ECES-352
Winter 2013
Homework #6
DUE: Feb. 19

Reading: Chapter 5 on FIR Filters

PROBLEM 6.1:

For each of the following systems, determine if they are (1) linear; (2) time-invariant; (3) causal.

(a) $y[n] = 3x[n-1] + x[n] + 3x[n+1]$

(b) $y[n] = x[n]\cos(.3\pi n)$

(c) $y[n] = |x[-n]|$

(d) $y[n] = x[n-2] + 2x[n] + x[n+2]$

(e) $y[n] = nx[n]$

(f) $y[n] = (x[-n])^2$

6.1 a) $y[n] = 3x[n-1] + x[n] + 3x[n+1]$

LTI AND NON CAUSAL

LTI BECAUSE THIS IS IN THE FORM OF A GENERAL FIR. SEE SECTION 5.3.3 OF DSP FIRST. NON CAUSAL BECAUSE DEPENDS UPON $[n+1]$.

6.1 b) $y[n] = x[n] \cos(0.3\pi n)$

LINEAR, NOT TIME INVARIANT, AND CAUSAL

PROOFS

LINEARITY TEST (SEE FIG 5.17 OF DSP FIRST FOR BLOCK DIAGRAM DESCRIBING $w[n]$ & $y[n]$)

$w[n] = (\alpha x_1[n] + \beta x_2[n]) \cos(0.3\pi n)$

$y[n] = \alpha x_1[n] \cos(0.3\pi n) + \beta x_2[n] \cos(0.3\pi n)$

BY INSPECTION: $w[n] = y[n]$, $\therefore$ LINEAR

TIME-INVARIANCE TEST (SEE FIG 5.16 OF DSP FIRST FOR BLOCK DIAGRAM DESCRIBING $w[n]$ & $y[n]$)

$w[n] = x[n-n_0] \cos(0.3\pi n)$

$y[n-n_0] = x[n-n_0] \cos(0.3\pi (n-n_0))$

BY INSPECTION: $w[n] \neq y[n-n_0]$

$\therefore$ NOT TIME-INVARIANT

CAUSAL TEST DEPENDS ONLY ON PRESENT VALUE ON n. $\therefore$ CAUSAL

6.1 c) $y[n] = |x[-n]|$

NOT LINEAR, NOT TIME-INVARIANT, NON-CAUSAL

LINEARITY TEST (SEE FIG. 5.17) $w[n] = |\alpha x_1[n] + \beta x_2[n]|$

$y[n] = \alpha |x_1[n]| + \beta |x_2[n]|$

BY INSPECTION: $w[n] \neq y[n]$, USE EXAMPLE TO SHOW $x_1[n] = 1$, $x_2[n] = -1$, $\alpha > \beta > 0$

$w[n] = |a_1 - a_2| = a_1 - a_2$; $y[n] = a_1 + a_2$

6.1 c) CONTINUED

TIME-INVARIANCE TEST (SEE FIG. 5.16)

$$w[n] = |x[-(n)-n_0]| = |x[-n-n_0]|$$

$$y[n-n_0] = |x[-(n-n_0)]| = |x[-n+n_0]|$$

$w[n] = y[n-n_0]$, ∴ NOT TIME-INVARIANT

CAUSALITY TEST

$y[n]$ DEPENDS UPON FUTURE VALUES OF n FOR $n<0$, ∴ NON-CAUSAL.

6.1 d) $y[n] = x[n-2] + 2x[n] + x[n+2]$

LTI AND NON-CAUSAL

LTI BECAUSE IS IN GENERAL FIR FORM, AND NON-CAUSAL BECAUSE DEPENDS UPON $[n+2]$.

6.1 e) $y[n] = nx[n]$

LINEAR, NOT TIME-INVARIANT, & CAUSAL

LINEARITY TEST (SEE FIG. 5.17 OF DSP FIRST)

$$w[n] = n\{\alpha x_1[n] + \beta x_2[n]\}$$

$$w[n] = \alpha n x_1[n] + \beta n x_2[n]$$

but, $y[n] = \alpha n x_1[n] + \beta n x_2[n]$

$w[n] = y[n]$, ∴ LINEAR

TIME-INVARIANCE TEST (SEE FIG. 5.16)

$$w[n] = n x[n-n_0]$$

$$y[n-n_0] = (n-n_0)x[n-n_0]$$

$w[n] \neq y[n]$, ∴ NOT TIME-INVARIANT

CAUSALITY TEST

$y[n]$ IS CAUSAL BECAUSE IT DEPENDS ONLY ON PRESENT VALUES OF n.

6.1 f) $y[n] = (x[-n])^2$

NOT LINEAR, NOT TIME-INVARIANT, NON-CAUSAL

LINEARITY TEST (SEE FIG. 5.17 OF DSP FIRST)

$$w[n] = (\alpha x_1[-n] + \beta x_2[-n])^2$$

$$y[n] = \alpha (x_1[-n])^2 + \beta (x_2[-n])^2$$

$w[n] \neq y[n]$, BECAUSE OF CROSS TERMS IN $w[n]$. $\therefore$ NOT LINEAR.

TIME-INVARIANCE TEST (SEE FIG. 5.16 OF DSP FIRST)

$$w[n] = (x[-n-n_0])^2$$

$$y[n-n_0] = (x[-(n-n_0)])^2 = (x[-n+n_0])^2$$

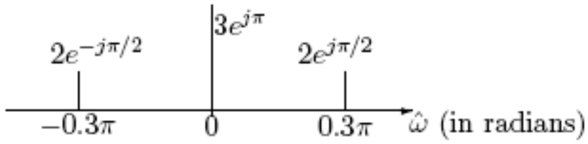
$w[n] \neq y[n-n_0]$, $\therefore$ NOT TIME-INVARIANT.

CAUSALITY TEST

$y[n]$ IS NOT-CAUSAL BECAUSE IT DEPENDS UPON FUTURE n VALUES FOR $n<0$.

PROBLEM 6.2

A discrete-time signal $x[n]$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x[n]$. Make sure to express $x[n]$ as a real-valued signal.

(b) Determine the formula for the output signal $y[n]$.

6.2 a) FROM LEFT TO RIGHT, SPECTRAL LINES ARE:
$2e^{-j\pi/2}e^{-j0.3\pi n}$, $3e^{j\pi}$, $2e^{j\pi/2}e^{j0.3\pi n}$

OR IN COSINE FORM

$x[n] = -3 + 4\cos(0.3\pi n + \pi/2)$

b) 9 PT FILTER $\Rightarrow h[n] = \frac{1}{9}\sum_{k=0}^{8}\delta[n-k]$

IN GENERAL $y[n] = h[n] * x[n]$, AND THIS PROBLEM CAN BE SOLVED IN THE TIME DOMAIN BY PERFORMING THE CONVOLUTION $h[n] * x[n]$, OR IN THE FREQUENCY DOMAIN USING $\mathcal{H}(\hat{\omega})$ DEVELOPED IN CHAPTER 6 OF DSP FIRST. IT WILL BE SOLVED HERE USING $\mathcal{H}(\hat{\omega})$ METHOD.

6.2 a) CONTINUED

$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{8} \left(\frac{1}{9}\right) e^{-j\hat{\omega}k}$$

USING $\displaystyle\sum_{k=0}^{L-1} \alpha^k = \frac{1-\alpha^L}{1-\alpha}$; LET $\alpha = e^{-j\hat{\omega}}$; $L = 9$

$$\mathcal{H}(\hat{\omega}) = \frac{1}{9}\left(\frac{1-e^{-j9\hat{\omega}}}{1-e^{-j\hat{\omega}}}\right) = \frac{1}{9}\,\frac{e^{-j\frac{9\hat{\omega}}{2}}}{e^{-j\hat{\omega}/2}}\left[\frac{e^{j\frac{9\hat{\omega}}{2}}-e^{-j\frac{9\hat{\omega}}{2}}}{e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}}\right]$$

$$\mathcal{H}(\hat{\omega}) = \frac{1}{9} e^{-j4\hat{\omega}} \, \frac{\sin(9\hat{\omega}/2)}{\sin(\hat{\omega}/2)}$$

COMPARE TO EQN. 6.7.4 DSP FIRST

NEXT EVALUATE FOR

$\hat{\omega} = 0$, $\hat{\omega} = 0.3\pi$

$$\mathcal{H}(\hat{\omega}=0) = \frac{1}{9}\lim_{\hat{\omega}\to 0} \frac{\sin(9\hat{\omega}/2)}{\sin(\hat{\omega}/2)} = \frac{\cos(0)}{\cos(0)} = 1$$

$$\mathcal{H}(\hat{\omega}=0.3\pi) = \frac{1}{9} e^{-j1.2\pi} \, \frac{\sin\left(\frac{2.7\pi}{2}\right)}{\sin\left(\frac{0.3\pi}{2}\right)}$$

$$= e^{-j1.2\pi}\,\frac{1}{9}\,\frac{(-0.891)}{0.454}$$

$$= e^{-j1.2\pi} e^{+j\pi}(0.2181)$$

$$\mathcal{H}(\hat{\omega}=0.3\pi) = 0.2181\, e^{-j0.2\pi}$$

NOW CAN WRITE

$$y[n] = -3\cdot|\mathcal{H}(0)| + 4\cdot|\mathcal{H}(0.3\pi)| \cos\left(0.3\pi n + \frac{\pi}{2} + \angle\mathcal{H}(0.3\pi)\right)$$

$$y[n] = -3 + 0.8722 \cos\left(0.3\pi n + \frac{\pi}{2} - 0.2\pi\right)$$

$$\boxed{y[n] = -3 + 0.8722 \cos(0.3\pi n + 0.3\pi)}$$

PROBLEM 6.3:

A second discrete-time system is defined by the input/output relation

$$y[n] = (x[n+1])^3. \tag{1}$$

(a) Determine whether or not the system defined by (1) is (i) linear; (ii) time-invariant; (iii) causal.

(b) For the system of Equation (1), determine the output $y_1[n]$ when the input is

$$x_1[n] = 2\cos(0.6\pi n) = e^{j0.6\pi n} + e^{-j0.6\pi n}.$$

Express your answer in terms of cosine functions. Do not leave any powers of cosine functions in your answers. Note that this system produces output contain frequencies that are not present in the input signal. Explain how this system might cause "aliasing" of sinusoidal components of the input.

6.3 a) $y[n] = (x[n+1])^3$

NON-LINEAR, TIME-INVARIANT, NON-CAUSAL

LINEARITY TEST (SEE FIG. 5.17 OF DSP FIRST FOR BLOCK DIAGRAM)

$$w[n] = (\alpha x_1[n+1] + \beta x_2[n+1])^3$$
$$y[n] = \alpha(x_1[n+1])^3 + \beta(x_2[n+1])^3$$

$w[n] \neq y[n]$ BECAUSE OF CROSS TERMS

$\therefore$ NOT LINEAR

TIME INVARIANCE TEST (SEE FIG 5.16 OF DSP FIRST FOR BLOCK DIAGRAM)

$$w[n] = (x[n+1-n_0])^3$$
$$y[n-n_0] = (x[(n-n_0)+1])^3 = (x[n-n_0+1])^3$$

$\therefore$ TIME-INVARIANT

CAUSALITY TEST

NON-CAUSAL SINCE $y[n]$ DEPENDS UPON FUTURE VALUES OF n.

6.3 b) $x_1[n] = 2\cos(0.6\pi n) = e^{j0.6\pi n} + e^{-j0.6\pi n}$

OR $y_1[n] = [e^{j0.6\pi(n+1)} + e^{-j0.6\pi(n+1)}]^3$

LET $z = e^{j0.6\pi(n+1)}$ AND REWRITE $y_1[n]$

$$y_1[n] = (z+z^{-1})^3 = z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3}$$
$$y_1[n] = (z^3+z^{-3}) + 3(z+z^{-1})$$

SUBST BACK FOR z

$$y_1[n] = [e^{j1.8\pi(n+1)} + e^{-j1.8\pi(n+1)}] + 3[e^{j0.6\pi(n+1)} + e^{-j0.6\pi(n+1)}]$$

$$y_1[n] = 2\cos(1.8\pi(n+1)) + 6\cos(0.6\pi(n+1))$$

6.3 b) CONTINUED

SINCE 1.8π IS OUTSIDE THE RANGE $-\pi < \hat{\omega} \leq \pi$, FOLDED ALIASES WILL RESULT:

$$e^{j1.8\pi m} e^{j1.8\pi} \longrightarrow e^{-j0.2\pi m} e^{j1.8\pi}$$

$$e^{-j1.8\pi m} e^{-j1.8\pi} \longrightarrow e^{j0.2\pi m} e^{-j1.8\pi}$$

OR THE RECONSTUCTED OUTPUT BECOMES

$$y_1[m] = 2\cos(0.2\pi m - 1.8\pi) + 6\cos(0.6\pi(m+1))$$

WHICH MAY BE REWRITEN AS

$$\boxed{y_1[m] = 2\cos(0.2\pi m + 0.2\pi) + 6\cos(0.6\pi(m+1))}$$

PROBLEM 6.4

The *unit step* sequence, denoted by $u[n]$, is defined as

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

(a) Make a plot of $u[n]$ for $-5 \leq n \leq 10$. Describe the plot of $u[n]$ outside this range.

(b) We can use the unit step sequence as a convenient representation for sequences that are given by formulas over a range of values. Make a plot of the sequence

$$x[n] = u[n] - u[n-5])$$

for $-5 \leq n \leq 10$.

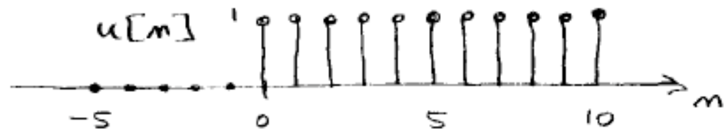
(c) Now make a plot of the sequence

$$x[n] = (.9)^n(u[n] - u[n-5])$$

for $-5 \leq n \leq 10$.

(d) Suppose that $x[n]$ in part (c) is the input to a 4-point running average system. Compute and plot $y[n]$, the output of the system for $-5 \leq n \leq 10$.

6.5 a) PLOT OF $u[n]$, $-5 \le n \le 10$



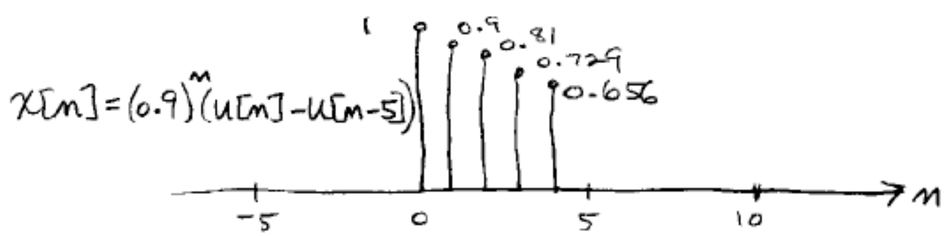
NOTE: $u[n] = 1$, ALL $n > 10$ TO RIGHT
$= 0$, ALL $n < -5$ TO LEFT

6.5 b) PLOT $x[n] = u[n] - u[n-5]$

6.5c) PLOT $x[n] = (0.9)^n (u[n] - u[n-5])$

TABLE OF X[N] VALUES

n	<0	0	1	2	3	4	>4
$u[n]-u[n-5]$	0	1	1	1	1	1	0
0.9^n		1	0.9	0.81	0.729	0.6561	0
$x[n]$	0	1	0.9	0.81	0.729	0.6561	0



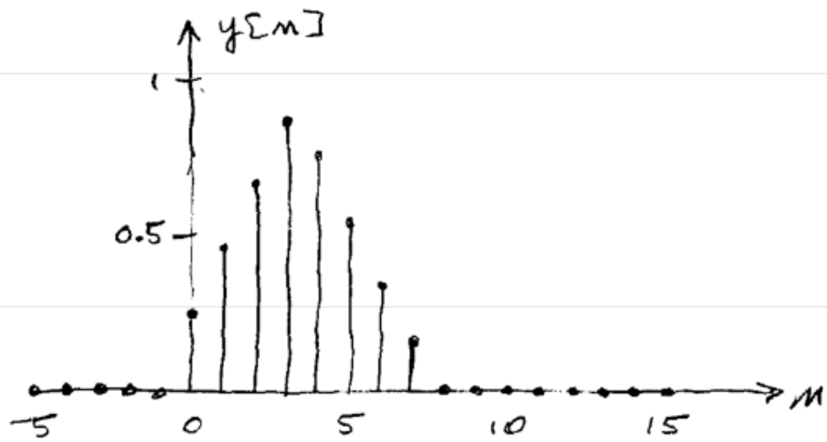
6.5d) FOUR PT. RUNNING AVERAGE FORM

$$y[n] = \sum_{k=0}^{3} \left(\frac{1}{4}\right) x[n-k]$$

6.5 d) COMPUTE $y[n] = \sum_{k=0}^{3} \frac{1}{4} x[n-k]$

n	0	1	2	3	4	5	6	7
$x[n]$	1	.9	.81	.729	.6561			
$h[n]$	.25	.25	.25	.25				
$4 \cdot h[0]x[n]$	1	.9	.81	.729	.6561			
$4 \cdot h[1]x[n-1]$		1	.9	.81	.729	.6561		
$4 \cdot h[2]x[n-2]$			1	.9	.81	.729	.6561	
$4 \cdot h[3]x[n-3]$				1	.9	.81	.729	.6561
$4 \cdot y[n]$	1	1.9	2.71	3.439	3.095	2.1951	1.3851	.6561
$y[n]$	.25	.475	.6775	.8598	.7738	.5488	.3463	.164

PLOT OF $y[n]$



PROBLEM 6.5

This problem is concerned with finding the output of an FIR filter for a given input signal. A linear time-invariant system is described by the difference equation

$$y[n] = \sum_{k=0}^{4} (2-k)x[n-k]$$

(a) Determine the filter coefficients $\{b_k\}$ of this FIR filter.

(b) Find the impulse response, $h[n]$, for this FIR filter. The impulse response is a discrete-time signal, so make a (stem) plot of $h[n]$ versus n.

(c) Use the above difference equation to compute the output $y[n]$ when the input is

$$x[n] = \begin{cases} 0 & n < 0 \\ 5 & 0 \le n \le 5 \\ 1 & 6 \le n \le 10 \\ 0 & n \ge 11. \end{cases}$$

Make a plot of both $x[n]$ and $y[n]$ vs. n. (Hint: you might find it useful to check your results with MATLAB's `conv()` function.)

6.6 $\quad y[n] = \sum_{k=0}^{4} (2-k)\, x[n-k]$

a) Finding filter coefficients $b_k = (2-k)$

$b_0 = 2,\ b_1 = 1,\ b_2 = 0,\ b_3 = -1,\ b_4 = -2$

or $\{b_k\} = \{2, 1, 0, -1, -2\}$

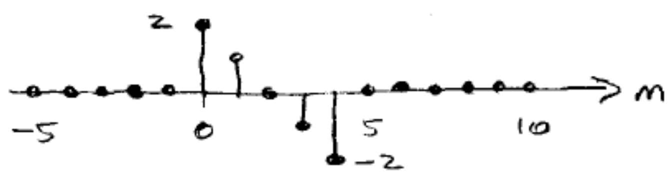
b) Find impulse response

$$h[n] = \sum_{k=0}^{4} (2-k)\, \delta(n-k)$$

$$h[n] = \sum_{k=0}^{4} b_k\, \delta(n-k) = \begin{cases} b_n, & n = 0,1,2,3,4 \\ 0 & \text{otherwise} \end{cases}$$

$h[n] = [2, 1, 0, -1, -2]$

h[n]



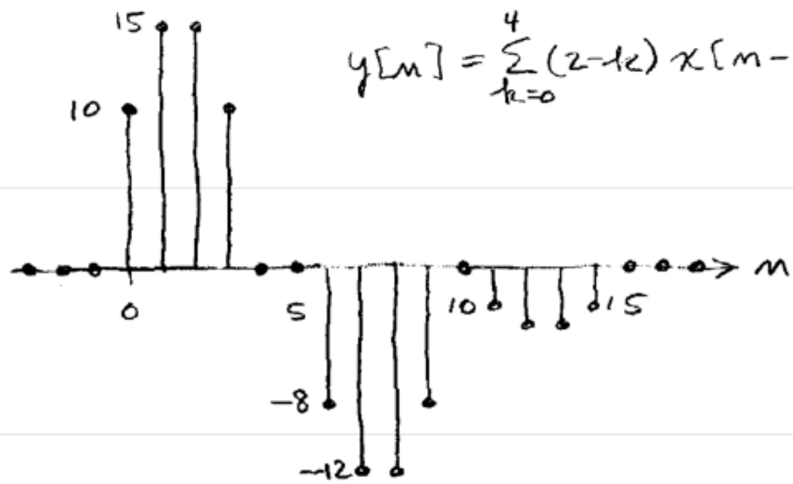
c) compute y[n] for x[n] below

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
x[n]	5	5	5	5	5	5	1	1	1	1	1	0	0	0	0
h[n]	2	1	0	-1	-2										
h[0]x[n]	10	10	10	10	10	10	2	2	2	2	2				
h[1]x[n-1]		5	5	5	5	5	5	1	1	1	1	1			
h[2]x[n-2]			0	0	0	0	0	0	0	0	0	0	0		
h[3]x[n-3]				-5	-5	-5	-5	-5	-5	-1	-1	-1	-1	-1	
h[4]x[n-4]					-10	-10	-10	-10	-10	-10	-2	-2	-2	-2	-2
y[n]	10	15	15	10	0	0	-8	-12	-12	-8	0	-2	-3	-3	-2

6.6 c) CONTINUED

$$x[m] = \begin{cases} 0 & m < 0 \\ 5 & 0 \le m \le 5 \\ 1 & 6 \le m \le 10 \\ 0 & m \ge 11 \end{cases}$$

$$y[m] = \sum_{k=0}^{4} (2-k)\, x[m-k]$$



PROBLEM 6.6

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

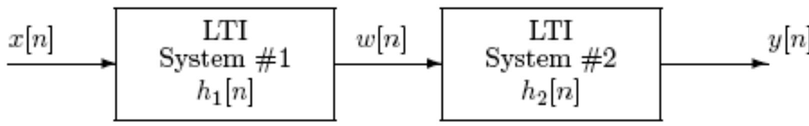


Figure 1: Cascade connection of two LTI systems.

(a) Suppose that LTI System #1 is described by the difference equation

$$w[n] = x[n] - 0.2x[n-1].$$

Determine the impulse response $h_1[n]$ of the first system.

(b) The LTI System #2 is described by the impulse response

$$h_2[n] = (0.2)^n(u[n] - u[n-L]) = \sum_{k=0}^{L-1}(0.2)^k\delta[n-k] = \begin{cases} (0.2)^n & n = 0, 1, \ldots, L-1 \\ 0 & \text{otherwise.} \end{cases}$$

For the special case of $L = 10$, use convolution to show that the impulse response sequence of the overall cascade system is

$$h[n] = h_1[n] * h_2[n] = \delta[n] - (0.2)^{10}\delta[n-10].$$

(c) Generalize your result in part (b) for the general case of L any integer value.

(d) Obtain a single difference equation that relates $y[n]$ to $x[n]$ in Fig. 1.

(e) How would you choose L so that $y[n] = x[n]$ in Figure 1; i.e., how would you choose L so that the second system "undoes" the effect of the first system?

6.7 a) Determine impulse response for

$$w[n] = x[n] - 0.2\,x[n-1]$$

$$h_1[n] = \delta[n] - 0.2\,\delta[n-1]$$

b) For system 2, impulse response is

$$h_2[n] = 0.2^n (u[n] - u[n-L])$$

or $$h_2[n] = \sum_{k=0}^{L} (0.2)^k \delta(n-k) = \begin{cases} (0.2)^n, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

For $L=10$ compute $h[n] = h_1[n] * h_2[n]$

6.7.b) CONTINUED

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$h_1[n]$	1	$-.2$											
$h_2[n]$	1	$.2$	$.2^2$	$.2^3$	$.2^4$	$.2^5$	$.2^6$	$.2^7$	$.2^8$	$.2^9$			
$h_1[0]h_2[n]$	1	$.2$	$.2^2$	$.2^3$	$.2^4$	$.2^5$	$.2^6$	$.2^7$	$.2^8$	$.2^9$			
$h_1[1]h_2[n-1]$		$-.2$	$-.2^2$	$-.2^3$	$-.2^4$	$-.2^5$	$-.2^6$	$-.2^7$	$-.2^8$	$-.2^9$	$-.2^{10}$		
$h[n]$	1	0	0	0	0	0	0	0	0	0	$-.2^{10}$		

or $h[n] = \delta[n] - 0.2^{10}\delta[n-10]$

c) From above result, can see for any integer L, will have only first term ($n=0$) and last term ($n = L+1$), thus have for L any integer

$$\underline{h[n] = \delta(n) - 0.2^L \delta(n-L)}$$

d)
$$y[n] = h[n] * x[n]$$
$$y[n] = [\delta(n) - 0.2^L \delta(n-L)] * x[n]$$
$$y[n] = x[n] - 0.2^L x[n-L]$$

e) Choose L very large so that 0.2^L is a negligable term, then
$$y[n] \simeq x[n].$$

PROBLEM 6.7

A linear time-invariant discrete-time system is described by the difference equation

$$y[n] = x[n] - 2x[n-1] + x[n-2] + 3x[n-4].$$

(a) Draw a block diagram that represents this system in terms of unit-delay elements, coefficient multipliers, and adders as in Figure 5.13 in the text.

(b) Determine the impulse response $h[n]$ for this system. Express your answer as a sum of scaled and shifted unit impulse sequences.

(c) Use convolution to determine the output due to the input

$$x[n] = \delta[n] + \delta[n-1] - \delta[n-2]$$

Plot the output sequence $y[n]$ for $-2 \le n \le 10$.

(d) Now consider another LTI system whose impulse response is

$$h_d[n] = \delta[n] + \delta[n-1] - \delta[n-2].$$

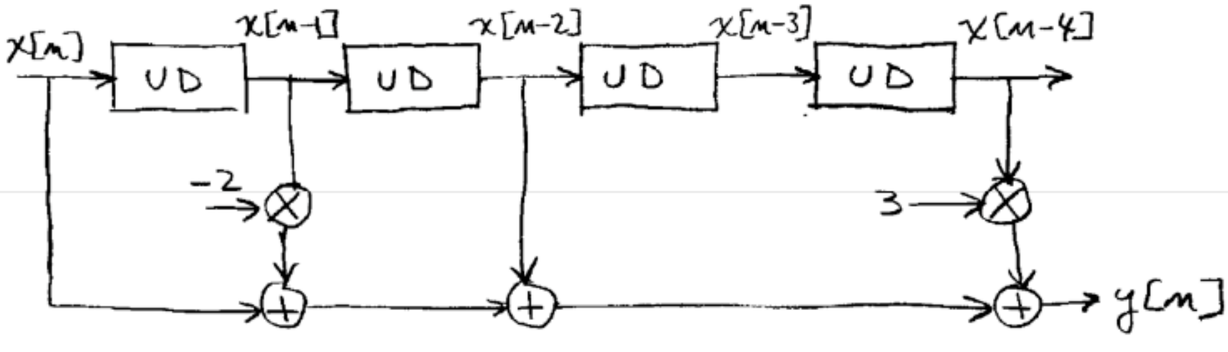
Use convolution again to determine $y_d[n] = x_d[n] * h_d[n]$, the output of this system when the input is

$$x_d[n] = \delta[n] - 2\delta[n-1] + \delta[n-2] + 3\delta[n-4].$$

How does your answer compare to the answer in part (c)? This example illustrates the general commutative property of convolution; i.e., $x[n] * h[n] = h[n] * x[n]$.

6.8 $y[n] = x[n] - 2x[n-1] + x[n-2] + 3x[n-4]$

a) Block Diagram:



b) $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2] + 3\delta[n-4]$

c) For input $x[n] = \delta[n] + \delta[n-1] - \delta[n-2]$
find output $y[n] = h[n] * x[n]$

n	0	1	2	3	4	5	6	7	8	9	10	>10
$x[n]$	1	1	-1									
$h[n]$	1	-2	1	0	3							
$h[0]x[n]$	1	1	-1									
$h[1]x[n-1]$		-2	-2	2								
$h[2]x[n-2]$			1	1	-1							
$h[3]x[n-3]$				0	0	0						
$h[4]x[n-4]$					3	3	-3					
$y[n]$	1	-1	-2	3	2	3	-3	0	0	0	0	0