MATH4460 SPRING 2023
FINAL EXAM

THURSDAY, MAY 11, 2023

Name: ________________________________

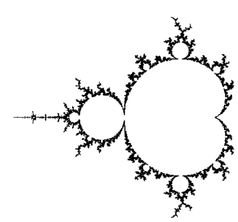
This exam is open notes, but calculators are not allowed. There are 100 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1.

(a) (5 points) Decompose $\dfrac{3z^2}{z^3+1}$ into partial fractions.

(b) (5 points) Let $f(z) = z^2 + i$. Let $a_0 = 0$ and $a_n = f(a_{n-1})$ for $n \geq 1$. Prove that the sequence $(a_n)_{n\geq 0}$ is bounded.

Bonus (1 point). What does part (b) say about the Mandelbrot set?



Problem 2.

(a) (5 points) Prove or give a counterexample: If $f(z)$ and $g(z)$ have a pole of order 2 at $z = a$, then $f(z) + g(z)$ also has a pole of order 2 at $z = a$.

(b) (5 points) Prove or disprove: There exists a sequence $(z_n)_{n\geq 0}$ of complex numbers such that both of the following hold:

- $e^{1/z_n} = -2023$ for all n.
- $\lim_{n\to\infty} z_n = 0$.

Problem 3. Let γ be the square with corners $\pm 2023 \pm 2023i$, oriented counterclockwise.

(a) (3 points) Find
$$\int_\gamma \frac{1}{z^2 + 2024^2}\, dz.$$

(b) (3 points) Find
$$\int_\gamma \frac{1}{(z^2 + 2024^2)(z - 2022)^2}\, dz.$$

(c) (4 points) Find
$$\int_\gamma \frac{e^z}{e^z - 2}\, dz.$$

You may use the fact that $\frac{2023}{2\pi} \approx 321.97$.

Problem 4.

(a) (5 points) Sketch, in the complex plane, the set $\{e^z : z \in \mathbb{C} \text{ and } \operatorname{Re} z = 2\}$.

(b) (5 points) Let

$$f(z) = \frac{e^{\frac{1}{z+8}}}{(z-1)(z^2-1)(z^3-1)(z^4-1)}.$$

Classify the singularities of f in the extended complex plane $\widetilde{\mathbb{C}}$, and give the order of each non-essential singularity.

Problem 5.

(a) (4 points) Solve the equation $z^4 + z^2 + 1 = 0$. Express the solutions as powers of $\omega := e^{\pi i/3}$.

(b) (6 points) Calculate

$$\int_{-\infty}^{\infty} \frac{1}{x^4 + x^2 + 1} \, dx.$$

Problem 6.

(a) (4 points) What is the Laurent series of $e^z/(z-2)^4$ around $z=2$?

Hint: $e^z = e^2 \cdot e^{z-2}$.

(b) (1 point) Based on part (a), what is the residue of $e^z/(z-2)^4$ at $z=2$?

(c) (5 points) What is the disk of convergence of the power series of $f(z) = 1/(z^4+16)$ around $z=-1$?

Protip: Don't compute the series. The first few terms are $\frac{1}{17} + \frac{4}{289}(z+1) - \frac{86}{4913}(z+1)^2 + \cdots$ but this will not help.

Problem 7 (10 points)**.** Liouville's theorem states that a bounded entire function is constant. Prove the stronger statement that a bounded holomorphic function on $\mathbb{C} - \{a_1, a_2, \ldots, a_n\}$, where $\{a_1, a_2, \ldots, a_n\}$ is a finite set of complex numbers, must be constant.

Problem 8 (10 points)**.** Let γ be the curve

$$z(t) = e^{10(\cos t + i \sin t)}, 0 \leq t \leq 2\pi.$$

Prove, non-graphically, that the winding number of γ around the origin is 0.

Problem 9.

(a) (5 points) Prove that for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s).$$

(b) (5 points) Find $\zeta(-3)$. Note that part (a) does not help here, as the series given there does not converge for $s = -3$.

Problem 10. The infinite product expression for $\Gamma(z)$ is below:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(\left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}} \right).$$

The *digamma function* $\psi(z)$ is defined as the logarithmic derivative of $\Gamma(z)$, that is,

$$\psi(z) \coloneqq \frac{\Gamma'(z)}{\Gamma(z)}.$$

(a) (5 points) Derive the infinite partial fraction expansion of $\psi(z)$.

(b) (5 points) What are the poles of $\psi(z)$, as well as the singular part and residue at each pole, of $\psi(z)$? (Read this question carefully to make sure you get everything.)