ECES-352
Summer 2014
Homework #7
Solutions

PROBLEM 7.1:

Suppose that a discrete-time system is described by the input-output relation

$$y[n] = (x[n])^3$$

(a) Determine the output when the input is the complex exponential signal

$$x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$$

(b) Is the output of the form

$$y[n] = \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

If not, why not?

7.1 $y[n] = (x[n])^3$

(a) $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$

$\Rightarrow \quad y[n] = \left(Ae^{j\phi}e^{j\hat{\omega}n}\right)^3 = A^3 e^{j3\phi} e^{j3\hat{\omega}n}$

(b) The output cannot be expressed in the form

$$y[n] = \mathcal{H}(\hat{\omega})Ae^{j\phi}e^{j\hat{\omega}n}$$

since the output frequency is $3\hat{\omega}$ instead of $\hat{\omega}$. This occurs because the system is nonlinear.

PROBLEM 7.2*:

A discrete-time system is defined by the input/output relation

$$y[n] = x[n+1] - 2x[n] + x[n-1]. \quad (1)$$

(a) Determine whether or not the system defined by Equation (1) is (i) linear; (ii) time-invariant; (iii) causal. Explain your answers.

(b) Obtain an expression for the frequency response of this system.

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(d) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.25\pi n) = e^{j0.25\pi n} + e^{-j0.25\pi n}.$$

Express your answer in terms of cosine functions.

(e) For the system of Equation (1), determine the output $y_2[n]$ when the input is

$$x_2[n] = 1 + \cos(0.25\pi(n-1)).$$

Hint: use the linearity and time-invariance properties.

7.2

$$y[n] = x[n+1] - 2x[n] + x[n-1]$$

(a) The system is:

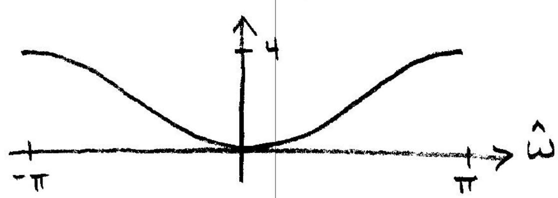
- linear, because the output is computed by taking a linear combination of inputs;
- time-invariant, because the coefficients of the linear combination are all constants;
- non-causal, because the present value of the output depends on future input values.

(b) $x[n] = A e^{j\phi} e^{j\hat{\omega} n} \Rightarrow y[n] = \mathcal{H}(\hat{\omega}) A e^{j\phi} e^{j\hat{\omega} n}$

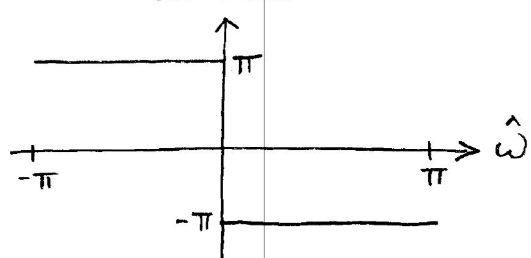
where $\mathcal{H}(\hat{\omega}) = e^{j\hat{\omega}} - 2 + e^{-j\hat{\omega}}$

(c) $\mathcal{H}(\hat{\omega}) = 2\cos\hat{\omega} - 2 = 2(1-\cos\hat{\omega})\, e^{\pm j\pi}$

$|\mathcal{H}(\hat{\omega})|$



$\angle \mathcal{H}(\hat{\omega})$



(d)

$x_1[n] = 2\cos\left(\frac{\pi}{4} n\right)$

$\mathcal{H}\left(\frac{\pi}{4}\right) = 2\left(1 - \frac{1}{\sqrt{2}}\right) e^{j\pi}$

$y_1[n] = 4\left(1 - \frac{1}{\sqrt{2}}\right) \cos\left(\frac{\pi}{4} n + \pi\right)$

(e)

$x_2[n] = 1 + \cos\left(\frac{\pi}{4}(n-1)\right)$

$\mathcal{H}(0) = 0$

$y_2[n] = \frac{1}{2} y_1[n-1] = 2\left(1 - \frac{1}{\sqrt{2}}\right) \cos\left(\frac{\pi}{4} n + \frac{3\pi}{4}\right)$

PROBLEM 7.3:

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

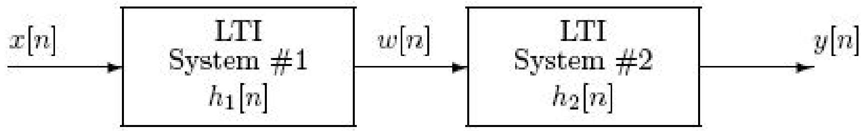


Figure 1: Cascade connection of two LTI systems.

(a) Suppose that the two LTI systems are described by the impulse responses

$$h_1[n] = \delta[n] - \delta[n-1] \qquad \text{and} \qquad h_2[n] = u[n] - u[n-10].$$

(b) Determine $H_1(\hat{\omega})$, the frequency response of the first system.

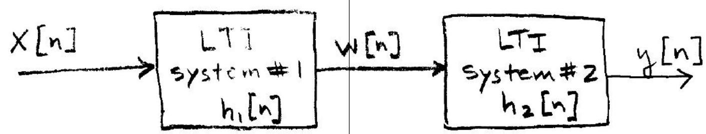
(c) Determine $H_2(\hat{\omega})$, the frequency response of the second system.

(d) By using numerical convolution, show that $h[n] = h_1[n] * h_2[n] = \delta[n] - \delta[n-10]$.

(e) From $h[n]$ determine $H(\hat{\omega})$ the frequency response of the overall system (from $x[n]$ to $y[n]$).

(f) Show that your result in part (d) is the product of the results in parts (a) and (b); i.e., $H_1(\hat{\omega})H_2(\hat{\omega}) = H(\hat{\omega})$.

7.3



(a) $h_1[n] = \delta[n] - \delta[n-1]$ $\qquad$ $h_2[n] = u[n] - u[n-10]$

$\Downarrow$ $\qquad$ $\Downarrow$

$\{b_0, b_1\}_{\#1} = \{1, -1\}$ $\qquad$ $\{b_0, \ldots, b_9\}_{\#2} = \{1, \ldots, 1\}$

(b) $\mathcal{H}_1(\hat{\omega}) = 1 - e^{-j\hat{\omega}}$

(c) $\mathcal{H}_2(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + \cdots + e^{-j9\hat{\omega}}$

(d)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$h_2[n]$	1	1	1	1	1	1	1	1	1	1			
$h_1[n]$	1	-1											
	1	1	1	1	1	1	1	1	1	1			
		-1	-1	-1	-1	-1	-1	-1	-1	-1	-1		
$h[n]$	1	0	0	0	0	0	0	0	0	0	-1	0	0

$h[n] = \delta[n] - \delta[n-10] \Rightarrow \{b_0, \ldots, b_{10}\} = \{1, 0, \ldots, 0, -1\}$

(e) $\mathcal{H}(\hat{\omega}) = 1 - e^{-j10\hat{\omega}}$

(f)
$$\begin{aligned}
\mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega}) &= (1 - e^{-j\hat{\omega}})(1 + e^{-j\hat{\omega}} + \cdots + e^{-j9\hat{\omega}}) \\
&= 1 + e^{-j\hat{\omega}} + \cdots + e^{-j9\hat{\omega}} - (e^{-j\hat{\omega}} + \cdots + e^{-j10\hat{\omega}}) \\
&= 1 - e^{-j10\hat{\omega}}
\end{aligned}$$

PROBLEM 7.4*:

Suppose that three systems are hooked together in "cascade." In other words, the output of $\mathcal{S}_1$ is the input to $\mathcal{S}_2$, and the output of $\mathcal{S}_2$ is the input to $\mathcal{S}_3$. The three systems are specified as follows:

$$\begin{aligned}
\mathcal{S}_1: &\qquad \mathcal{H}_1(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} \\
\mathcal{S}_2: &\qquad y_2[n] = x_2[n] + x_2[n-2] \\
\mathcal{S}_3: &\qquad y_3[n] = 2x_3[n-1] + 2x_3[n-2]
\end{aligned}$$

NOTE: the output of $\mathcal{S}_i$ is $y_i[n]$ and the input is $x_i[n]$.

The objective in this problem is to determine the equivalent system that is a single operation from the input $x[n]$ (into $\mathcal{S}_1$) to the output $y[n]$ which is the output of $\mathcal{S}_3$. Thus $x[n]$ is $x_1[n]$ and $y[n]$ is $y_3[n]$.

(a) Determine the difference equation for $\mathcal{S}_1$, i.e., express $y_1[n]$ in terms of $x_1[n]$, $x_1[n-1]$, $x_1[n-2]$, etc.

(b) Determine the frequency response of the other two systems: $\mathcal{H}_i(\hat{\omega})$ for $i = 2, 3$.

(c) Determine the frequency response of the overall cascaded system.

(d) Write *one difference equation* that defines the overall system in terms of $x[n]$ and $y[n]$ only.

7.4

$$x_1[n] \longrightarrow \boxed{S_1} \xrightarrow{y_1[n]=x_2[n]} \boxed{S_2} \xrightarrow{y_2[n]=x_3[n]} \boxed{S_3} \longrightarrow y_3[n]$$

(a) S_1: $\mathcal{H}_1(\hat{\omega}) = e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}$

$$\Rightarrow \{b_k\}_{\#1} = \{0, 1, -1\}$$

$$y_1[n] = x_1[n-1] - x_1[n-2]$$

(b) S_2: $y_2[n] = x_2[n] + x_2[n-2]$

$$\Rightarrow \{b_k\}_{\#2} = \{1, 0, 1\}$$

$$\mathcal{H}_2(\hat{\omega}) = 1 + e^{-j2\hat{\omega}}$$

S_3: $y_3[n] = 2x_3[n-1] + 2x_3[n-2]$

$$\Rightarrow \{b_k\}_{\#3} = \{0, 2, 2\}$$

$$\mathcal{H}_3(\hat{\omega}) = 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}}$$

(c)

$$\begin{aligned}\mathcal{H}(\hat{\omega}) &= \mathcal{H}_1(\hat{\omega})\,\mathcal{H}_2(\hat{\omega})\,\mathcal{H}_3(\hat{\omega}) \\ &= \left(e^{-j\hat{\omega}} - e^{-j2\hat{\omega}}\right)\left(1 + e^{-j2\hat{\omega}}\right)\left(2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}}\right) \\ &= 2\left(e^{-j2\hat{\omega}} - e^{-j6\hat{\omega}}\right)\end{aligned}$$

(d) $\{b_k\} = \{0, 0, 2, 0, 0, 0, -2\}$

$$y[n] = 2x[n-2] - 2x[n-6]$$

The frequency response of a linear time-invariant filter is given by the formula

$$H(\hat{\omega}) = (1 + e^{-j\hat{\omega}})(1 - e^{-j\pi/3}e^{-j\hat{\omega}})(1 - e^{j\pi/3}e^{-j\hat{\omega}}). \tag{2}$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. *Hint:* Multiply out the factors to obtain a sum of powers of $e^{-j\hat{\omega}}$.

(b) What is the impulse response of this system?

(c) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of $-\pi \le \hat{\omega} \le \pi$ will $y[n] = 0$ for all n?

(d) Use superposition to determine the output of this system when the input is

$$x[n] = 2 - 3\delta[n-4] + 7\cos(\pi/3n) \qquad \text{for } -\infty < n < \infty$$

Hint: Divide the input into three parts and find the outputs separately each by the easiest method and then add the results.

Problem 7.5

a)

$$H(w) = \left(1 - e^{-j\frac{\pi}{3}} e^{-jw} + e^{-jw} - e^{-j\frac{\pi}{3}} e^{-j2w}\right)\left(1 - e^{j\frac{\pi}{3}} e^{-j}\right.$$

$$= 1 - e^{-j\frac{\pi}{3}} e^{-jw} + e^{-jw} - e^{-j\frac{\pi}{3}} e^{-j2w}$$

$$- e^{j\frac{\pi}{3}} e^{-jw} + e^{-j2w} - e^{j\frac{\pi}{3}} e^{-j2w} + e^{-j3w}$$

$$= 1 + \left(-e^{-j\frac{\pi}{3}} + 1 - e^{j\frac{\pi}{3}}\right) e^{-jw}$$

$$+ \left(-e^{-j\frac{\pi}{3}} - e^{+j\frac{\pi}{3}} + 1\right) e^{-j2w} + e^{-j3w}$$

$$= 1 + \left(1 - 2\cos\frac{\pi}{3}\right) e^{-jw} + \left(1 - 2\cos\frac{\pi}{3}\right) e^{-jw} + e^{-j3w}$$

$$= 1 + e^{-j3w}$$

b) $h[n] = \delta[n] + \delta[n-3]$

c) $y[n] = 0 \Rightarrow Y(w) = 0 \Rightarrow Y(w) = X(w)H(w) = 0$

$\Rightarrow$ $X(w)$ is not zero, so $H(w)$ must be zero

$$(1 + e^{-jw})(1 - e^{-j\frac{\pi}{3}} e^{-jw})(1 - e^{j\frac{\pi}{3}} e^{-jw}) = 0$$

so one of these 3 terms must be zero.

$$e^{-j\omega} + 1 = 0 \Rightarrow \boxed{\omega = \pm \pi}$$

or

$$1 - e^{-j\frac{\pi}{3}} e^{-j\omega} = 0 \Rightarrow \boxed{\omega = -\pi/3}$$

or

$$1 - e^{j\frac{\pi}{3}} e^{-j\omega} = 0 \Rightarrow \boxed{\omega = \pi/3}$$

d) $$x[n] = \underbrace{2}_{x_1} \underbrace{- 3\delta[n-4]}_{x_2} + \underbrace{\frac{7}{2} e^{j\frac{\pi}{3}n}}_{x_3} + \underbrace{\frac{7}{2} e^{-j\frac{\pi}{3}n}}_{x_4}$$

$$y_1[n] = 2 \times H(0) = 4$$

$$y_2[n] = -3\delta[n-4] - 3\delta[n-7]$$

$$y_3[n] = x_3[n]\, H\left(\frac{\pi}{3}\right) = \frac{7}{2} e^{j\frac{\pi}{3}n} \overset{0}{\cancel{\left(1 + e^{-j3\frac{\pi}{3}}\right)}} = 0$$

$$y_4[n] = x_4[n]\, H\left(-\frac{\pi}{3}\right) = \frac{7}{2} e^{j\frac{\pi}{3}n} \overset{0}{\cancel{\left(1 + e^{j3\frac{\pi}{3}}\right)}} = 0$$

$$\boxed{y[n] = 4 - 3\delta[n-4] - 3\delta[n-7]}$$