MATH4460 SPRING 2023
PROBLEM SET 9

This problem set is due on Wednesday, April 26 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Redo any parts of Problem 2 in Exam 2 that you missed, for up to half points back on the exam.

Problem 2. Let $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$, the famous Fibonacci sequence. Let $F(z) = \sum_{n=0}^{\infty} f_n z^n$.

(a) Prove that $z^2F(z)$ can be expressed in terms of $zF(z)$ and $F(z)$. From this, deduce that $F(z)$ is a rational function.

(b) From this, use partial fractions to deduce a closed form for f_n. Very cool!

Problem 3. Repeat the derivation done in class of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6}$.

Problem 4.

(a) Decompose $3z^2/(z^3 - n^3)$ into partial fractions.

Hint: $3z^2/(z^3 - n^3)$ is of the form f'/f so it has a very simple partial fraction decomposition.

(b) Using this, find a closed form for

$$\sum_{n=-\infty}^{\infty} \frac{3z^2}{z^3 - n^3},$$

and hence a closed form for

$$\sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}.$$

Test your closed form on $z = 2^{1/3}$ with a computer and verify that you get

$$\sum_{n=-\infty}^{\infty} \frac{1}{2 - n^3} \approx 1.75976.$$

Problem 5. Show that

$$\Gamma\left(\frac{1}{6}\right) = 2^{-\frac{1}{3}} \left(\frac{3}{\pi}\right)^{\frac{1}{2}} \Gamma\left(\frac{1}{3}\right)^2.$$

Problem 6. What are the residues of $\Gamma(z)$ at the poles $z = -n$?

Hint: The recursive equation for $\Gamma(z)$ is all you need. No integral or product formulas needed.

Problem 7. This is a space to reflect on something about this problem set. You can mention if you found any problems particularly difficult, or particularly easy. You can also mention problems you liked, or problems that took a long time, etc. (Please write something here to get credit!)