MATH4460 SPRING 2023
PRACTICE MIDTERM 1

Most of these problems are from a past MATH4460 exam.

Problem 1. For each of the following statements, determine whether it is true or false. No explanations need to be given.

(a) If f is holomorphic at a point z_0, then f is differentiable at z_0.

(b) The imaginary part of $\sqrt{17}\left(\cos\left(\frac{65\pi}{38}\right) + i\sin\left(\frac{65\pi}{38}\right)\right)^{19}$ is negative.

(c) The function $f(z) = \overline{z}^2$ is continuous on $\mathbb{C}$.

(d) If $\operatorname{Im}(z) > 0$, then $\operatorname{Im}(1/z) < 0$.

(e) If $\pi/4 < \operatorname{Arg}(z) < \pi/2$, then $3\pi/4 < \operatorname{Arg}(z^3) < 3\pi/2$.

Problem 2. Given the function $U(x, y) = xy - 8x$ from $\mathbb{R}^2$ to $\mathbb{R}$, find a holomorphic function $f(z)$ whose real part is equal to U, or prove no such function exists.

Problem 3. Let $A = \{z \in \mathbb{C} : |z - 2 + i| > 3\}$.

(a) Graph A in the complex plane.

(b) Is A open?

(c) Determine the boundary of A.

Problem 4. Let z, w be complex numbers with $|w| < 1$ and $|z| < 1$.

(a) Do you expect $(1 - w\overline{w})(1 - z\overline{z})$ to be positive or negative, or not enough information?

(b) Use the result from part (a) to show that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1.$$

Problem 5. Find all solutions of the complex equation $2z^3 + i = 0$. Leave your answers in exponential form.

Problem 6. Find the image of the line $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ under the mapping $f(z) = 2/z$. Draw the image in the complex plane as well.

Problem 7. Let $f(z)$ be an analytic function defined on an open set $D \subseteq \mathbb{C}$. Suppose that $e^{f(z)}$ is a real-valued function (i.e. no imaginary part). Prove that $f(z)$ is a constant.

1. Remarks

Remark 1.

(a) True

(b) False. The number can be rewritten as $\sqrt{17}(e^{65\pi i/38})^{19} = \sqrt{17}e^{\pi i/2}$ which points up (it's $\sqrt{17}i$).

(c) True. $z \mapsto \overline{z}$ is continuous since it's $(x, y) \mapsto (x, -y)$ when interpreted as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, and squaring is continuous, and compositions of continuous functions are continuous.

(d) True. Direct computation.

(e) Would be true if Arg took values in $[0, 2\pi)$. Since Arg takes values in $(\pi, \pi]$ in thie class, it's false.

Remark 2. If $V = \operatorname{Im} f$ then CR equations imply $V_x = -x$ and $V_y = y - 8$. Your favorite multivariable method to find a potential function with gradient $(-x, y - 8)$ gives $V(x) = -\frac{1}{2}x^2 + \frac{1}{2}y^2 - 8y + C$, and any choice of C works. Let's take $C = 0$ and write $U + iV$ as a function of z. Here's how I do this last step. We know $z = x + yi$ and $z^2 = (x^2 - y^2) + 2xyi$. Based on this, we can see that $8x$ is the real part of $8z$ and xy is the real part of $-\frac{1}{2}iz^2$. So then $-\frac{1}{2}iz^2 + 8z$ is our desired function.

Remark 3. We didn't rigorously define open sets in this class so there is really nothing to do in this problem. The picture of A in the complex plane is the complement of a closed disk of radius 3 centered at $2 - i$. The boundary is the circle of radius 3 centered at $2 - i$.

Remark 4. $w\overline{w} = |w|$ and similarly with z, and so $0 < 1 - w\overline{w} < 1$ and $0 < 1 - z\overline{z} < 1$ meaning the answer to part (a) is positive.

Part (b) is some algebraic manipulation.

Remark 5. $2z^3 = -i = e^{3\pi i/2} \implies z^3 = \frac{1}{2}e^{3\pi i/2} \implies z = \frac{1}{\sqrt[3]{2}}e^{\pi i/2 + (2\pi i/3)k}$, $k \in \mathbb{Z}$.

Remark 6. Expand out $2/(x + i)$, look at real and imaginary parts, and trace out the curve (where x is the time parameter) in the plane.

Remark 7. Work it out with CR equations. Before diving in you might notice that $f(z)$ must be real-valued up to integer multiples of $2\pi i$, but since f is continuous and the set of integer multiples of $2\pi i$ is discrete, the imaginary part of f must be a constant multiple of $2\pi i$. Then it's a lot easier to show that the real part of f must be constant as well.