Closed form for Fibonacci Sequence (advertisement for vector spaces and eigendecomposition)

MATH2212 Spring 2022

January 26, 2022

Theorem 1 (Binet's formula). *Let $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Now let $\phi = (1+\sqrt{5})/2$. Then*

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\left(-\frac{1}{\phi}\right)^n \quad \textit{for all } n \geq 0.$$

Proof using vector spaces. Let us consider the set V of all real-valued sequences $(a_n)_{n=0}^{\infty}$ satisfying $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. We can see that V is a vector space over $\mathbb{R}$ because the sum of two sequences satisfying the recurrence still satisfies the recurrence, and any scalar multiple of a sequence satisfying the recurrence still satisfies the recurrence. Moreover, a sequence in V is determined by its first two terms a_0 and a_1, which can be freely chosen, so V is a 2-dimensional vector space over $\mathbb{R}$.

Let ϕ be the positive root of $x^2 - x - 1$ (i.e. the golden ratio), so its negative root is $-1/\phi$. The sequences $(1, \phi, \phi^2, \dots)$ and $(1, -1/\phi, (-1/\phi)^2, \dots)$ are both in V and are linearly independent, and therefore form a basis of V since V is 2-dimensional. Hence $(F_0, F_1, F_2, \dots) = c_1(1, \phi, \phi^2, \dots) + c_2(1, -1/\phi, (-1/\phi)^2, \dots)$ for some c_1 and c_2. This yields an infinite sequence of linear equations involving c_1 and c_2. Using the first two of these to solve for c_1 and c_2 we get $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$, thus obtaining the formula

$$F_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\left(-\frac{1}{\phi}\right)^n \quad \text{for all } n \geq 0.$$

□

A bit more detail on how to solve for c_1 and c_2. In what follows, I have renamed ϕ to r and $-1/\phi$ to s. As above, the sequences $(1, r, r^2, r^3, \dots)$ and $(1, s, s^2, s^3, \dots)$ are both in V and are linearly independent, so they form a basis of V. That is, every element of V is a real linear combination of these two sequences. In particular, $(0, 1, 1, 2, 3, \dots)$, the Fibonacci sequence, is a linear combination of $(1, r, r^2, r^3, \dots)$ and $(1, s, s^2, s^3, \dots)$. So there are unique real numbers c_1 and c_2 such that

$$(0, 1, 1, 2, 3, \dots) = c_1(1, r, r^2, \dots) + c_2(1, s, s^2, \dots).$$

If we expand what this says, this says that

$$\begin{aligned} F_0 &= 0 = c_1 + c_2 \\ F_1 &= 1 = c_1 r + c_2 s \\ &\vdots \\ F_n &= c_1 r^n + c_2 s^n \qquad \text{for all } n \geq 2. \end{aligned}$$

The first two lines allow us to solve for c_1 and c_2: the first line says $c_2 = -c_1$, and this combined with the second line implies that $c_1 = \frac{1}{r-s} = -\frac{1}{\sqrt{5}}$ (because $r - s = \sqrt{5}$). $\square$

Proof by eigendecomposition. Let $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then one can verify (by induction) that

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \quad \text{for all } n \geq 1.$$

Now M has eigenvalues $\phi = (1+\sqrt{5})/2$ and $-1/\phi = (1-\sqrt{5})/2$ because the characteristic polynomial of M is $x^2 - x - 1$. The matrix M has eigendecomposition

$$\begin{aligned} M &= \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{pmatrix} \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & -\frac{1}{\phi} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\phi} \\ -1 & \phi \end{pmatrix}. \end{aligned}$$

The purpose of this eigendecomposition is that powers of M now have a closed form:

$$\begin{aligned} M^n &= \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \left(-\frac{1}{\phi}\right)^n \end{pmatrix} \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi^n & 0 \\ 0 & \left(-\frac{1}{\phi}\right)^n \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\phi} \\ -1 & \phi \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \phi & -\frac{1}{\phi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \phi^n & \phi^{n-1} \\ -\left(-\frac{1}{\phi}\right)^n & \phi\left(-\frac{1}{\phi}\right)^n \end{pmatrix}. \end{aligned}$$

We only want F_n, so let's compute the top right entry of the product. We find that it is

$$\frac{1}{\sqrt{5}}\left(\phi^n - \left(-\frac{1}{\phi}\right)^n\right),$$

as promised. $\square$