MATH4460 SPRING 2023
PROBLEM SET 6

This problem set is due on Wednesday, March 22 at 11:59 pm. Each problem part is worth 3 points. Collaboration is encouraged. In all cases, you must write your own solutions, and and you must cite collaborators and resources used.

Problem 1. Prove that if f is a meromorphic function in the extended complex plane, meaning f has no essential singularities and only a finite number of poles in the extended complex plane, then f is a rational function.

Remark: If topology was a prerequisite for this class, I would also have you show that "a finite number of poles" is automatic and does not need to be listed as a hypothesis. It uses the fact that the extended complex plane is compact.

Problem 2. The following is a theorem proven on page 127 of Ahlfors: If f is holomorphic in a (connected) region $\Omega \subseteq \mathbb{C}$ and, for some $a \in \Omega$, $f^{(n)}(a) = 0$ for all integers $n \geq 0$, then f is identically zero in Ω.

(a) Show that the same is not true if you replace $\mathbb{C}$ with $\mathbb{R}$ and consider infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$. *Hint*: Use $f(x) = e^{-1/x^2}$ (extended to 0 at $x = 0$) and $a = 0$. For fun, plot this function if you are not sure what it looks like.

(b) Why does the same function $f(z) = e^{-1/z^2}$ (extended to 0 at $z = 0$) not give a counterexample to the theorem in the original setting?

Problem 3.

(a) Let γ be the circle of radius 0.999 centered at 1, traversed counterclockwise. What is
$$\int_\gamma \frac{12z^{11}}{z^{12}-1}\,dz?$$

(b) Let γ be the circle of radius 1.001 centered at 1, traversed counterclockwise. What is
$$\int_\gamma \frac{12z^{11}}{z^{12}-1}\,dz?$$

(c) Let γ be the circle of radius 10^{100} centered at 1, traversed counterclockwise. What is
$$\int_\gamma \frac{12z^{11}}{z^{12}-1}\,dz?$$

Problem 4. Prove that if $f(z)$ is a quadratic polynomial, $\Delta \subseteq \mathbb{C}$ is a disk, and f is not injective in Δ, then $f'(z) = 0$ for some $z \in \Delta$.

Hint: Use the fact that f is a quadratic polynomial. Start by supposing that $f(a) = f(b)$ for some $a, b \in \Delta$. What can you say about the polynomial $f(z) - f(a)$?

Problem 5. Let D be the unit disk in $\mathbb{C}$ centered at the origin. Prove that a holomorphic function $f\colon D \to D$ which is bijective and sends 0 to 0 must be of the form $f(z) = cz$ for some $c \in \mathbb{C}$ with $|c| = 1$.

Hint: Use the Schwarz Lemma.

Problem 6. This is a space to reflect on something about this problem set. You can mention if you found any problems particularly difficult, or particularly easy. You can also mention problems you liked, or problems that took a long time, etc. (Please write something here to get credit!)