# MATH4460 SPRING 2023
# EXAM 1

WEDNESDAY, MARCH 1, 2023

Name: ________________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

**Problem 1.**

(a) (3 points) Sketch, in the complex plane, the set of complex numbers $z$ such that $1 < \operatorname{Arg}(z) < 2$. You may want to recall that $\pi \approx 3.14$.

(b) (3 points) Let $f(z) = \frac{z^5-1}{z^2-5z+6}$. List all pairs $(a, \operatorname{ord}_a f)$ for $a$ in $\widetilde{\mathbb{C}}$ (the extended complex plane, aka the Riemann sphere) such that $\operatorname{ord}_a f \neq 0$. What is $\sum_{a\in\widetilde{\mathbb{C}}} \operatorname{ord}_a f$?

(c) (4 points) Let's call a complex number $z$ a *fizzbuzz* if $z$ is a 15th root of unity which is neither a 3rd root of unity nor a 5th root of unity. List all the fizzbuzzes, in polar form.

**Problem 2.**

(a) (5 points) Prove that $\overline{z}^2 = \overline{z^2}$ for all $z \in \mathbb{C}$, in any way you like.

(b) (5 points) Prove or disprove that $\overline{z}^2$ is a holomorphic function on $\mathbb{C}$.

**Problem 3** (10 points)**.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Prove that the function $f(z) = z^3$ maps $D$ to $D$, and that $f$ is a surjection.[1]

[1] A function $f$ from $X$ to $Y$ is a surjection if for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

**Problem 4.**

(a) (5 points) Prove that $\cos z \neq 0$ for any $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$.

(b) (5 points) What is $\log(\frac{1}{2}\exp(\frac{1}{2}\log(\frac{1}{2}\exp(7+11i))))$?

**Problem 5.** Let $f(z) = \dfrac{1}{(z-1)(z-5)^3}$.

(a) (5 points) There is a unique complex number $a$ such that we can write $f(z)$ as

$$f(z) = \frac{a}{z-1} + g(z)$$

where $g(z)$ is a rational function that does not have a pole at $z = 1$. What is $a$?

(b) (5 points) Let $\gamma$ be the circle $|z-1| = \frac{1}{2}$, traversed counterclockwise. With the help of part (a), find

$$\int_\gamma f(z)\, dz.$$

If you could not find the value of $a$ in part (a), express your answer in terms of $a$.