MATH4460 SPRING 2023
PROBLEM SET 4

This problem set is due on Wednesday, February 22 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Prove that $\frac{d}{dz}\log z = \frac{1}{z}$ on $\mathbb{C} - \{0\}$ using *only* the fact that log and exp are inverses and the differential equation $\frac{d}{dz}\exp(z) = \exp(z)$.

Problem 2. Compute $\int_\gamma \mathrm{Re}(z)\,dz$ where γ is the directed line segment from 0 to $1+i$.

Problem 3. Compute $\int_{|z|=r} \mathrm{Re}(z)\,dz$ where the line integral takes place counterclockwise around the circle (with start and end both at 1) in two ways:

(a) By use of a parametrization of the circle.

(b) Observing that $\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

Problem 4. Suppose that $f(z)$ is holomorphic on a closed curve γ. Show that

$$\int_\gamma \overline{f(z)} f'(z)\,dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)

Problem 5.

(a) Compute $\int_\gamma \frac{1}{z}\,dz$ where γ is the counterclockwise semicircle centered at the origin from 1 to -1.

(b) Compute $\int_\gamma \frac{1}{z}\,dz$ where γ is the curve that goes around the unit circle for one and a half full turns (which is another example of a curve that goes from 1 to -1).

(c) What is the value of the difference between your two answers? What is the connection to the logarithm function?

Problem 6. Prove that $\int_\gamma e^z\,dz = 0$ for any closed curve γ in $\mathbb{C}$.

Problem 7. Assume that $f(z)$ is holomorphic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for every closed curve γ in Ω. (The continuity of $f'(z)$ is taken for granted.)

Problem 8. This is a space to reflect on something about this problem set. You can mention if you found any problems particularly difficult, or particularly easy. You can also mention problems you liked, or problems that took a long time, etc. (Please write something here to get credit!)