Intro to DSP

Week 1

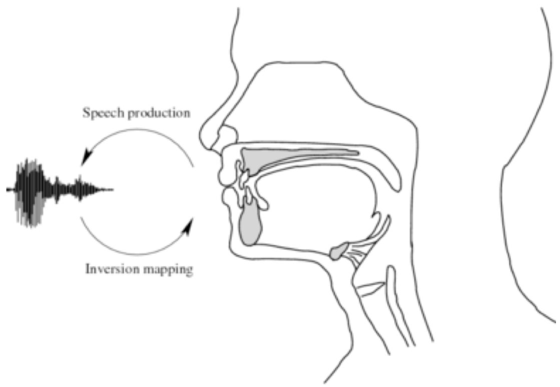
Instructor: Gail Rosen

Syllabus

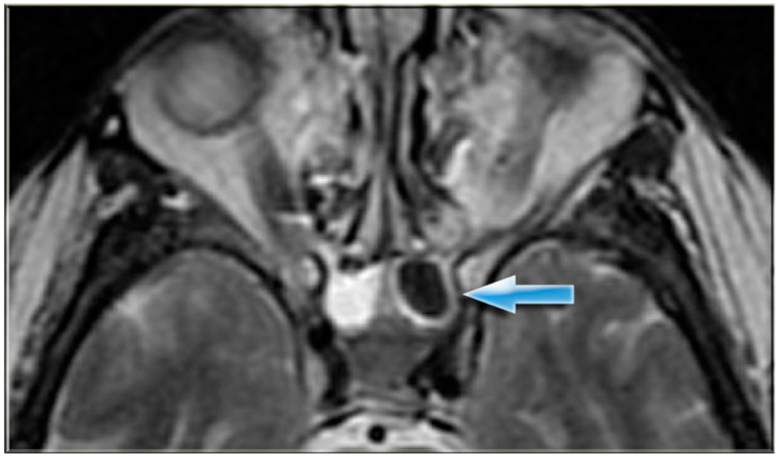
What is DSP?

- Digital Signal Processing?
- What are examples of signals?

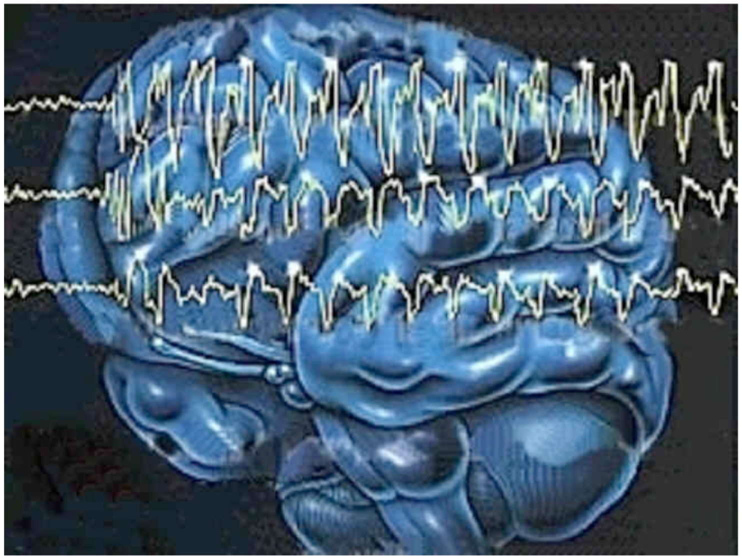
Examples of Signals



Speech



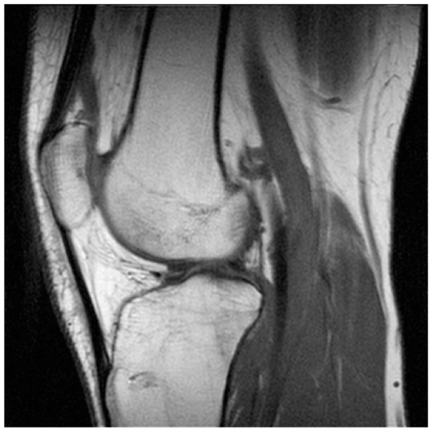
Images



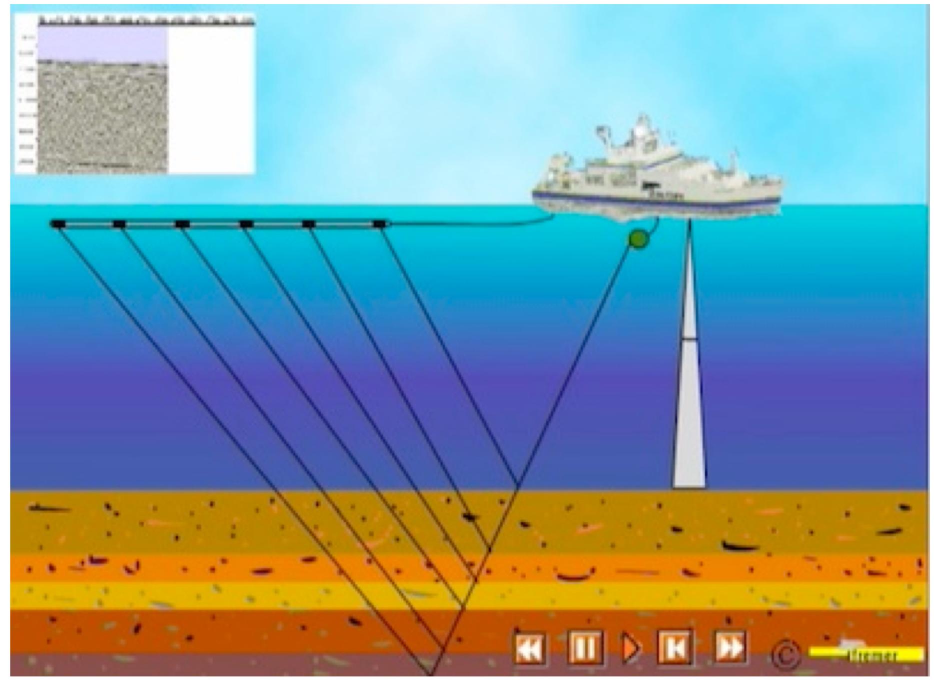
EEG

DNA

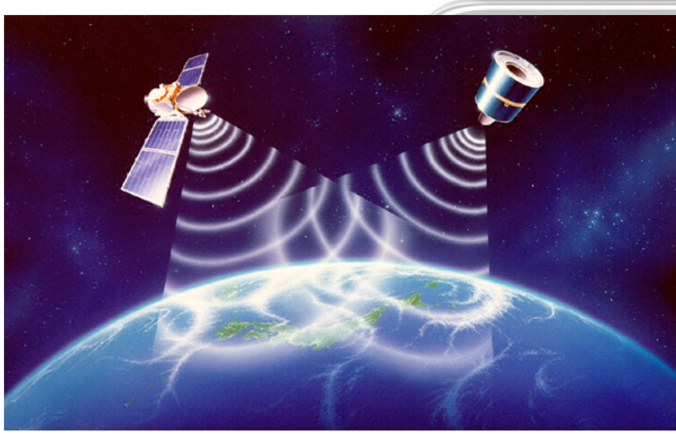
Where is DSP used?



MRI

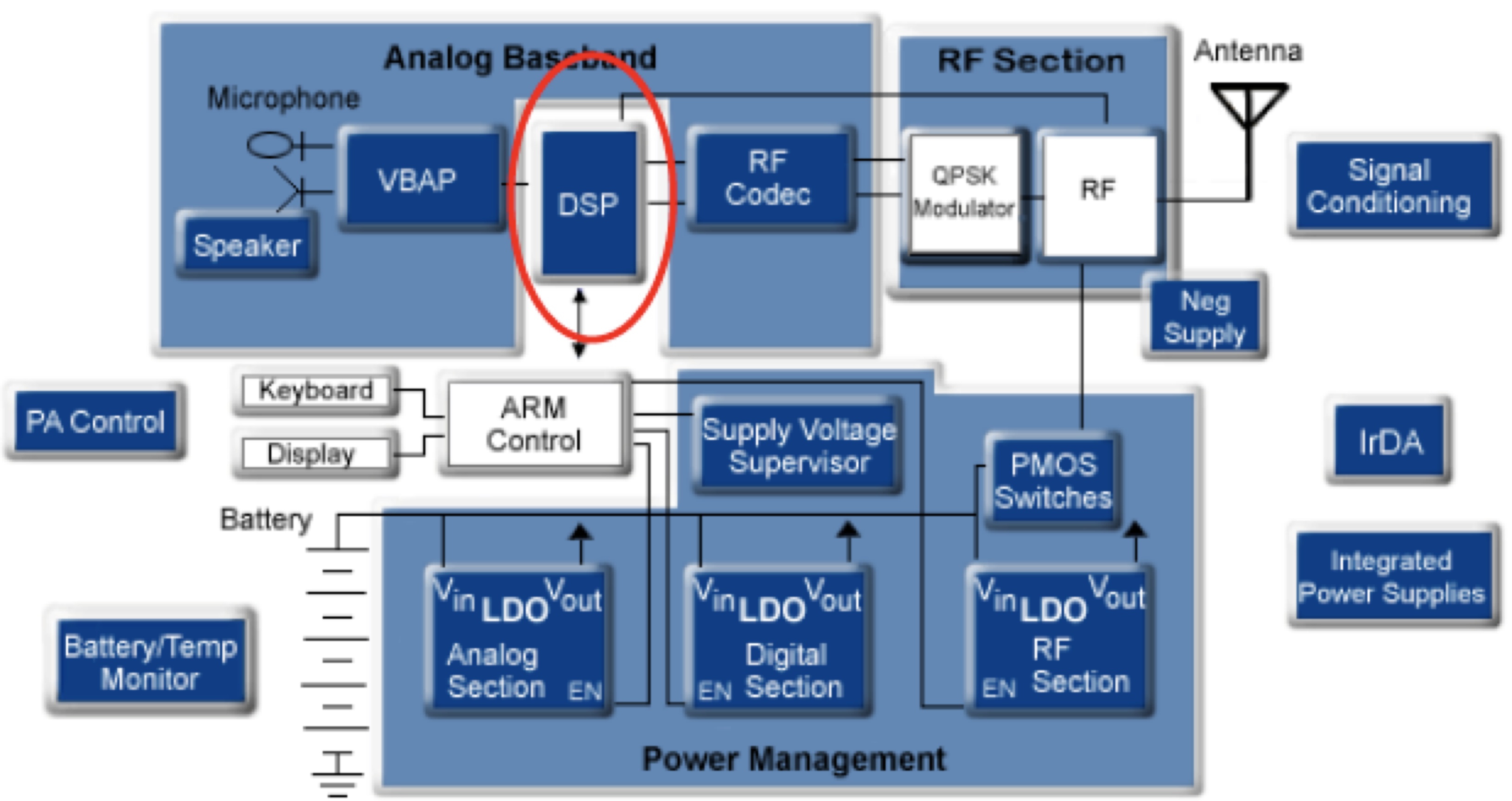


Seismic

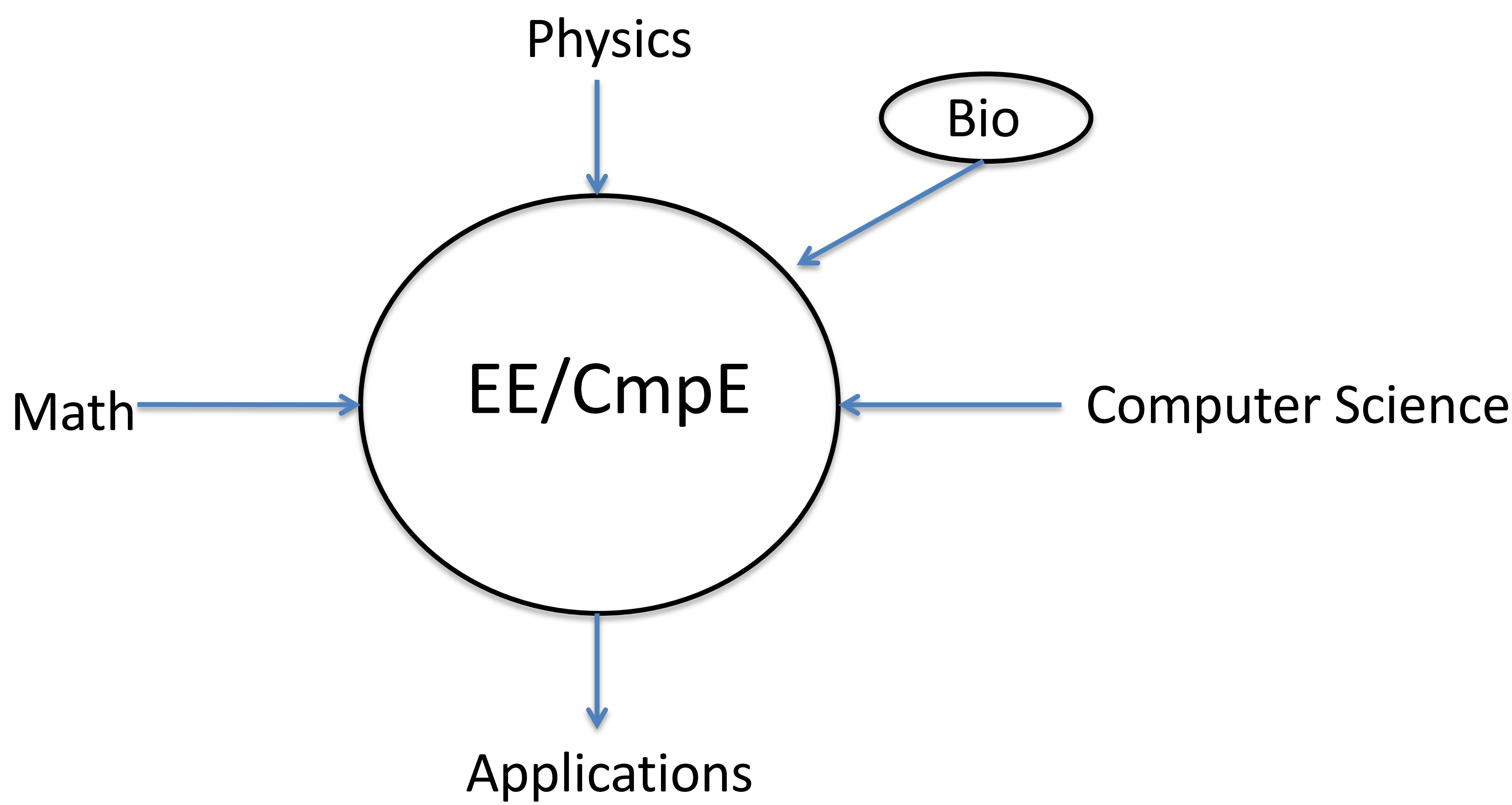


Telecommunications

“Phone” portion of Cellphone

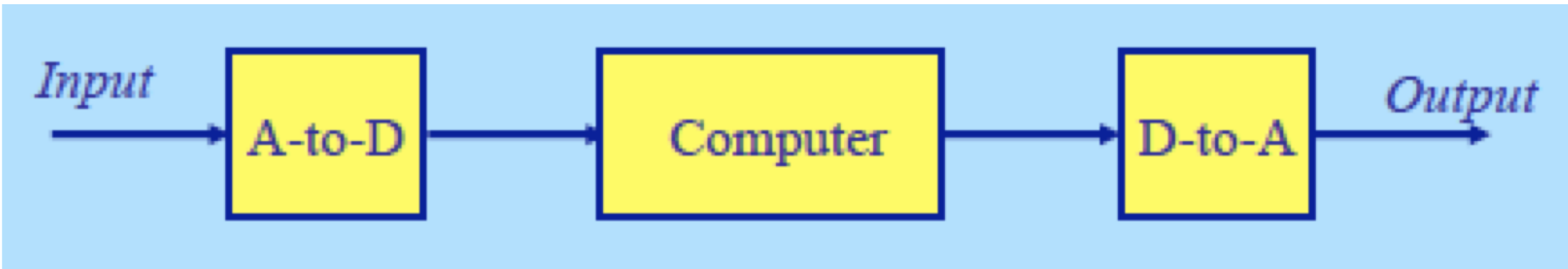


Convergence of Fields



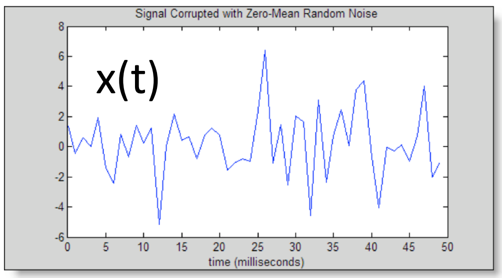
Physics, Acoustics, Medicine, Biology, Archaelogy, etc.

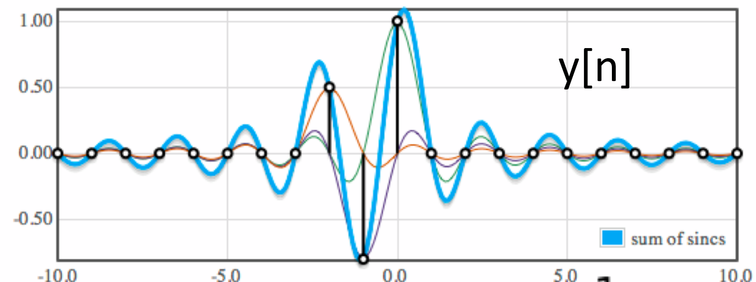
Why use DSP?

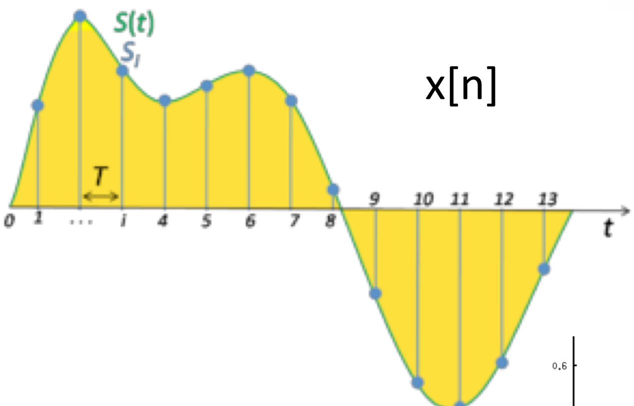


- Digital Signal Processing (DSP) is concerned with representation of signals in digital form and manipulating them with digital computation.
- Computer implementations are **flexible**
- Applications provide a **physical** context

x(t) x[n] h[n] y[n] y(t)

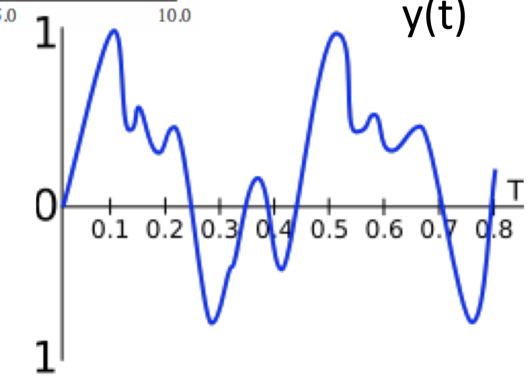




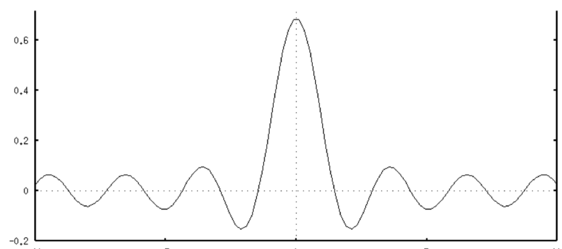


x[n]

y(t)



h[n]

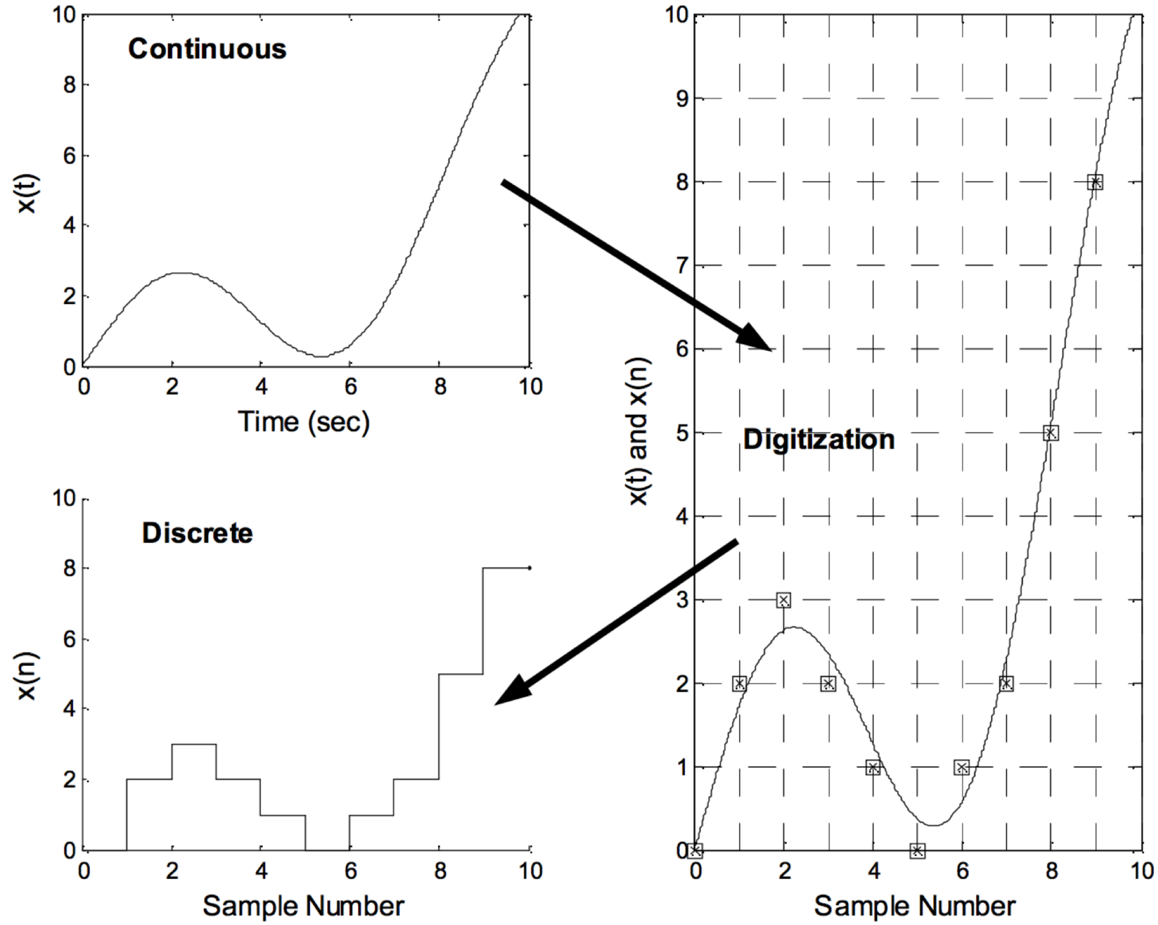


(all diagrams “symbolic” … does not represent the actual mathematical representations)

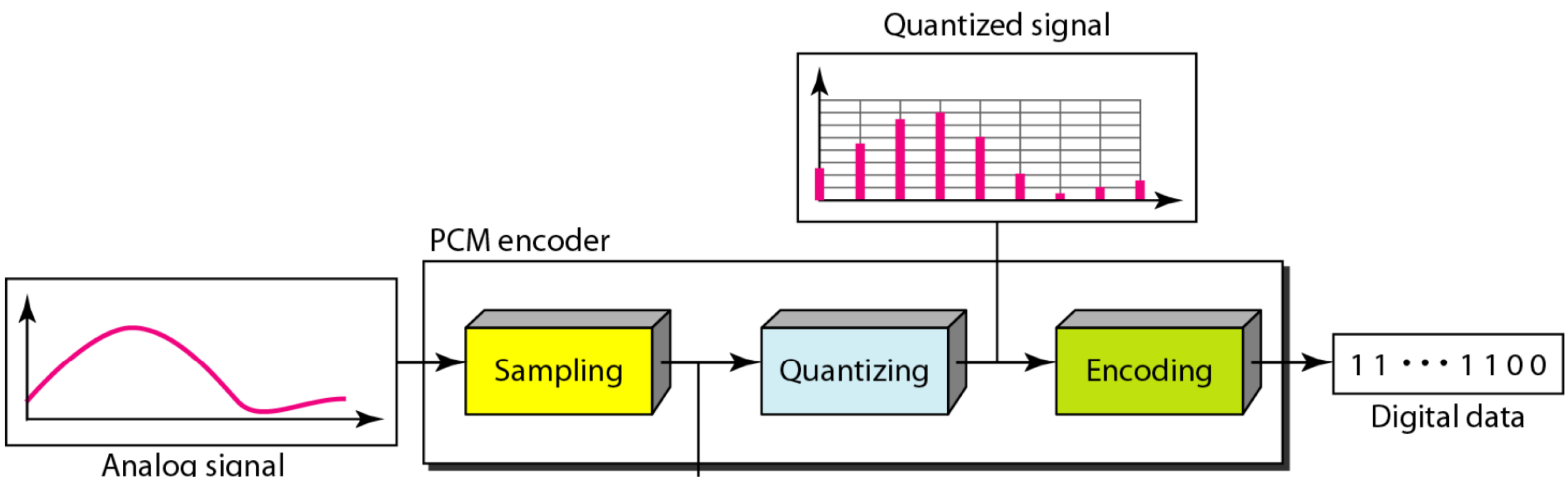
A-to-D different from ADC

Analog-to Digital Conversion

- ADC consists of four steps to digitize an analog signal:
  1. Filtering
  2. Sampling
  3. Quantization
  4. Binary encoding



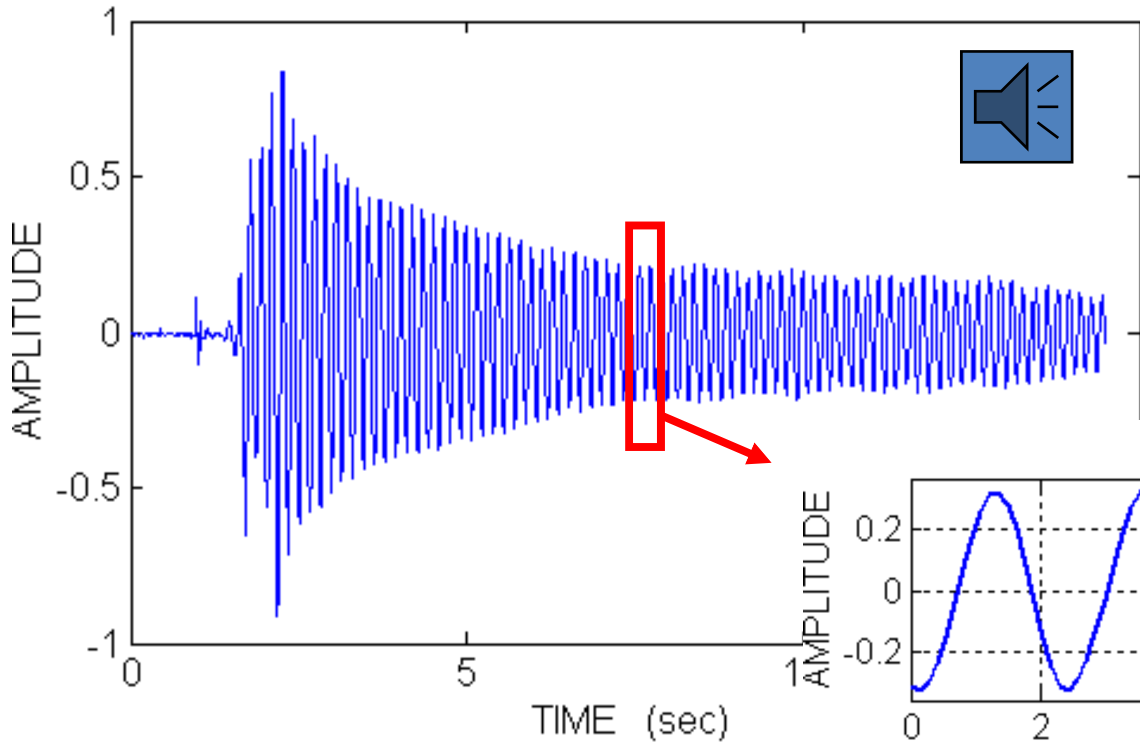
ADC


Quantized signal
PCM encoder
Sampling
Quantizing
Encoding
11 • • • 1100
Digital data
Analog signal

Storing Digital Sound

- ***x*[*n*]** is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
- 16-bit samples
- Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

Tuning Fork Example A-440 Hz



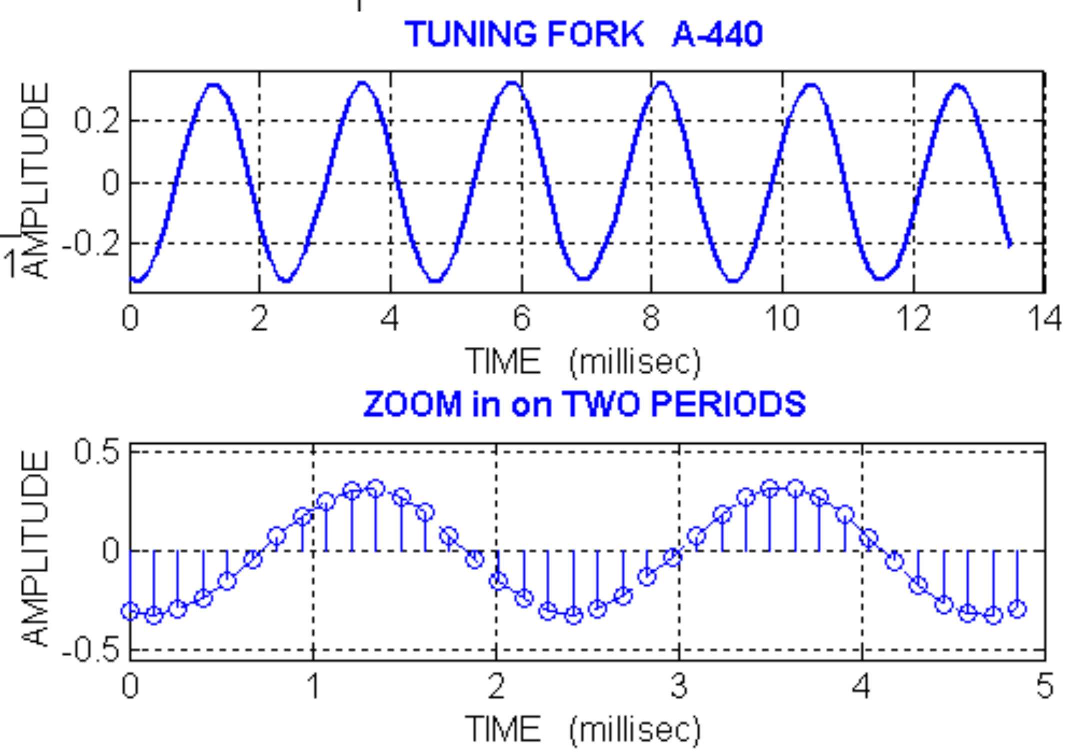
Estimate Period

Middle plot below gives

$T \approx 8.15\text{ms} - 5.85\text{ms} = 2.3\text{ms}$

Bottom plot below gives

$T \approx (2 + 2/7)\text{ ms} = 2.28\text{ms}$

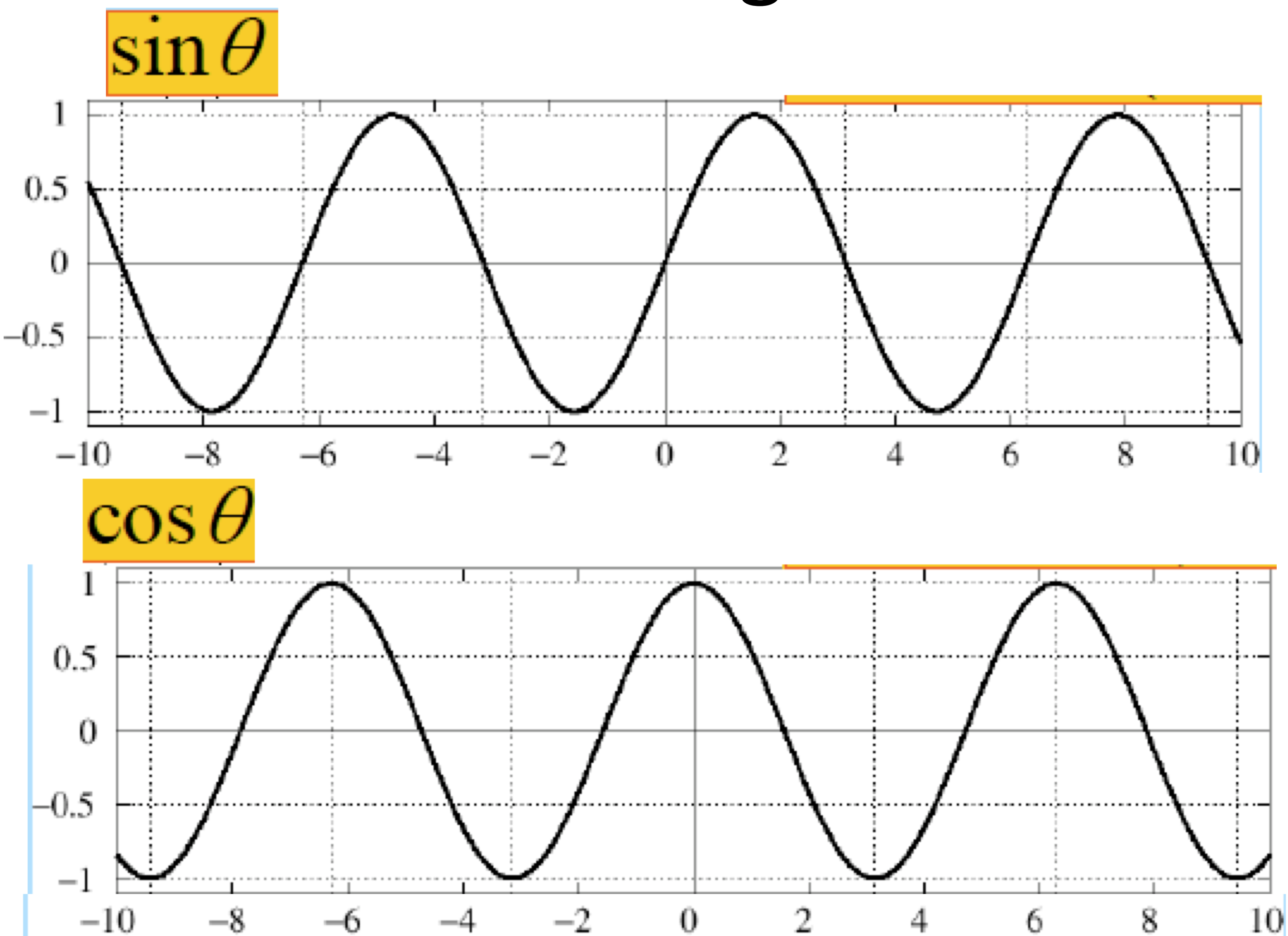


Estimate Frequency

$f_0 = 1 / T = 1 / (2.3\text{ms}) = 435\text{ Hz}$

$f_0 = 1 / T = 1 / (2.28\text{ms}) = 438\text{ Hz}$

Sines and cosines as a function of angle



Sine and Cosine Signals

- Make $\theta = \omega t + \varphi = \theta(t)$
- Always use the COSINE FORM

$$\boxed{\cos(\omega t + \varphi)}$$

- Sine is a special case:

$$\sin(\omega t) = \cos(\omega t - \pi/2)$$

Sinusoidal Signal

$$x(t) = A\cos(\omega t + \varphi)$$

- **FREQUENCY** $\boxed{\omega}$
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega = (2\pi)f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

- **AMPLITUDE** $\boxed{A}$
  - Magnitude

- **PHASE** $\boxed{\varphi}$

Make a plot

- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines A, ω, and ϕ

$$\boxed{\begin{aligned} A &= 5 \\ \omega &= 0.3\pi \\ \varphi &= 1.2\pi \end{aligned}}$$

Plotting Cosine signal from formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

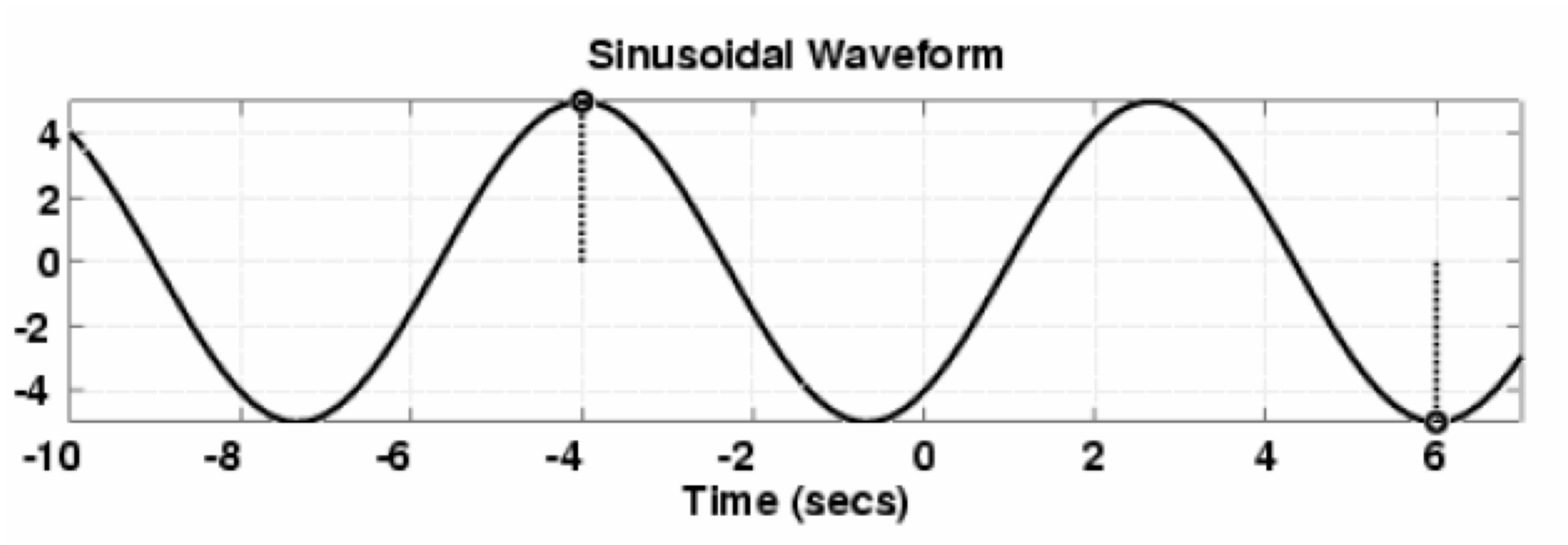
$$\boxed{T = 2\pi / \omega = 2\pi / 0.3\pi = 20 / 3}$$

- Determine a **peak** location by solving

$$\boxed{(\omega t + \varphi) = 0}$$

- **Zero** crossing is T/4 before or after
- Positive & Neg. peaks spaced by T/2

Result

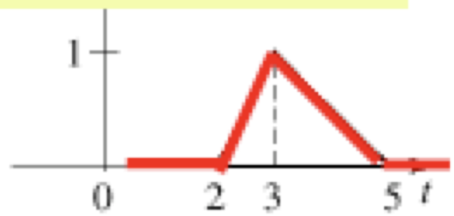


Time-Shift

- In a mathematical formula replace t with $t-t_d$ $\quad x(t) = s(t - t_d)$

$s(t)$

$x_1(t) = s(t-2)$



$x_2(t) = s(t+1$

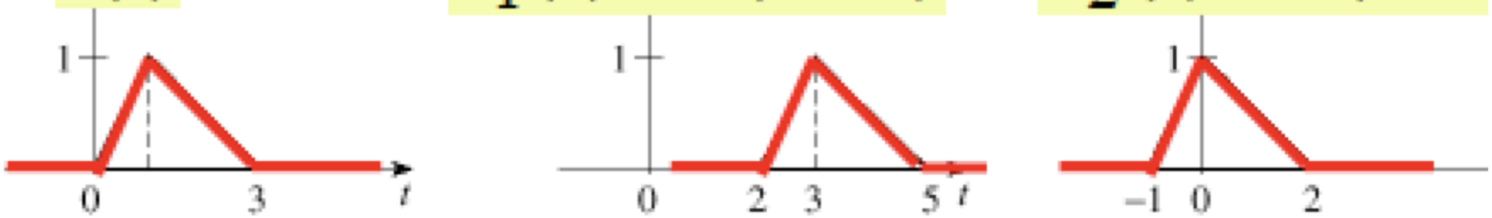
- Now try it on a cosine signal

$$x(t - t_d) = A\cos(\omega(t - t_d))$$

Time-Shift

- In a mathematical formula replace t with $t\text{-}t_d$ $\quad x(t) = s(t - t_d)$

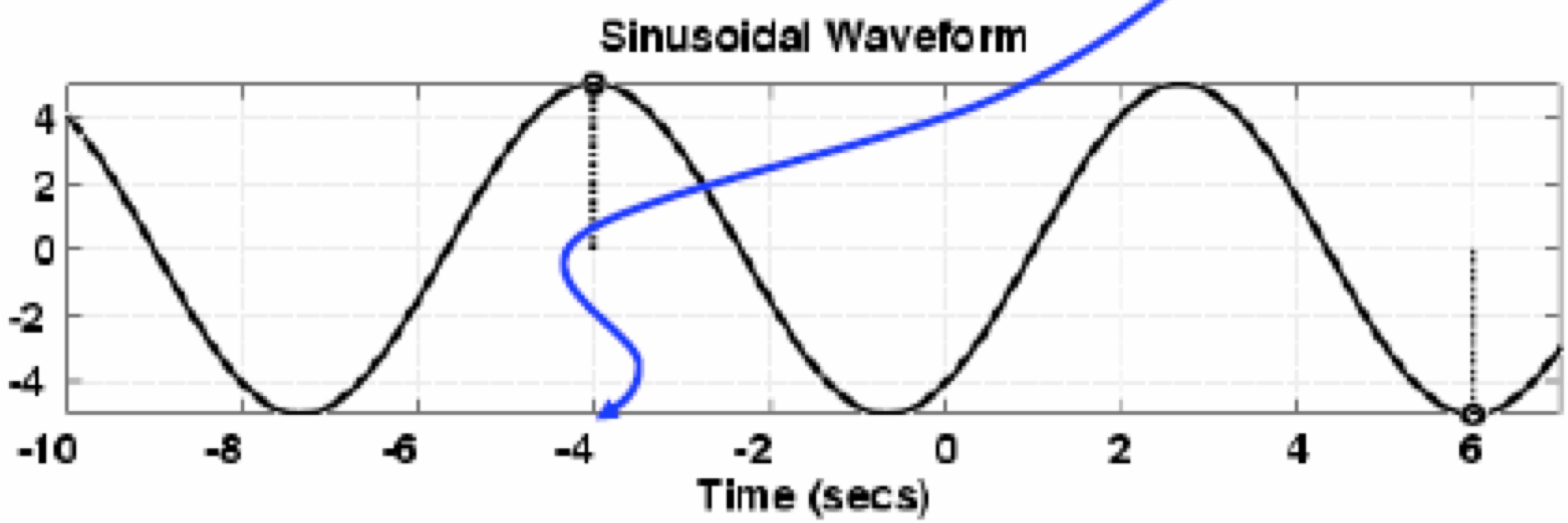
$s(t)$ $\quad x_1(t) = s(t-2)$ $\quad x_2(t) = s(t+1$



Time-shifted Sinusoid

- Then the $t=0$ point moves to $t=t_d$

$$x(t+4)=5\cos(0.3\pi(t+4))=5\cos(0.3\pi(t-(-4)))$$



Phase <-> Time Shift

- Equate the formulas:

$$A\cos(\omega(t - t_d)) = A\cos(\omega t + \varphi)$$

- and we obtain: $\boxed{-\omega t_d = \varphi}$

- or, $\boxed{t_d = \frac{-\varphi}{\omega}}$

Ex: Time-Shift from Phase

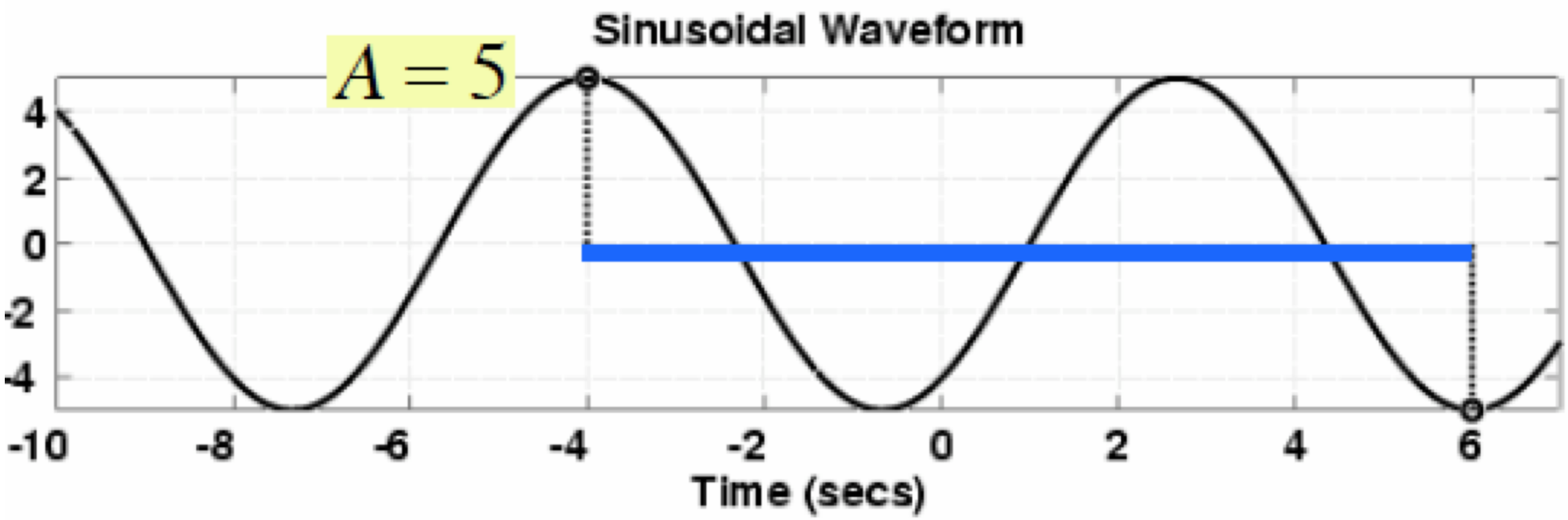
- Frequency : $\omega = .3\pi$ rad/s
- Phase: $\phi = 1.2\pi$ radians
- What is the time shift?
  - Also called the "time delay"
  - $t_d = -\phi/\omega = -(1.2\pi)/.3\pi$
  - **$t_d = -4$ sec.**
  - Note: $T = 2\pi/\omega = 20/3$ sec. (period)

Sinusoid from Plot

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of peak: t_d
  - Compute phase: $\phi = -\omega\, t_d$
- Measure height of positive peak: A

3 steps

(A, ω, ϕ) from a PLOT



$T = 10/(1.5) = 20/3 \longrightarrow \omega = 2\pi/T = 0.3\pi$

$t_d = -4 \longrightarrow \phi = -(-4)(0.3\pi) = 1.2\pi$

Phase is Ambiguous

- The cosine signal is periodic
  - Period is 2π

$$A\cos(\omega t+\varphi+2\pi)=A\cos(\omega t+\varphi)$$

  - Thus adding any multiple of 2π leaves $x(t)$ unchanged
  -

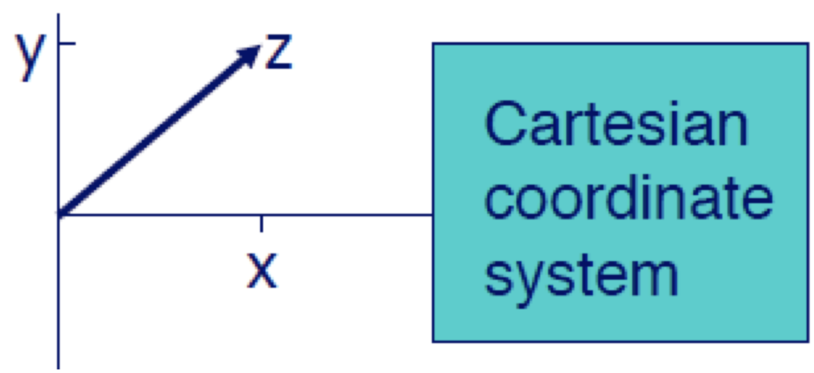
$$\text{if } t_d=\frac{-\varphi}{\omega}, \text{ then}$$

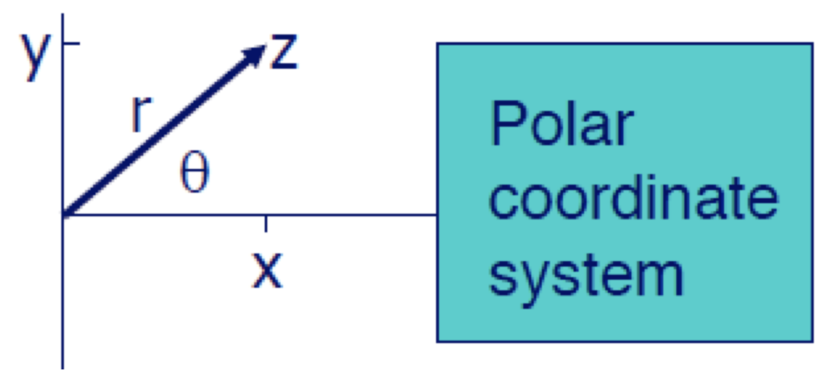
$$t_{d_2}=\frac{-(\varphi+2\pi)}{\omega}=\frac{-\varphi}{\omega}-\frac{2\pi}{\omega}=t_m-T$$

Complex Numbers

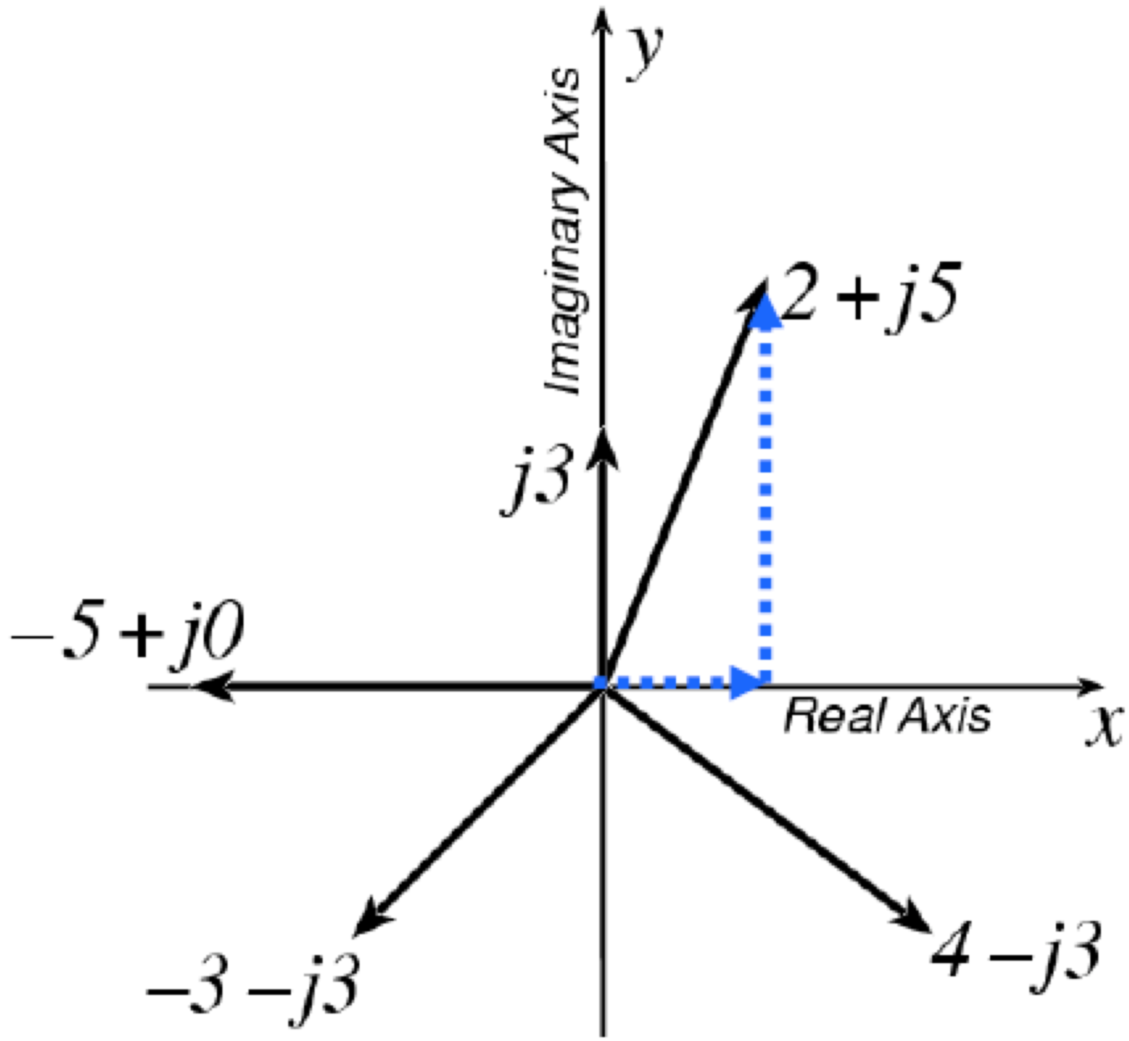
- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = (x,y) = x + \boldsymbol{j}y$ (cartesian)

  $Z = re^{j\theta}$ (polar)





Plot Complex Numbers

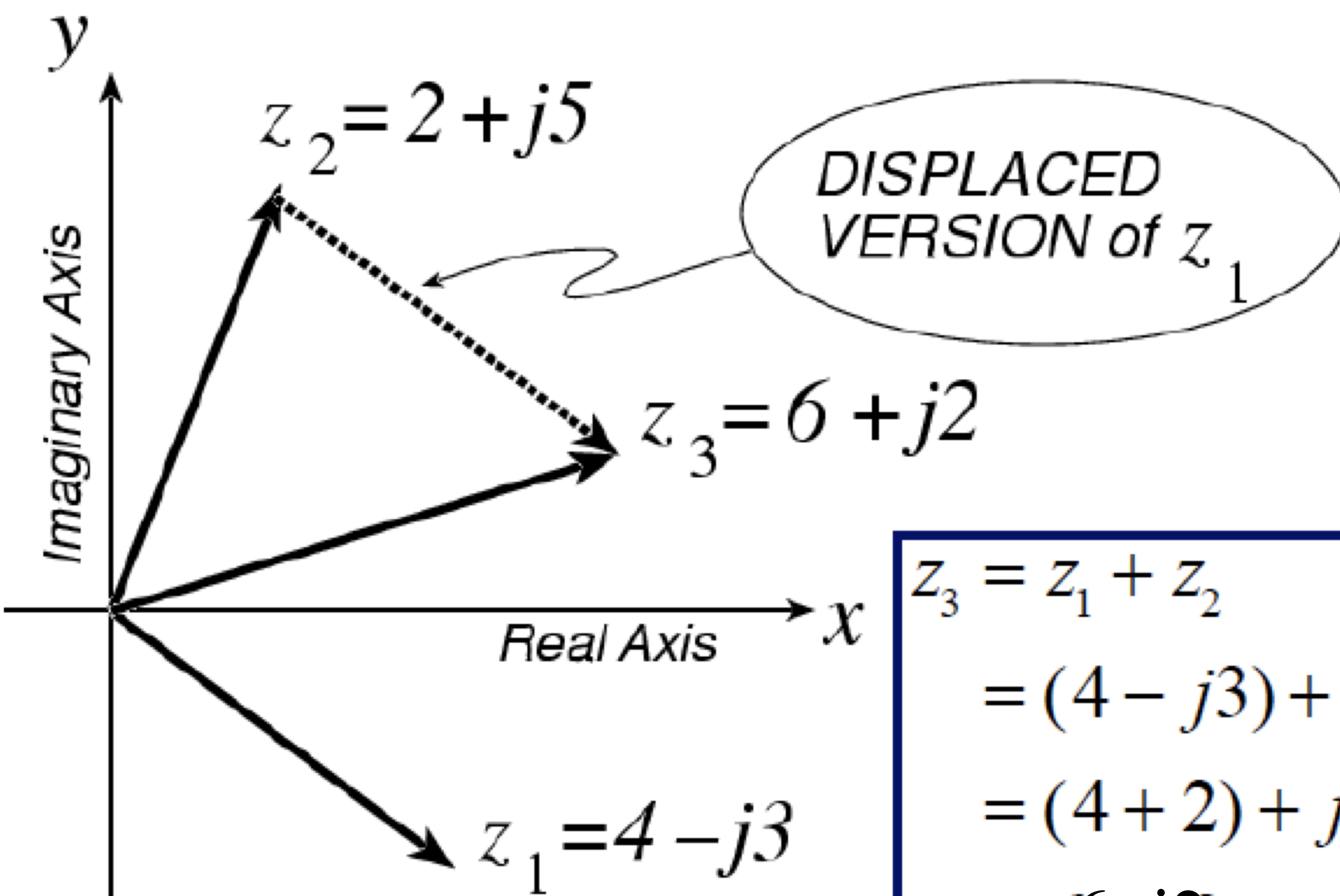


Complex Addition

- $z = z1+z2 = (a + jb) + (c + jd)$

  $= (a + c) + j(b + d)$

Complex Addition = Vector Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5)
\end{aligned}
$$

6+j2

Polar Coordinates

- Complex number can also be expressed as:

$$z(x,y) \leftrightarrow z(r,\theta)$$

r is the vector length

ϑ is the angle between the vector and the x-axis

Polar Coordinates (Cont.)

- Let's relate (x, y) to (r, θ)

$$z(x,y) = x + jy = z(r,\theta) = re^{j\theta}$$

$$r^2 = x^2 + y^2$$

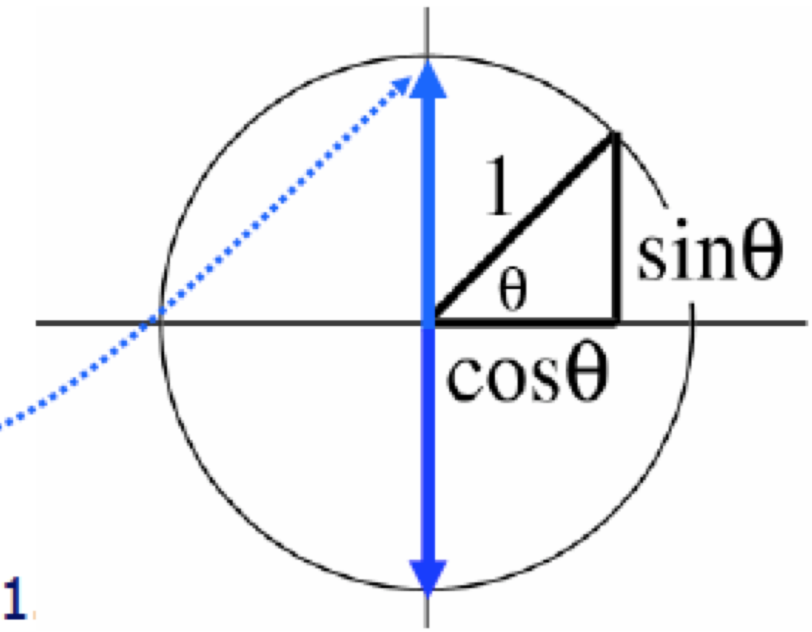
$$\theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Polar Form

- Vector Form
  - Length =1
  - Angle = θ
- Common Values
  - $\mathbf{j}$ has angle of 0.5π
  - -1 has angle of π
  - $-\mathbf{j}$ has angle of 1.5π
  - also, its angle is $-0.5\pi = 1$
  - AMBIGUOUS PHASE



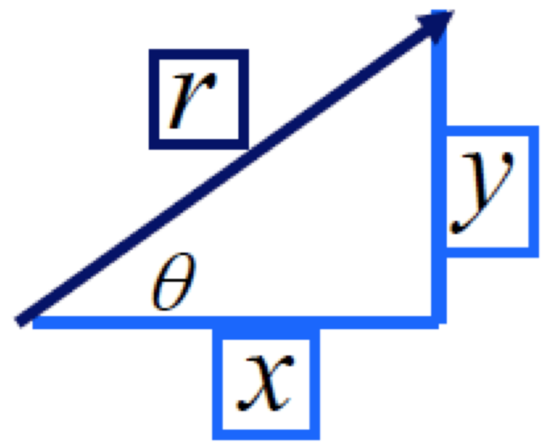
Polar <-> Rectangular

Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = x + jy = re^{j\theta}$$

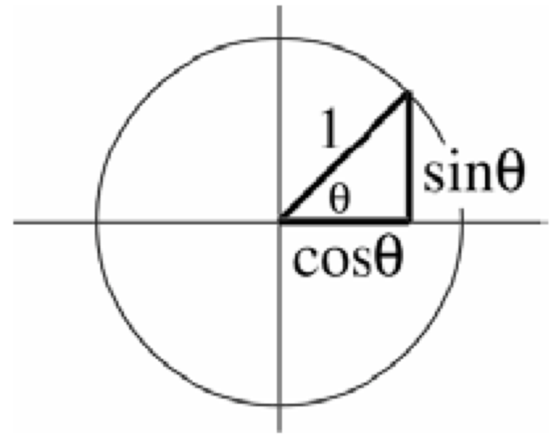


$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Euler's Formula

- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

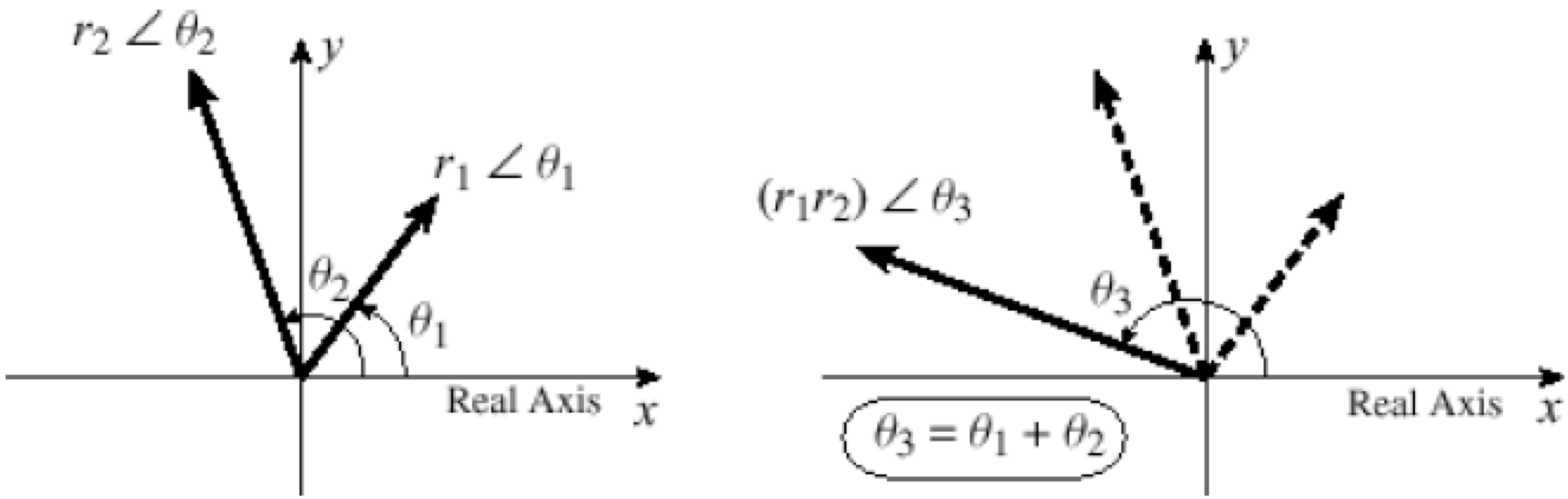
$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

Multiplication

- The polar form works best for multiplying complex numbers

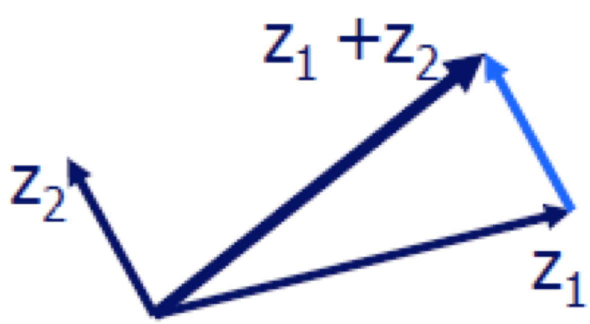
$$z_1 = r_1 e^{j\theta_1} \qquad z_2 = r_2 e^{j\theta_2}$$

$$z = z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}$$



Add Complex Numbers

- **VECTOR** Addition is necessary



- Example: $z = 4 - j3$, $w = 2 + j5$
  - $z + w = (4+2) + j(-3+5) = 6 + j2$
- Add sinusoids = add complex nums

Phasors for Complex exponential Signals

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids

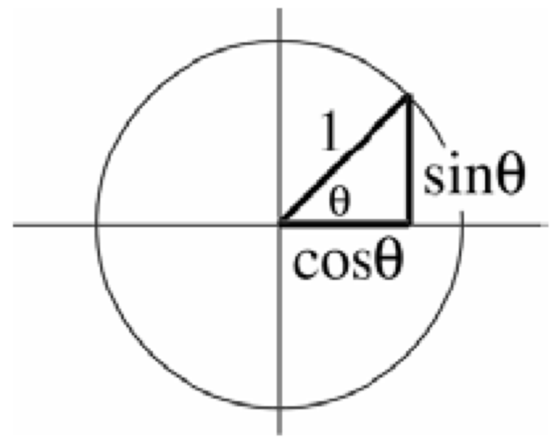
$$\boxed{z(t) = Xe^{j\omega t} = (Ae^{j\varphi})e^{j\omega t}}$$

- Develop the ABSTRACTION:
  - Adding Sinusoids = Complex Addition
  - **PHASOR ADDITION THEOREM**

Complex Exponential Signal

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

- **Rotating Vector**
  - Angle changes vs. time
  - $\theta = \omega \boldsymbol{t}$
  - ex: $\omega = 10\pi$
  - Rotates 0.1π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cosine = Real Part

Real Part of Euler's

$$\cos(\omega t) = \Re e\left\{e^{j\omega t}\right\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\left\{A e^{j(\omega t + \varphi)}\right\}$$

$$= \Re e\left\{A e^{j\varphi} e^{j\omega t}\right\}$$

Complex Amplitude

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Sinusoid = REAL PART of $(Ae^{j\phi})e^{j\omega t} = Ae^{j(\omega t+\phi)}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\{z(t)\}$$

Complex AMPLITUDE = *X* (PHASOR)

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

Avoid Trigonometry

- Algebra, even complex, is EASIER !!!
- Can you recall $\cos(\theta_1+\theta_2)$?
- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)$

$$e^{j(\theta_1+\theta_2)} = e^{j\theta_1} e^{j\theta_2}$$

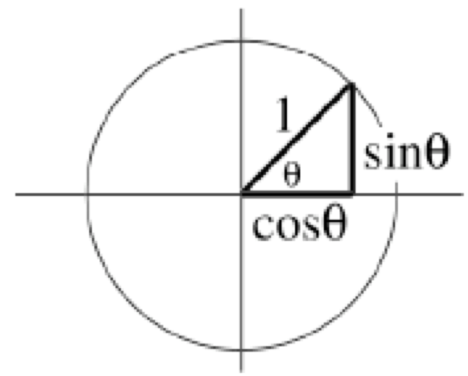
$$= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$

$$= \boxed{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)} + j(\ldots)$$

Phasor

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

- Interpret this as a **Rotating Vector**
  - $\boldsymbol{\theta = \omega t}$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Examples

- Find the COMPLEX AMPLITUDE for:

$$\boxed{x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)}$$

Solution

$$x(t) = \Re e\left\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\right\}$$

$$= \Re e\left\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\right\}$$

$$\boxed{X = \sqrt{3}e^{j0.5\pi}}$$

Add Sinusoids of Same Frequency

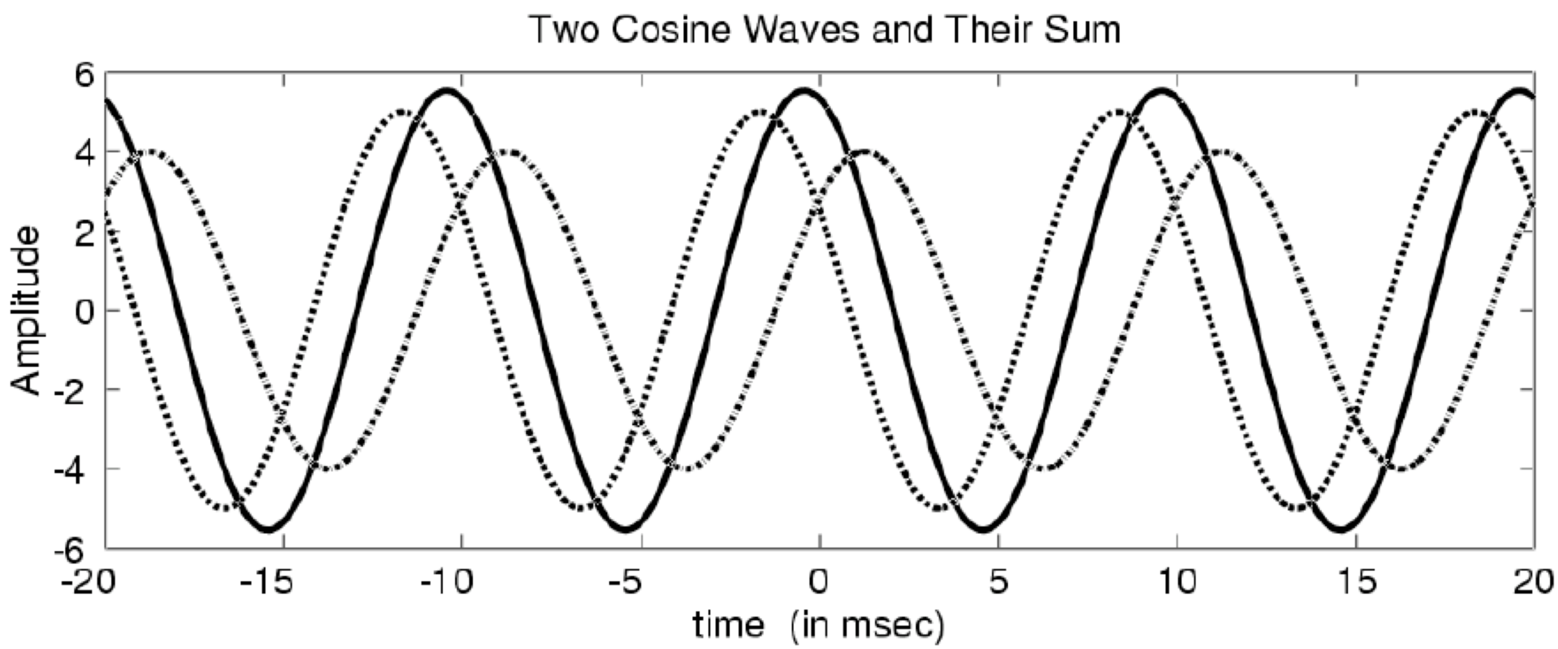
- ALL SINUSOIDS HAVE SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$\begin{aligned} x_3(t) &= x_1(t) + x_2(t) \\ &= 1.532\cos(2\pi(10)t + 141.79\pi/180) \end{aligned}$$

- Sum Sinusoid has **SAME** Frequency



Phasor AdditionRule

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= A\cos(\omega_0 t + \phi)$$

<u>Get the new complex amplitude by complex addition</u>

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi}$$

Phasor Addition Proof

$$
\begin{aligned}
\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) &= \sum_{k=1}^{N} \Re e\left\{A_k e^{j(\omega_0 t + \phi_k)}\right\} \\
&= \Re e\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\} \\
&= \Re e\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\} \\
&= \Re e\left\{\left(A e^{j\phi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)
\end{aligned}
$$

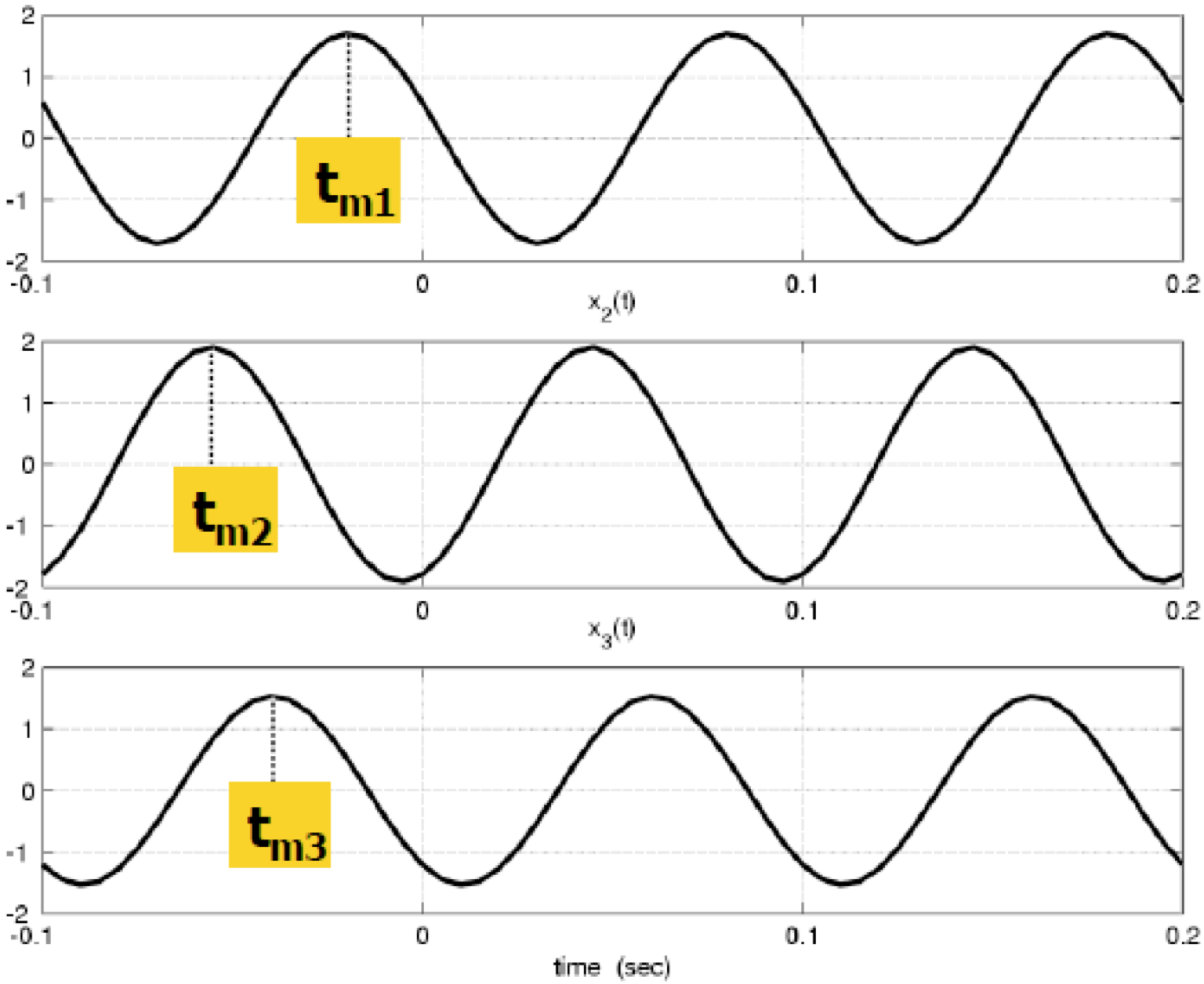
Example of Adding Two Complex Signals of the same frequency

- ADD THESE 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t)$$

$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

Amplitudes of Signals?



Convert Time-Shift to Phase

- Measure peak times:
  - t_{m1}=-0.0194, t_{m2}=-0.0556, t_{m3}=-0.0394
- Convert to phase (T=0.1)
  - $\phi_1 = -\omega t_{m1} = -2\pi(t_{m1}/T) = 70\pi/180$,
  - $\phi_2 = 200\pi/180$
- Amplitudes
  - A_1=1.7, A_2=1.9, A_3=1.532

Spectrum Representation for Sinusoids of Different Frequencies

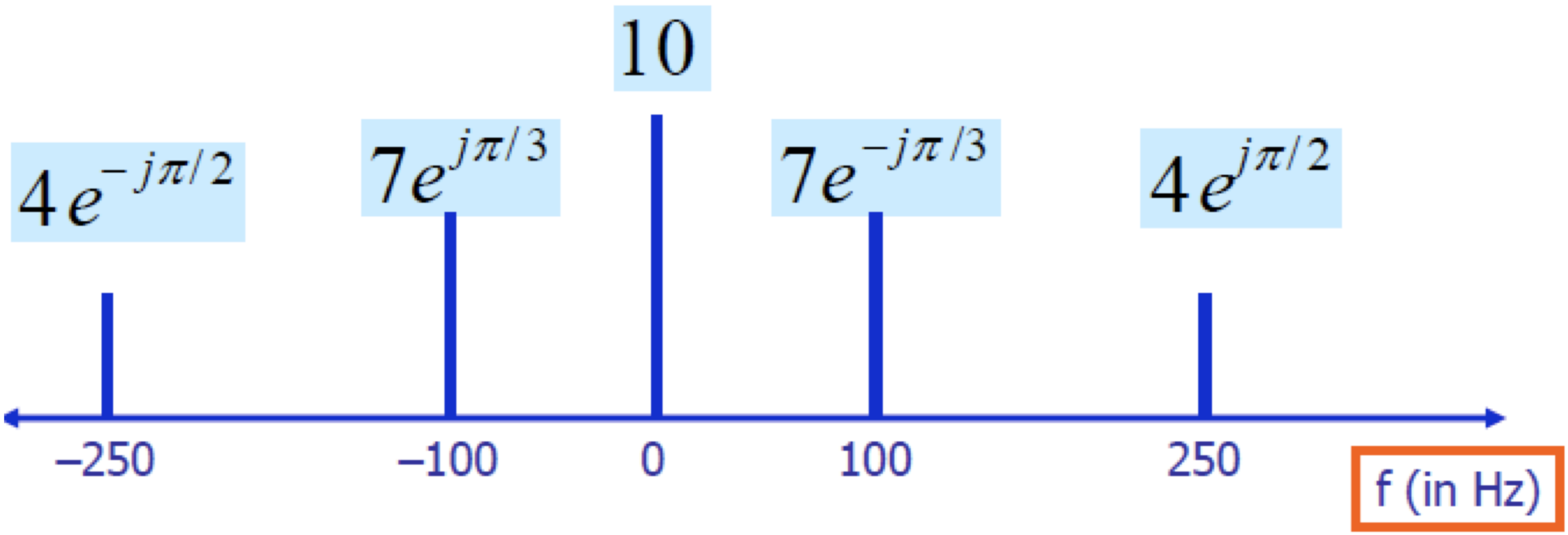
- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

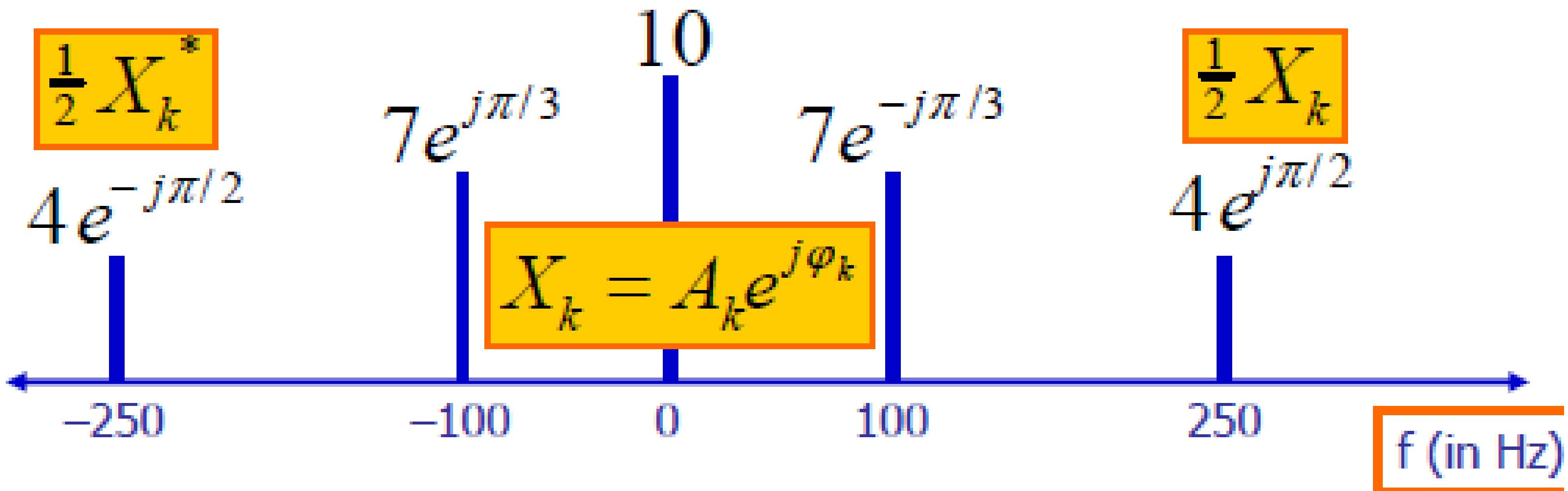
- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

Frequency Diagram

- Plot Complex Amplitude vs. Freq
- Example



Sum of Sinusoids (Cont.)



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Harmonic Signals

- All frequencies are the multiplication of the fundamental frequency f_0

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

- $\omega_0 = 2\pi f_0 t$

Another Frequency Diagram

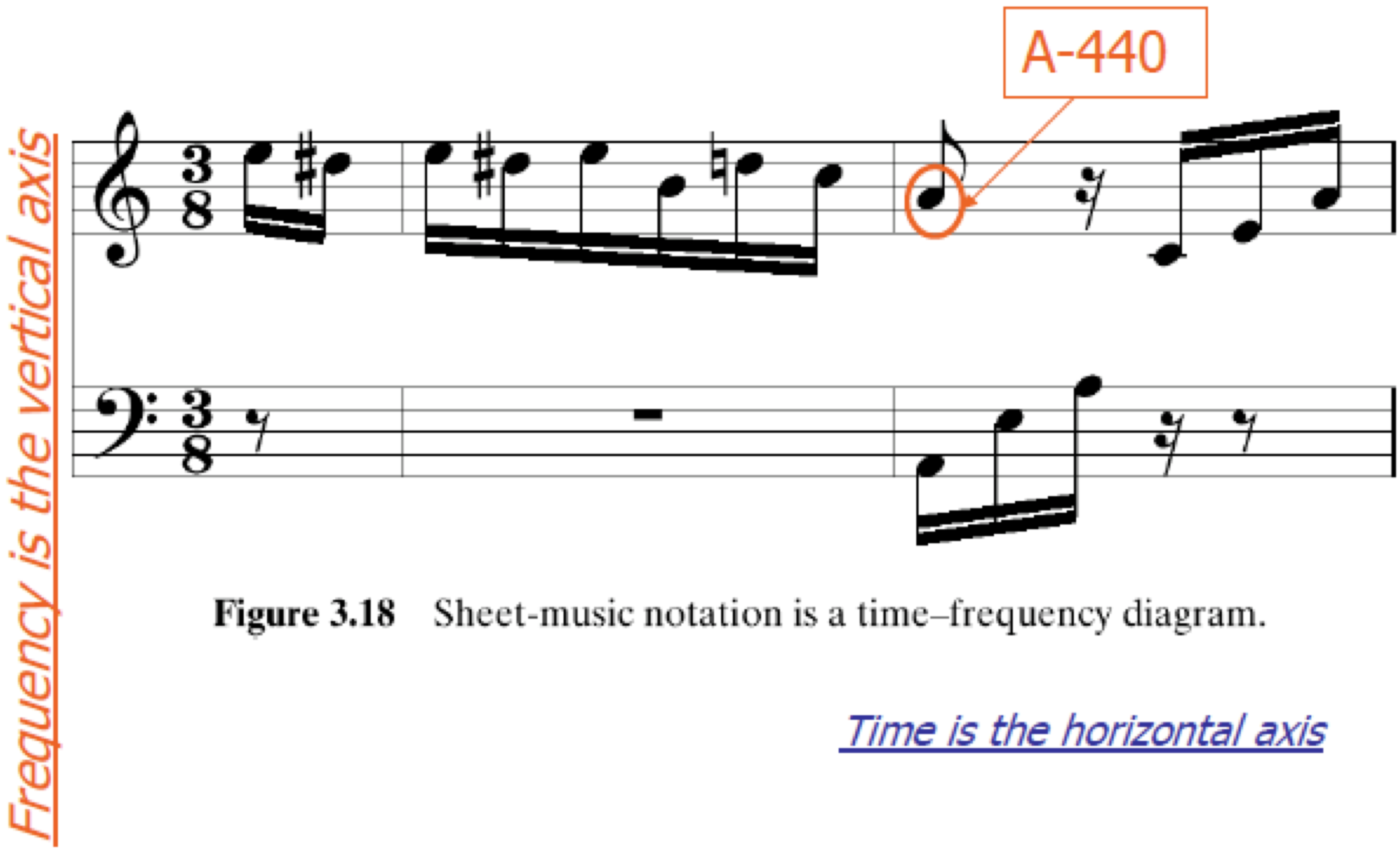


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

Summary

- Signal Definition
- Advances in DSP
- Complex numbers
- Vector addition
- Vector multiplication
- Use of Complex numbers to sum sinusoids
- Introduction to Spectrum representation