# MATH4460 SPRING 2023
# EXAM 2 REVIEW

The following problems are about series. Please prioritize these problems since you have not had a problem set about Taylor or Laurent series.

**Problem 1.** Calculate the first 3 nonzero terms of the Taylor series of the following:

(a) $\cos z$ around $z = 0$.

(b) $\cos z$ around $z = \pi$.

(c) $\cos z$ around $z = \pi/2$. Compare this to the Taylor series of $\sin z$ around $z = \pi$.

(d) $\sin z \cos z$ around $z = 0$.

**Problem 2.**

(a) If $f$ is holomorphic, $f(0) \neq 0$, and you know the Taylor series for $f(z)$ around 0, discuss how to find the first few terms of the Taylor series of $1/f(z)$ around 0. (*Hint*: Set up an equation of the form $f(z)g(z) = 1$, where $g(z)$ is the sought-after function, and solve for coefficients for $g(z)$ iteratively.)

(b) For practice, carry this out on $1/e^z$ and verify that you get the Taylor series for $e^{-z}$.

(c) Carry this out on $\sec z = 1/\cos z$.

**Problem 3.** What is the radius of convergence of the Taylor series of $\dfrac{\cos z}{z-5}$ around $z = i$? (*Hint*: Do not try to compute this series. It is very ugly.)

**Problem 4.** Find the Laurent series expansion for $f(z) = \dfrac{1}{z-3}$, around $z = 5$, that is valid in the region $|z - 5| > 2$.

**Problem 5.** Which coefficients in the Laurent series of $\csc z = 1/\sin z$ around $z = 0$ are nonzero? How do you know?

Compare with $1/\cos z$ from 2(c). Why did that one not have any negative exponents?

**Problem 6.** Suppose $f(z)$ is holomorphic on $\mathbb{C}$ except for isolated singularities. Let $a \in \mathbb{C}$ be a singularity of $f$ and suppose $f$ has a Laurent series

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

which converges in a punctured disk $0 < |z-a| < r_2$. Classify the type of singularity at $a$ (removable, pole, or essential) according to the numbers $c_n$. Give reasoning.

**Problem 7** (Ahlfors 5.1.3.1)**.** Write $\dfrac{1}{1+z^2}$ in powers of $z - a$, for $a$ a real number. Find the general coefficient, and for $a = 1$ further reduce this coefficient to simplest form.

*Hint*: Do partial fractions first, then use the geometric series method.

The following problems are review of concepts since the first exam.

**Problem 8.** True or false: The partial fraction decomposition of

$$\frac{2}{z^2(z+1)^2}$$

is of the form

$$\frac{A}{z^2} + \frac{B}{(z+1)^2},$$

where $A$ and $B$ are constants.

**Problem 9.** Classify zeros and singularities of $\frac{e^z-1}{(z-2)^7(z-4)}$ on the extended complex plane. More specifically, for each zero or singularity, give its order and, if it is a singularity, say what type of singularity it is (removable, pole, or essential).

**Problem 10.** What is the order of $z$ at $\infty$? What is the order of $z\,dz$ at $\infty$?

**Problem 11.** Give an example of a function on $\mathbb{C}$ which has an essential singularity at $z = 3$ and is holomorphic everywhere else, including at $\infty$. (Try to what remember what "holomorphic at $\infty$" means as well.)

**Problem 12.** State and prove the "coefficient of $(z-a)^{-1}$" rule for calculating residues at $z = a$.

*Hint*: The proof has to do with the fact that, for example, $(z-a)^{-2}$ has a well-defined single-valued antiderivative in any punctured open neighborhood of $a$.

**Problem 13.** Prove the "simple pole trick" for the calculation of residues. What was the condition to use it?

**Problem 14.**

(a) Let $f(z)$ be a polynomial of degree $n$, and consider an extremely large circle $C$ (radius larger than $10^{100}$ times the absolute values of all the coefficients of $f(z)$). What is $n(f(C), 0)$, the winding number of $f(C)$ around the origin? Explain this from the perspective that $f(z)$ is "basically" $z^n$ for such large $z$, and also from the perspective using the fact that this large circle contains all the roots of $f$.

(b) Let $f(z) = e^z$, and let $\gamma$ be a circle of radius 12 around $z = i$. What is the winding number of the image curve $f(\gamma)$ around the origin? Approach this question from at least 2 different perspectives.

**Problem 15.** Compute the residue of $\dfrac{1}{1+z^3}$ at $z = e^{i\pi/3}$.

**Problem 16.** Evaluate the integral

$$\int_\gamma \frac{z^5 - 10z^2 + 5}{(z-i)^4}\,dz$$

where $\gamma$ is any simple **clockwise** closed loop around $i$.

**Problem 17.** Evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 3\cos\theta}\,d\theta.$$

Next page has remarks. Stop reading if you don't want to be spoiled!

## 1. Remarks (to some problems)

**Remark 1.**

**Remark 2.**

**Remark 3.** The only pole of the function is at $z = 5$. So just find the largest open disk around $i$ that doesn't hit a pole. Use the Pythagorean theorem.

**Remark 4.** If you write an infinite sum of positive powers of $(z-5)$, this will converge in the disk $|z-5| < 2$. You need to write it as an infinite sum of negative powers of $(z-5)$ instead. Let $w = 1/(z-5)$ and try to massage $1/(z-3)$ into a geometric series involving positive powers of $w$.

**Remark 5.** The key difference between $1/\sin z$ and $1/\cos z$ is that $\cos 0 = 1$, meaning that $1/\cos z$ does not have a pole at $z = 0$.

**Remark 6.** Count the number of $c_n \neq 0, n < 0$. If infinite, essential. If finite, pole. If zero, removable.

**Remark 7.**

**Remark 8.** False. There should be $C/z$ and $D/(z+1)$ terms too.

**Remark 9.** c.f. exam 1 problem 2(b). The difference is this is not a rational function due to the presence of $e^z$.

**Remark 10.** $-1$ and $-3$, respectively. (Remember how I said that $dz$ adds $-2$ to the order of any function at $\infty$!)

Advanced explanation ahead (cover your eyes): The element $z\,dz$ is no longer a function but a section of a line bundle on $\mathbb{P}^1$ (the (complex) projective line, what we called $\widetilde{\mathbb{C}}$ in this class), called the cotangent bundle, aka the bundle of differential forms. A line bundle is essentially a generalization of a function space, and a section of a line bundle is a generalization of a function. The cotangent bundle on $\mathbb{P}^1$ is also denoted by $\mathcal{O}(-2)$, which indicates the fact that there is a "twisting" of $-2$ in this line bundle. This explains the shifting of order at $\infty$ by $-2$. In fact, the element $dz$ itself is holomorphic everywhere except at $\infty$, and has a pole of order 2 at $\infty$.

Cotangent bundles are important because it turns out that the "true" type of things you integrate on any space are not functions, but differential forms.

**Remark 11.** Start with $e^{1/z}$ and do some linear shift of variable.

**Remark 12.**

**Remark 13.**

**Remark 14.** Keyword: argument principle.

**Remark 15.**

**Remark 16.** I got $20\pi i$ using the residue theorem. The residue of the integrand at $i$ is $-10$, we multiply by $2\pi i$ and then by $-1$ since the orientation is opposite of the default counterclockwise orientation.

**Remark 17.**