Review of Fourier Series and Fourier Transform

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Thinking about First Exam

- January 31?

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k t/T_0}\,dt$$

FUNDAMENTAL FREQ: $f_0=1/T_0$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

Integral of exp(j*k*w0*t)

x(t)=1

➤ INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi m t/T_0}\,dt = \frac{T_0}{-j2\pi m} e^{-j2\pi m t/T_0}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t}\,dt = 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

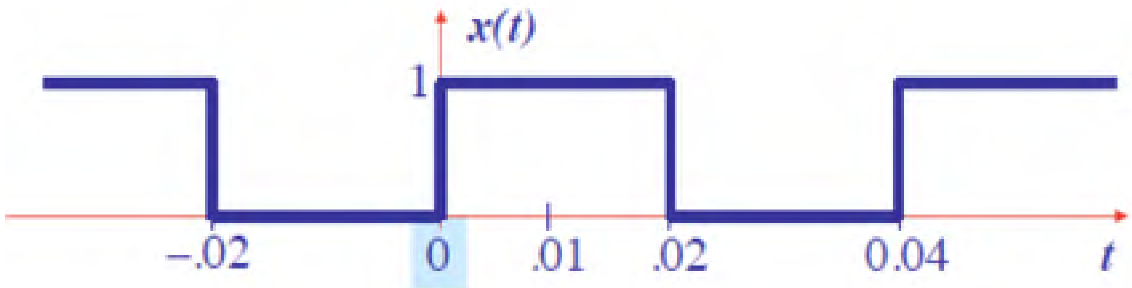
Orthogonality of exp(j*w0*k*t)

- INTEGRATE over ONE PERIOD

$$\boxed{\frac{1}{T_0}\int_0^{T_0} e^{j2\pi l t/T_0} e^{-j2\pi k t/T_0} dt = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(l-k)t/T_0} dt$$

Square Wave

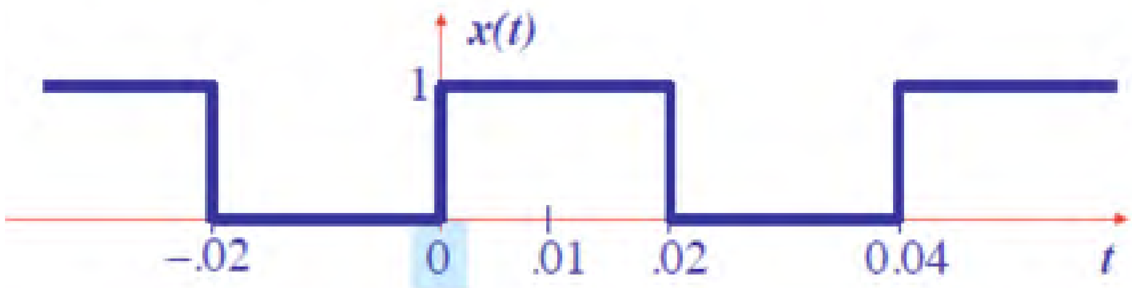


- Period?

Square Wave Example

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04\,\text{sec}$:



FS for a Square Wave

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j2\pi kt/.04}dt = \frac{1}{.04(-j\pi k/.02)}e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \frac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k t/T_0}\,dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{AREA})$$

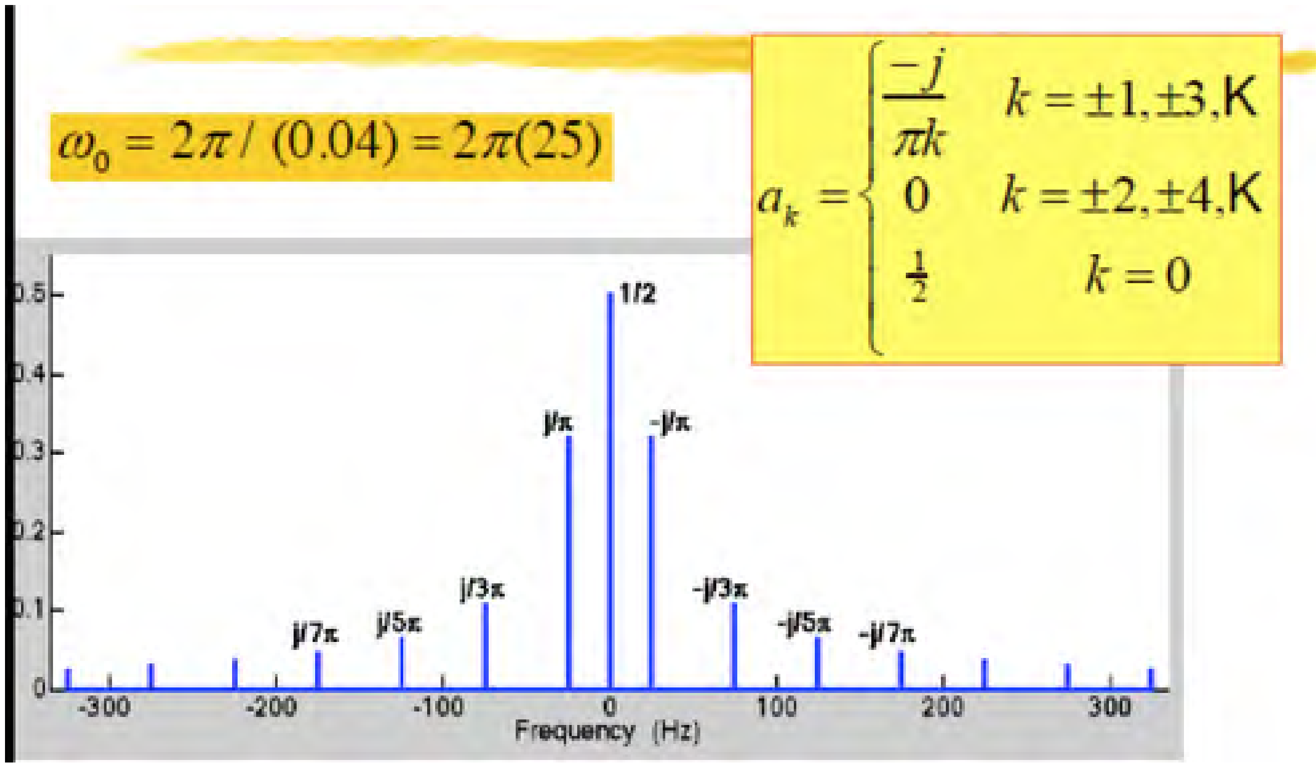
$$a_0 = \tfrac{1}{.04}\int_0^{.02} dt = \tfrac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients, a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

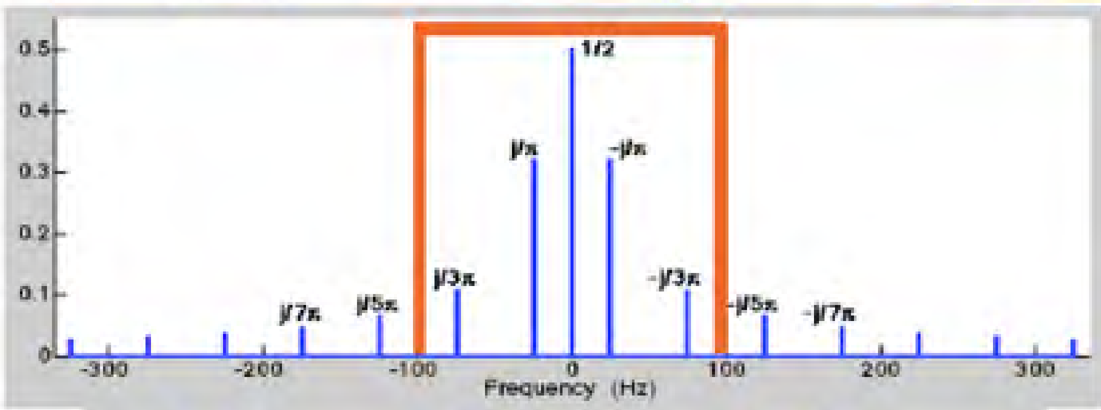
$$a_k = \frac{1 - e^{-j\pi k}}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

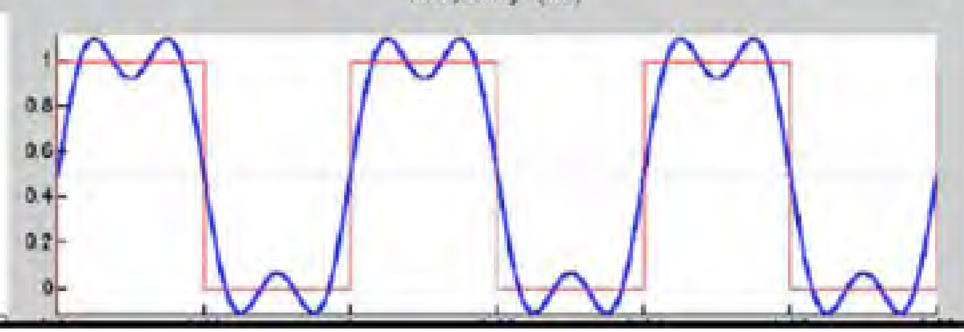
Spectrum from Fourier Series



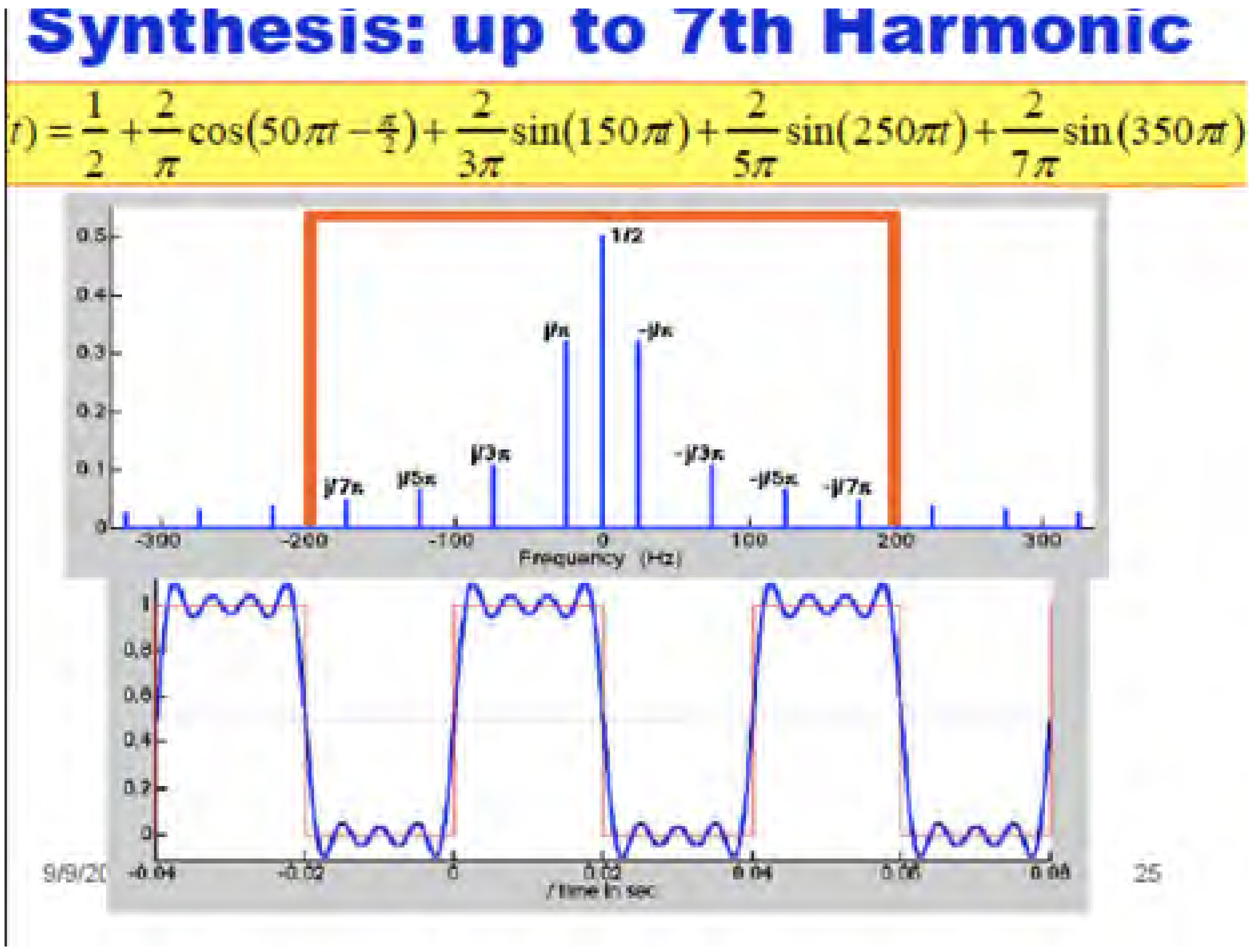
Synthesis: 1st and 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$



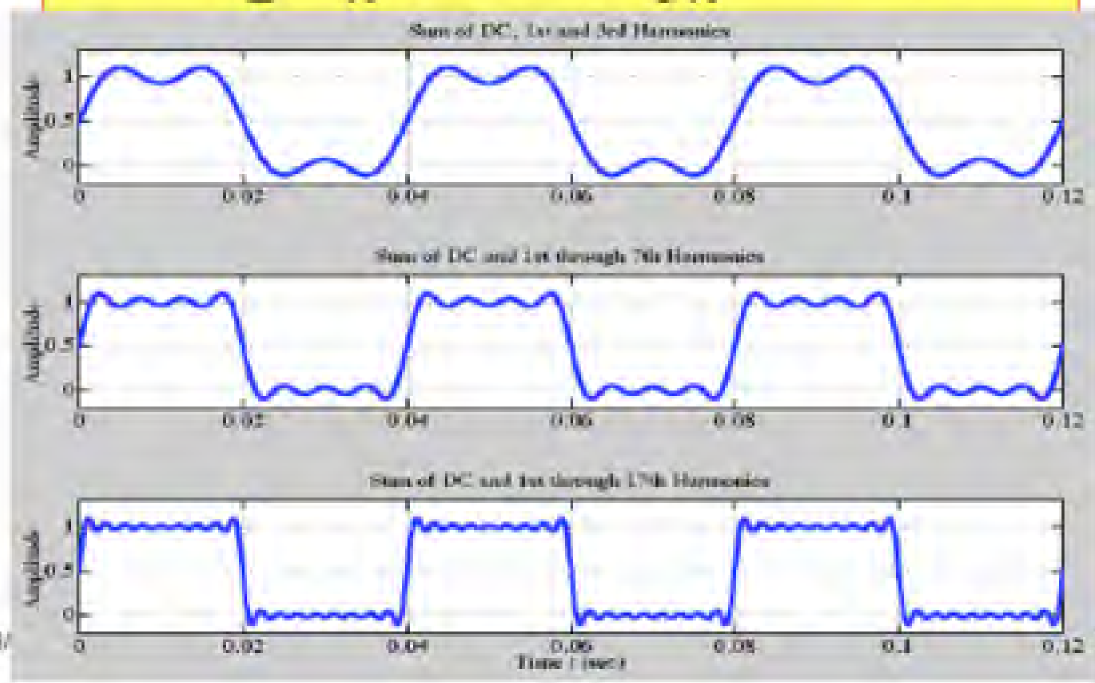


Synthesis: up to 7th Harmonic



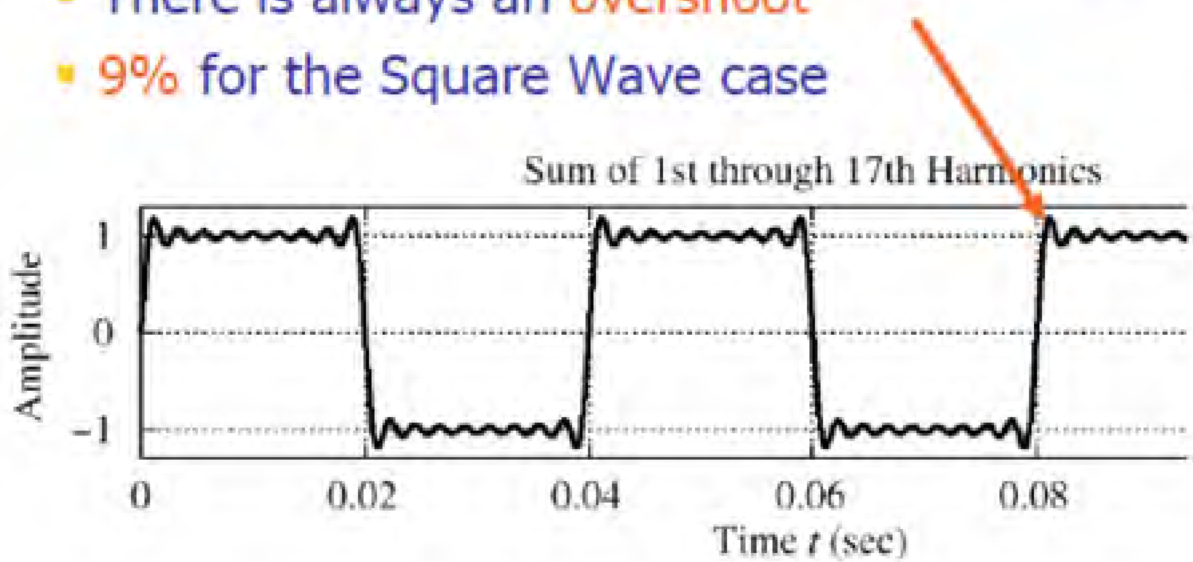
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \mathrm{K}$$

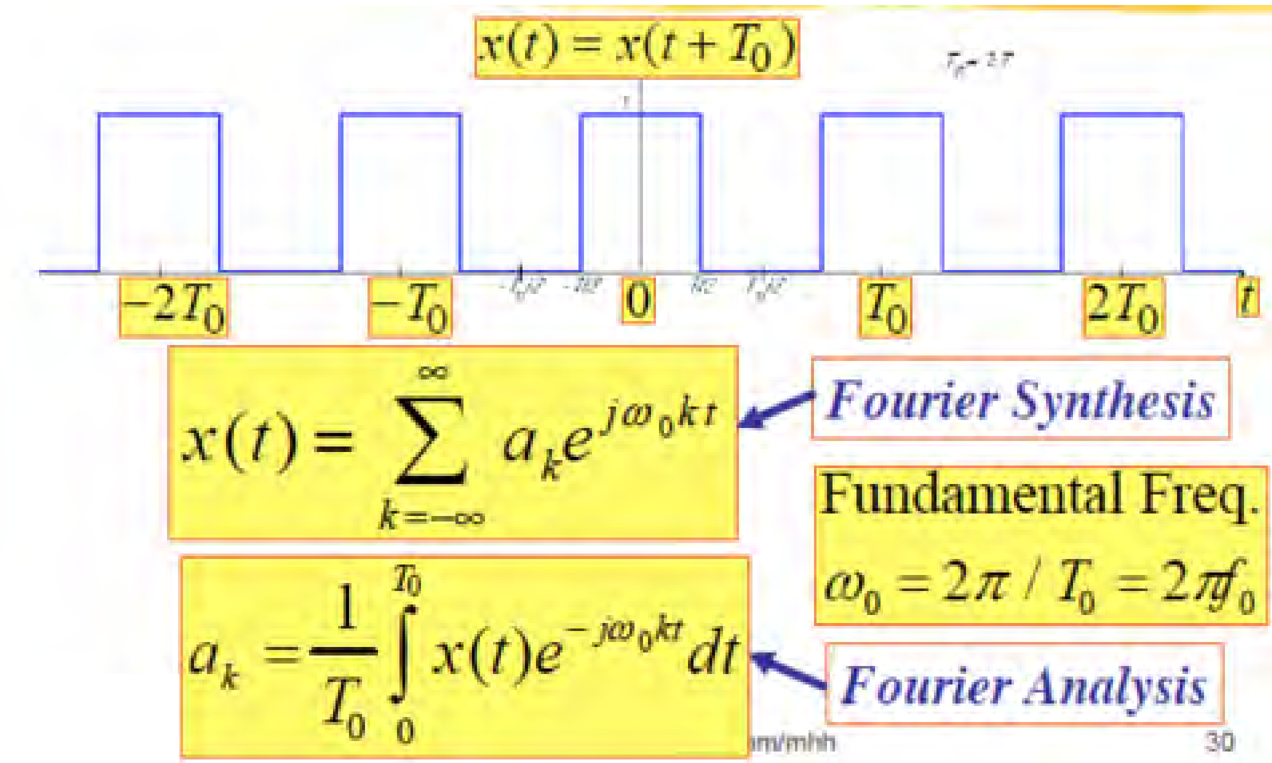


Gibb's Phenomenon

- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case

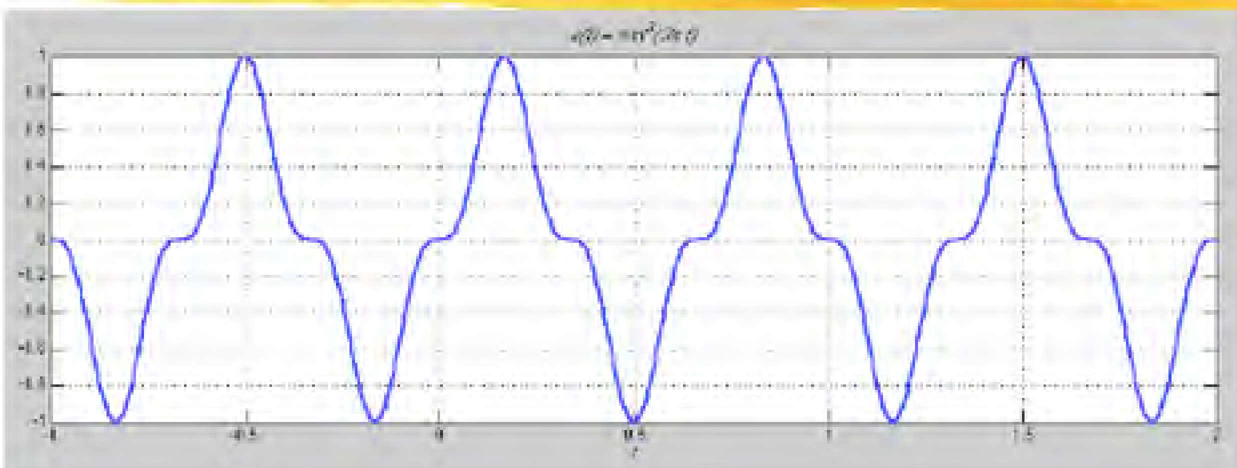


General Periodic Signals



Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

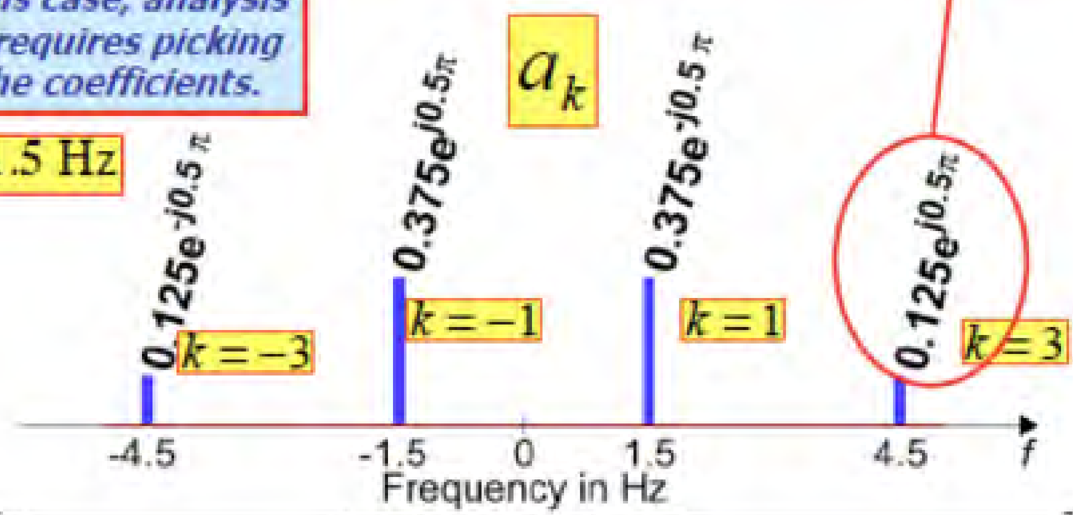
Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{-j}{8}\right)e^{-j9\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{j}{8}\right)e^{j9\pi t}$$

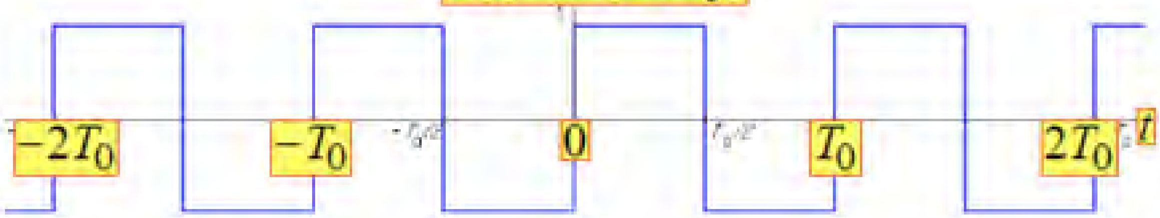
In this case, analysis just requires picking off the coefficients.

$f_0 = 1.5$ Hz



Square Wave Signal

$$x(t) = x(t + T_0)$$



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1)e^{-j\omega_0 kt}\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1)e^{-j\omega_0 kt}\,dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\,\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

Su

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$X_0 = A_0 e^{j0}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Bandlimited Signals

- A bandlimited signal has all its frequencies below a certain limit ω_N.
  - A square wave is *not* a bandlimited signal since its non-zero spectrum components go all the way up to infinity.
  - Bandlimited signals are very smooth.
  - Bandlimited signals can be sampled and then reconstructed exactly. This is the basis for all of modern communications and signal processing.

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

$$\text{Time - domain} \Leftrightarrow \text{Frequency - domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Example

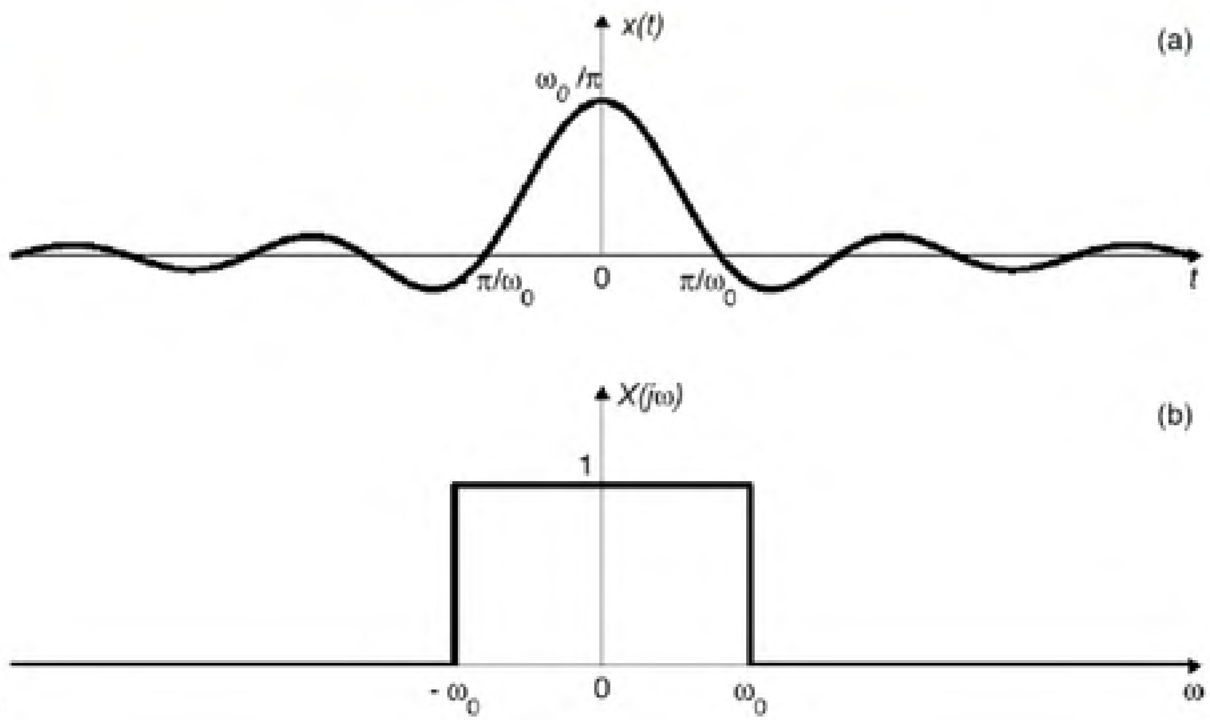
$$X(j\omega)=\begin{cases}1 & |\omega|<\omega_0\\ 0 & |\omega|>\omega_0\end{cases}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\,d\omega=\frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}e^{j\omega t}\,d\omega$$

$$x(t)=\frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_0}^{\omega_0}=\frac{1}{2\pi}\frac{e^{j\omega_0 t}-e^{-j\omega_0 t}}{jt}$$

$$\boxed{x(t)=\frac{\sin(\omega_0 t)}{(\pi t)}}$$

$$x(t)=\frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega)=\begin{cases}1 & |\omega|<\omega_0\\ 0 & |\omega|>\omega_0\end{cases}$$



Example

$$\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0}$$

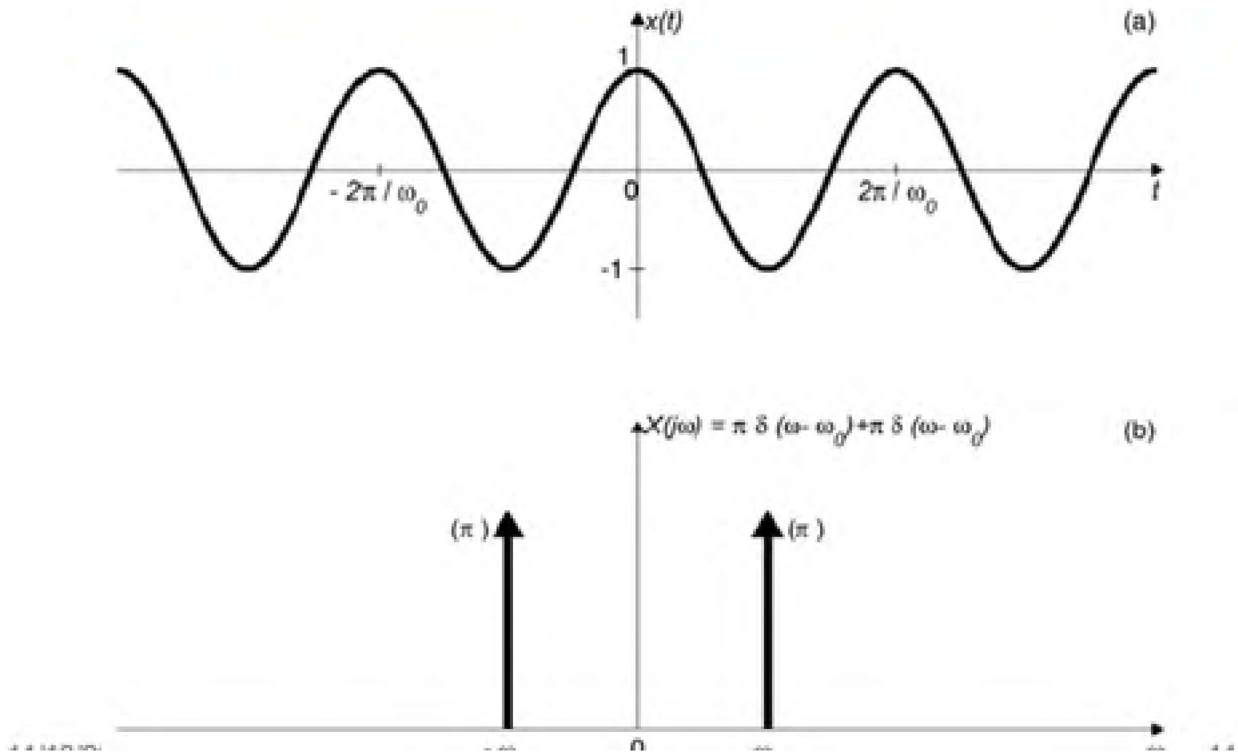
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

$$\boxed{x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$\boxed{x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)}$$

$$\boxed{\begin{aligned} &x(t) = \cos(\omega_0 t) \Leftrightarrow \\ &\qquad X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}}$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform of a Periodic Signal

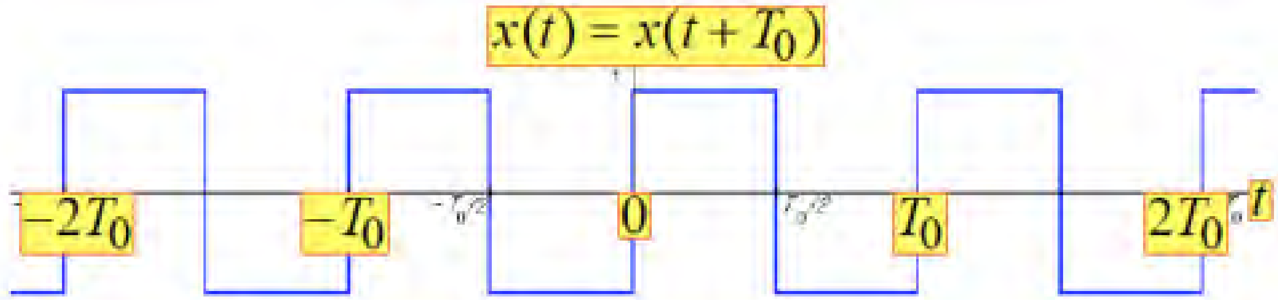
⌘If $x(t)$ is periodic with period T_0,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$\boxed{\text{Therefore, since } e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)}$$

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

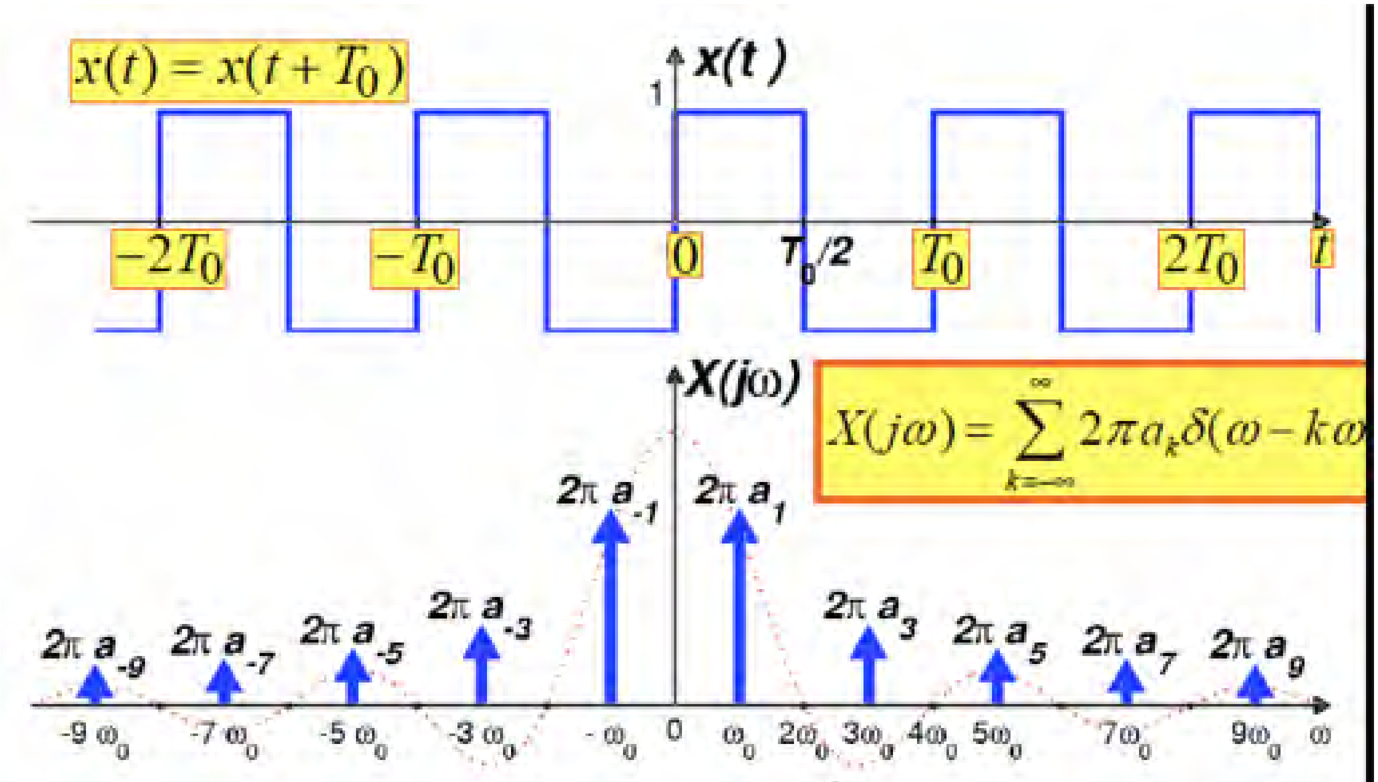
Square Wave Signal



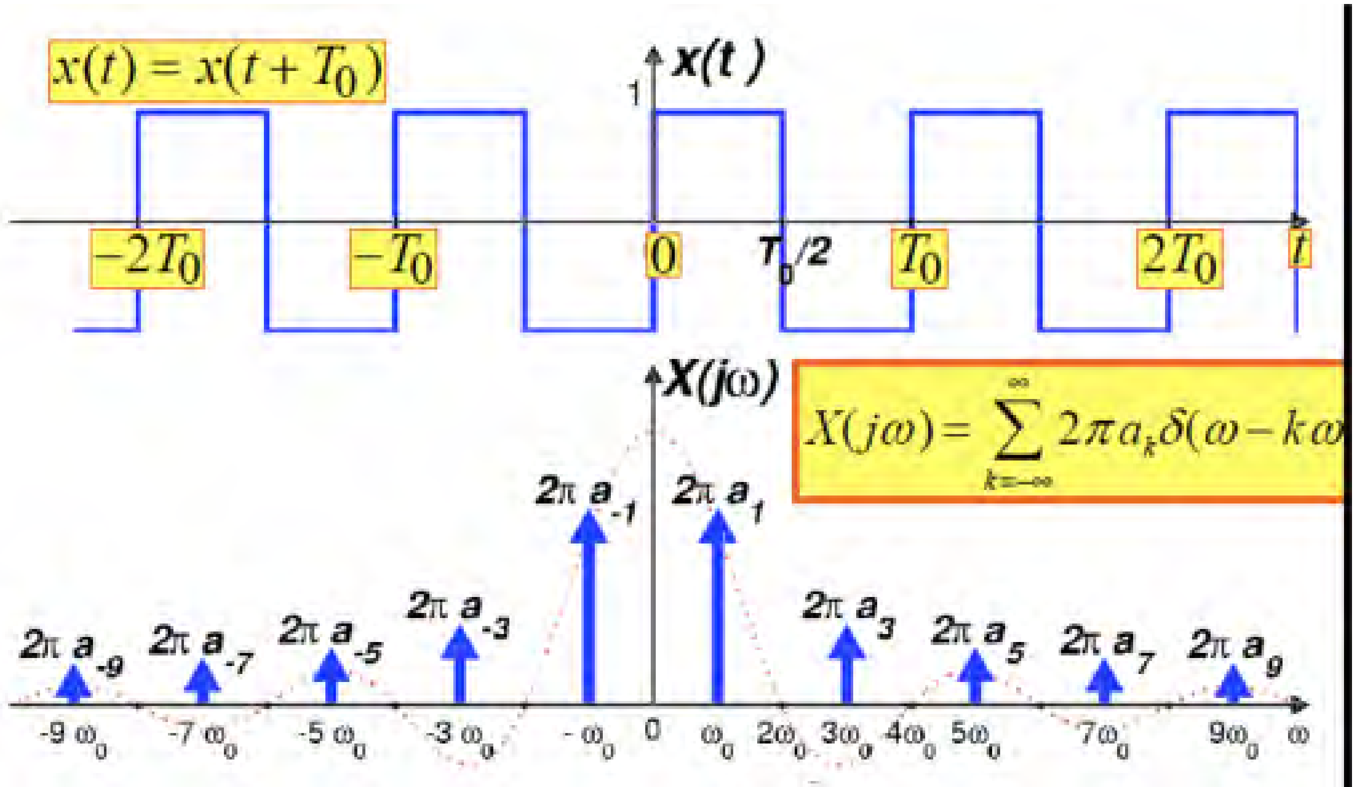
$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform



FT of Impulse Train



FT of impulse train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\boxed{\omega_0 = 2\pi / T_0}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 t} dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\boxed{\therefore \quad P(j\omega) = \left(\frac{2\pi}{T_0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)}$$

Table of Some FT Properties

Linearity Property

$$a x_1(t) + b x_2(t) \Leftrightarrow a X_1(j\omega) + b X_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

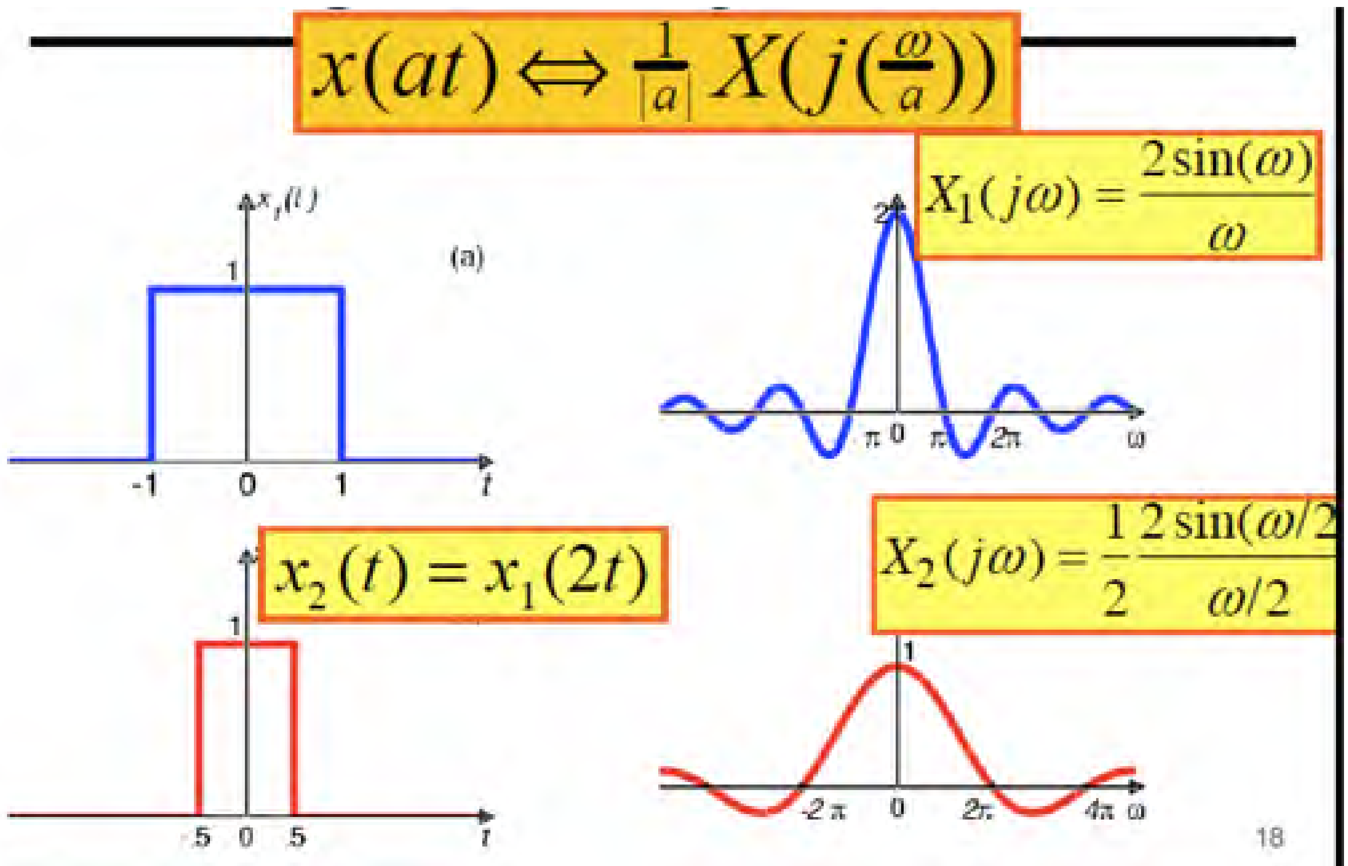
Delay Property

$$\boxed{x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t - t_d) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_d)} d\tau \\
&= e^{-j\omega t_d} X(j\omega)
\end{aligned}$$

For example, $e^{-a(t-5)} u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a + j\omega}$

Scaling Property



Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

More FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

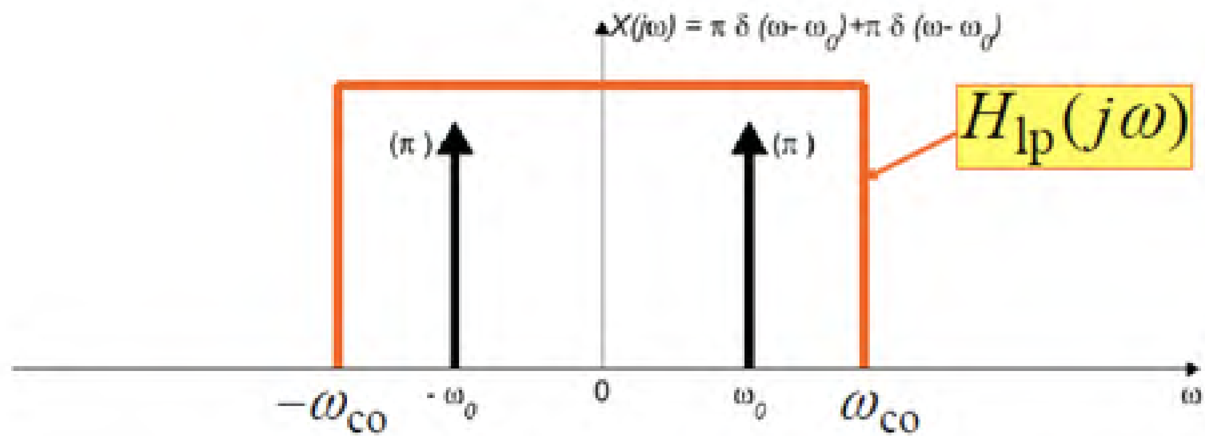
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

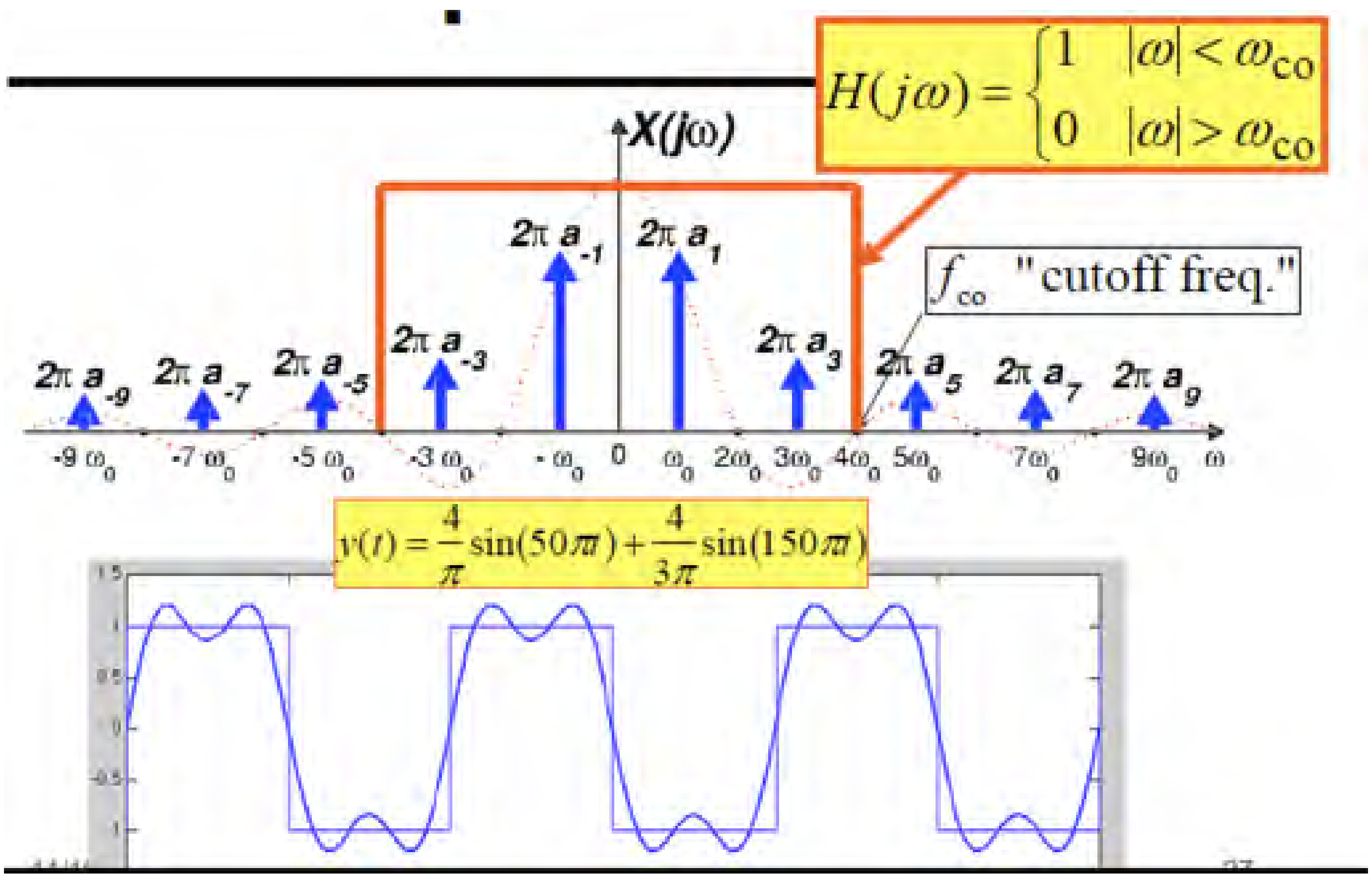
Ideal Low-pass Filter



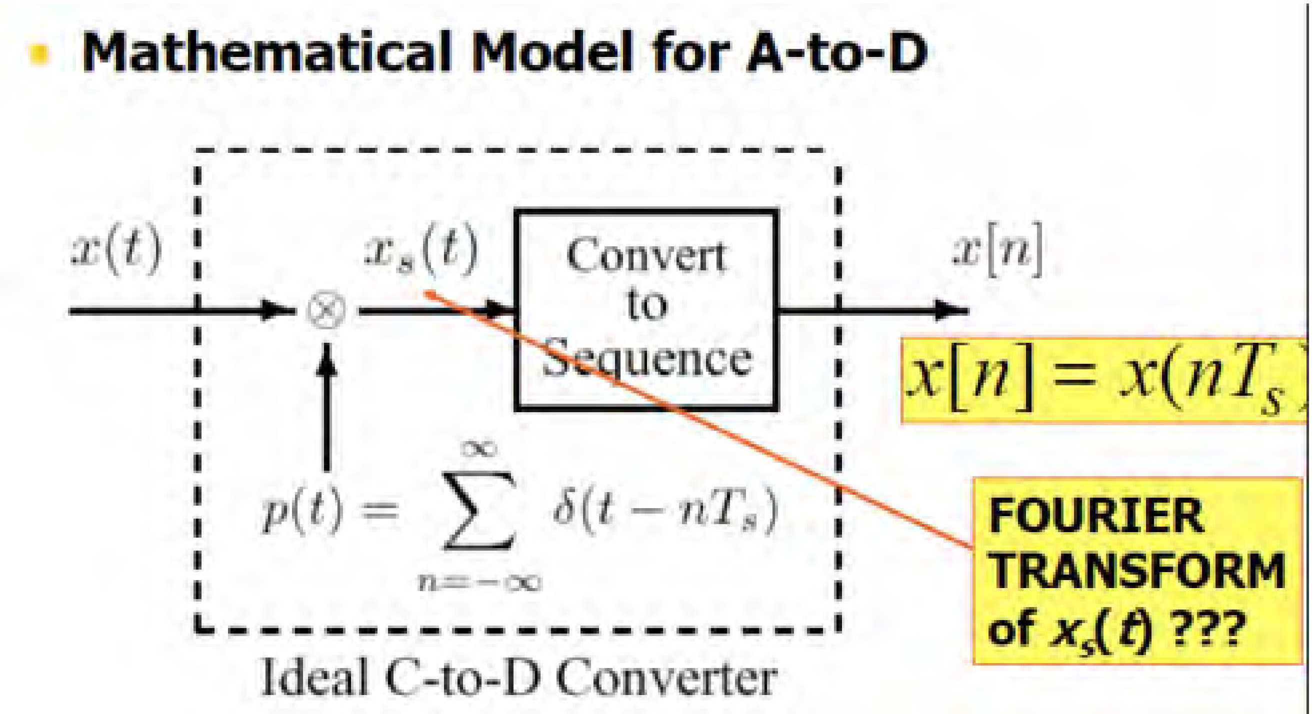
$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\rm co}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\rm co}$$

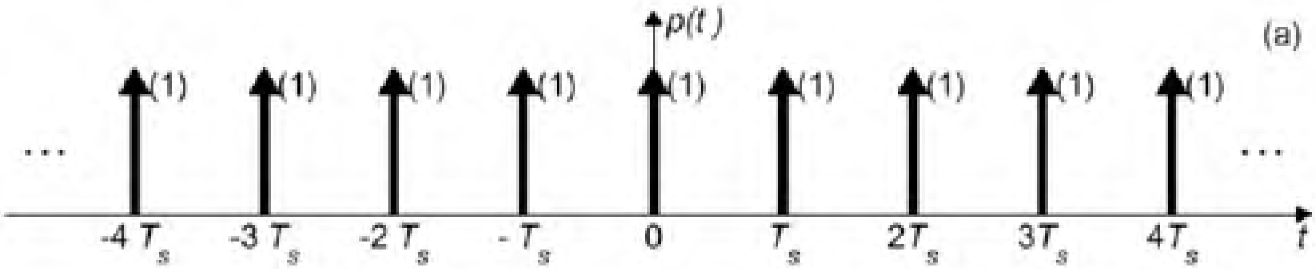
Ideal Lowpass Filter



Review of Sampling, assuming Fourier Transform



Periodic Impulse Train

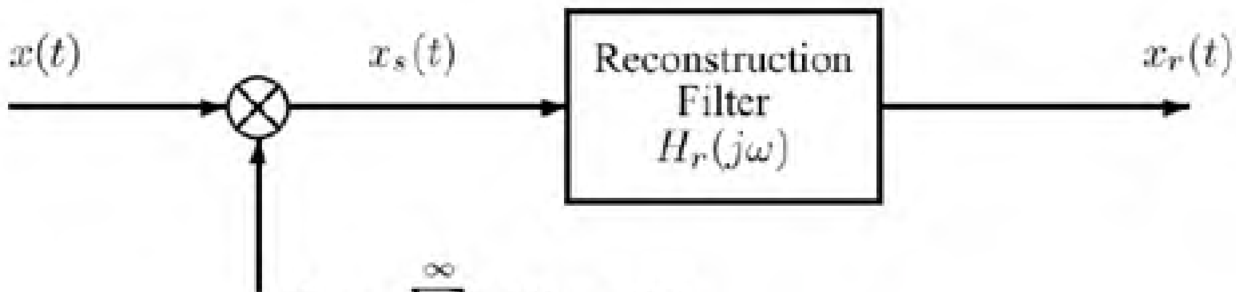


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

$$\boxed{\omega_s = \frac{2\pi}{T_s}}$$

Impulse Train Sampling

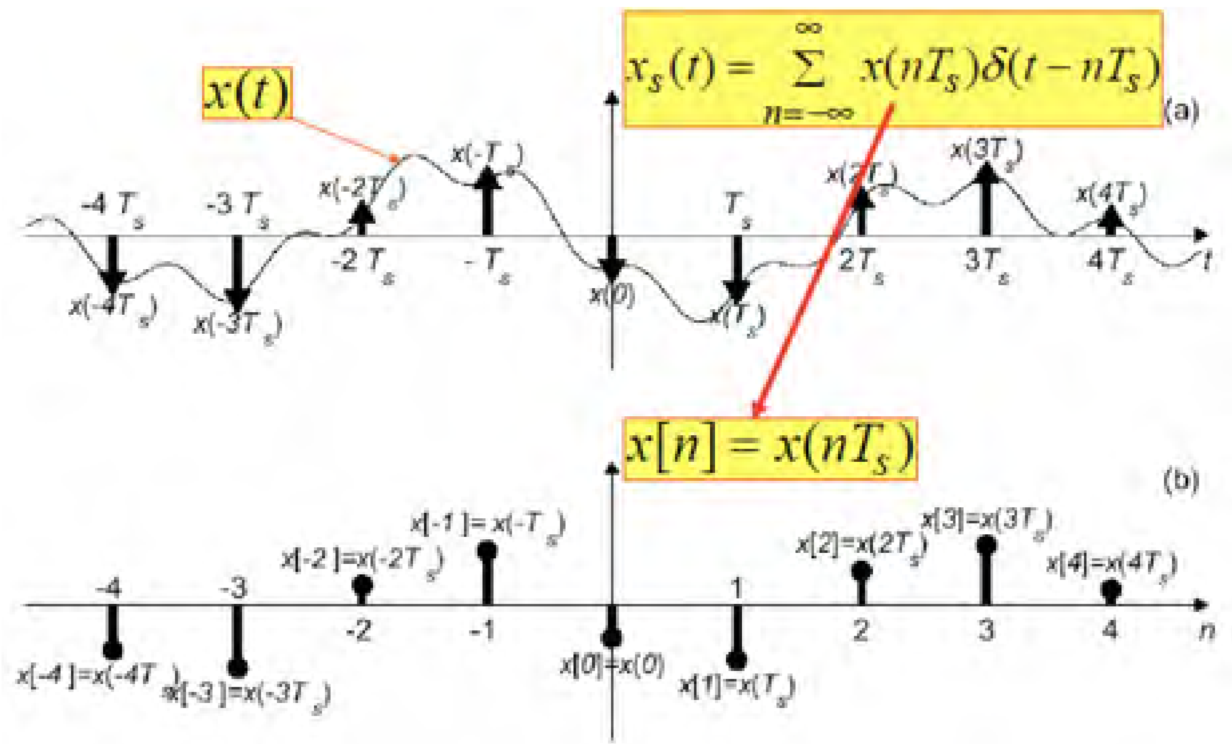


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

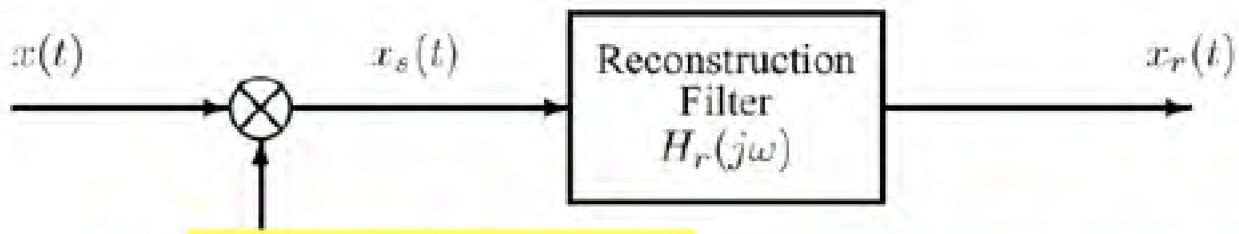
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \underline{x(t)\delta(t - nT_s)}$$

$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}}$$

Illustration of Sampling



Sampling Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT A LOT OF FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

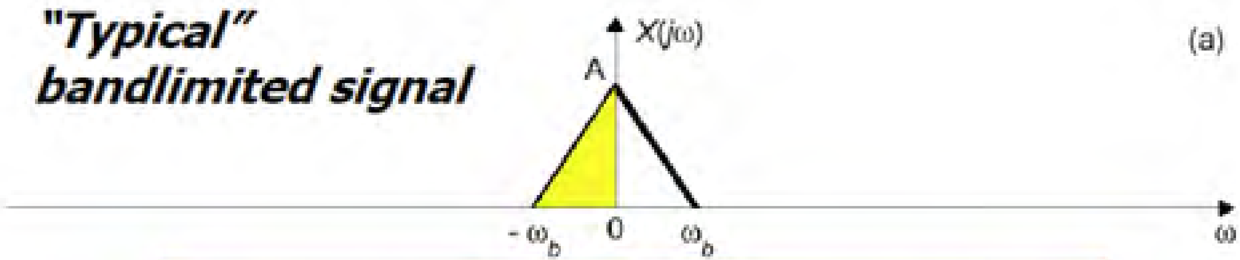
$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\frac{1}{T_s}e^{jk\omega_s t} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \underline{x(t)e^{jk\omega_s t}}$$

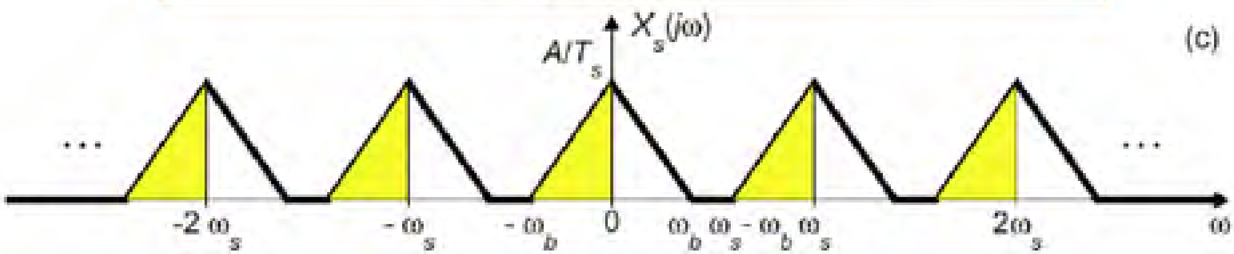
$$\boxed{X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \underline{X(j(\omega - k\omega_s))}}$$

$$\boxed{\omega_s = \frac{2\pi}{T_s}}$$

Frequency Domain Representation of Sampling

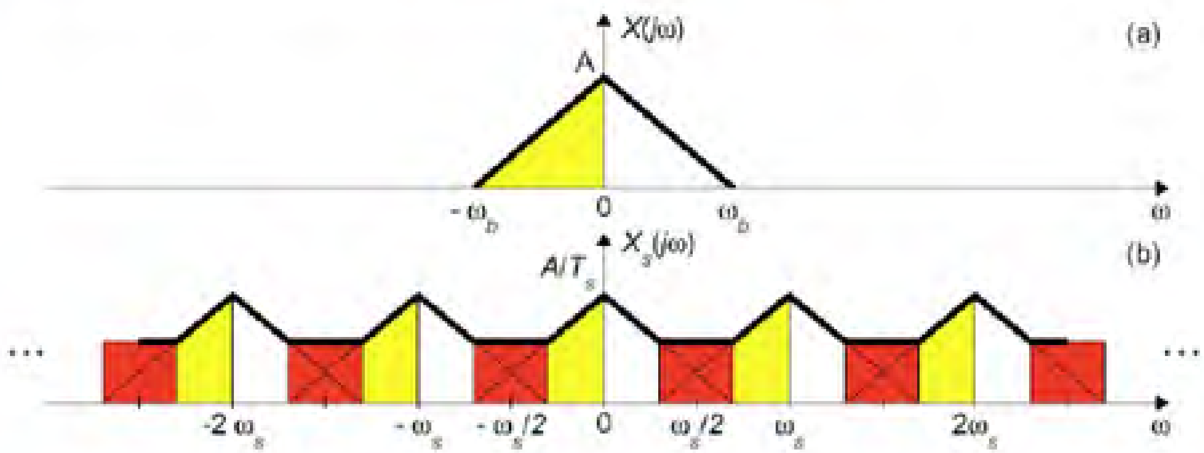


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



Aliasing Distortion

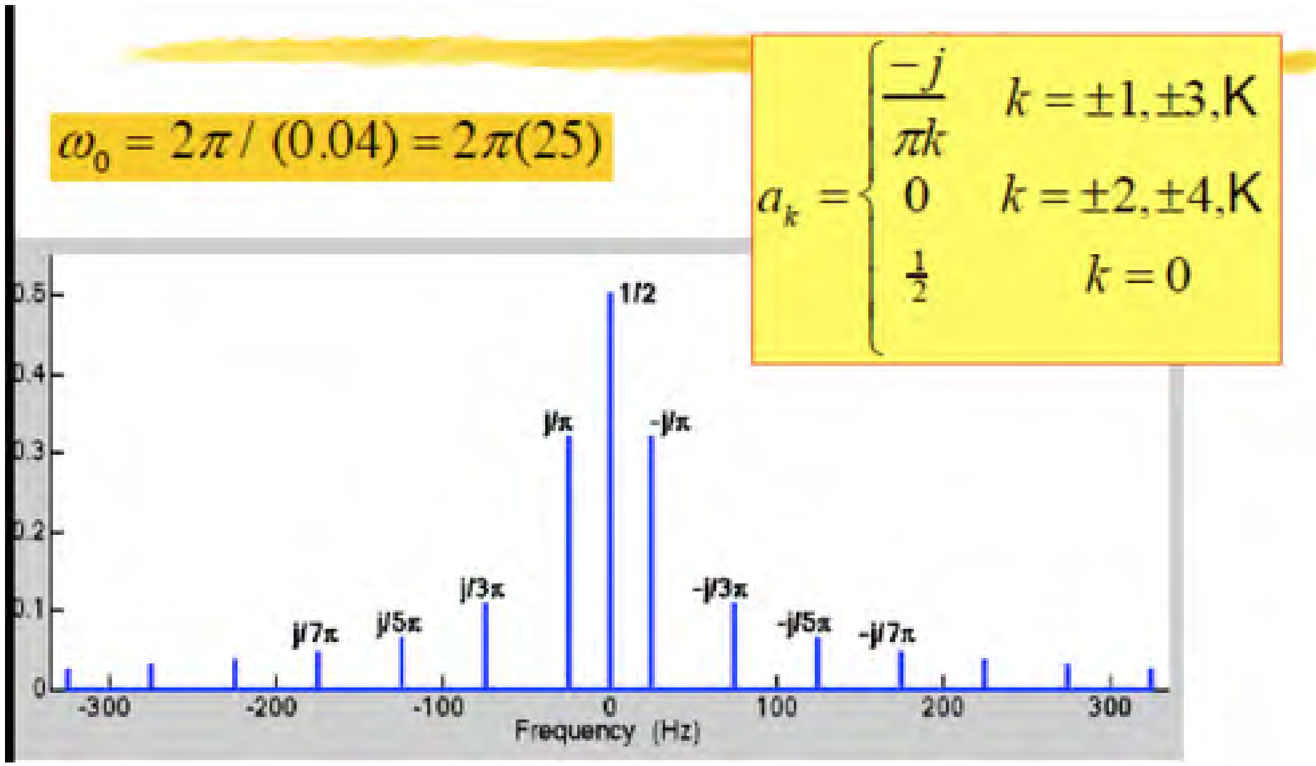
⌘If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



From our Book

- Assumes you don’t know Fourier Transform yet

Spectrum from Fourier Series



Bandlimited Signals

- A bandlimited signal has all its frequencies below a certain limit ω_N.
  - A square wave is *not* a bandlimited signal since its non-zero spectrum components go all the way up to infinity.
  - Bandlimited signals are very smooth.
  - Bandlimited signals can be sampled and then reconstructed exactly. This is the basis for all of modern communications and signal processing.

Signal Types

- A-to-D
  - Convert x(t) to numbers stored in memory
- D-to-A
  - Convert y[n] back to a "continuous-time" signal, y(t)
  - y[n] is called a "discrete-time" signal

Sampling

- SAMPLING can cause ALIASING
  - Sampling Theorem
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

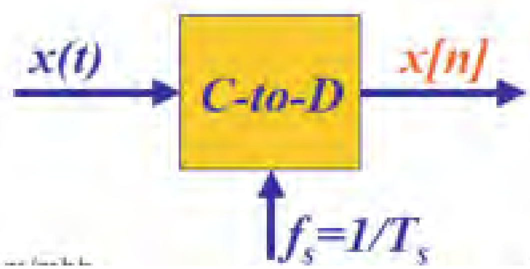
Signals

- ANALOG/ELECTRONIC:

  - Circuits: resistors, capacitors, op-amps
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)
- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory

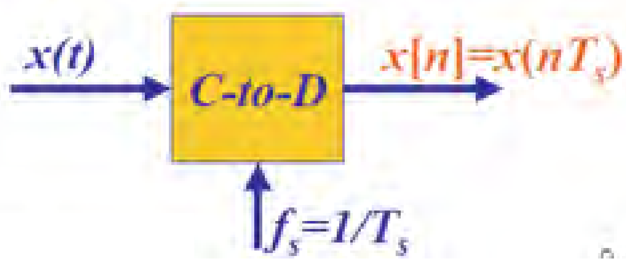
Sampling x(t)

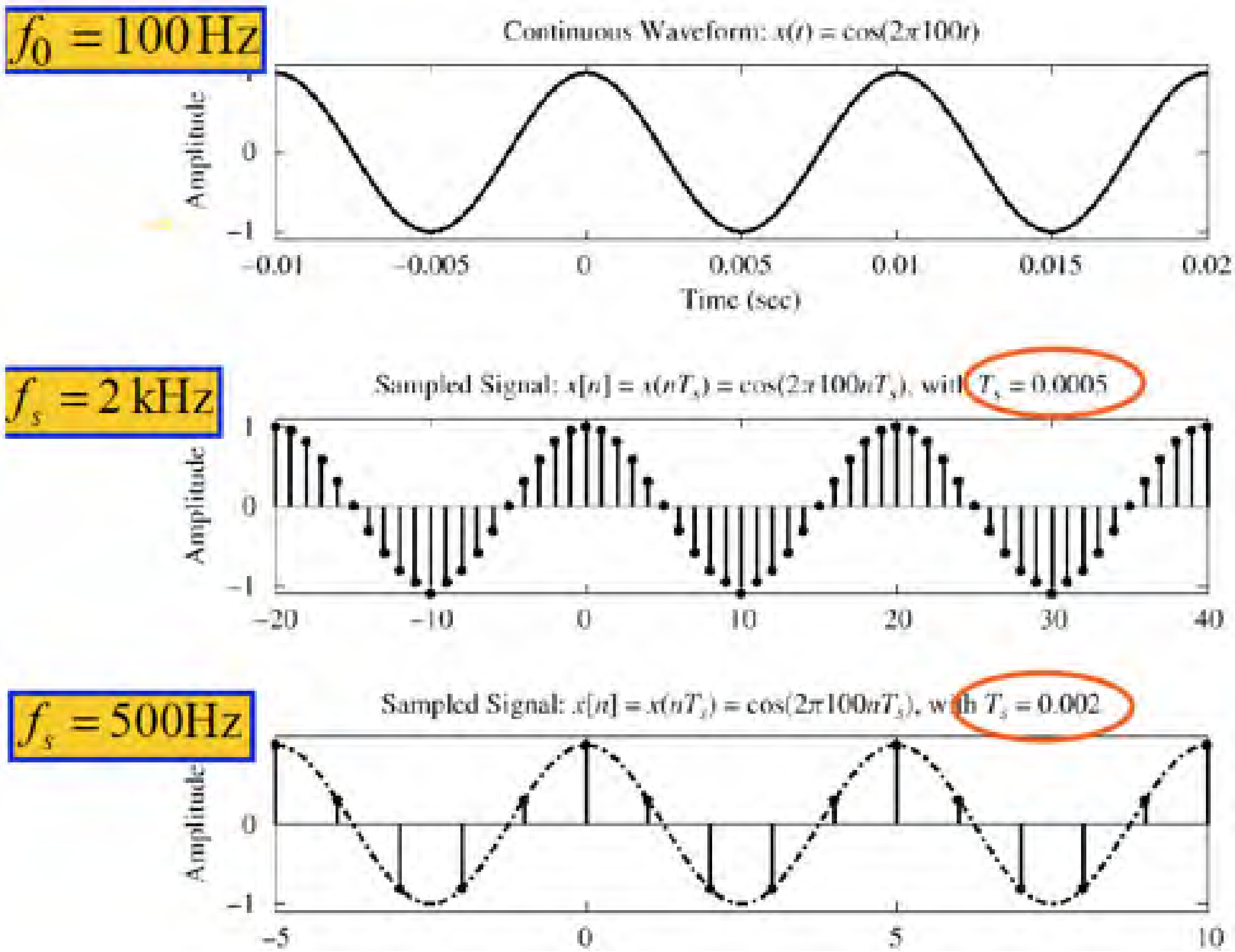
- SAMPLING PROCESS
  - Convert $x(t)$ to numbers $x[n]$
  - "n" is an integer; $x[n]$ is a sequence
  - "n" is the storage address in memory
  - We've already been doing this is in lab.
- UNIFORM SAMPLING at times $t_n = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



Sampling Rate, Fs

- SAMPLING RATE (f_s)
  - $1/T_s$ = NUMBER of SAMPLES PER SECOND
  - T_s = 125 microsec --> f_s = 8000 samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$





$f_0 = 100\,\text{Hz}$
Continuous Waveform: $x(t) = \cos(2\pi 100 t)$
Amplitude
Time (sec)
$f_s = 2\,\text{kHz}$
Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100 n T_s)$, with $T_s = 0.0005$
Amplitude
$f_s = 500\text{Hz}$
Sampled Signal: $x[n] = x(nT_s) = \cos(2\pi 100 n T_s)$, with $T_s = 0.002$
Amplitude

Band-limited Signals

- A bandlimited sum of sinusoids has the form

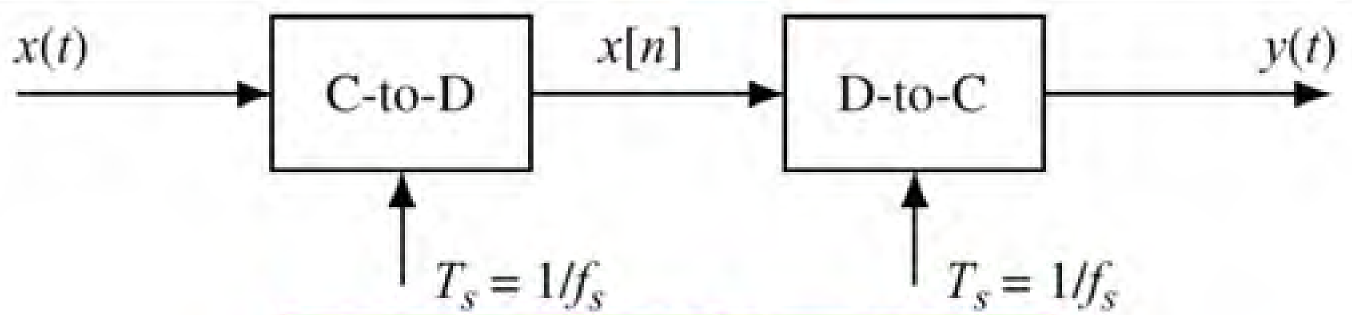
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) \text{ where } f_k \leq f_{\max}$$

- Our study of Fourier series has shown that:
  - Square waves are not bandlimited
  - Bandlimited signals are very smooth

Sampling Theorem

Shannon Sampling Theorem

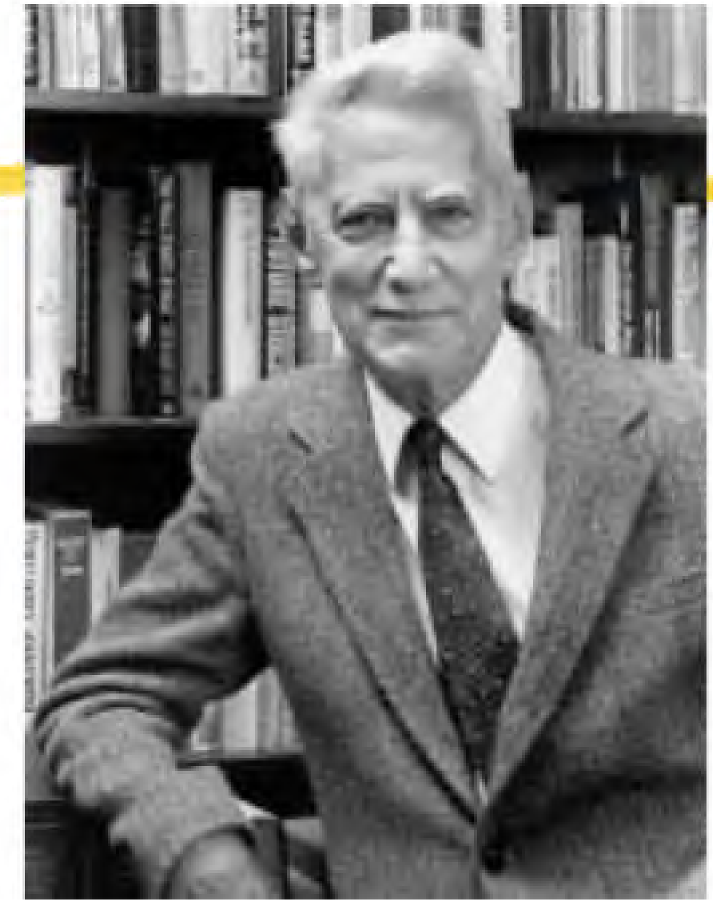
A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.



$$y(t) = x(t) \quad \text{if } f_s > 2f_{\max}$$

Claude Shannon

Founder of the theory of information



Nyquist Rate

- "Nyquist Rate" Sampling
  - f_s = **TWICE** THE HIGHEST FREQUENCY in x(t)
  - "Sampling above the Nyquist rate"
- BANDLIMITED SIGNALS
  - DEF: x(t) has a HIGHEST FREQUENCY COMPONENT in its SPECTRUM
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED

Storing Digital Sound

- $x[n]$ is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

Discrete-Time Sinusoid

- Change x(t) into x[n] **DERIVATION**

$$x(t) = A\cos(\omega_0 t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega_0 n T_s + \varphi)$$

$$x[n] = A\cos((\omega_0 T_s) n + \varphi)$$

$$x[n] = A\cos(\hat{\omega}_0 n + \varphi)$$

$$\hat{\omega}_0 = \omega_0 T_s$$

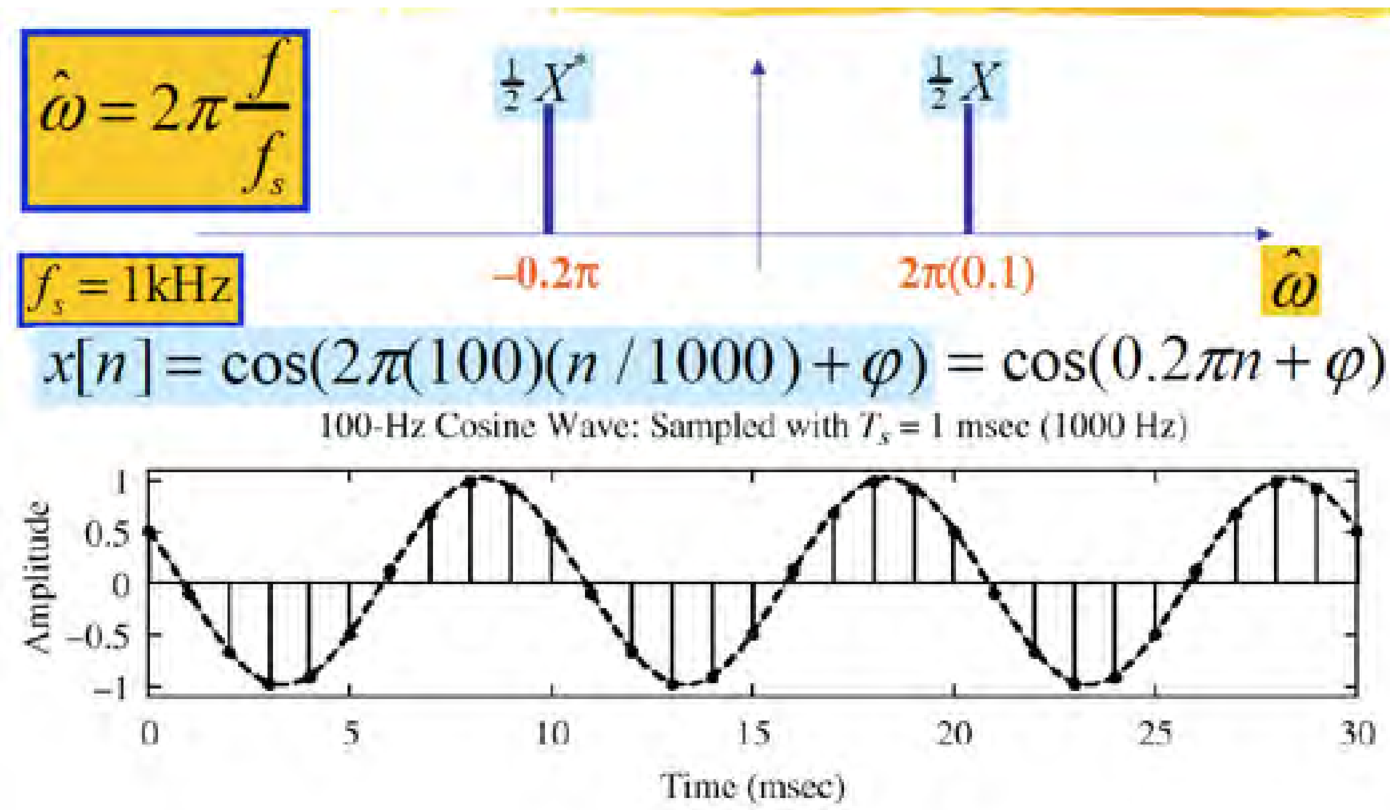
DEFINE DIGITAL FREQUENCY

Digital Frequency, $\hat{\omega}$

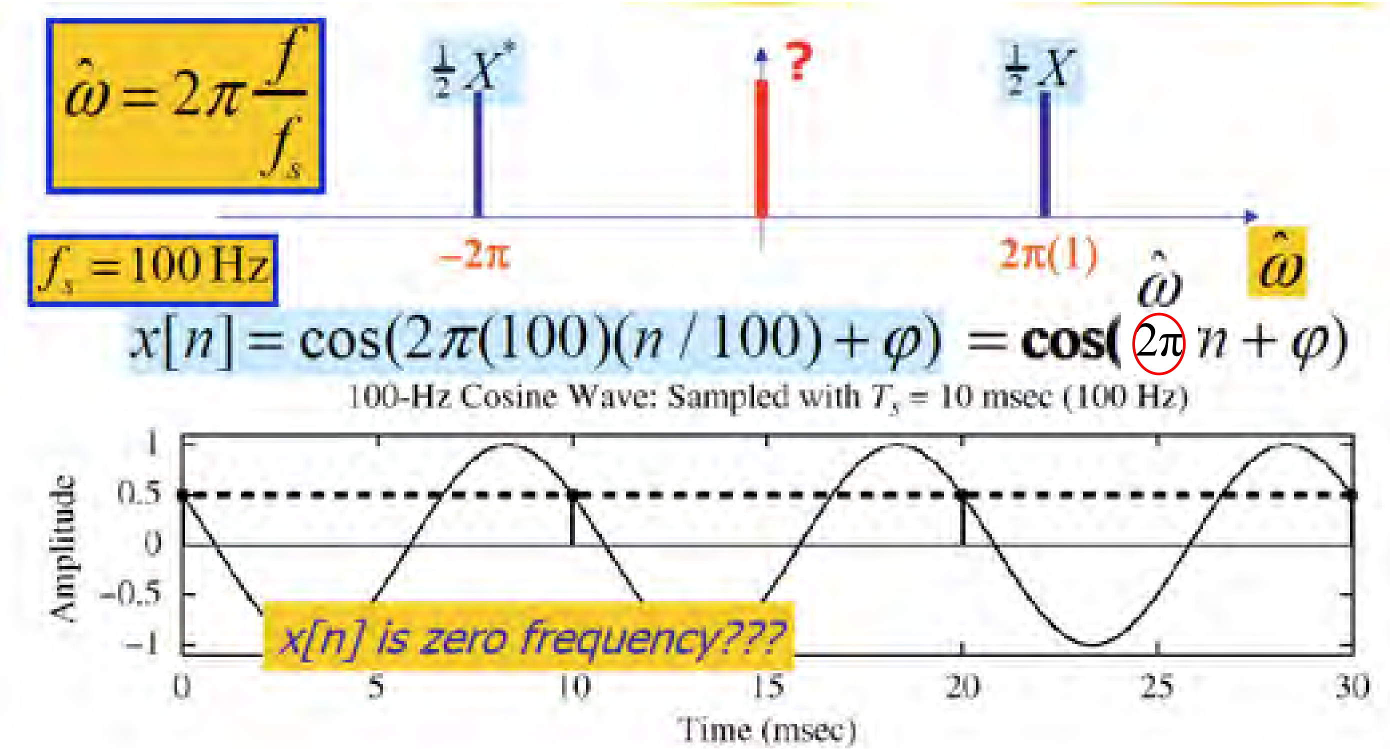
- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- DIGITAL FREQUENCY is NORMALIZED
- UNITS are radians, **not** rad/sec

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}}$$

Spectrum (Digital)



Spectrum (Digital) ???



Rest of Story

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**
- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega}n+\varphi)=A\cos((\hat{\omega}+2\pi)n+\varphi)$$

Four Frequency Axes

- ANALOG FREQUENCY: f, ω
- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s$$

$$\boxed{\hat{\omega} = 2\pi \frac{f}{f_s}}$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

Aliasing Derivation

- Other Frequencies give the same $\hat{\omega}$

If $x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$

and we substitute: $t \leftarrow \frac{n}{f_s}$

then: $x[n] = A\cos(2\pi(f + \ell f_s)\frac{n}{f_s} + \varphi)$

or, $x[n] = A\cos(2\pi\frac{f}{f_s}n + 2\pi\ell n + \varphi)$

Aliasing Derivation

- Other Frequencies give the same $\boxed{\hat{\omega}}$

If $x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$ $\quad \boxed{t \leftarrow \frac{n}{f_s}}$

and we want : $x[n] = A\cos(\hat{\omega}n + \varphi)$

then : $\hat{\omega} = \dfrac{2\pi(f + \ell f_s)}{f_s} = \dfrac{2\pi f}{f_s} + \dfrac{2\pi \ell f_s}{f_s}$

$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} \boxed{+2\pi\ell}$$

Aliasing Conclusions

- ADDING f_s or $2f_s$ or $-f_s$ TO THE FREQ of $x(t)$ gives exactly the same $x[n]$
- The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES

- GIVEN $x[n]$, WE CAN'T DISTINGUISH f FROM $(f + f_s)$ or $(f - f_s)$ or $(f + 2f_s)$, etc.

Normalized Frequency

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \boxed{+2\pi\ell}$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

Spectrum for x[n]

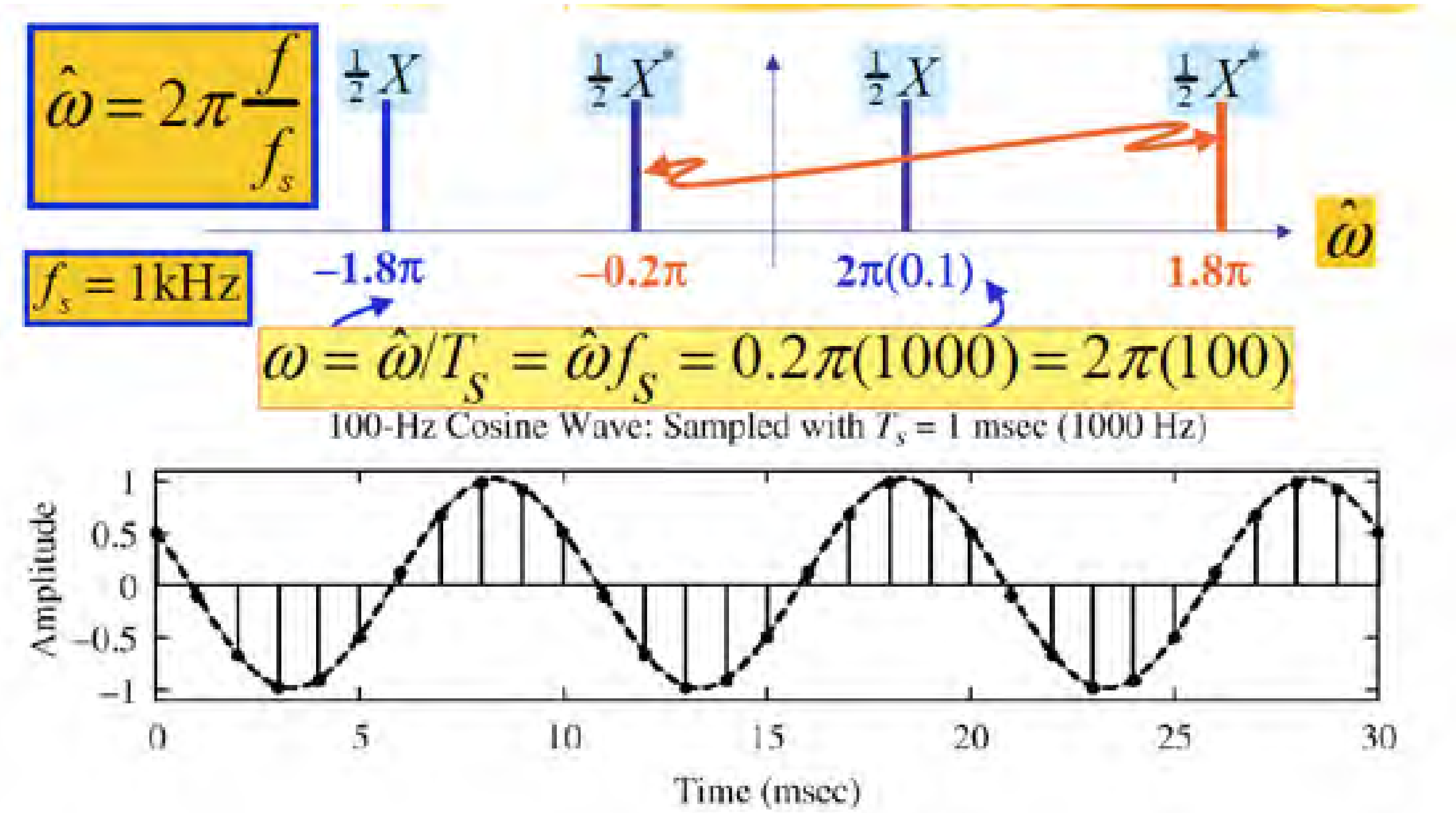
- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD INTEGER MULTIPLES of 2π and -2π
  - FOLDED ALIASES
    - ALIASES of NEGATIVE FREQS
- PLOT versus NORMALIZED FREQUENCY
  - i.e., DIVIDE f_o by f_s

$$\hat{\omega} = 2\pi\frac{f}{f_s} + 2\pi\ell$$

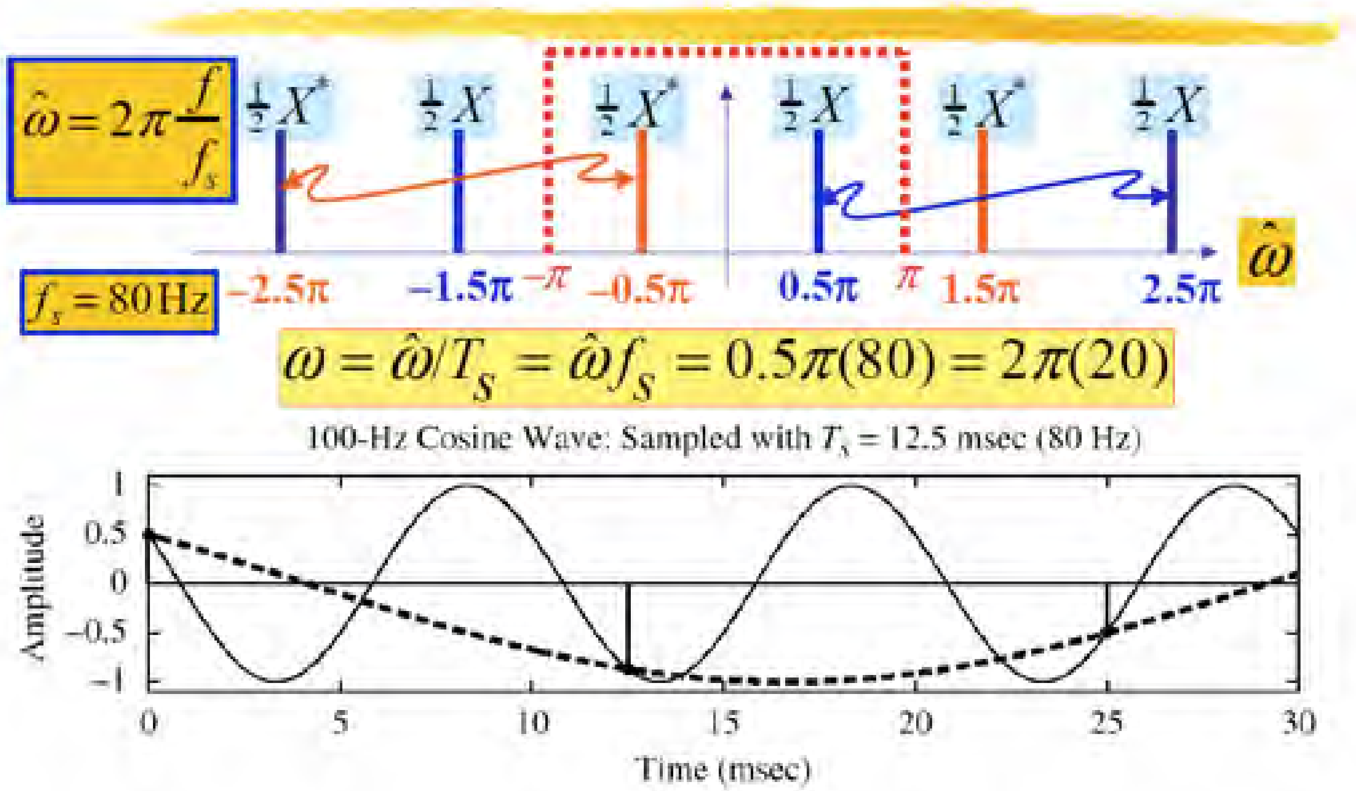
Example: Spectrum

- $x[n] = A\cos(0.2\pi n+\phi)$
- FREQS @ 0.2π and -0.2π
- ALIASES:
  - $\{2.2\pi, 4.2\pi, 6.2\pi, \ldots\}$ & $\{-1.8\pi, -3.8\pi, \ldots\}$
  - EX: $x[n] = A\cos(4.2\pi n+\phi)$
- ALIASES of NEGATIVE FREQ:
  - $\{1.8\pi, 3.8\pi, 5.8\pi, \ldots\}$ & $\{-2.2\pi, -4.2\pi \ldots\}$

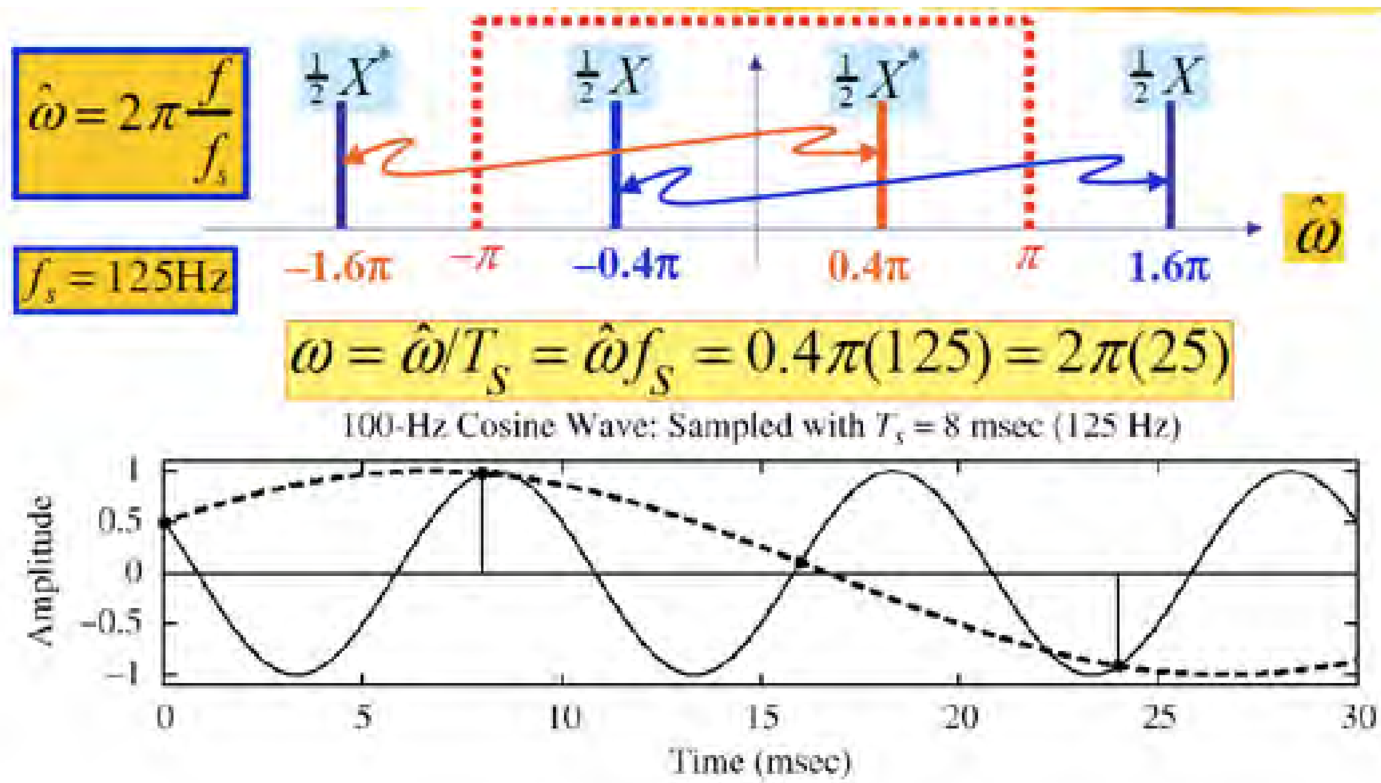
Spectrum (More Lines)



Spectrum (Aliasing Case)



Spectrum (Folding Case)



Folding Derivation

- Negative Freqs can give the same $\hat{\omega}$

$$x(t) = A\cos(2\pi(\underline{-f + \ell f_s})t - \varphi)$$

$$x[n] = x(nT_s) = A\cos(2\pi(-f + \ell f_s)nT_s - \varphi)$$

$$x[n] = A\cos((-2\pi f T_s)n + (2\pi \ell f_s T_s)n - \varphi)$$

$$x[n] = A\cos((2\pi f T_s)n - \underline{2\pi \ell n} + \varphi) \quad \boxed{\cos(-\theta) = \cos\theta}$$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

SAME DIGITAL SIGNAL

Folding (a type of aliasing)

- EXAMPLE: 3 different x(t); same x[n]
- 900 Hz "folds" to 100 Hz when f_s=1kHz

$$f_s = 1000\text{Hz}$$

$$\boxed{\hat{\omega} = 2\pi \frac{100}{1000} = 2(0.1)}$$

$$\cos(2\pi(100)t) \rightarrow \cos[2\pi(0.1)n]$$

$$\cos(2\pi(1100)t) \rightarrow \cos[2\pi(1.1)n] = \cos[2\pi(0.1)n]$$

$$\cos(2\pi(900)t) \rightarrow \cos[2\pi(0.9)n]$$

$$= \cos[2\pi(0.9)n - 2\pi n] = \cos[2\pi(-0.1)n] = \cos[2\pi(0.1)n]$$

Digital Frequency

Normalized Radian Frequency

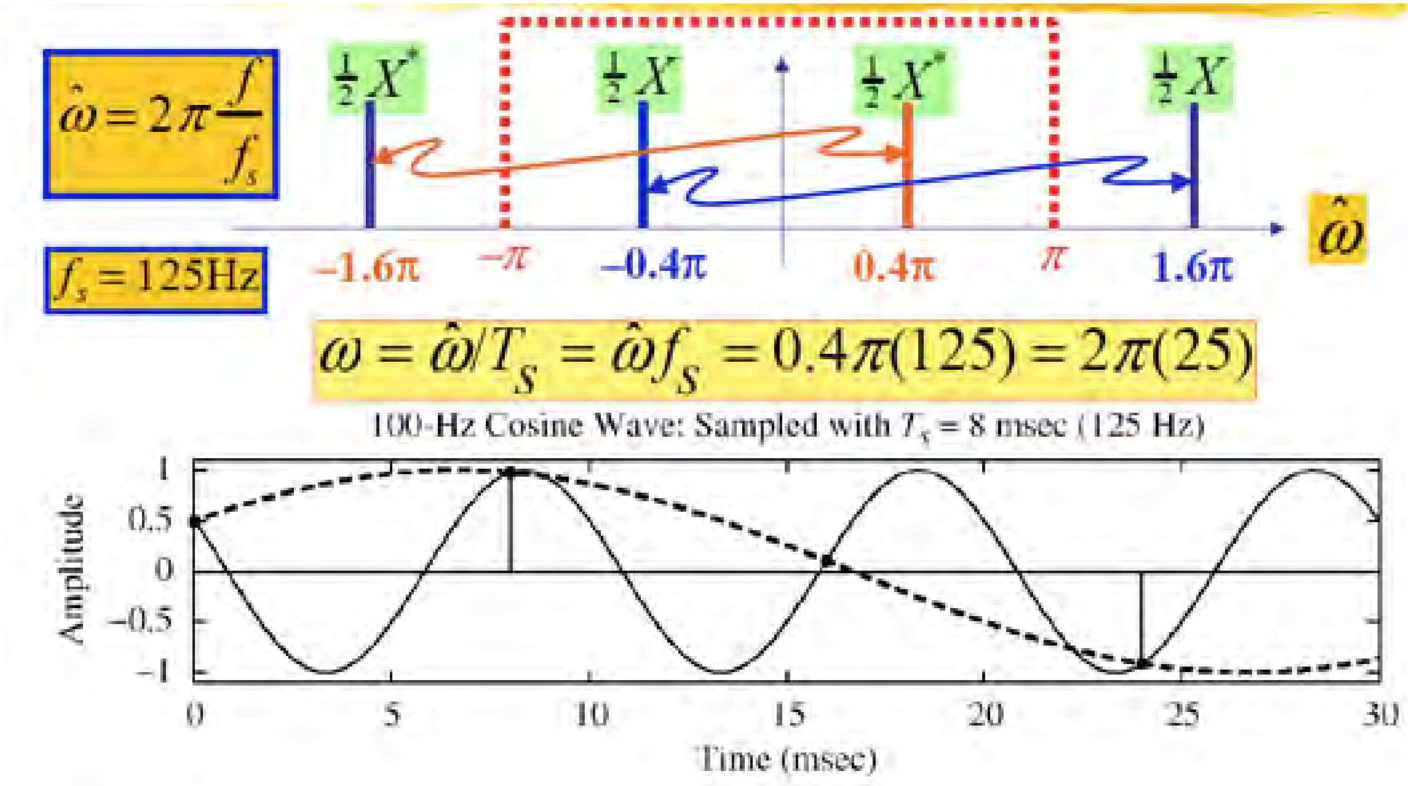
$$\boxed{\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}} + 2\pi\ell$$

ALIASING

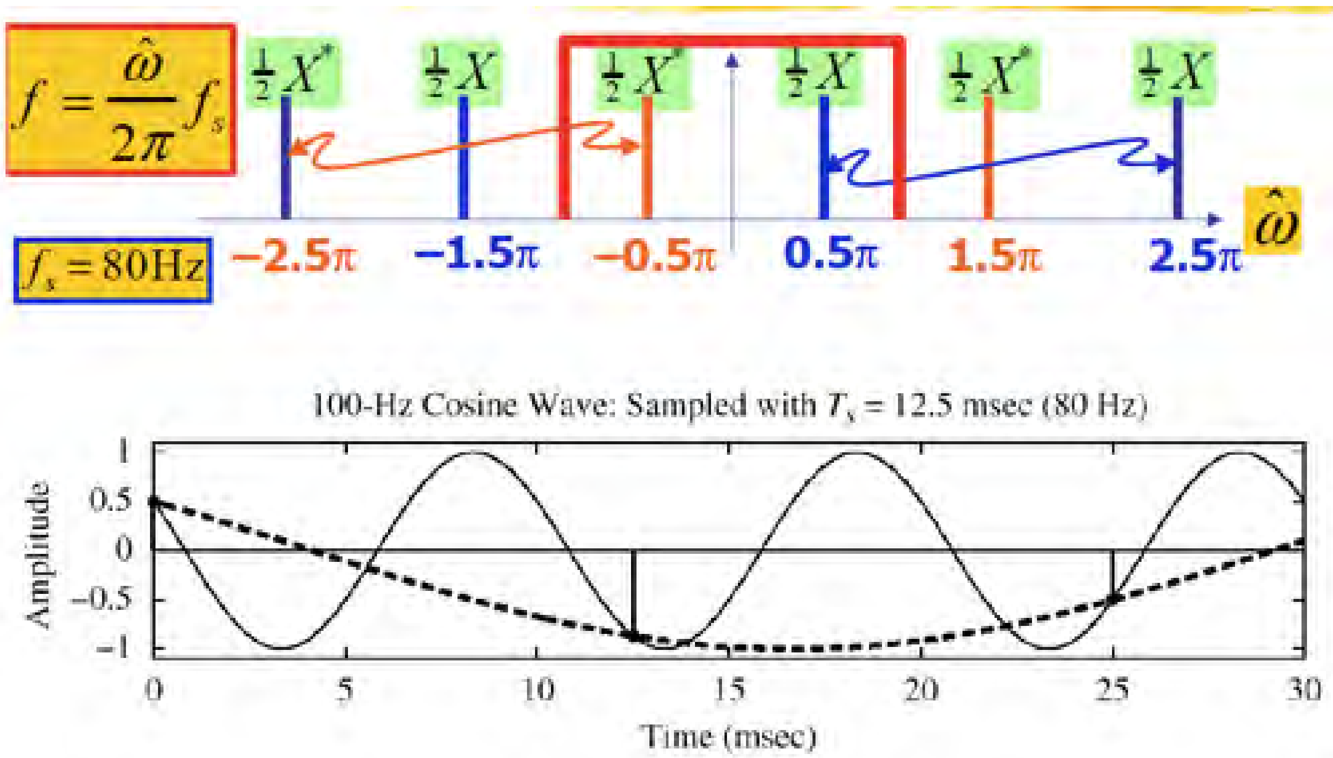
$$\hat{\omega} = \omega T_s = -\frac{2\pi f}{f_s} + 2\pi\ell$$

FOLDED ALIAS

Spectrum (Folding Case)



Spectrum (Aliasing Case)



Folding Diagram

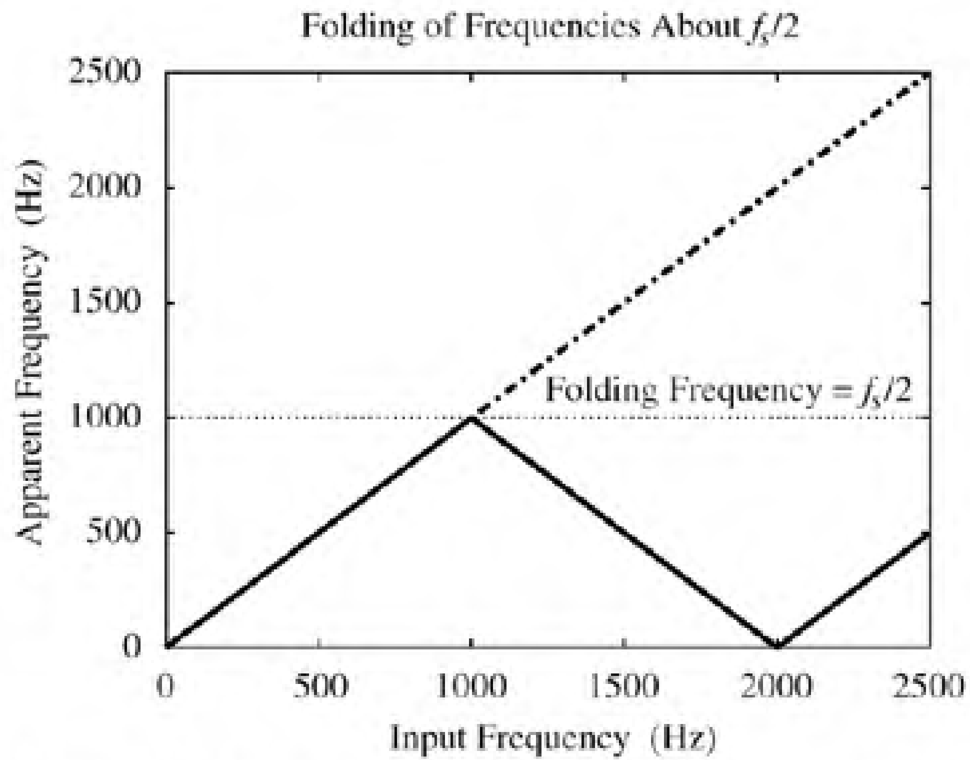


Figure 4.4 Folding of a sinusoid sampled at $f_s = 2000$ samples/sec. The apparent frequency is the lowest frequency of a sinusoid that has exactly the same samples as the input sinusoid.

Aliasing & Folding

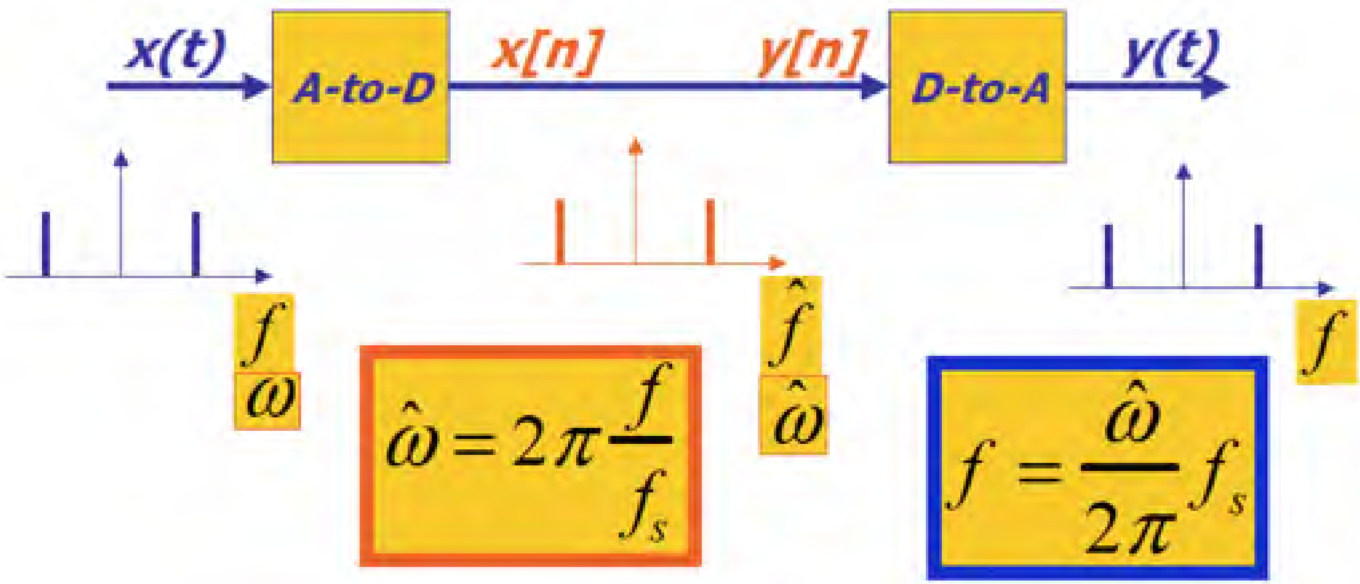
$x(t)$ = SINUSOID @ f_o
SAMPLED SIGNAL: $x[n] = x(n/ f_s)$

"over"

- **ALIASING**:
  - $x[n]$ COULD HAVE COME FROM
  - $(f_o + f_s)$
  - or $(f_o - f_s)$
  - or $(f_o + 2f_s)$
  - or $(f_o - 2f_s)$, etc.

- **FOLDING**:
  - A type of ALIASING
  - $x[n]$ COULD BE FROM:
  - $(-f_o + f_s)$
  - or $(-f_o - f_s)$
  - or $(-f_o + 2f_s)$
  - or $(-f_o - 2f_s)$, etc.

Frequency Domains



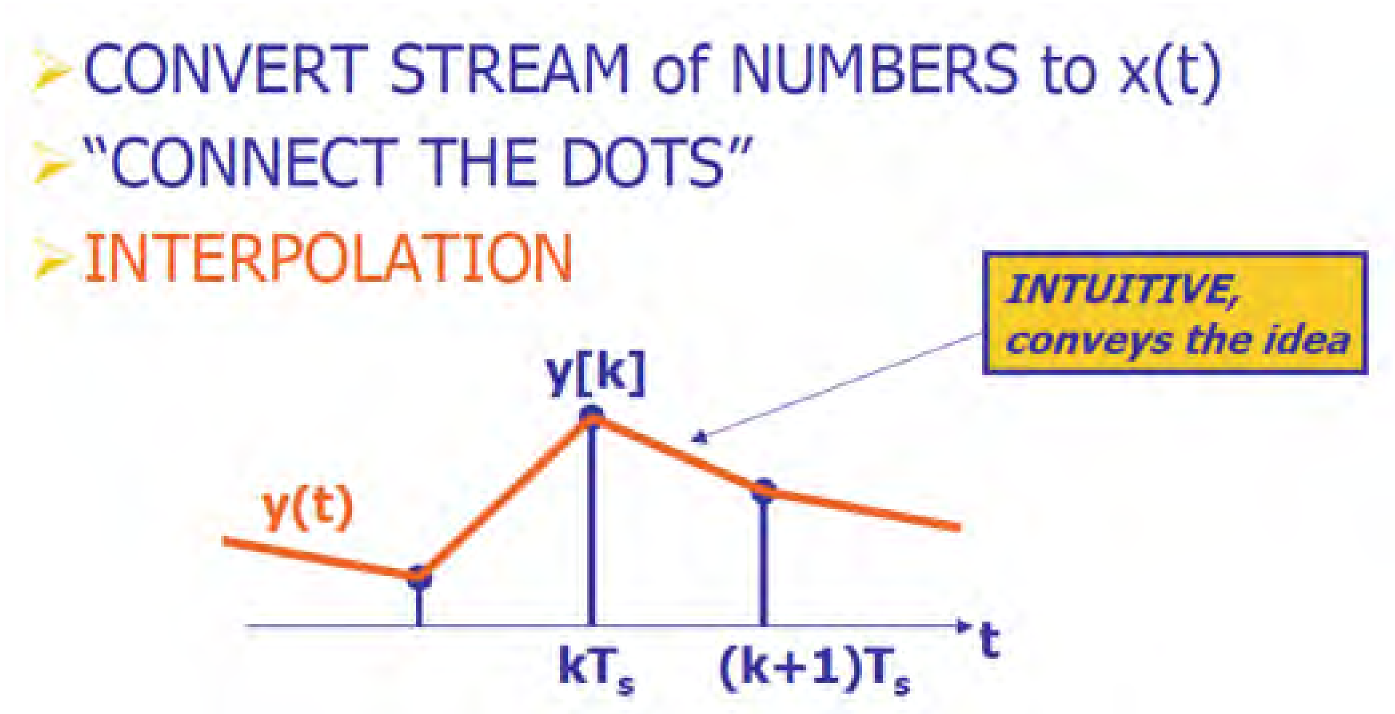
D-to-A Reconstruction

- Create continuous y(t) from y[n]
  - **IDEAL**
    - If you have formula for y[n]
  - Replace n in y[n] with $f_s t$
  - $y[n] = A\cos(0.2\pi n + \phi)$ with $f_s = 8000$ Hz
  - $y(t) = A\cos(2\pi(800)t + \phi)$

D-to-A Ambiguous!

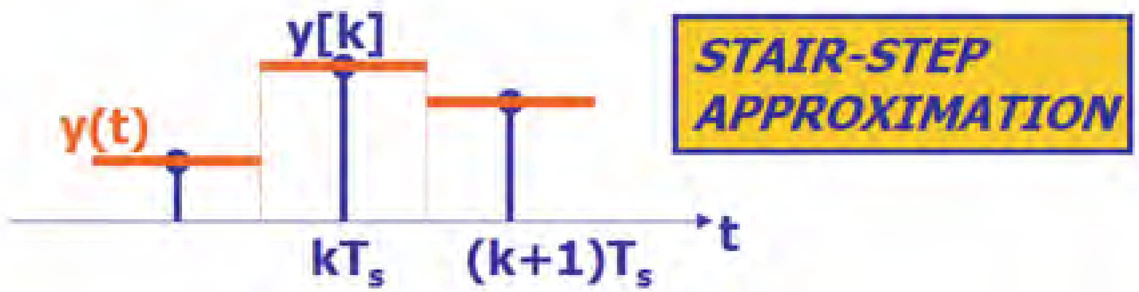
- ALIASING
  - Given y[n], which y(t) do we pick ? ? ?
  - INFINITE NUMBER of y(t)
    - PASSING THRU THE SAMPLES, y[n]
  - D-to-A RECONSTRUCTION MUST CHOOSE ONE OUTPUT
- RECONSTRUCT THE SMOOTHEST ONE
  - THE LOWEST FREQ, if y[n] = sinusoid

Reconstructive (D-to-A)

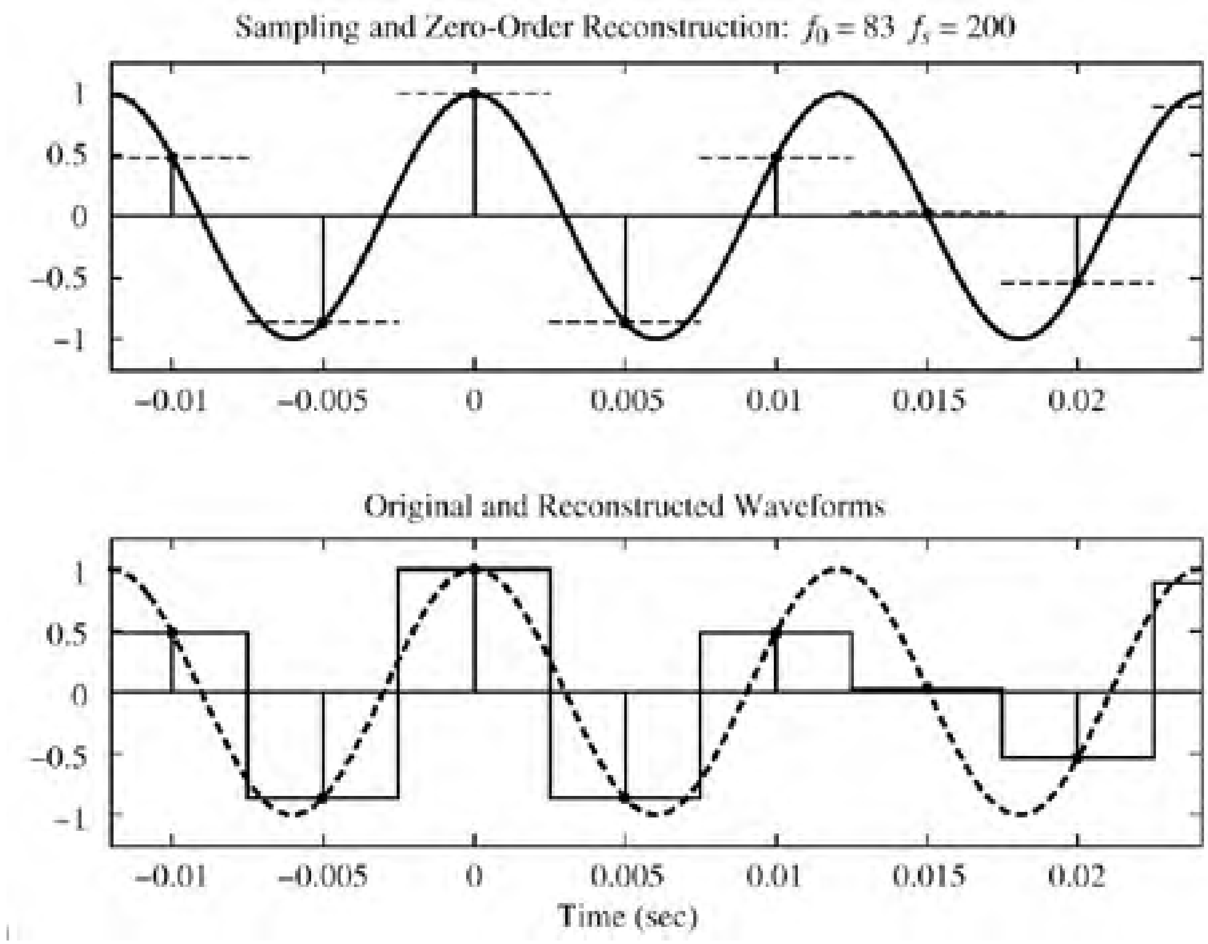


Sample & Hold Device

- CONVERT y[n] to y(t)
  - y[k] should be the value of y(t) at $t = kT_s$
  - Make y(t) equal to y[k] for
    - $kT_s - 0.5T_s < t < kT_s + 0.5T_s$



Square Pulse Case



Math Model D-to-A

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

SQUARE PULSE:

$$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

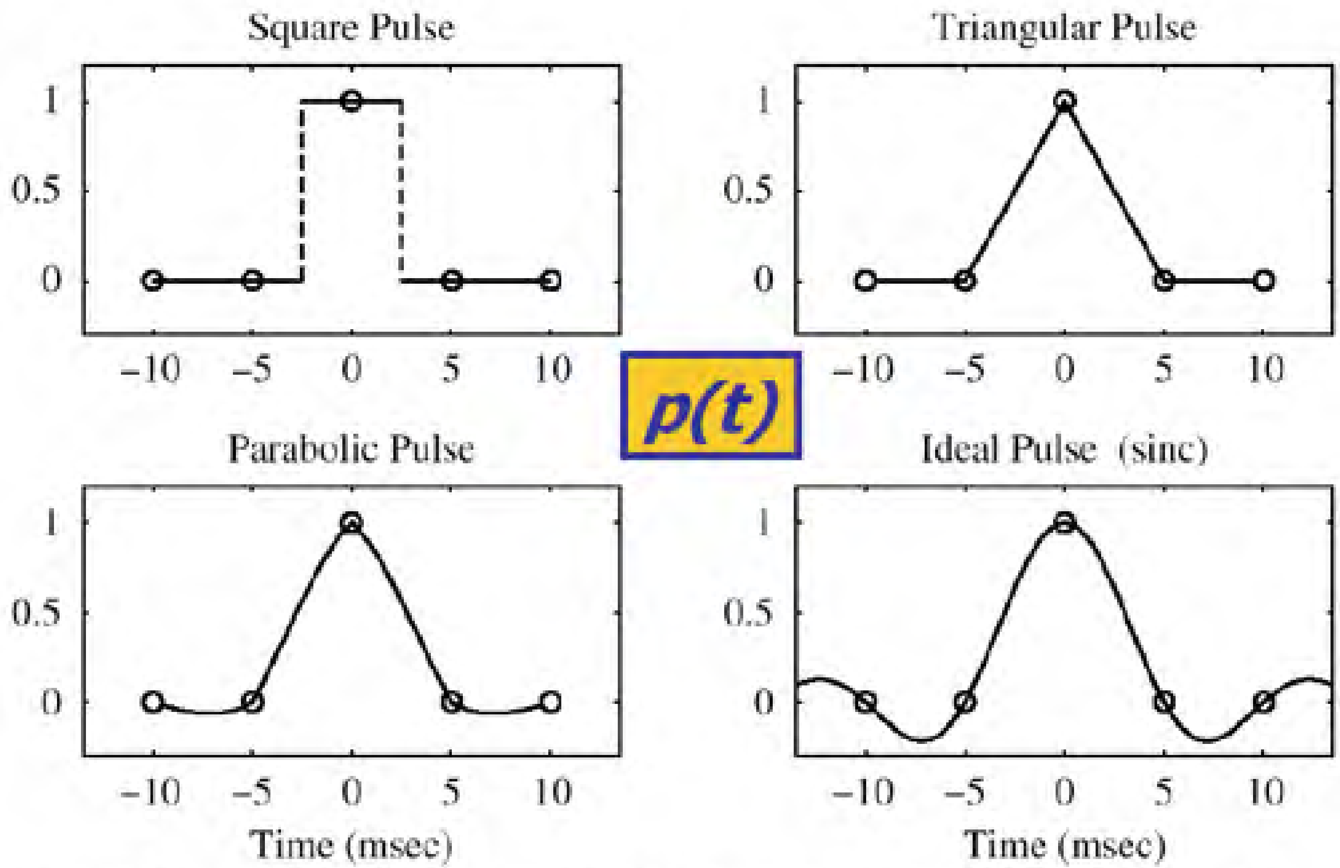


Figure 4.17 Four different pulses for D-to-C conversion. The sampling period is $T_s = 0.005$, i.e., $f_s = 200$ Hz. Note that the duration of each pulse is approximately one or two times T_s.

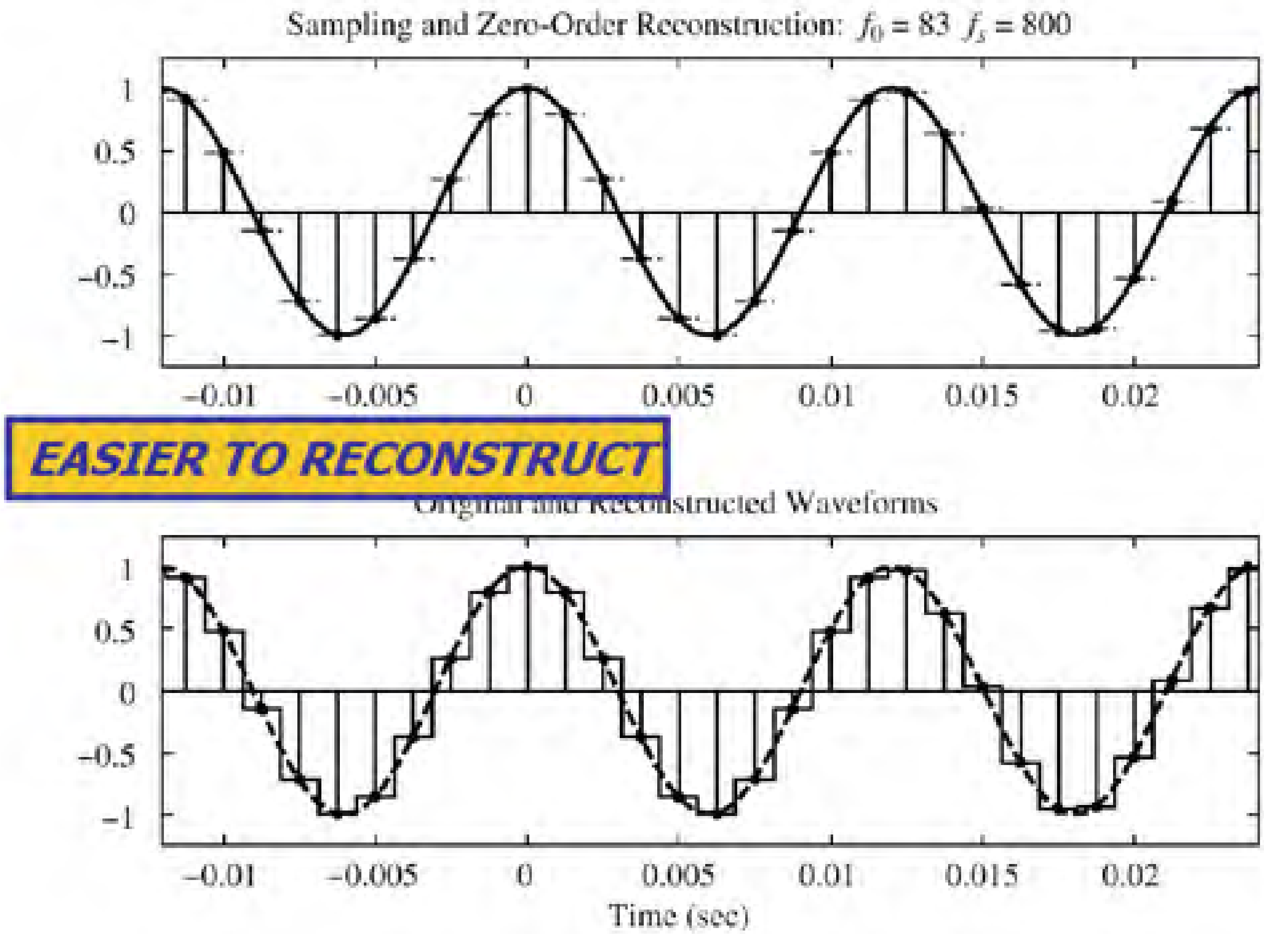
Expand the Summation

$$\sum_{n=-\infty}^{\infty} y[n]p(t-nT_s) =$$

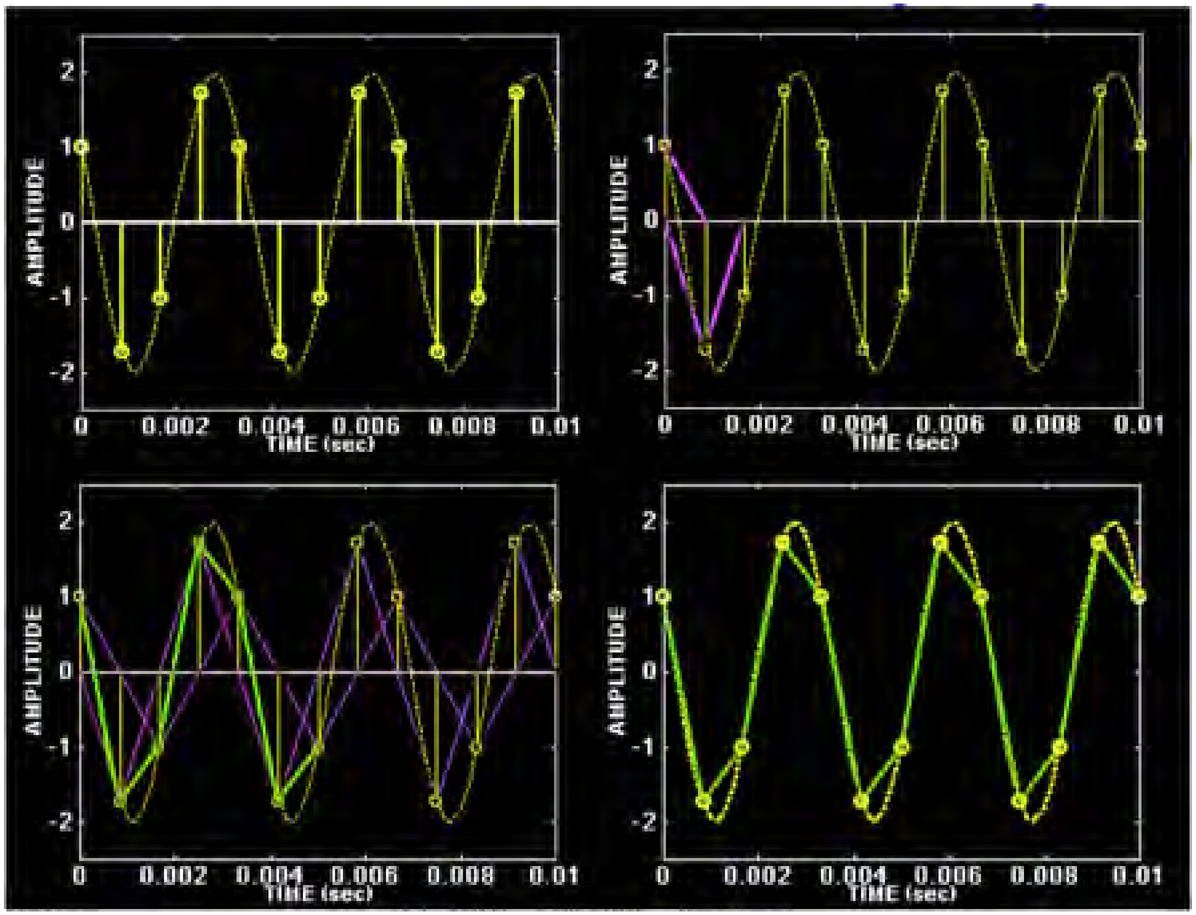
$$\ldots + y[0]p(t) + y[1]p(t-T_s) + y[2]p(t-2T_s) + \ldots$$

- SUM of SHIFTED PULSES $p(t\text{-}nT_s)$
  - "WEIGHTED" by y[n]
  - CENTERED at $t=nT_s$
  - SPACED by T_s
    - RESTORES "REAL TIME"

Over-Sampling Case



Triangular Pulse

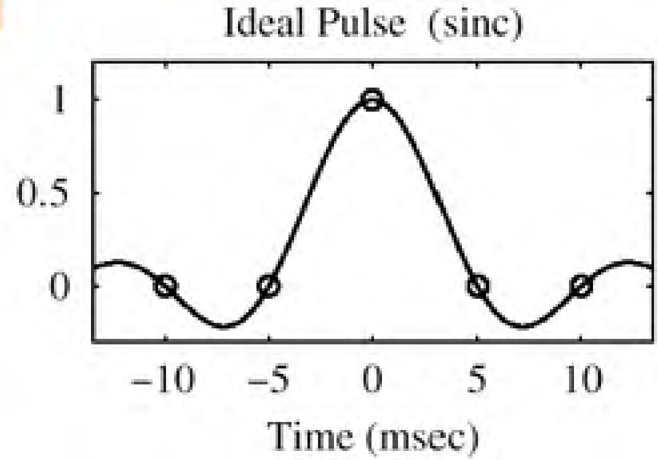


D-to-A Reconstruction

- Create continuous y(t) from y[n]
  - **REALISTIC CONSTRAINT:** SMOOTH y(t)
    - Use the lowest possible frequency
  - y[n] is a list of numbers
  - How fast?
  - In MATLAB: `soundsc(yy,fs)`

Optimal Pulse

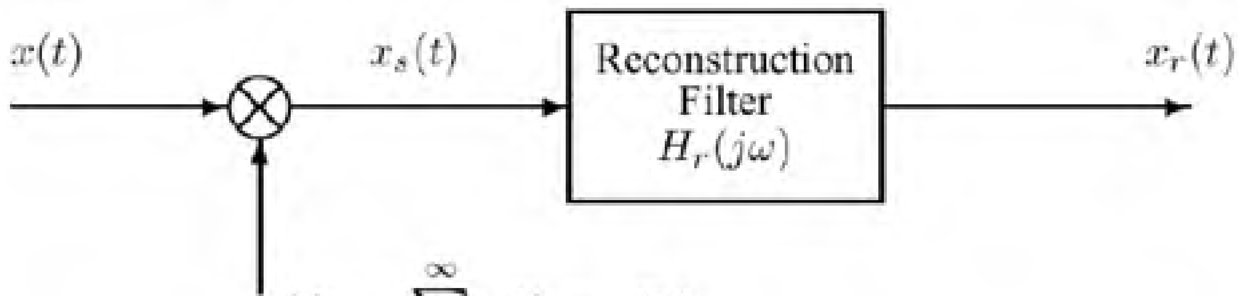
CALLED "BANDLIMITED INTERPOLATION"



$$p(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t} \qquad \text{for } -\infty < t < \infty$$

$$p(t) = 0 \qquad \text{for } t = 0, \pm T_s, \pm 2T_s$$

Reconstruction of x(t)



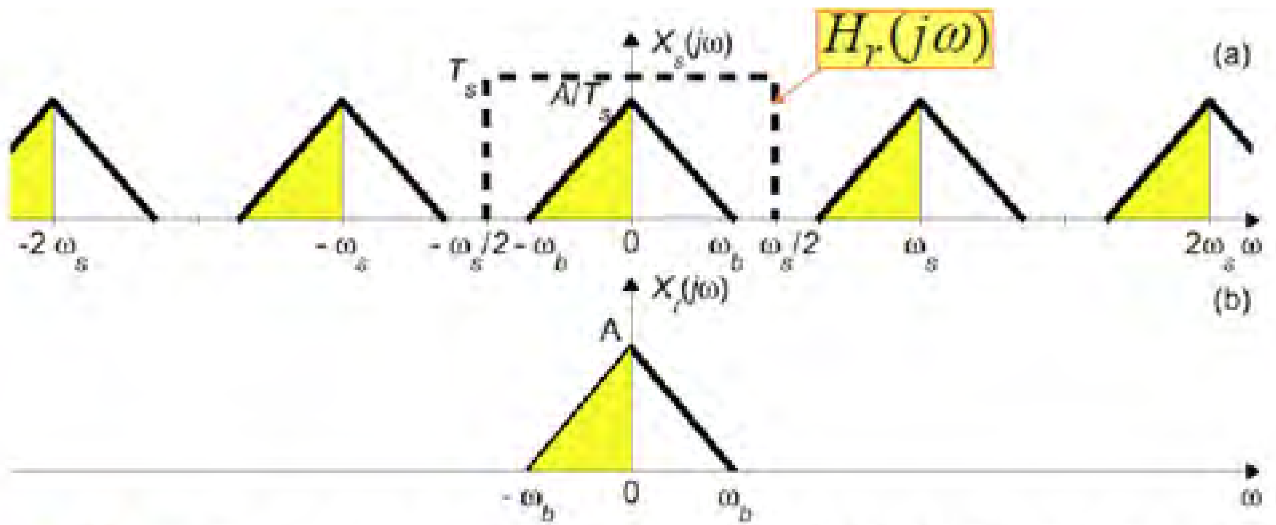
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

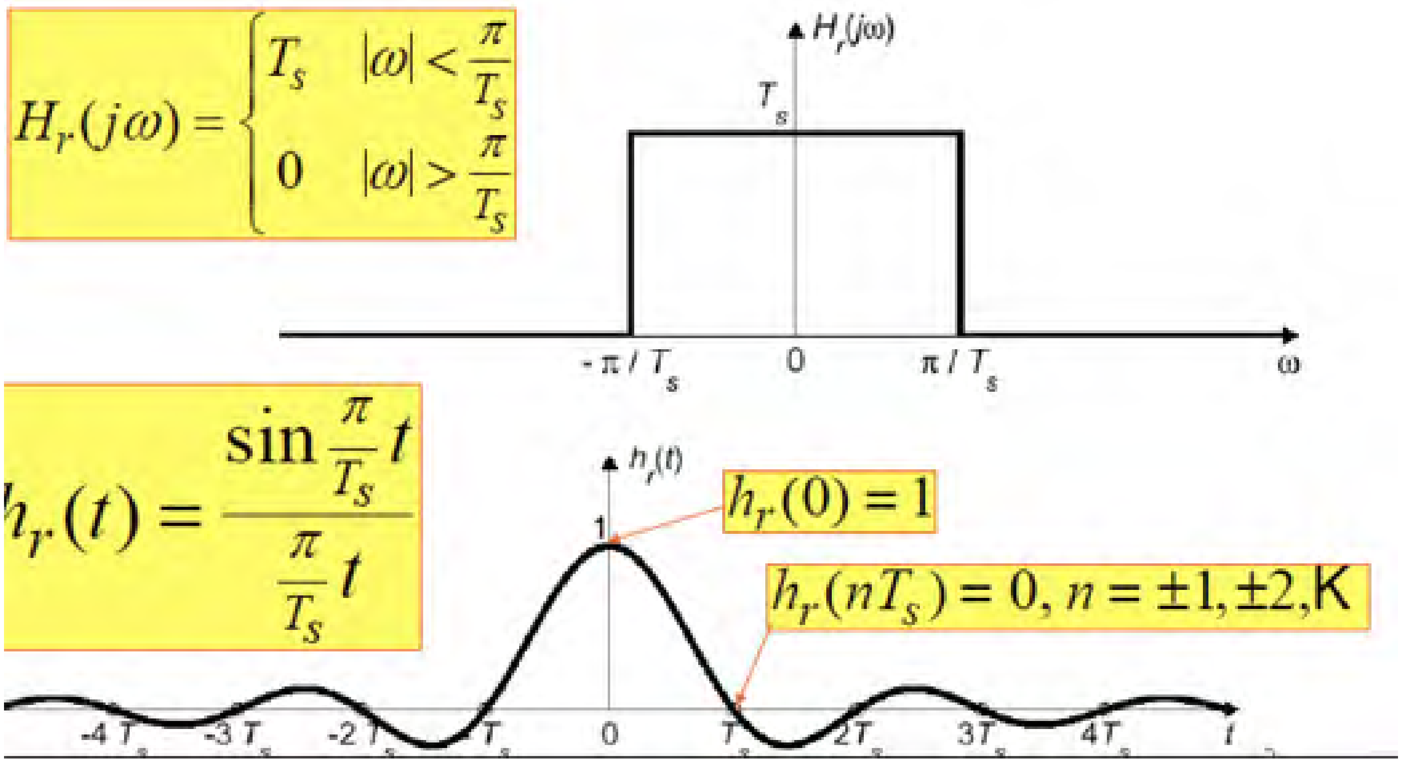
$$\boxed{X_r(j\omega) = H_r(j\omega)X_s(j\omega)}$$

Reconstruction in the Frequency-Domain



- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)\ X_s(j\omega)$.

Ideal Reconstruction Filter



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}}$$

Ideal bandlimited interpolation formula

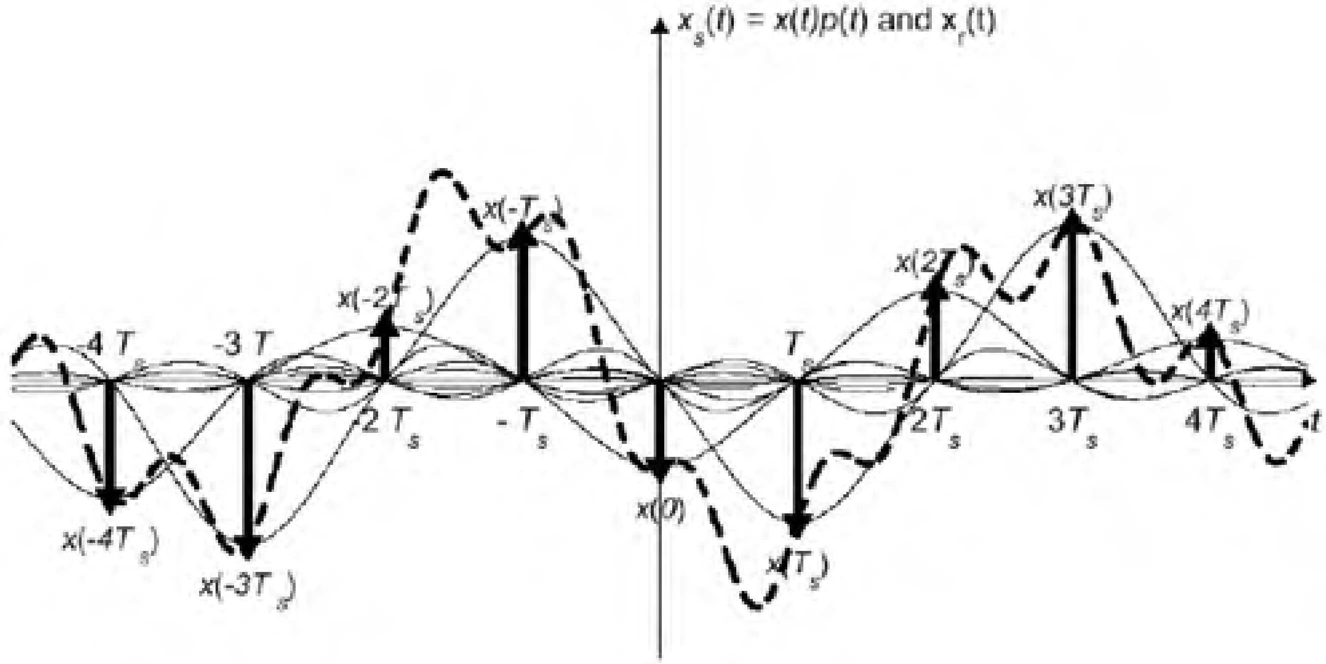
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

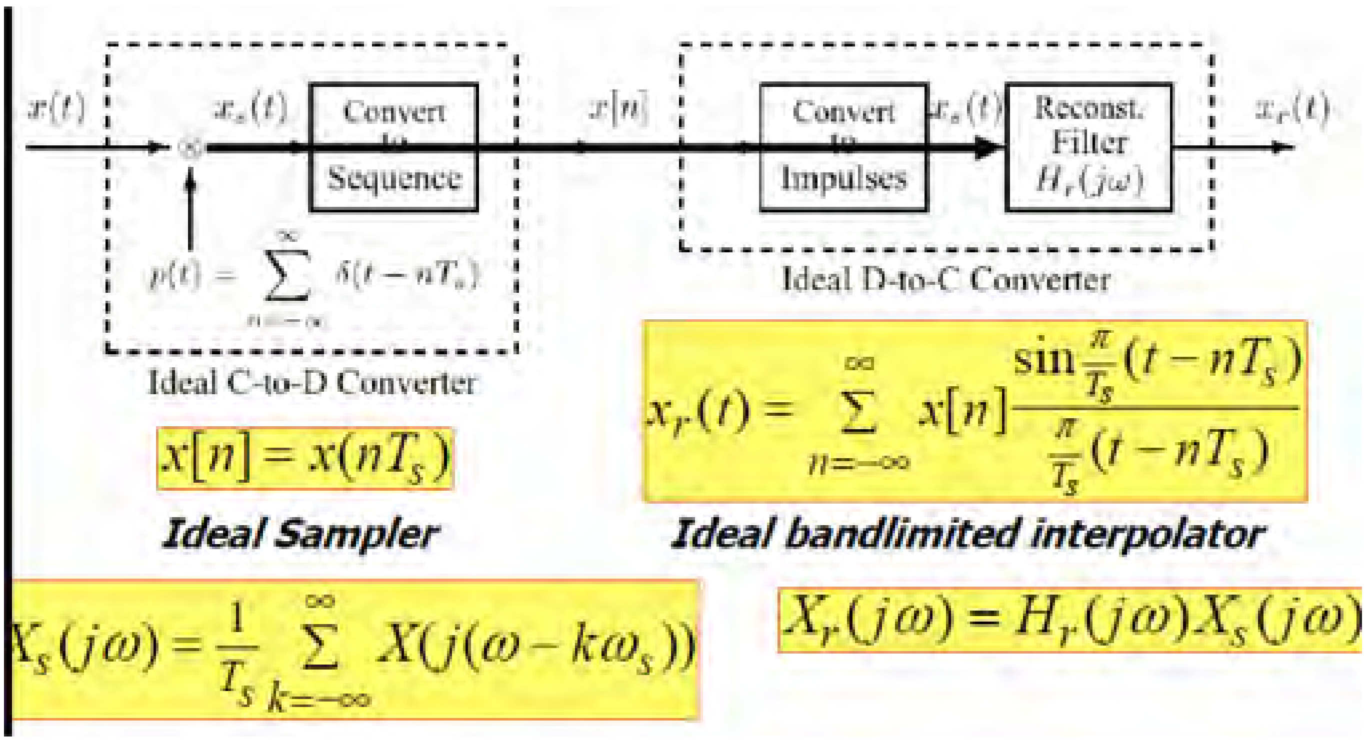
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

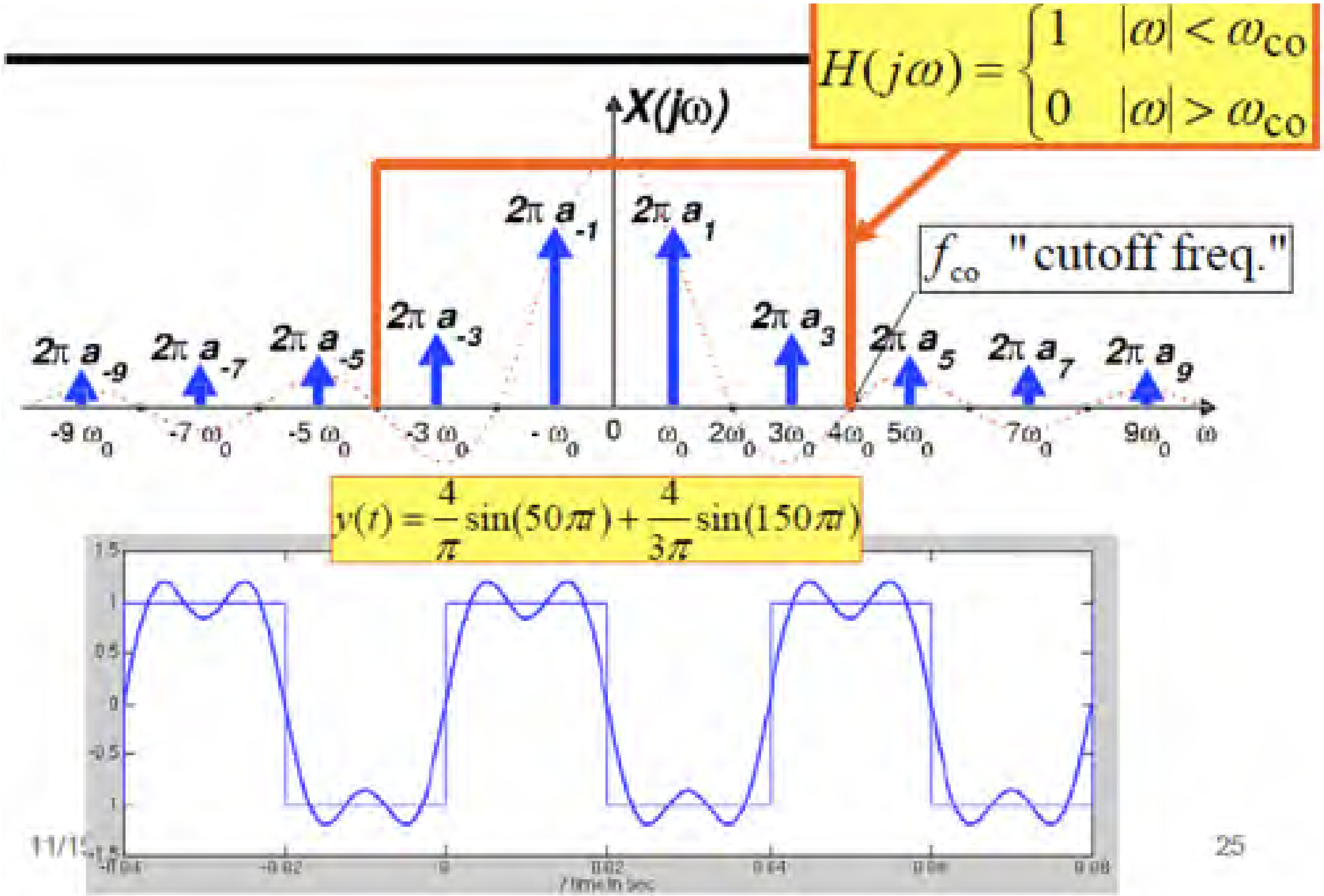
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in the Time-Domain



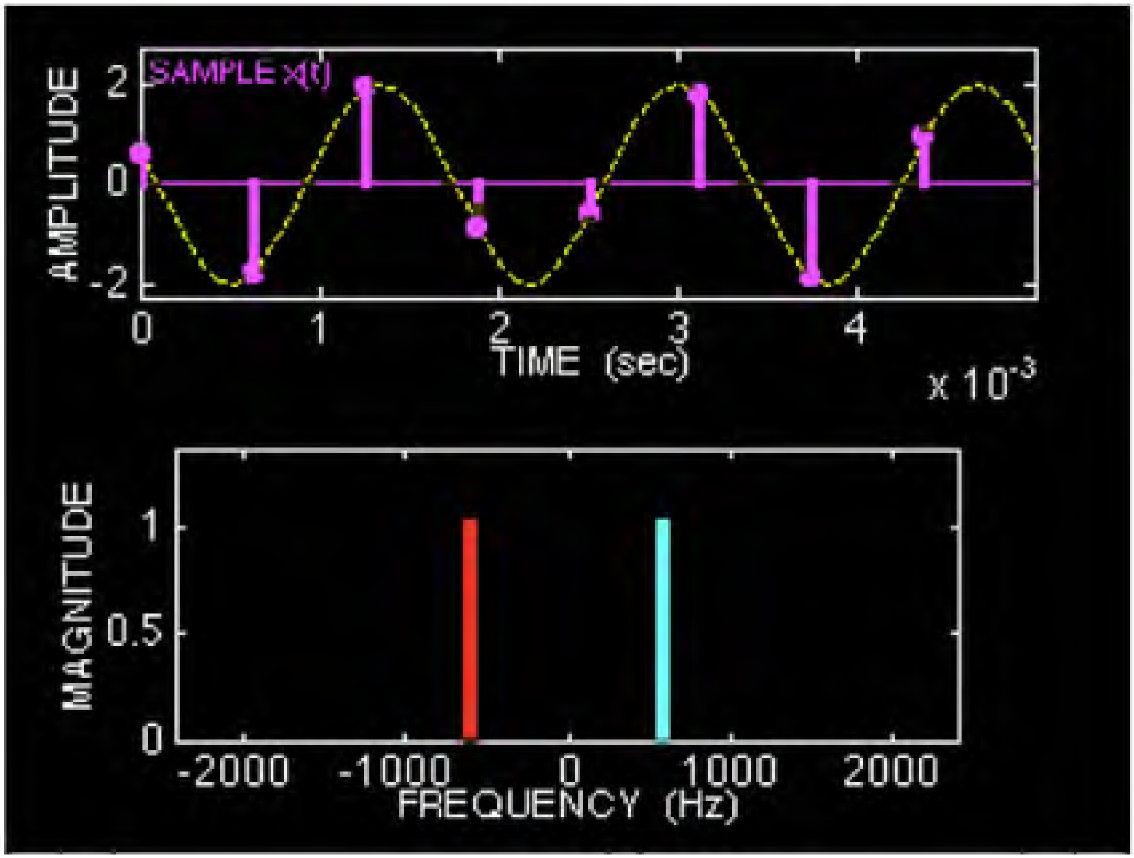
Ideal C-to-D and D-to-C



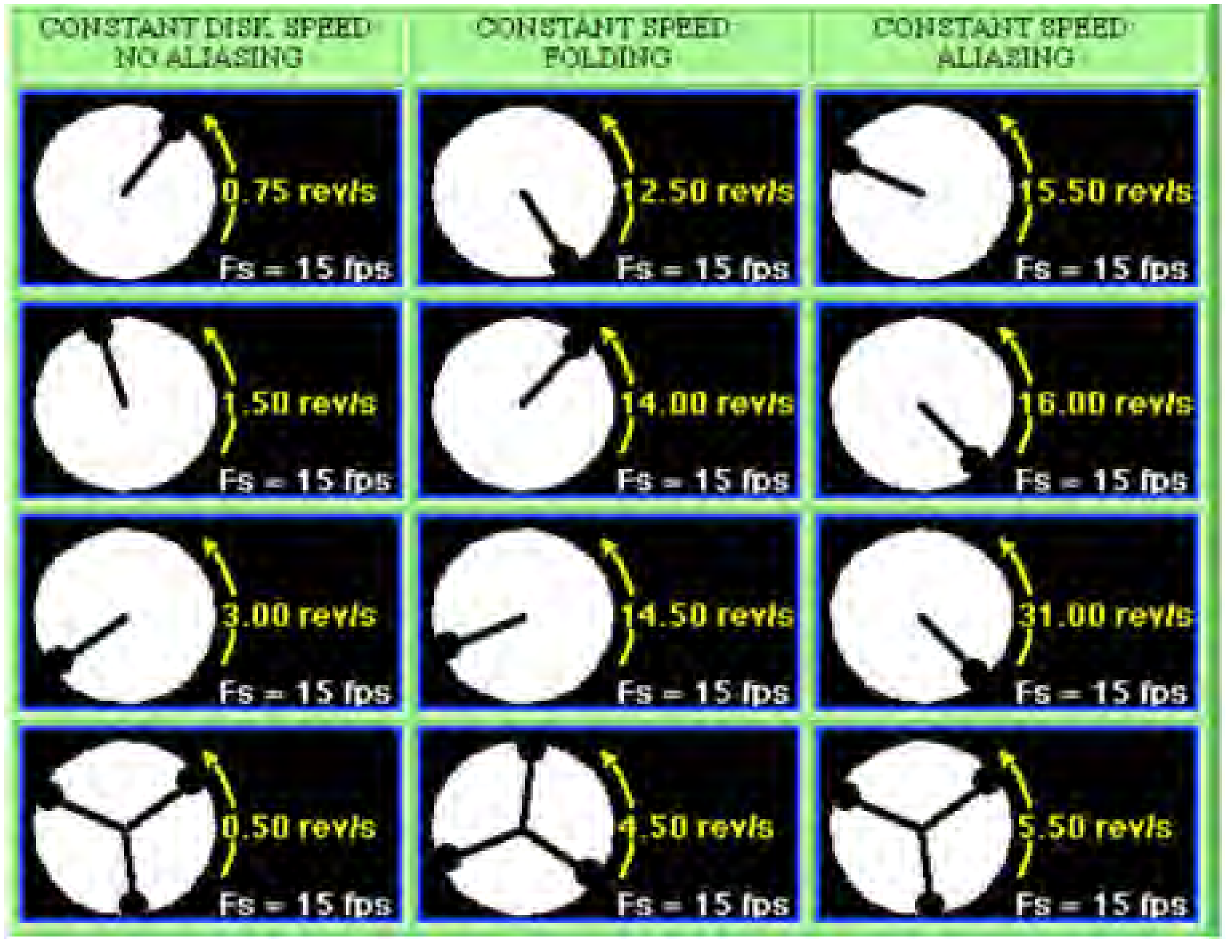

$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$
$X(j\omega)$
$2\pi a_{-9}$ $2\pi a_{-7}$ $2\pi a_{-5}$ $2\pi a_{-3}$ $2\pi a_{-1}$ $2\pi a_{1}$ $2\pi a_{3}$ $2\pi a_{5}$ $2\pi a_{7}$ $2\pi a_{9}$
f_{co} "cutoff freq."
$-9\omega_0$ $-7\omega_0$ $-5\omega_0$ $-3\omega_0$ $-\omega_0$ 0 ω_0 $2\omega_0$ $3\omega_0$ $4\omega_0$ $5\omega_0$ $7\omega_0$ $9\omega_0$ ω
$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$

Sampling Problem Start to End

Continuous-Discrete Sampling Demo



Strobe Demo



CONSTANT DISK SPEED
NO ALIASING
CONSTANT SPEED
FOLDING
CONSTANT SPEED
ALIASING
0.75 rev/s
Fs = 15 fps
12.50 rev/s
Fs = 15 fps
15.50 rev/s
Fs = 15 fps
1.50 rev/s
Fs = 15 fps
14.00 rev/s
Fs = 15 fps
16.00 rev/s
Fs = 15 fps
3.00 rev/s
Fs = 15 fps
14.50 rev/s
Fs = 15 fps
31.00 rev/s
Fs = 15 fps
0.50 rev/s
Fs = 15 fps
4.50 rev/s
Fs = 15 fps
5.50 rev/s
Fs = 15 fps