so $r = 1$ and $\pi \equiv 4\theta \pmod{2\pi}$. This gives 4 possible values of θ , namely $\pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. This gives us the solutions

$$\begin{aligned} z &= e^{\frac{\pi i}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \\ z &= e^{\frac{3\pi i}{4}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, \\ z &= e^{\frac{5\pi i}{4}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \\ z &= e^{\frac{7\pi i}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i. \end{aligned}$$

- (b) Find all complex solutions to $z^3 = i$.

Solution

Suppose $z = re^{i\theta}$ is a solution. Then $e^{\frac{\pi i}{2}} = i = z^3 = r^3 e^{3i\theta}$, so $r = 1$ and $\frac{\pi}{2} \equiv 3\theta \pmod{2\pi}$. This gives 3 possible values of θ , namely $\pi/6, 5\pi/6$, and $5\pi/6 = 3\pi/2$. This gives us the solutions

$$\begin{aligned} z &= e^{\frac{\pi i}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i, \\ z &= e^{\frac{5\pi i}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, \\ z &= e^{\frac{3\pi i}{2}} = -i. \end{aligned}$$

Express your answers both in the form $re^{i\theta}$ and by giving the real and imaginary parts.

Problem 7. Suppose $z_1, z_2 \in \mathbb{C}$. Prove the *triangle inequality*

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Solution

As both sides of the inequality are nonnegative, squaring both sides preserves the inequality, so it is equivalent to prove that

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2.$$

Let $z_1 = a + bi$ and $z_2 = c + di$, for some $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} |z_1 + z_2|^2 &= |(a + c) + (b + d)i|^2 = (a + c)^2 + (b + d)^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2ac + 2bd. \end{aligned}$$

Meanwhile,

$$\begin{aligned} (|z_1| + |z_2|)^2 &= \left(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \right)^2 \\ &= a^2 + b^2 + c^2 + d^2 + 2\sqrt{(a^2 + b^2)(c^2 + d^2)}. \end{aligned}$$

Therefore, it suffices to prove that $ac + bd \leq \sqrt{(a^2 + b^2)(c^2 + d^2)}$. If $ac + bd \leq 0$, then we are done as the right hand side is always non-negative. So assume $ac + bd > 0$. Then we can square both sides, and it suffices to prove that $(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$.^a Equivalently, it suffices to prove

$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 \geq 0.$$

Let us expand everything:

$$\begin{aligned} (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - a^2c^2 - b^2d^2 - 2abcd \\ &= a^2d^2 + b^2c^2 - 2abcd \\ &= (ad - bc)^2 \\ &\geq 0. \end{aligned}$$

So we have proven the triangle inequality.

Remark: Interestingly, last equality can be written as

$$|z_1|^2|z_2|^2 = (\operatorname{Re}(z_1z_2))^2 + (\operatorname{Im}(z_1z_2))^2.$$

This is the Pythagorean theorem for the complex number z_1z_2 !

^aFor those with Olympiad experience, you might recognize the Cauchy-Schwarz inequality here.