MATH4460 SPRING 2023
EXAM 2

WEDNESDAY, APRIL 19, 2023

Name: ______________________________

This exam is open notes, but calculators are not allowed. There are 50 points total in this exam. If you do not manage to solve a problem, show a strategy you tried and a reflection on why it did not work, for partial credit.

Problem 1. For each of the following meromorphic functions, list **all singularities on the extended complex plane** $\widetilde{\mathbb{C}}$, and for each one state whether it is a removable singularity, a pole, or essential singularity. In addition, for each pole, state what order the pole is.

(a) (3 points) $\dfrac{1}{z}$

(b) (3 points) $\sec^3 z$

(c) (4 points) $\dfrac{z}{e^z - 1}$

Problem 2. Let $f(z) = \dfrac{1}{(z-1)(z-5)}$. Any series below should be given as a summation or with enough terms that the pattern is easily visible.

(a) (3 points) Give the Laurent series expansion of $f(z)$ around $z = 1$ which converges at $z = 1.001$, and state its region of convergence.

(b) (3 points) Give the Laurent series expansion of $f(z)$ around $z = 1$ which converges at $z = 100$, and state its region of convergence.

(c) (4 points) Find $\mathrm{Res}_\infty\, f(z)\, dz$ in at least 2 different ways.

Problem 3. Integrals. *Hint*: Neither problem should involve heavy computations if done right.

(a) (5 points) Compute $\displaystyle\int_{|z-1|=2} \frac{e^{2z}}{(z-1)(z-5)^n}\,dz$ in terms of the positive integer n.

(b) (5 points) Compute $\displaystyle\int_\gamma \cot z\,dz$ where γ is the rectangle with corners $\pm\frac{99\pi}{4} \pm 100i$, traversed counterclockwise.

Problem 4 (10 points)**.** Compute

$$\int_{-\infty}^{\infty} \frac{a}{x^2+b}\,dx$$

in terms of the real numbers $a, b > 0$.

Problem 5. Let $a \in \mathbb{C}$, let f be holomorphic at a, and let g be meromorphic with an isolated pole at a.

(a) (5 points) If the pole of g at a is simple, prove that

$$\mathrm{Res}_a(f(z)g(z)\,dz) = f(a) \cdot \mathrm{Res}_a\, g(z)\,dz,$$

i.e. that the residue of $f(z)g(z)$ at $z = a$ is equal to $f(a)$ times the residue of $g(z)$ at $z = a$.

(b) (5 points) Give, with proof, a counterexample to the equation in part (a) if we remove the hypothesis that g has a simple pole at a.