Week 2: Intro to DSP

Summary

- Spectrum Representation
- Harmonic Signals
- Time-varying signals
- Spectrogram
- Intro to FS

Motivation

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

Example of Time-varying Frequency

- Fur Elise

Figure 3.18 Sheet-music notation is a time–frequency diagram.

Frequency is the vertical axis

Time is the horizontal axis

Euler's Formula's Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)}$$

Inverse Euler's Formula

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

Spectrum Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big

Negative Frequency

- Is negative frequency real?
- Doppler Radar provides an example
  - Police radar measures speed by using the Doppler shift principle
  - Let’s assume 400Hz <---> 60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)

Spectrum of Sine

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

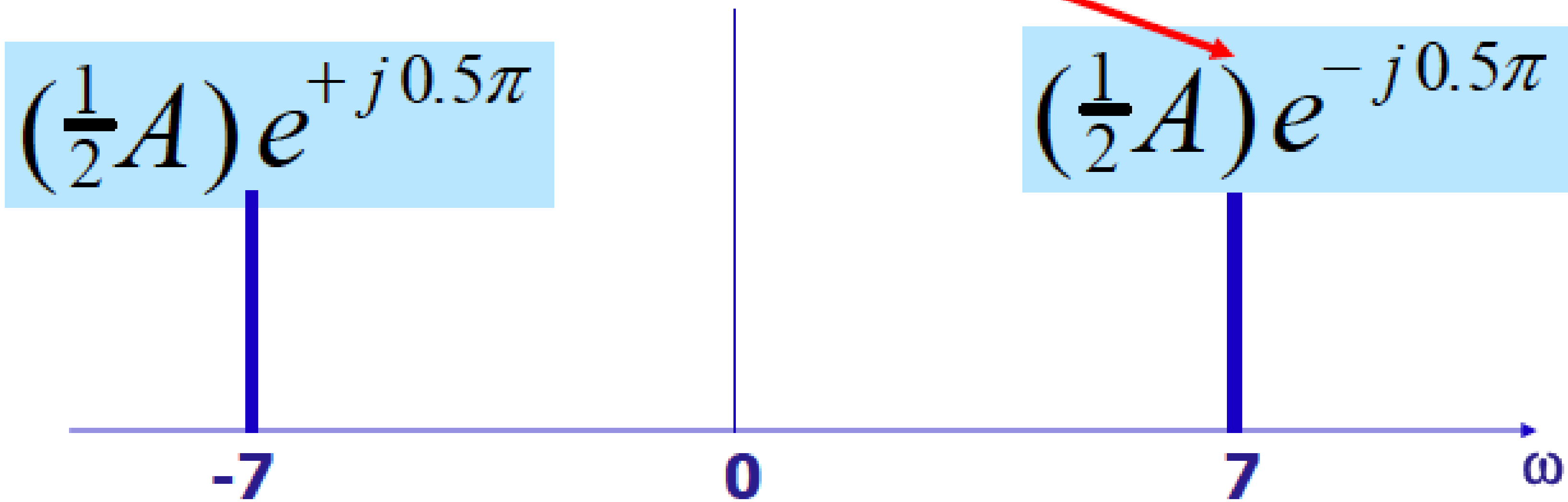
$$\frac{-1}{j} = j = e^{j0.5\pi}$$

  - Positive freq. has phase = -0.5π
  - Negative freq. has phase = +0.5π

Graphical Spectrum

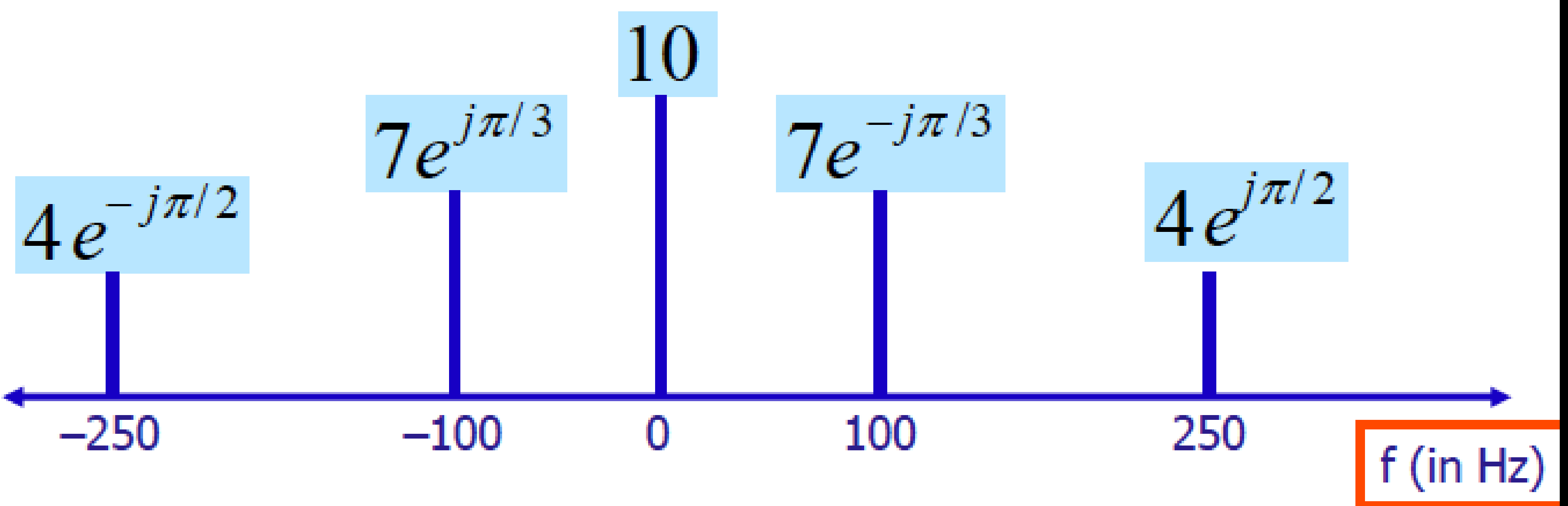
EXAMPLE of SINE

$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are shown

⌘Add the spectrum components:



What is the formula for the signal x(t)?

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

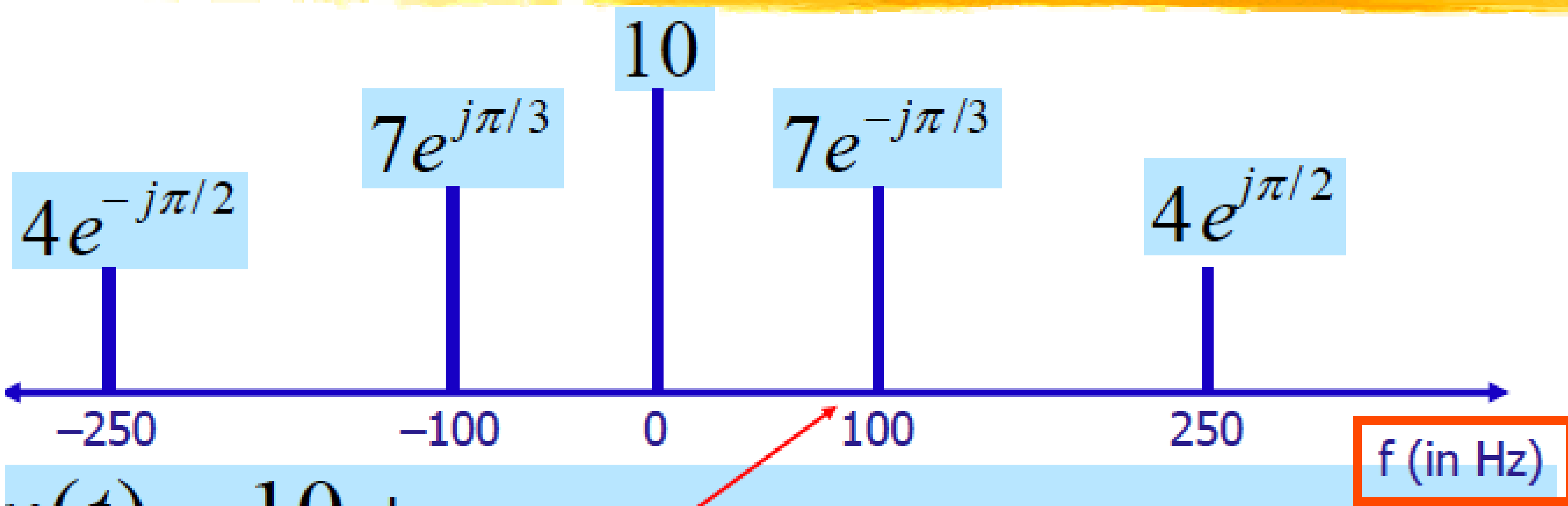
DC is another name for zero-freq component

DC component always has zero phase (for real ***x(t)***)

Add Spectrum Components-1

- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$+4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \frac{A}{2}e^{j\varphi}e^{j\omega t} + \frac{A}{2}e^{-j\varphi}e^{-j\omega t}$$

Final Answer

$$\boxed{\begin{aligned} x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2) \end{aligned}}$$

So, we get the general form:

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Harmonics and Frequency vs. time

- Signals with HARMONIC Frequencies
  - Add Sinusoids with $f_k = kf_0$

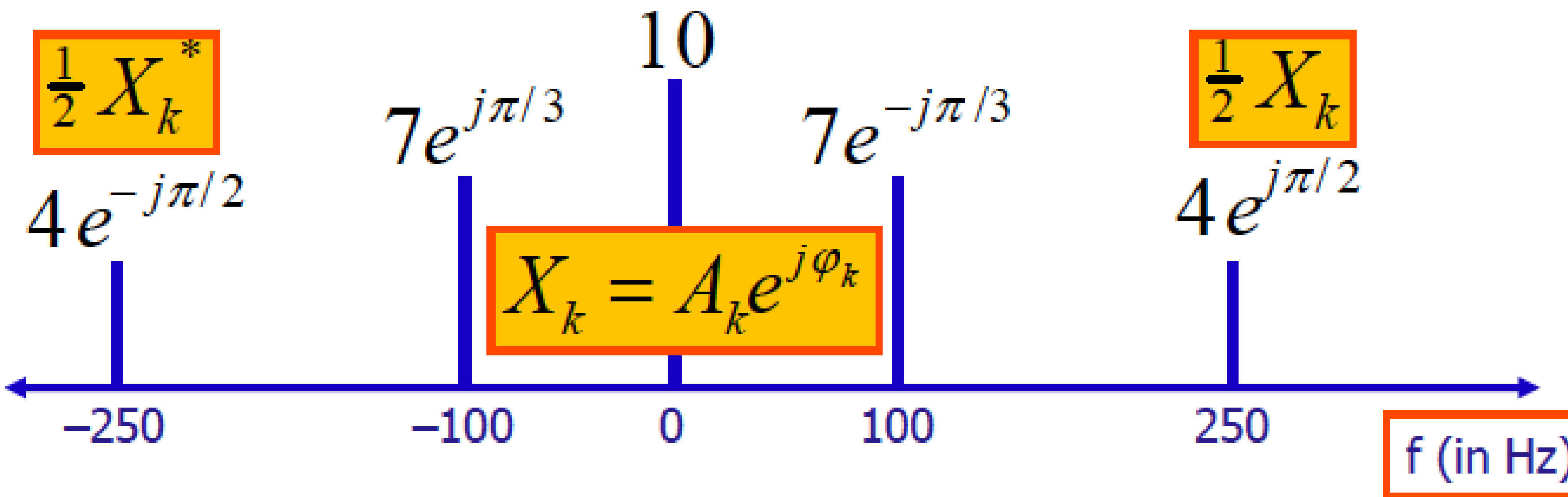
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

- FREQUENCY can change vs. TIME
  - Chirps: $x(t) = \cos(\alpha t^2)$
  - Introduce Spectrogram Visualization

`(specgram.m)` **`(plotspec.m)`**

Spectrum Diagram

- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Complex Signals General Form

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$X_0 = A_0 e^{j0}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

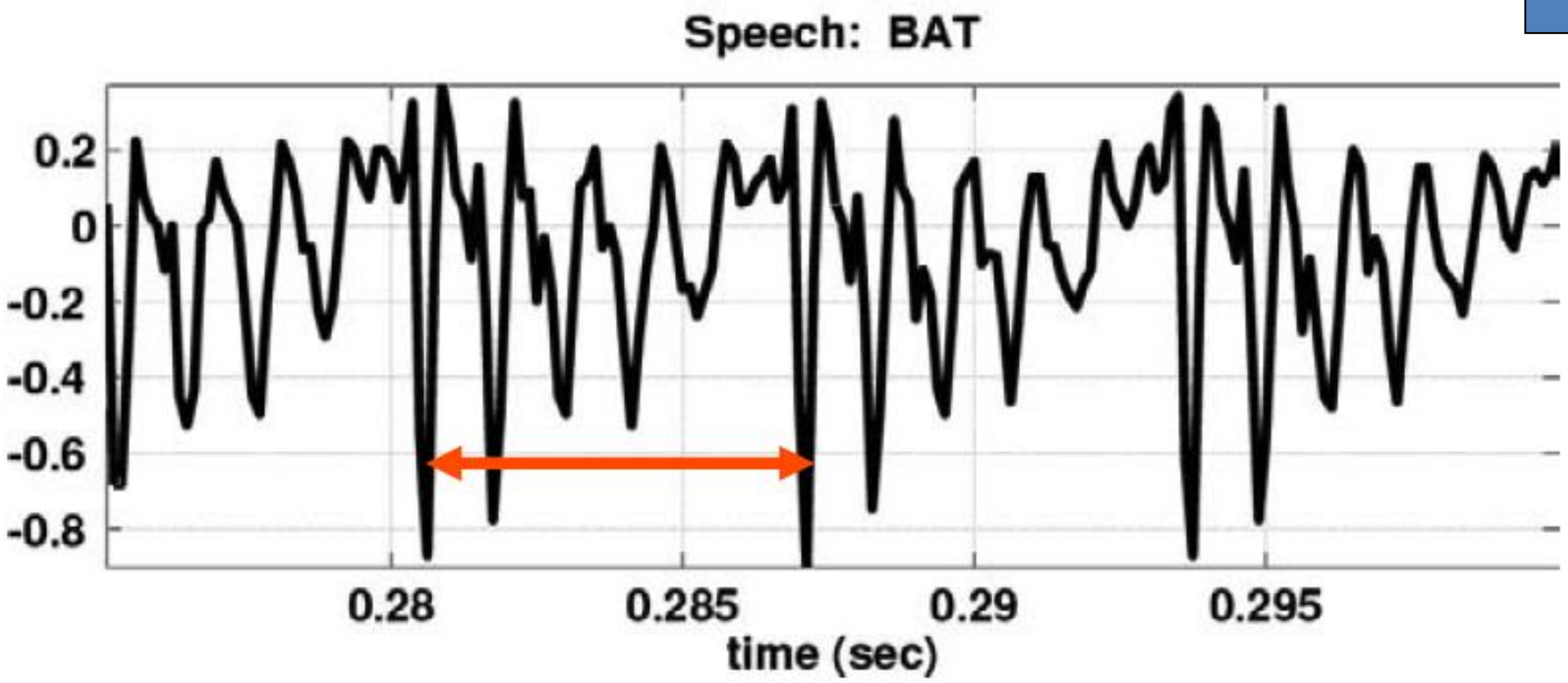
$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Example of Speech

- Nearly **Periodic** in Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Periodic Signals

- Repeat every T secs
  - Definition

$$\boxed{x(t) = x(t+T)}$$

  - Example:

$$x(t) = \cos^2(3t)$$

$T = ?$

$T = \frac{2\pi}{3}$ $T = \frac{\pi}{3}$

  - Speech can be "quasi-periodic"

Period of Complex Exponentials

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t)\ ?$$

Definition: Period is T

$$\cancel{e^{j\omega(t+T)}} = \cancel{e^{j\omega t}}$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$e^{j2\pi k} = 1$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k = \omega_0 k$$

k = integer

Harmonic Signal

$$\text{Periodic signal}: x(t) = x(t+T)$$

$$\text{Can only have } \textbf{\textit{harmonic}} \text{ freqs}: \ f_k = k f_0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$\boxed{f_0 T = 1}$$

$x(t)$ is periodic if

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

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Harmonic Signal Spectrum

Therefore, we can only have: $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$\boxed{X_k = A_k e^{j\varphi_k}}$$

$$f_0 = \frac{1}{T}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

Define Fundamental

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

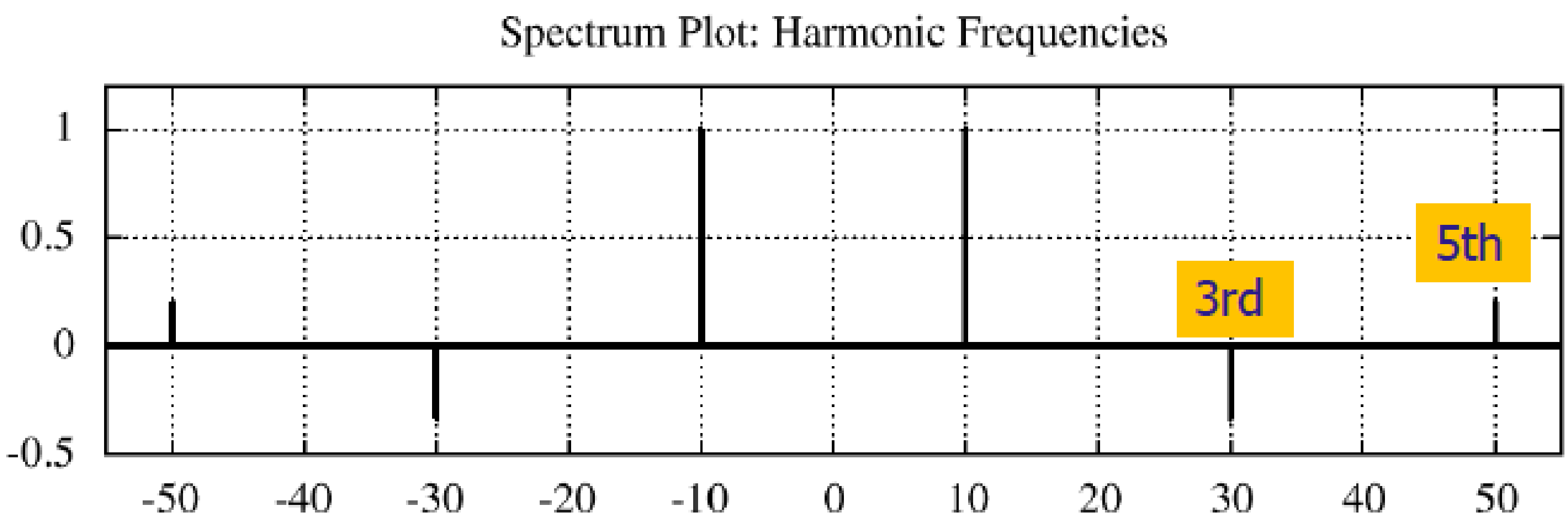
$$f_k = k\, f_0 \qquad (\omega_0 = 2\pi\, f_0)$$

$$f_0 = \text{fundamental frequency}$$

$$T_0 = \text{fundamental Period}$$

$$f_0 = \frac{1}{T_0}$$

Harmonic Signal (3 Freqs)

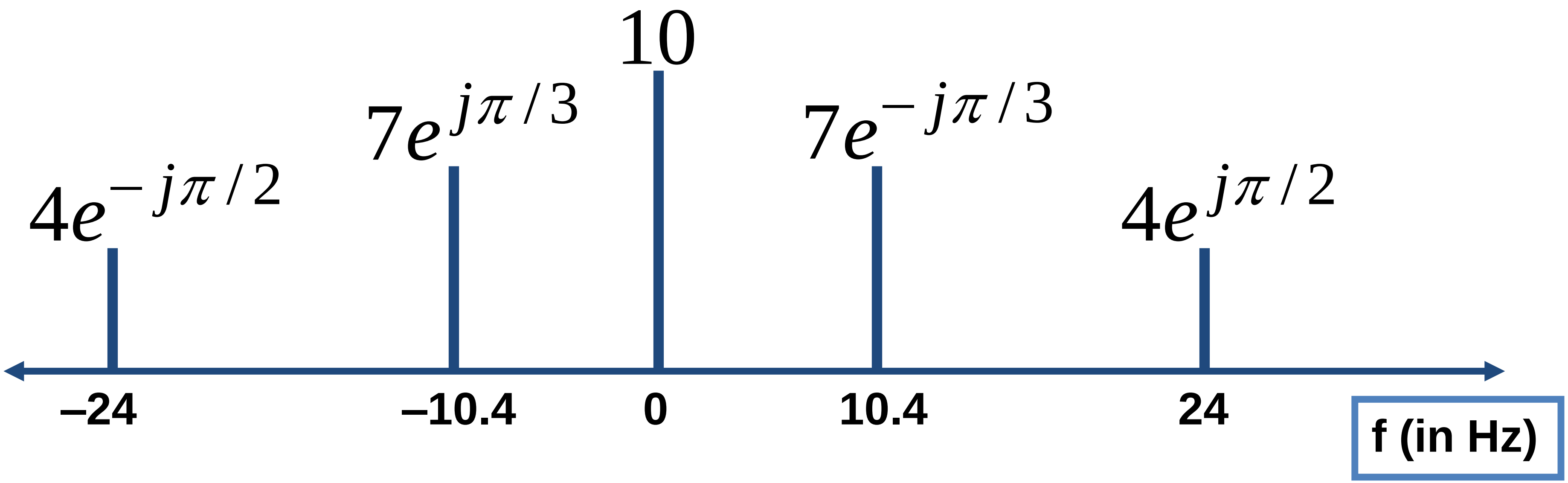


What is the fundamental frequency?

10 Hz

POP QUIZ: FUNDAMENTAL

- Here's another spectrum:



What is the fundamental frequency?

(0.1)GCD(104,240) = (0.1)(8)=0.8 Hz

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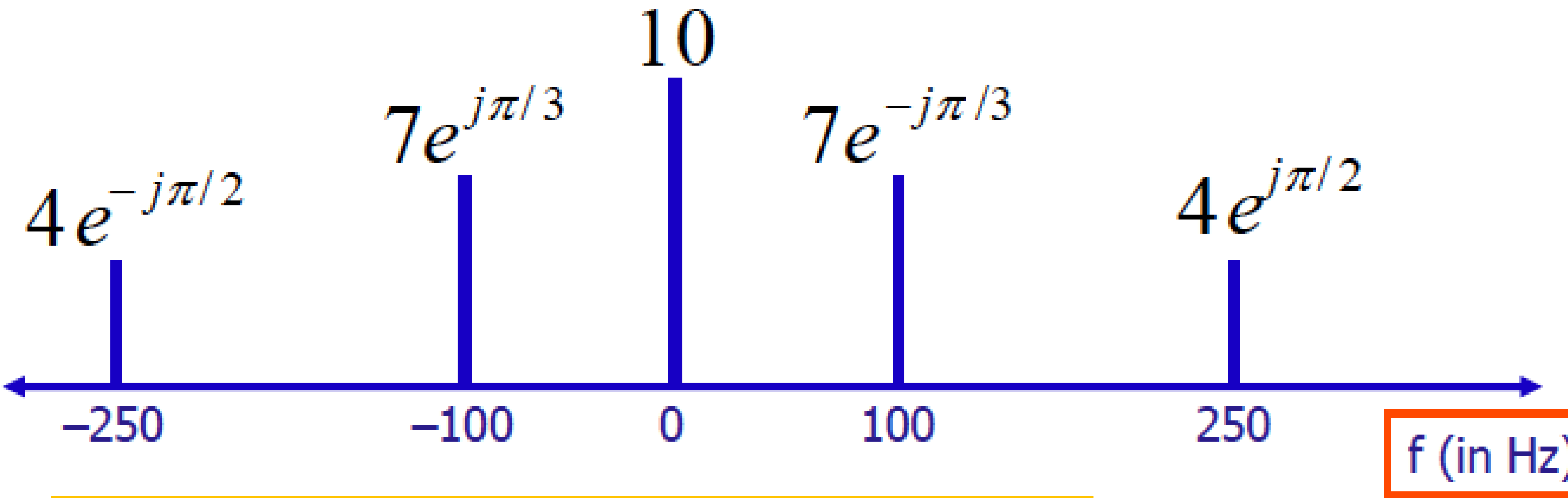
Find Fundamental Frequency

- Here's another spectrum:



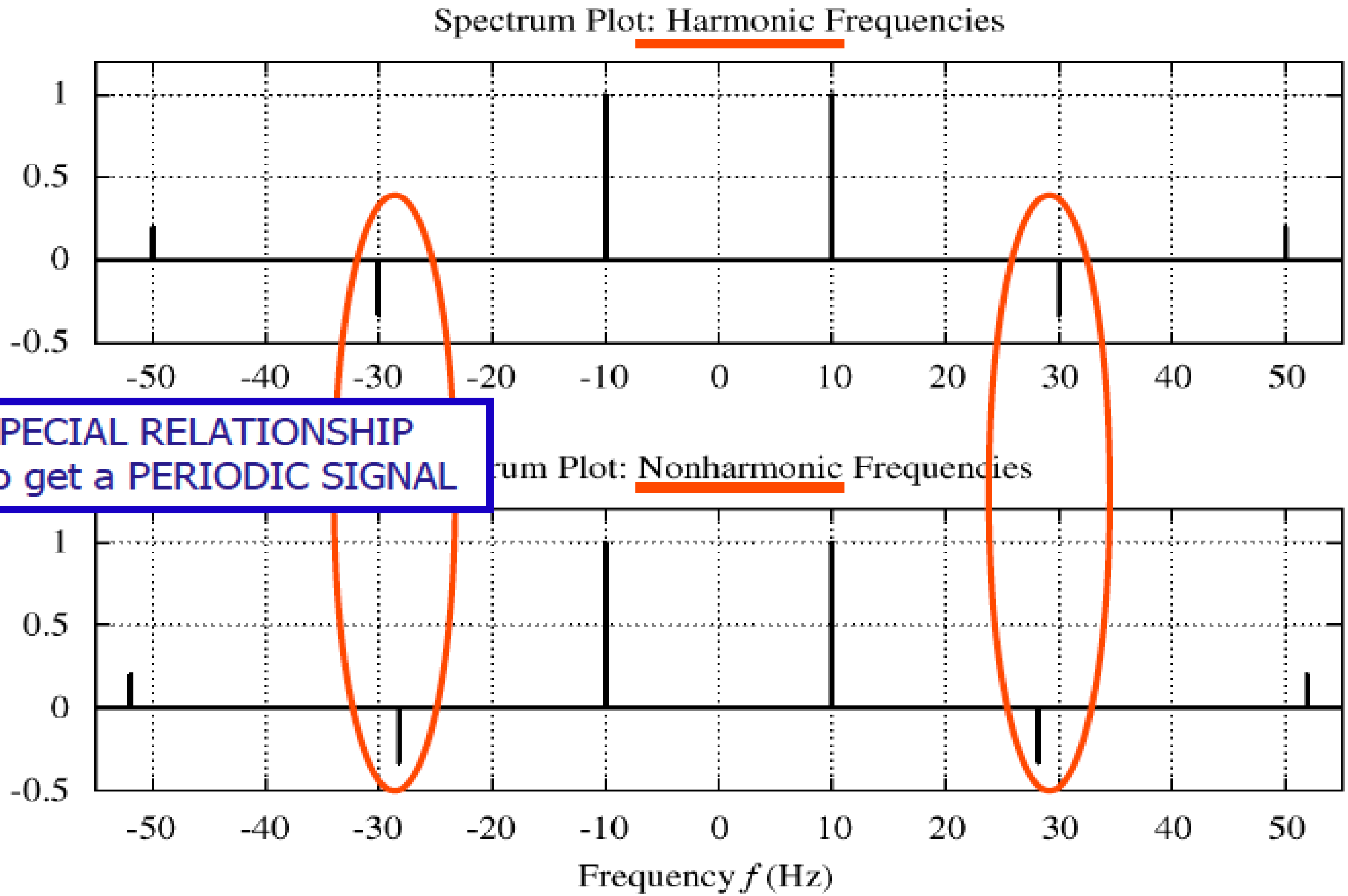
What is the fundamental frequency?

100 Hz ?

50 Hz ?

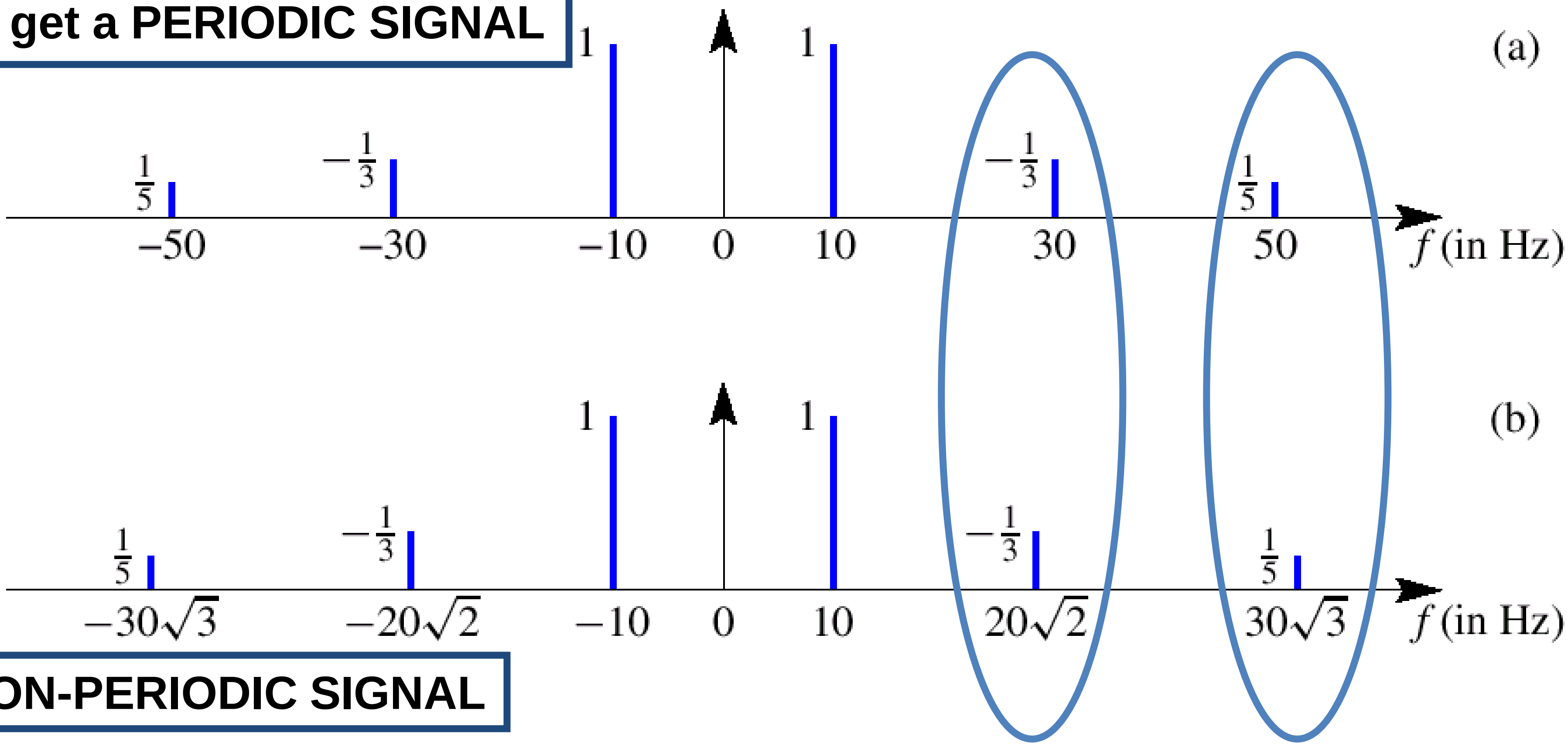
Greatest Common Factor

Need to be able to have multiple of a fundamental for all harmonics



IRRATIONAL SPECTRUM

SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL



NON-PERIODIC SIGNAL

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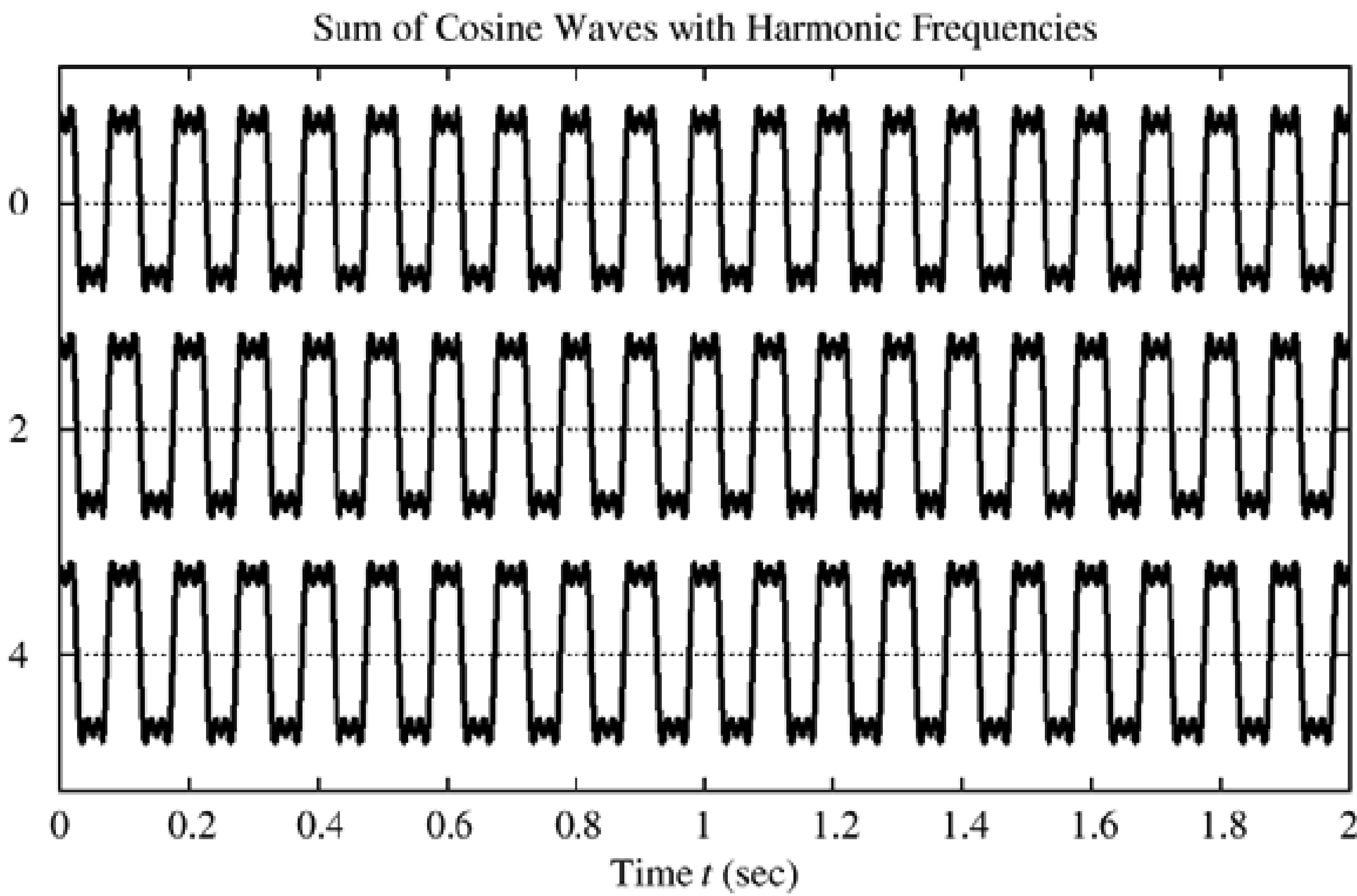
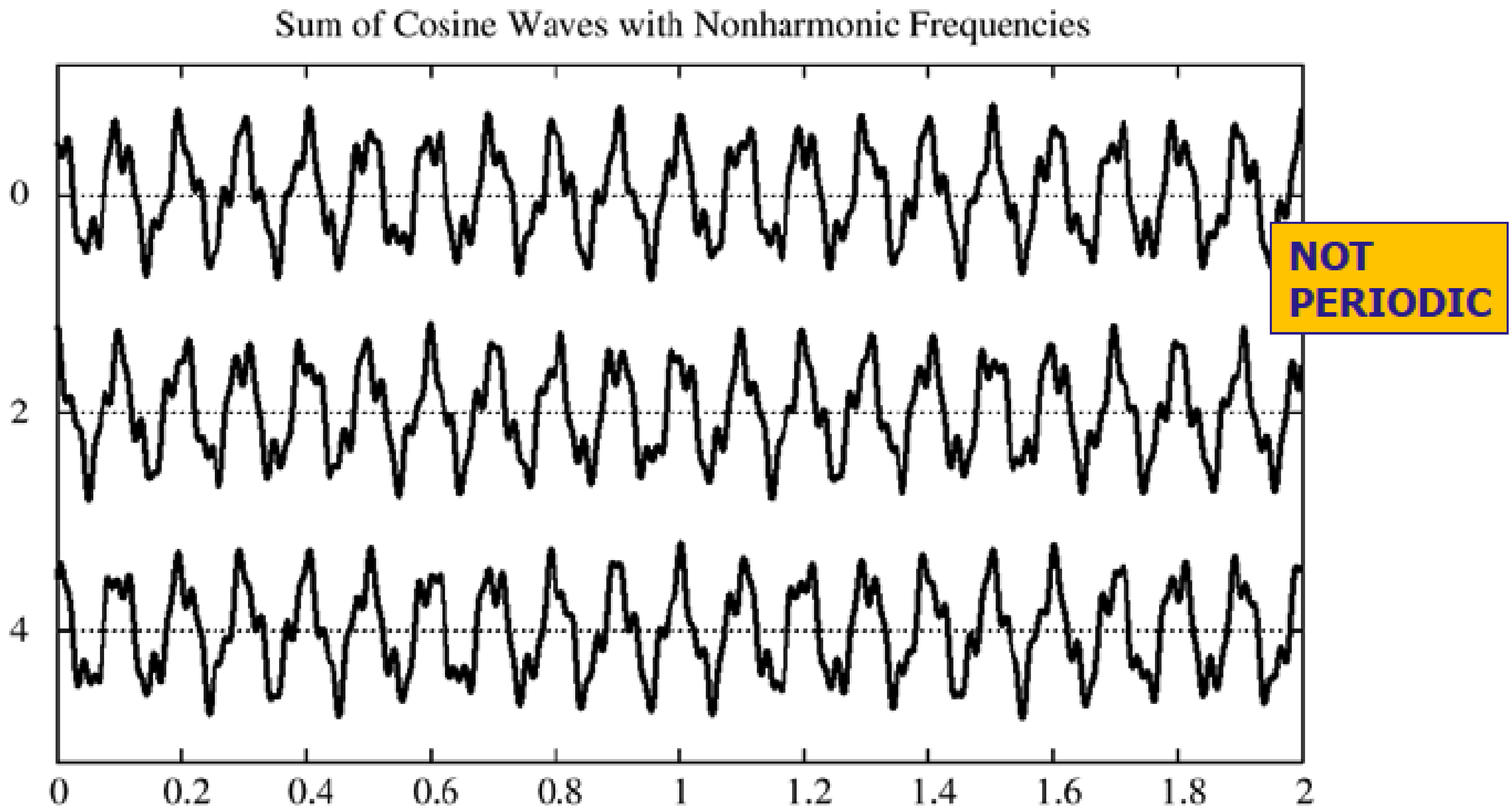
Harmonic Signal (3 Freqs)



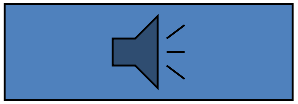
T=0.1

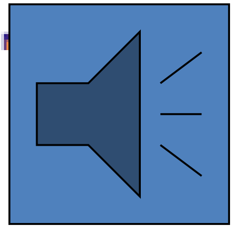
Non-Harmonic Signal



Example Application

- <u>**Now, a much HARDER problem**</u>
- Given a recording of a song, have the computer write the music





- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for x(t)
    - During short intervals

Time-Varying FREQUENCIES Diagram

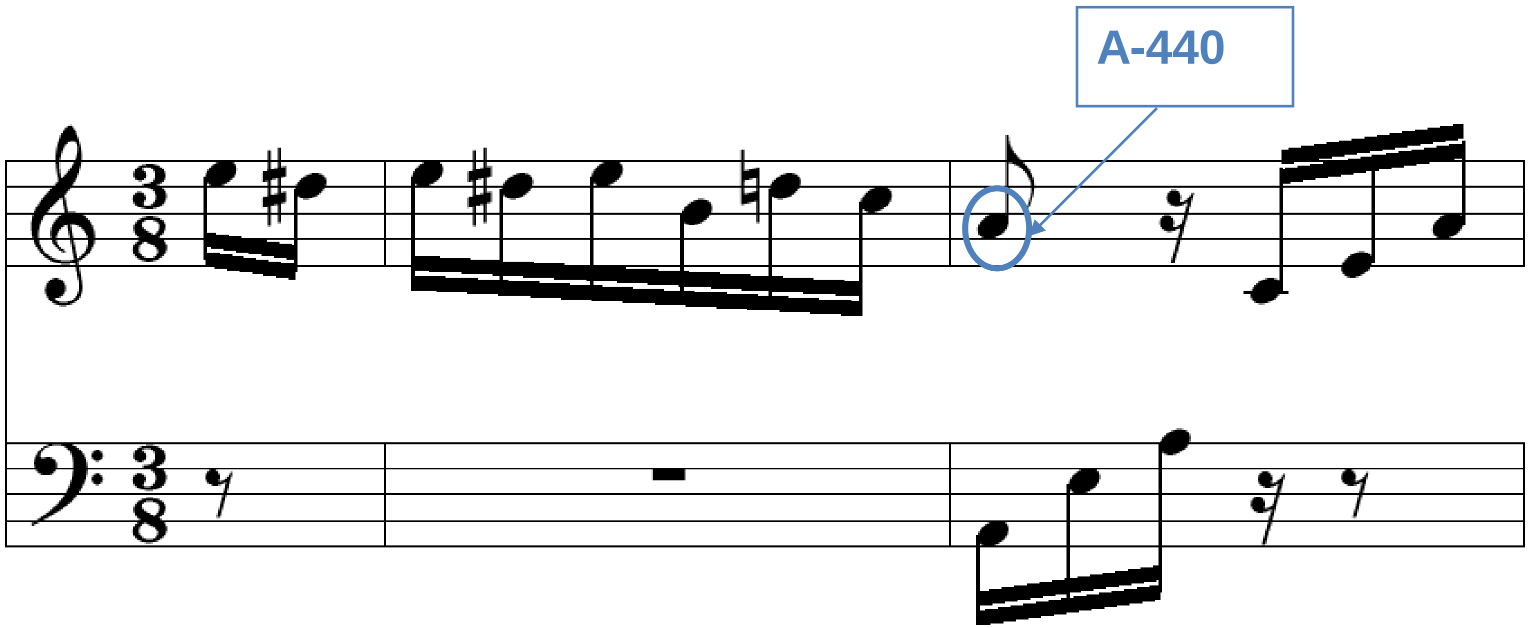


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Frequency is the vertical axis

Time is the horizontal axis

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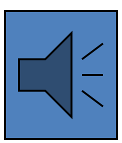
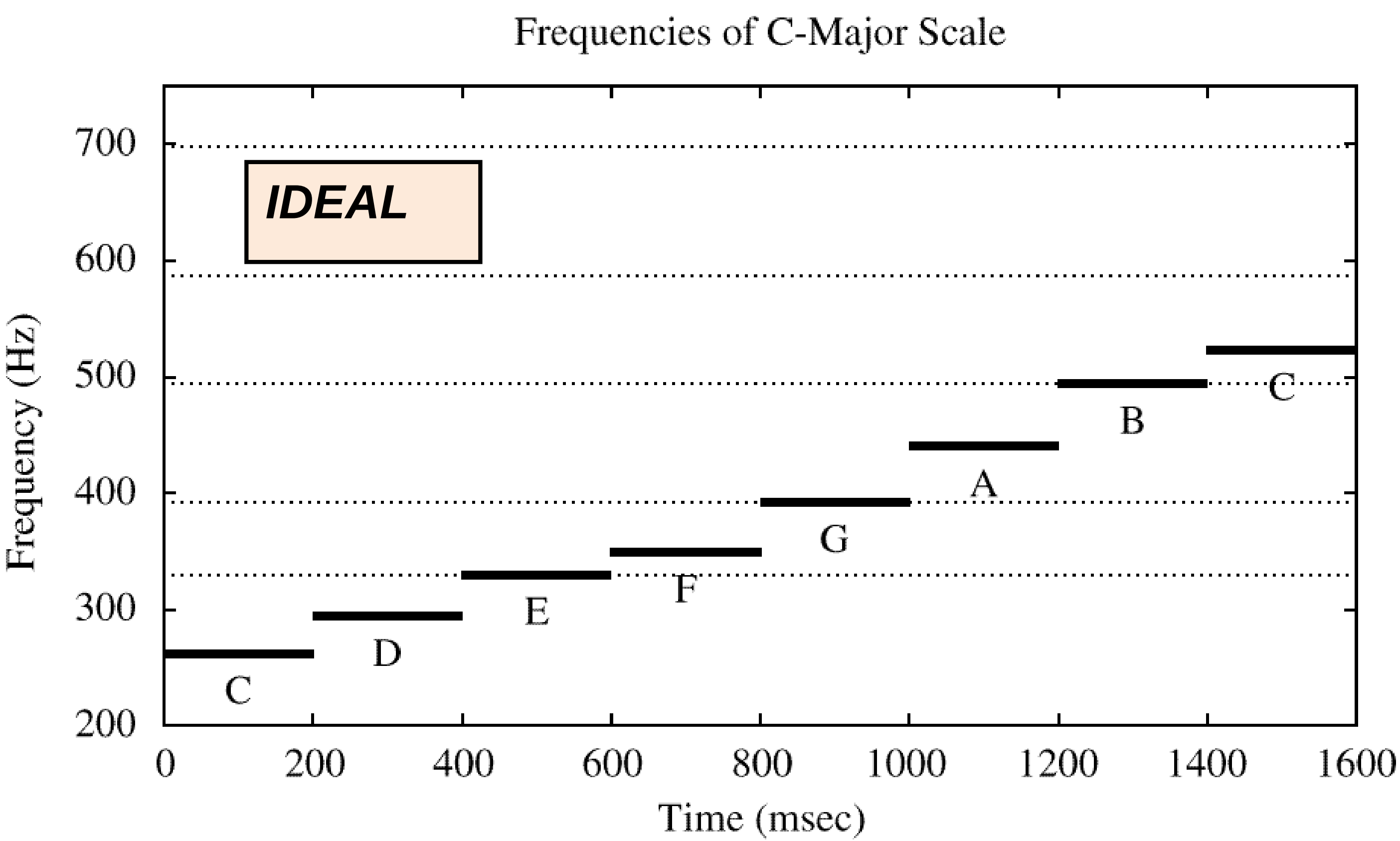
SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note





SPECTROGRAM (Short-time Fourier Transform)

- SPECTROGRAM Tool
  - MATLAB function is `spectrogram.m`
  - SP-First has `plotspec.m` & `spectgr.m`
- **ANALYSIS** program
  - Takes x(t) as input
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

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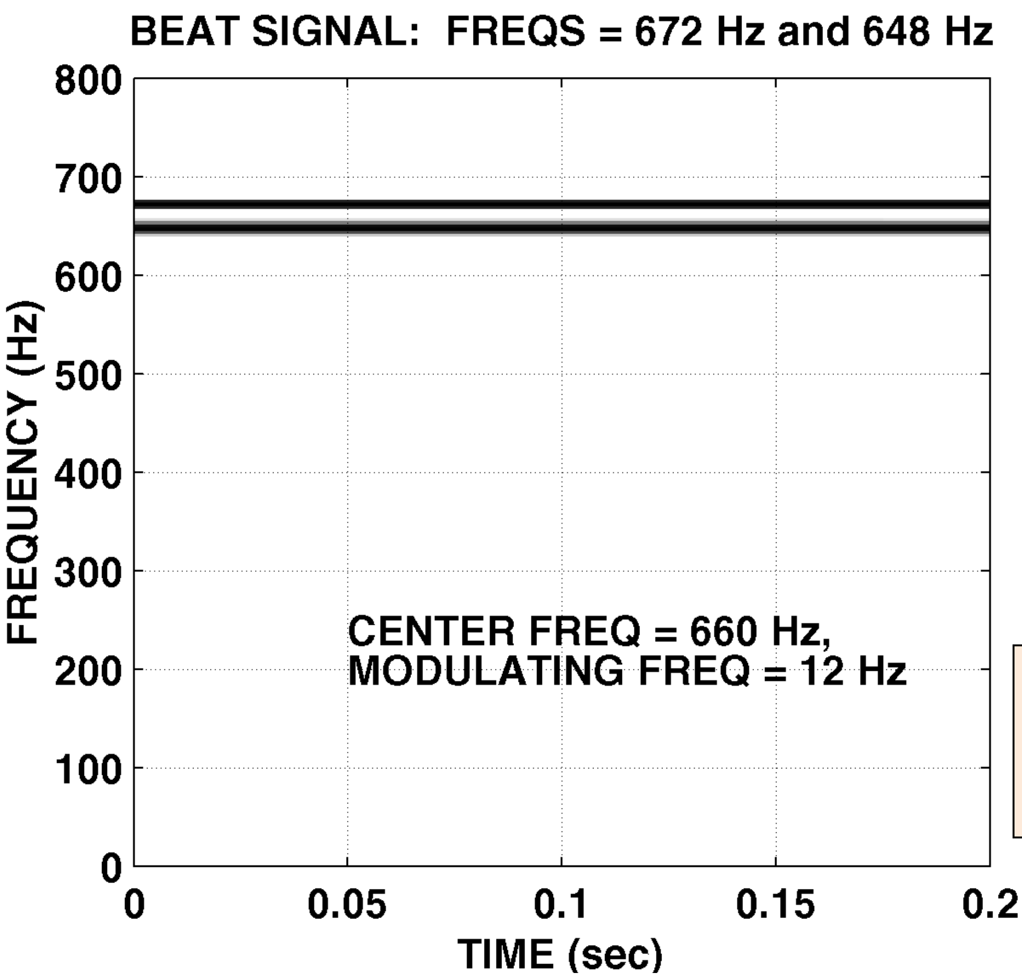
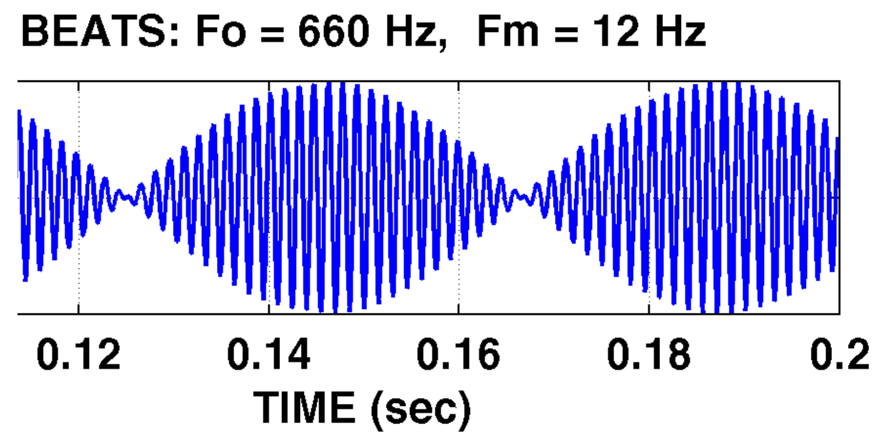
SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\cos(2\pi(672)t) + \cos(2\pi(648)t)$$
$$= 2\cos(2\pi(12)t)\cos(2\pi(660)t)$$

AM Radio Signal

- Same form as BEAT Notes, but higher in freq

$$\boxed{\cos(2\pi(\underline{660})t)\sin(2\pi(12)t)}$$

$$\frac{1}{2}\left(e^{j2\pi(660)t} + e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t} - e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t} - e^{-j2\pi(672)t} - e^{j2\pi(648)t} + e^{-j2\pi(648)t}\right)$$

$$\boxed{\tfrac{1}{2}\cos(2\pi(672)t - \tfrac{\pi}{2}) + \tfrac{1}{2}\cos(2\pi(648)t + \tfrac{\pi}{2})}$$

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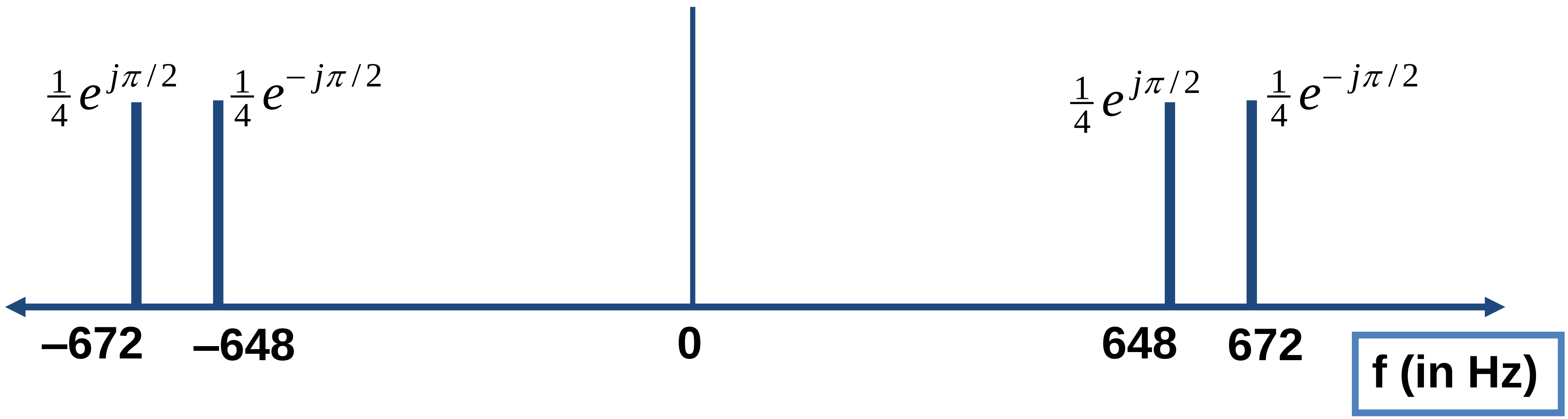
SPECTRUM of AM (Amplitude Modulation)

- **<u>SUM</u>** of 4 complex exponentials:



What is the fundamental frequency?

648 Hz ?

24 Hz ?

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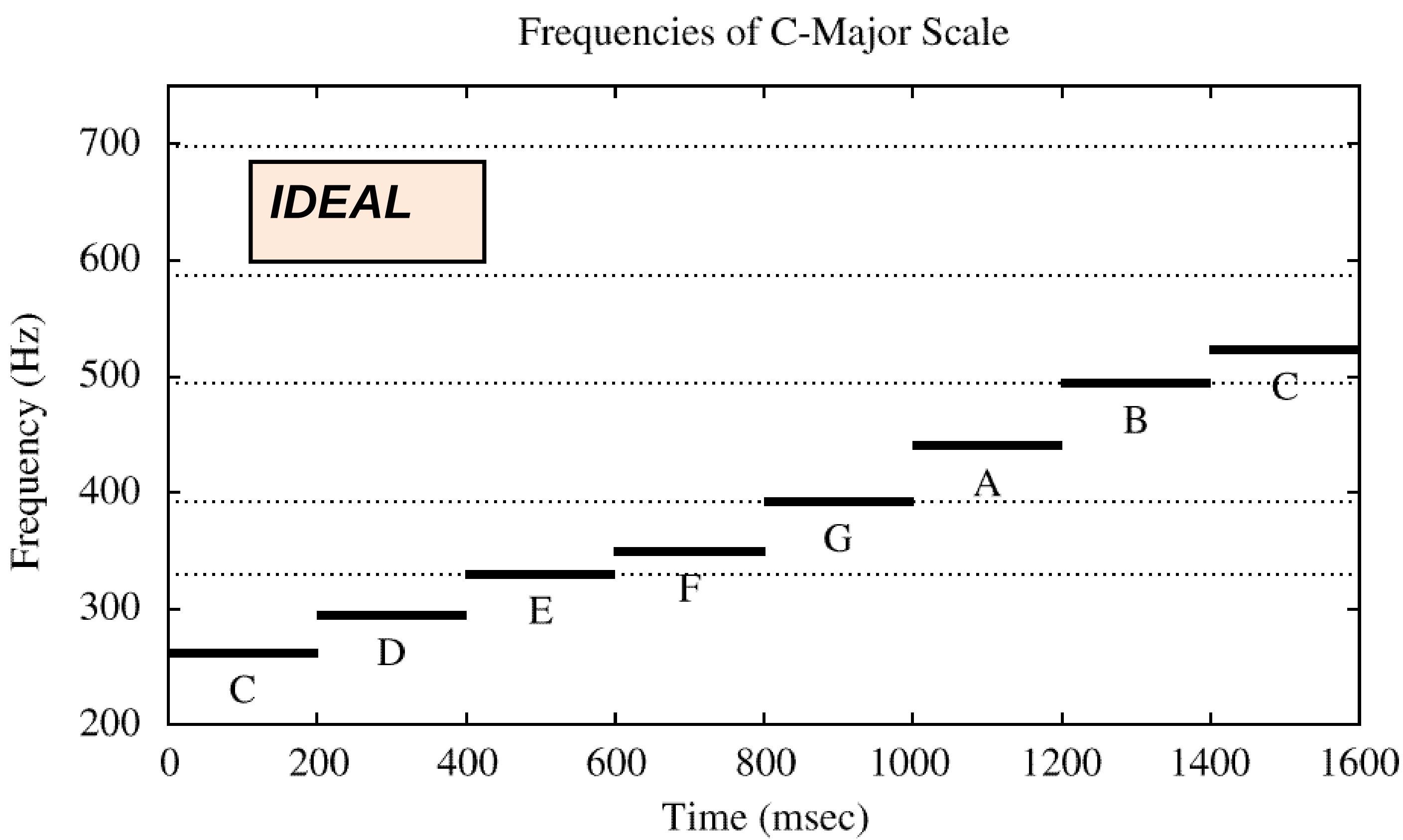
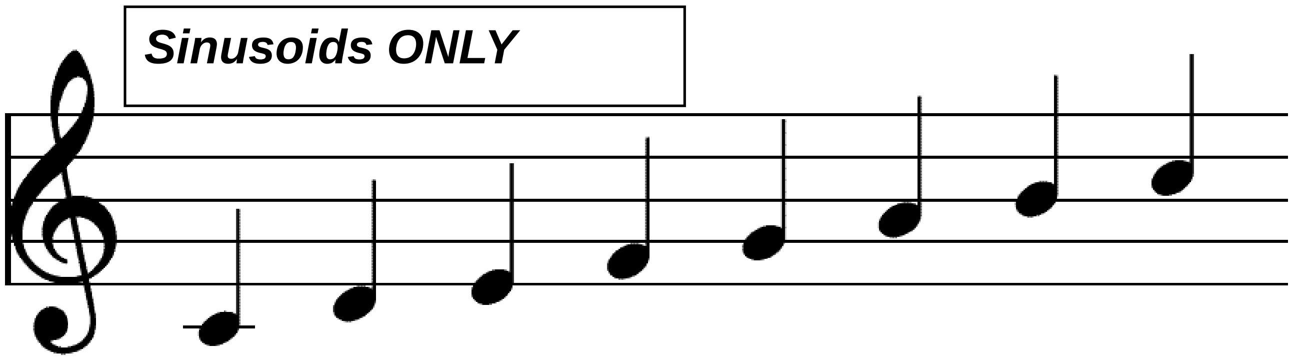
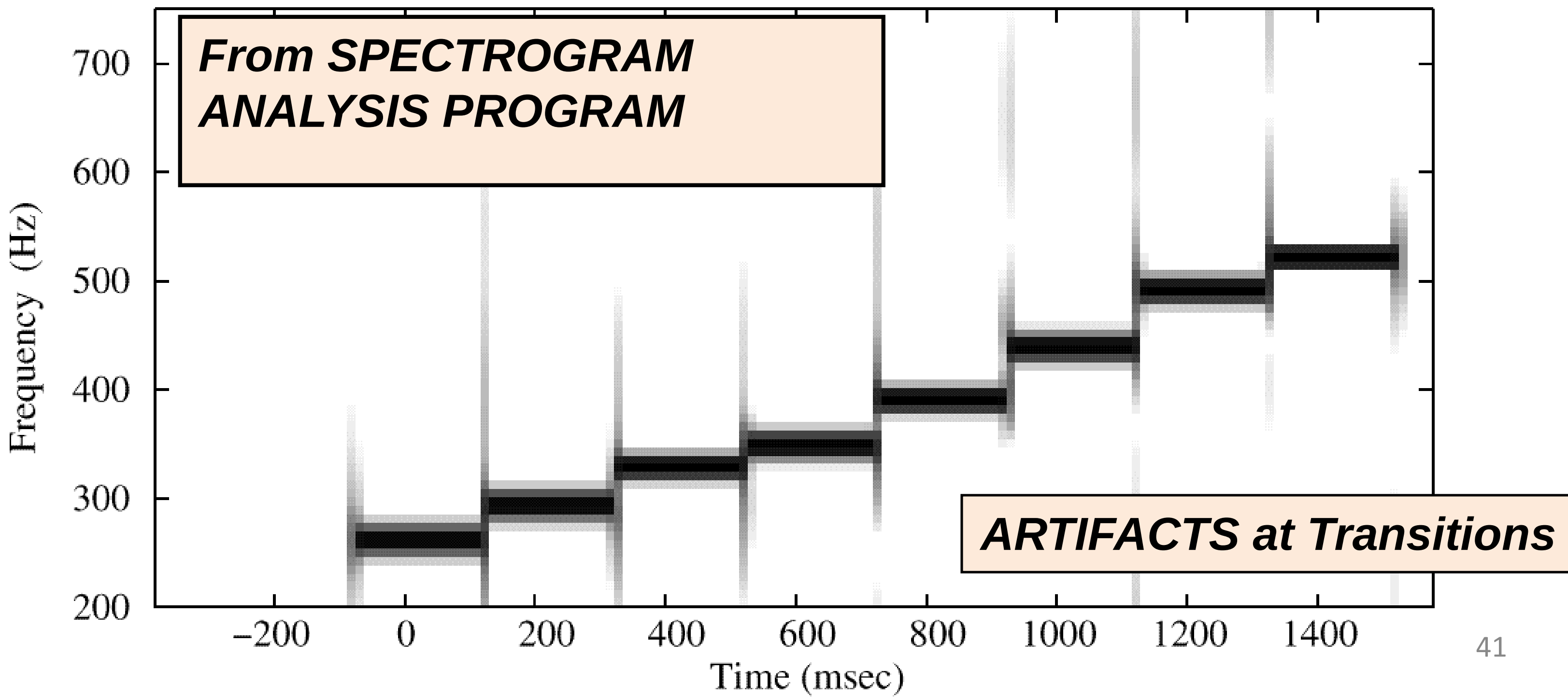
STEPPED FREQUENCIES

- C-major SCALE: successive sinusoids
  - Frequency is constant for each note

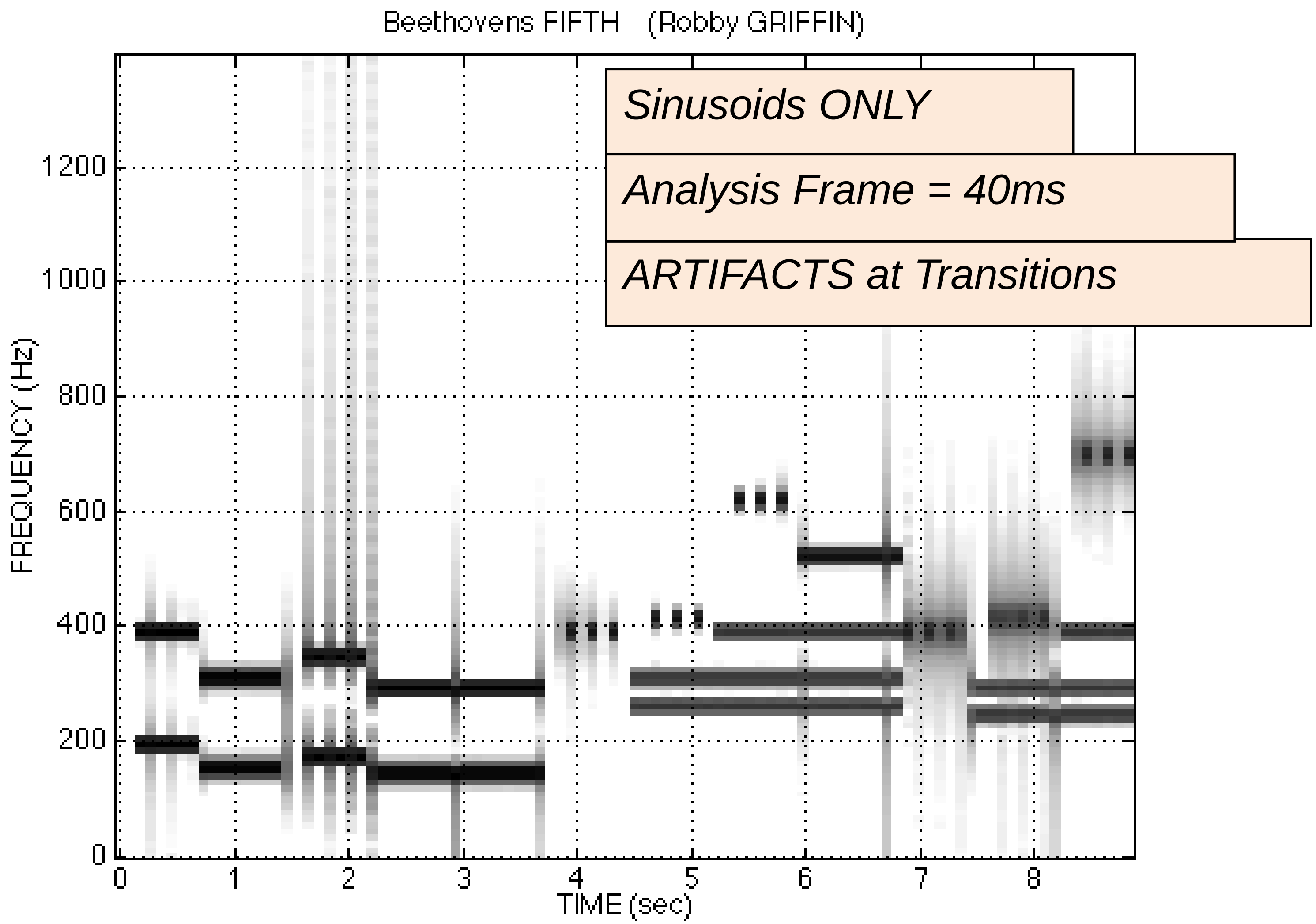


SPECTROGRAM of C-Scale



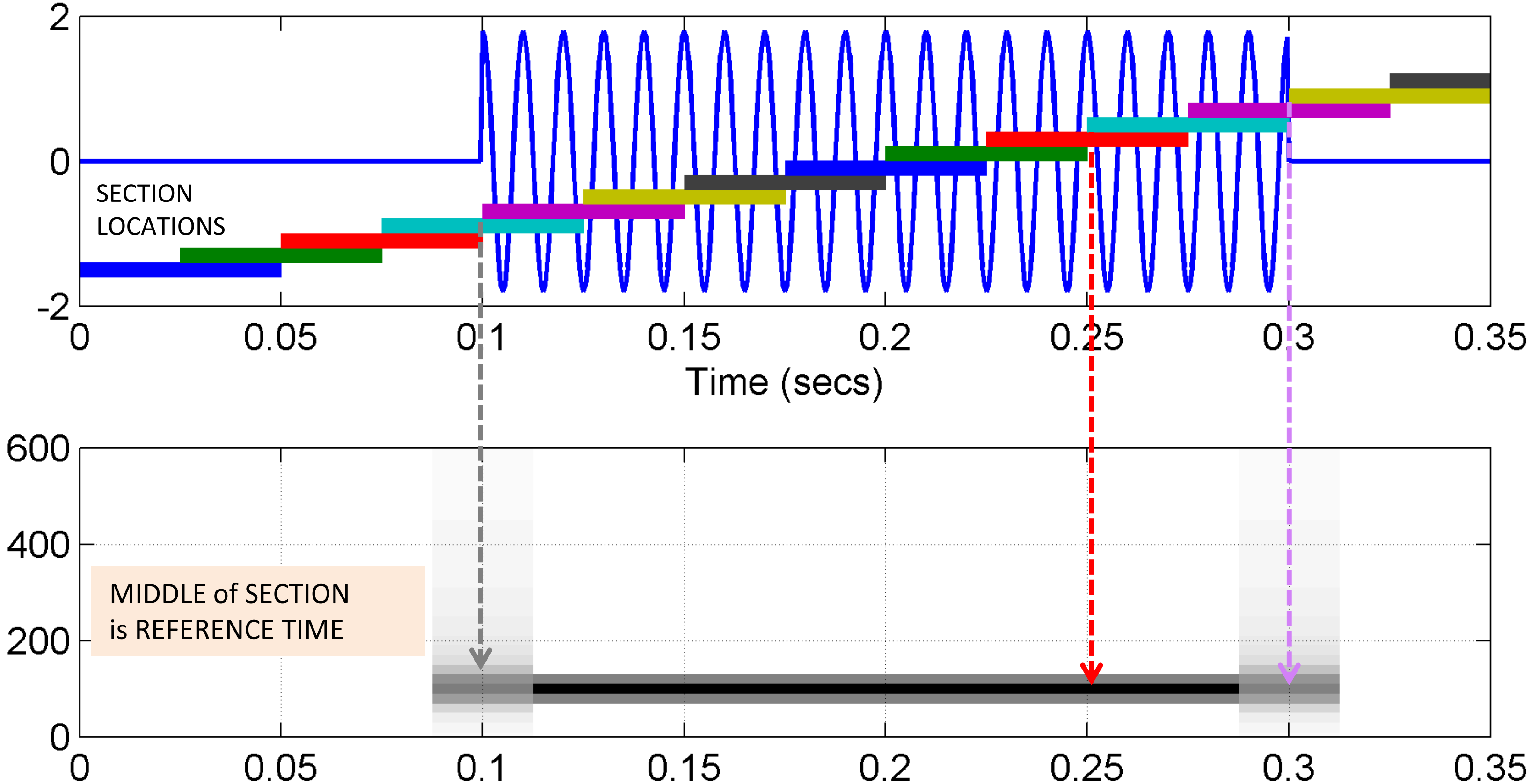


Spectrogram of a song



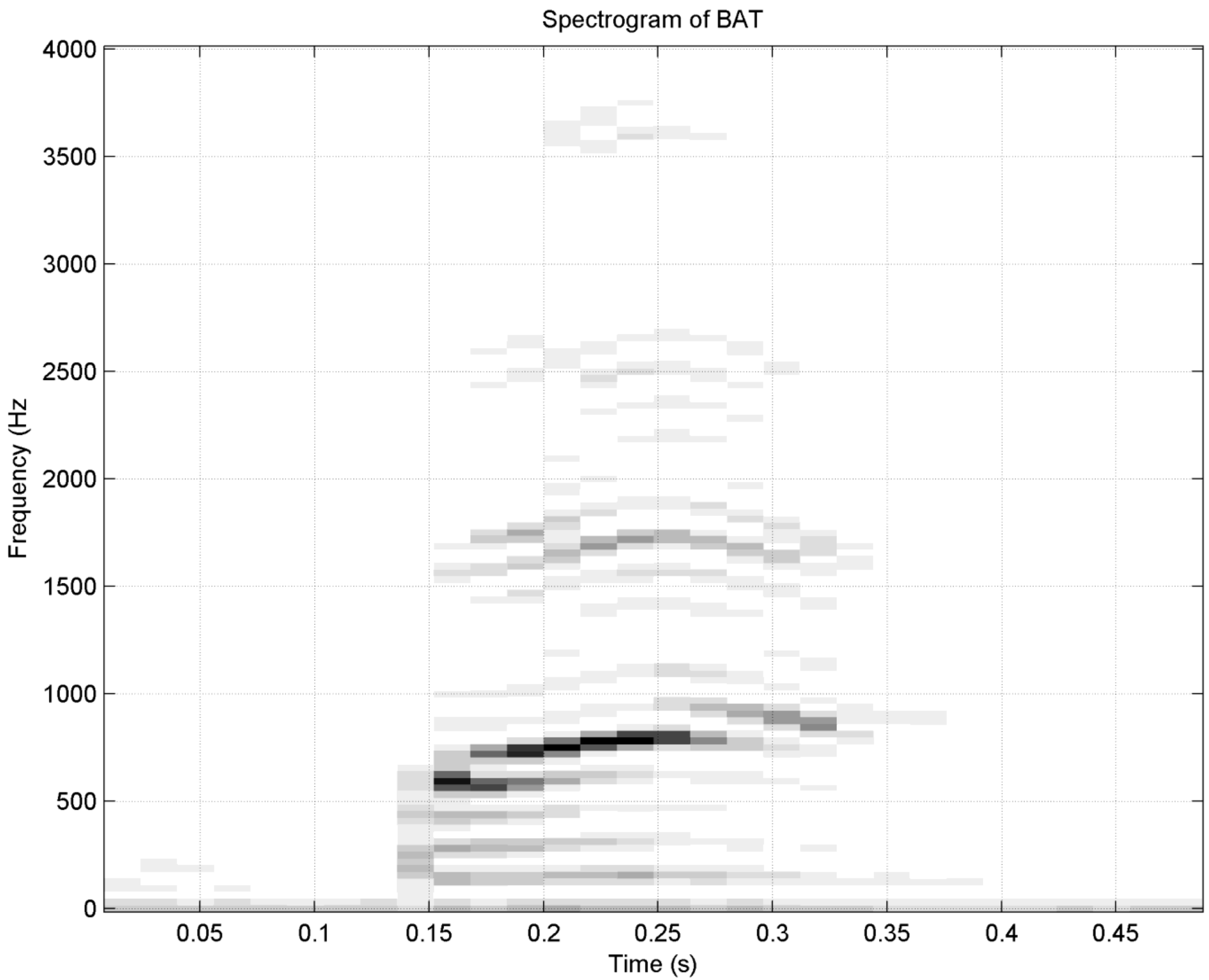
Overlapping Sections in Spectrograms (useful in Labs)

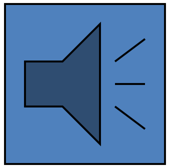
- 50% overlap is common
- Consider edge effects when analyzing a short sinusoid



Au

Spectrogram of BAT (plotspec)





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Time-Varying Frequency

- Frequency can change vs. time
  - **Continuously, not stepped**
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

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Chirp

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

Instantaneous Frequency

- Definition

$$x(t) = A\cos(\psi(t))$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$

$$\psi(t) = 2\pi f_0 t + \varphi$$

Makes sense

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\pi f_0$$

Instantaneous Frequency of Chirp

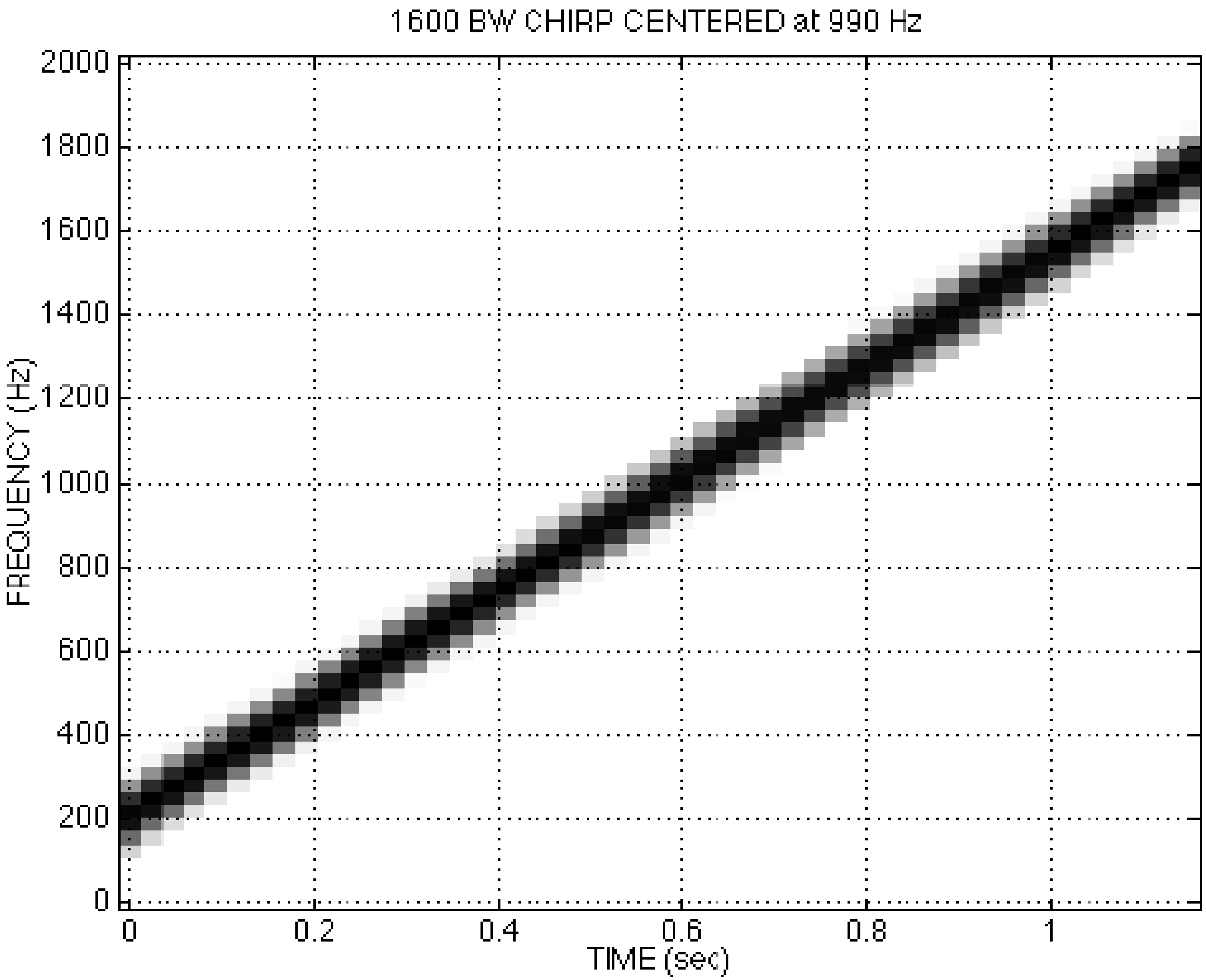
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

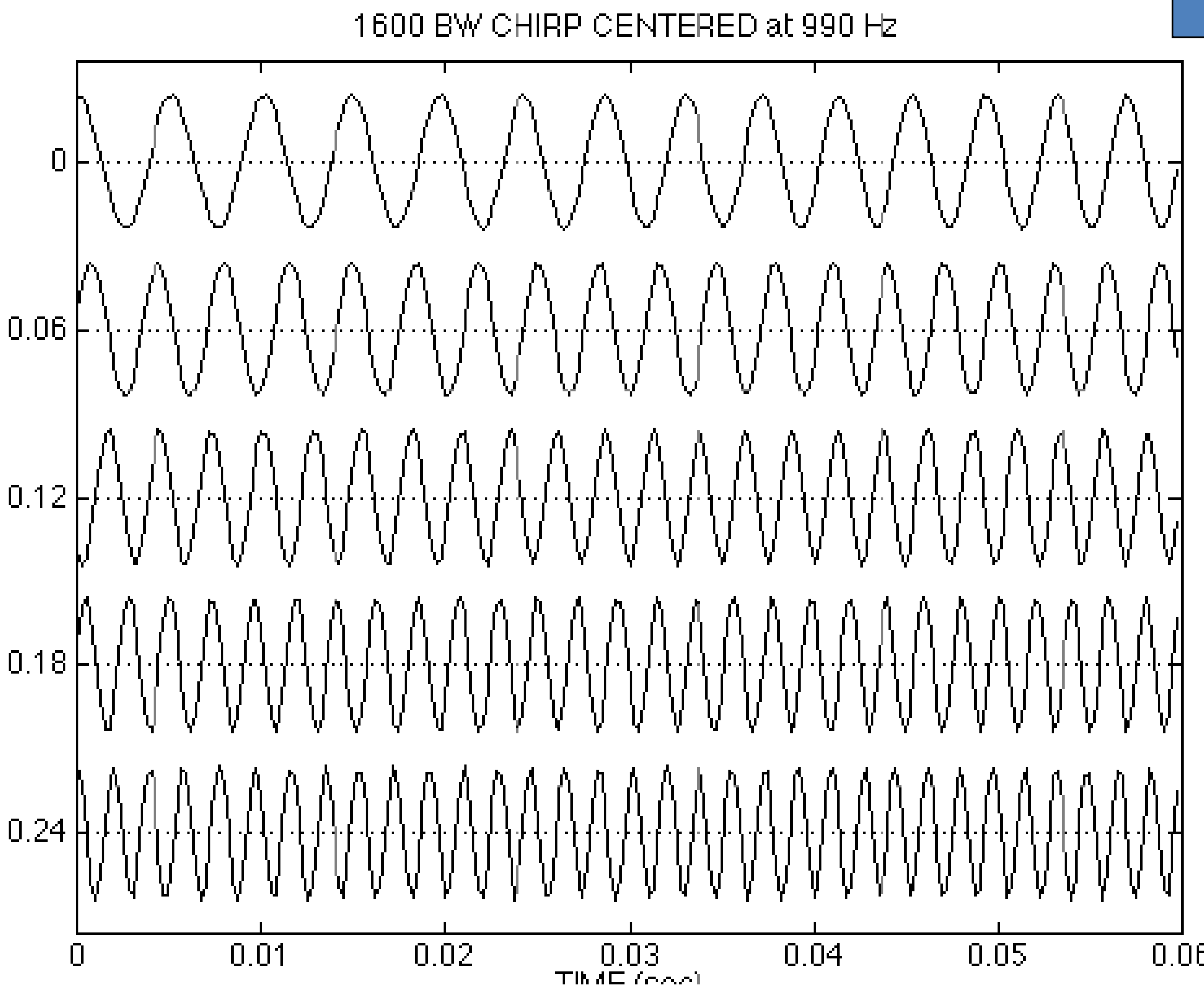
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$

Chirp Spectrogram



Chirp Waveform



Other Frequency Variation vs. time

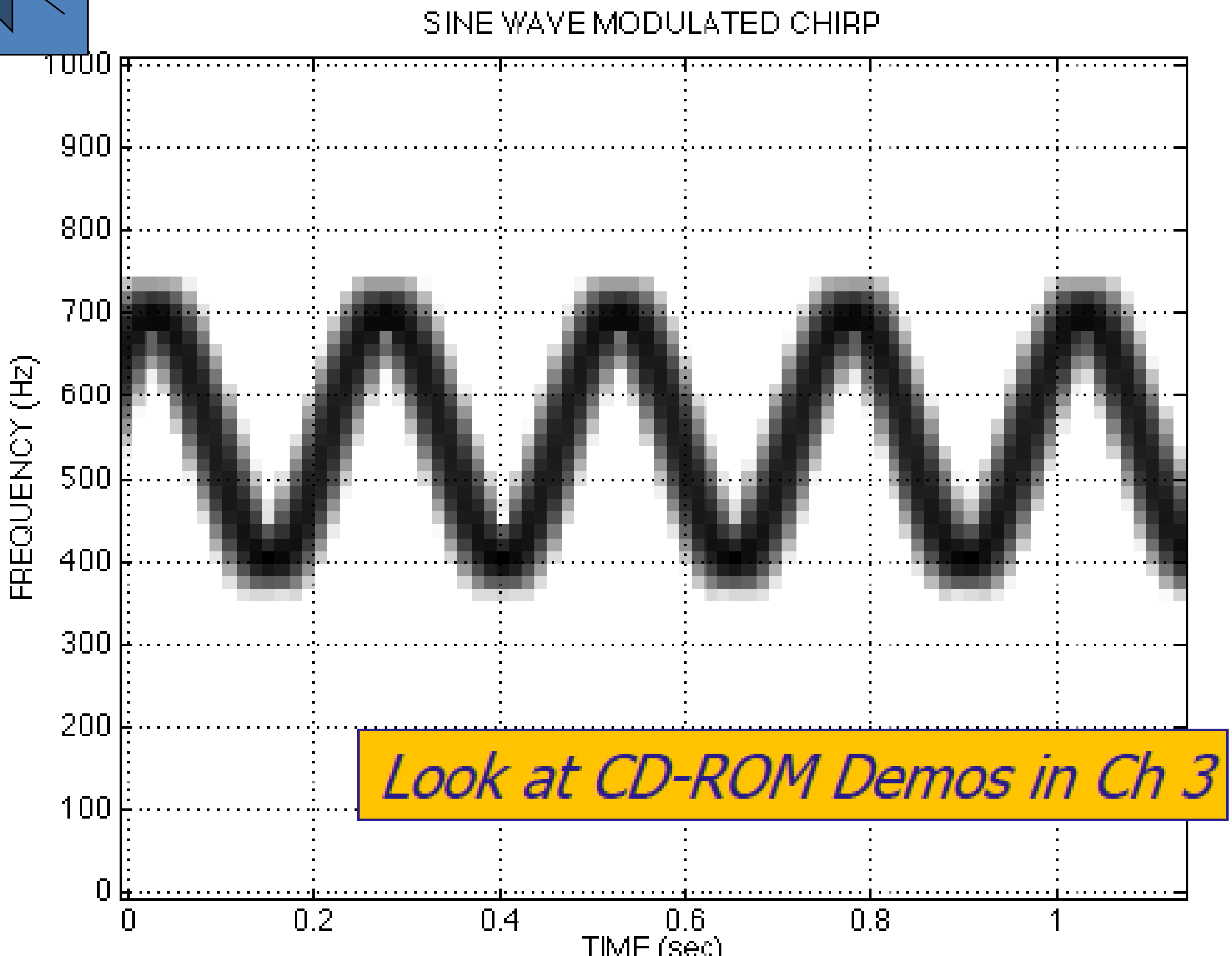
- $\psi(t)$ can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

- $\psi(t)$ could be speech or music:
  - FM radio broadcast

Sine-wave Freq. mod



Demo of Beat Freqs if have time

Fourier Series History

- Jean-Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat ?!
  - Napoleonic era

Lagrange

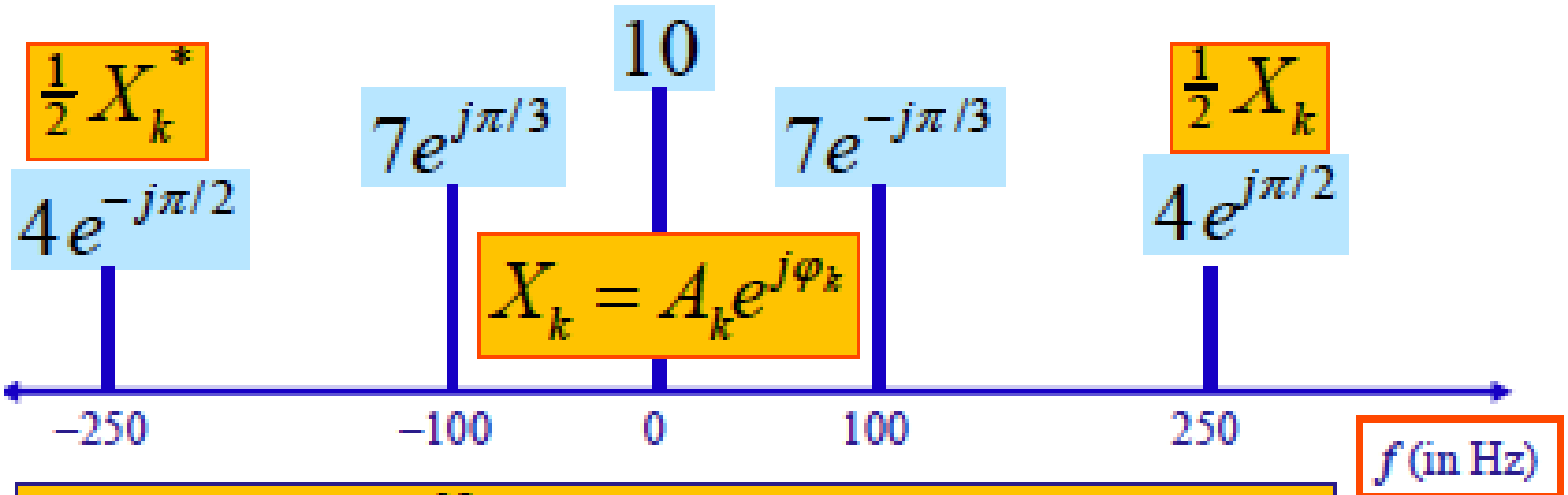
Jean-Baptiste Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Spectrum Diagram

- Recall Complex Amplitude vs. Freq



$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \frac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \frac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\omega_0 = \frac{2\pi k}{T_0} = \left(\frac{2\pi}{T_0}\right)k = 2\pi(f_0)k$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

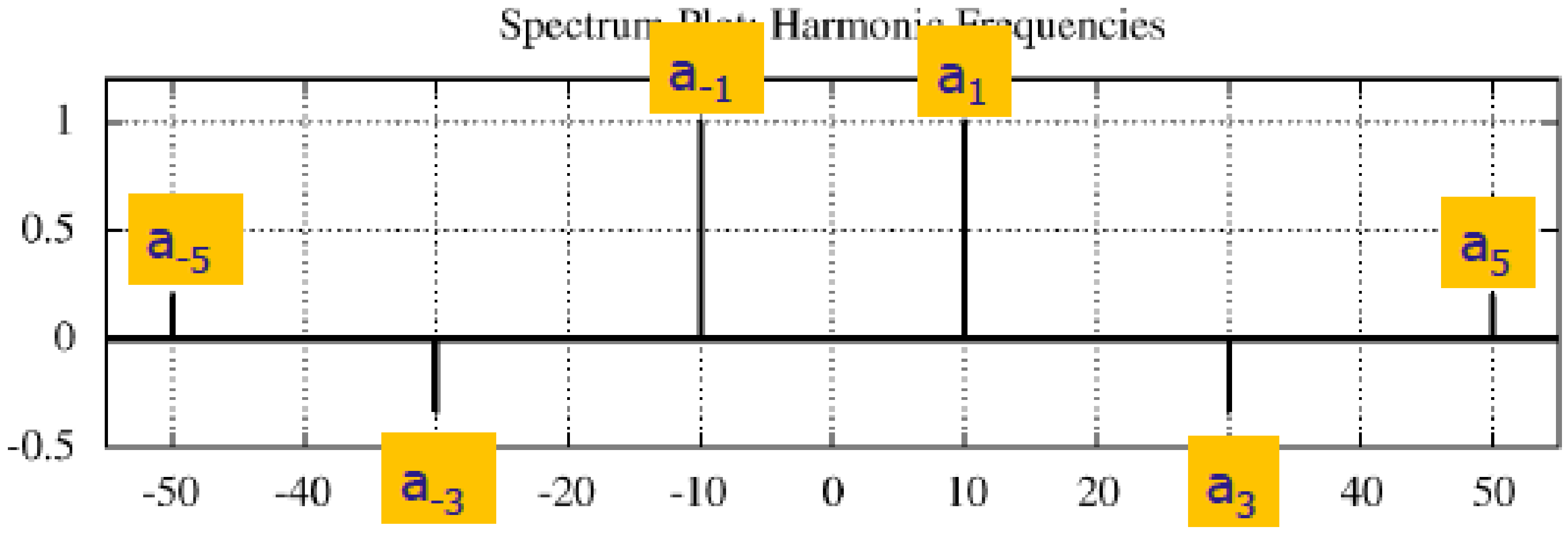
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}, k \neq 0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



a_k is the complex amplitude for kf_0

Synthesis vs. Analysis

- SYNTHESIS
  - Easy
  - Given (ω_k, A_k, ϕ_k) create $x(t)$
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good

- ANALYSIS
  - Hard
  - Given $x(t)$, extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

Strategy

- ANALYSIS
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - The answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k t/T_0}\,dt$$

FUNDAMENTAL FREQ: $f_0=1/T_0$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

Integral of exp(j*k*w0*t)

x(t)=1

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi m t/T_0}\,dt = \frac{T_0}{-j2\pi m} e^{-j2\pi m t/T_0}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t}\,dt = 0$$

$$\boxed{\omega_0 = \frac{2\pi}{T_0}}$$

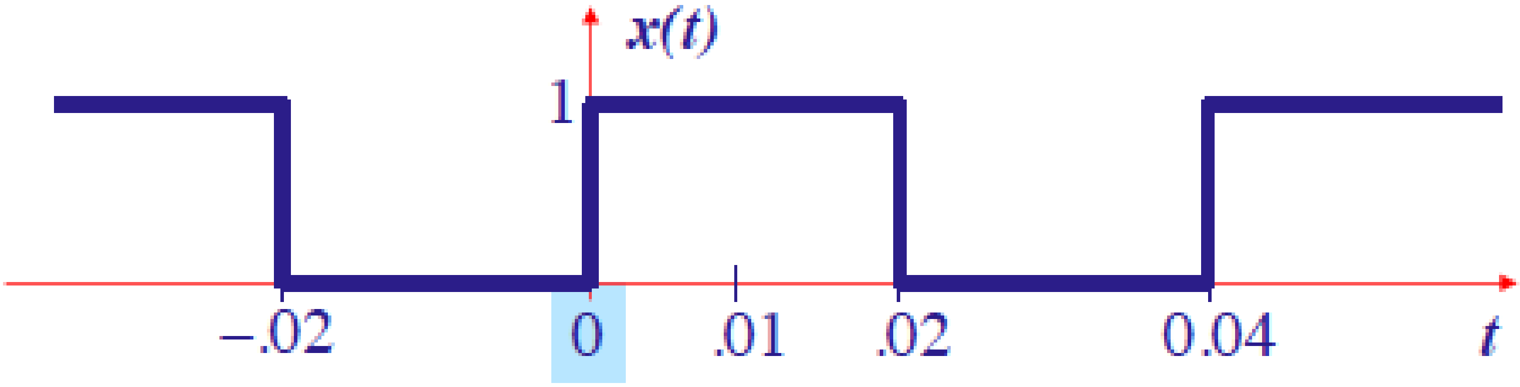
Orthogonality of exp(j*w0*k*t)

- INTEGRATE over ONE PERIOD

$$\boxed{\frac{1}{T_0}\int_0^{T_0} e^{j2\pi l t/T_0} e^{-j2\pi k t/T_0}\, dt = \begin{cases} 0 & k \neq l \\ 1 & k = l \end{cases}}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(l-k)t/T_0}\, dt$$

Square Wave

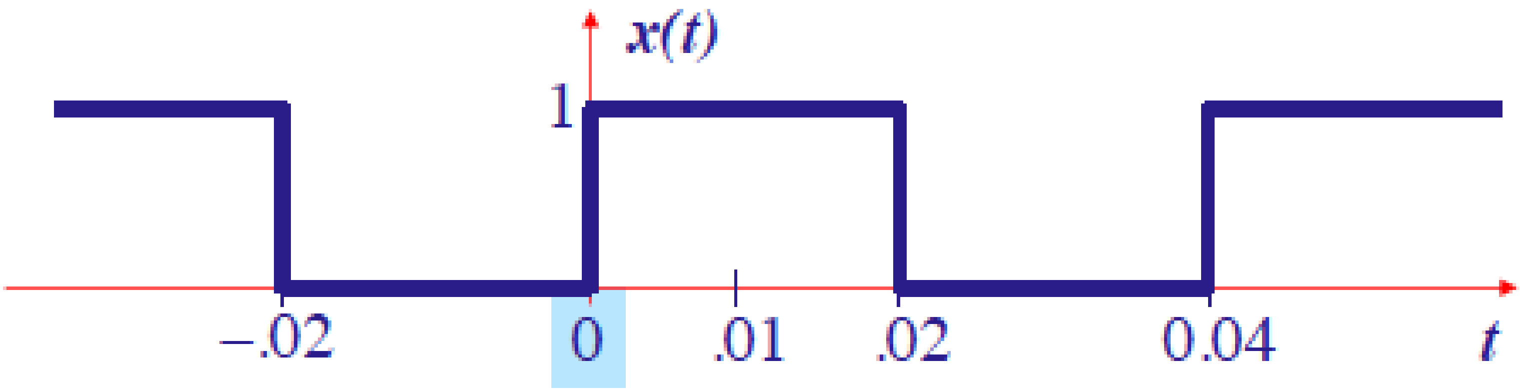


- Period?

Square Wave Example

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

$$\text{for } T_0 = 0.04 \sec:$$



FS for a Square Wave

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j2\pi kt/.04}\,dt = \frac{1}{.04(-j\pi k/.02)}\,e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \frac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k t/T_0}\,dt \qquad (k=0)$$

$$\boxed{a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\mathrm{AREA})}$$

$$a_0 = \tfrac{1}{.04}\int_0^{.02} dt = \tfrac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients, a_k

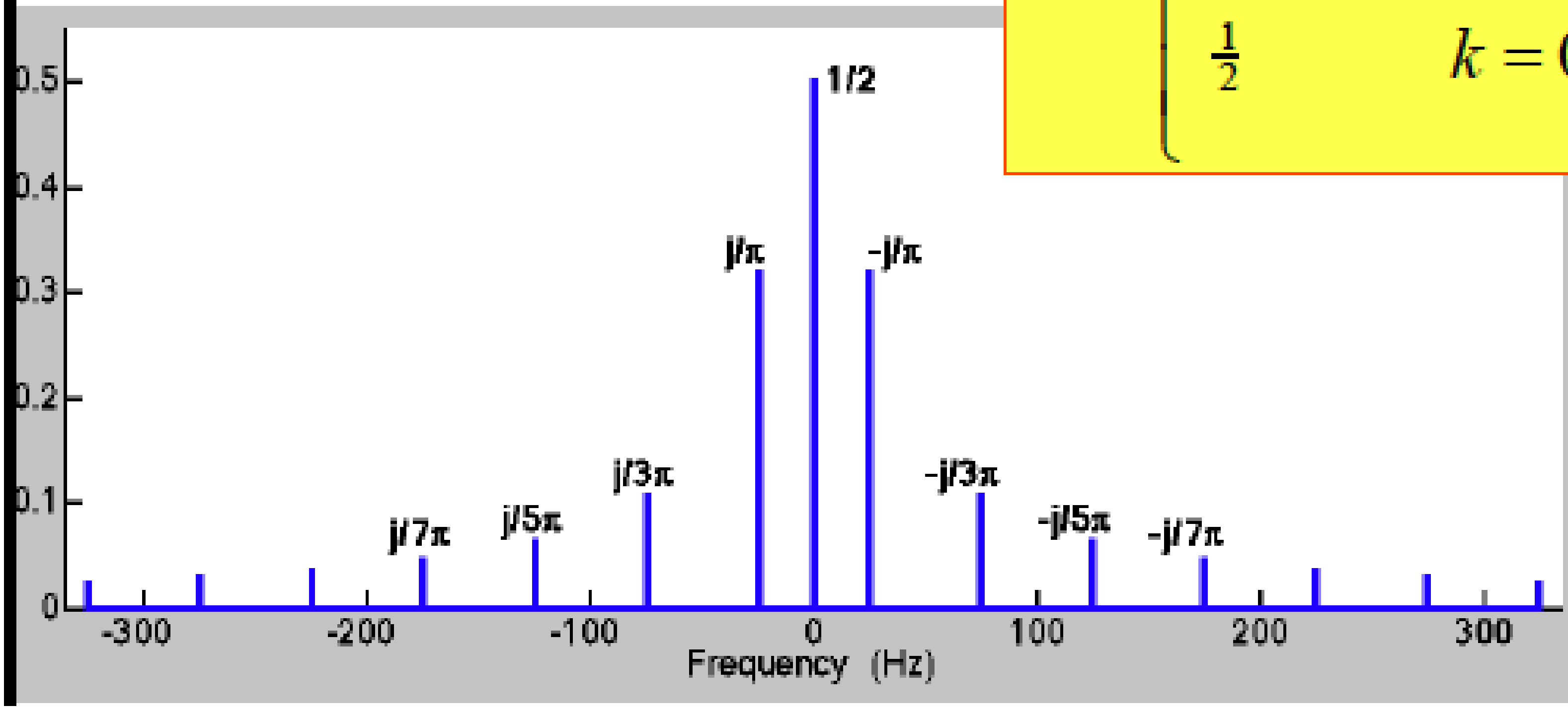
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - e^{-j\pi k}}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \mathrm{K} \\ 0 & k = \pm 2, \pm 4, \mathrm{K} \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

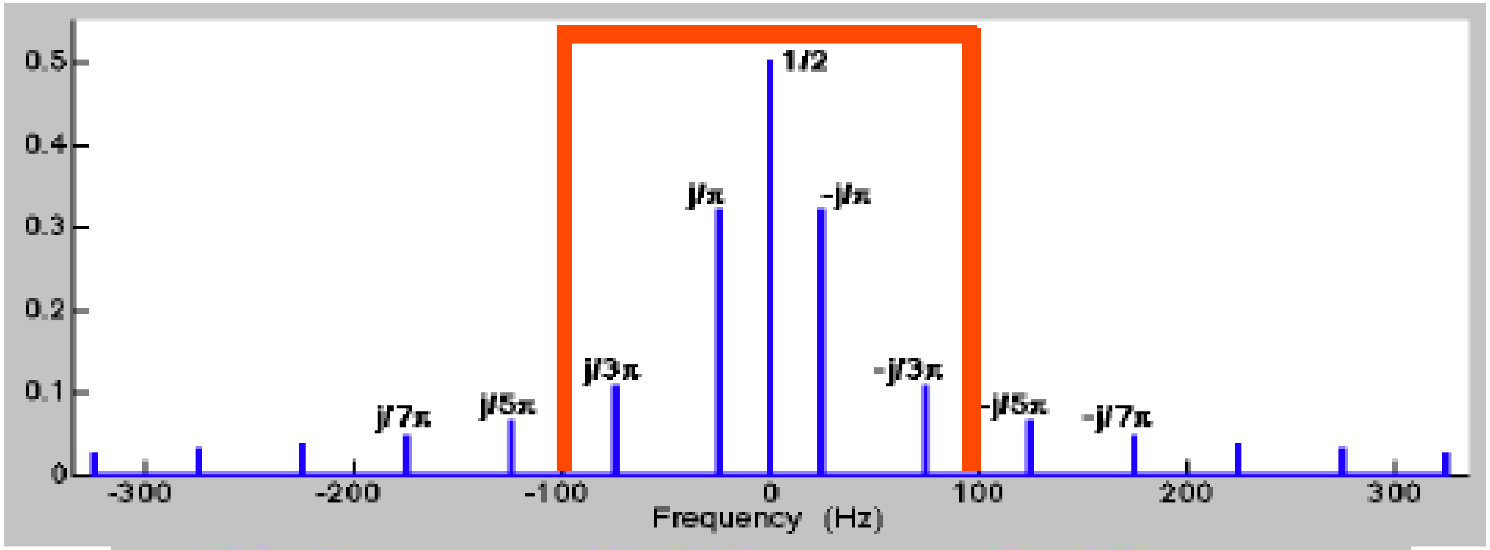
$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

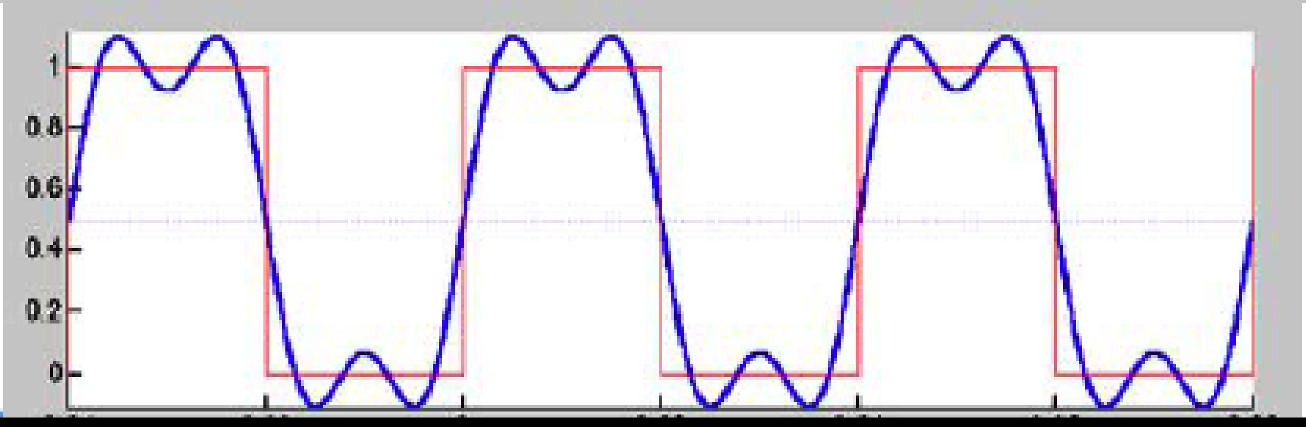
$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Synthesis: 1st and 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

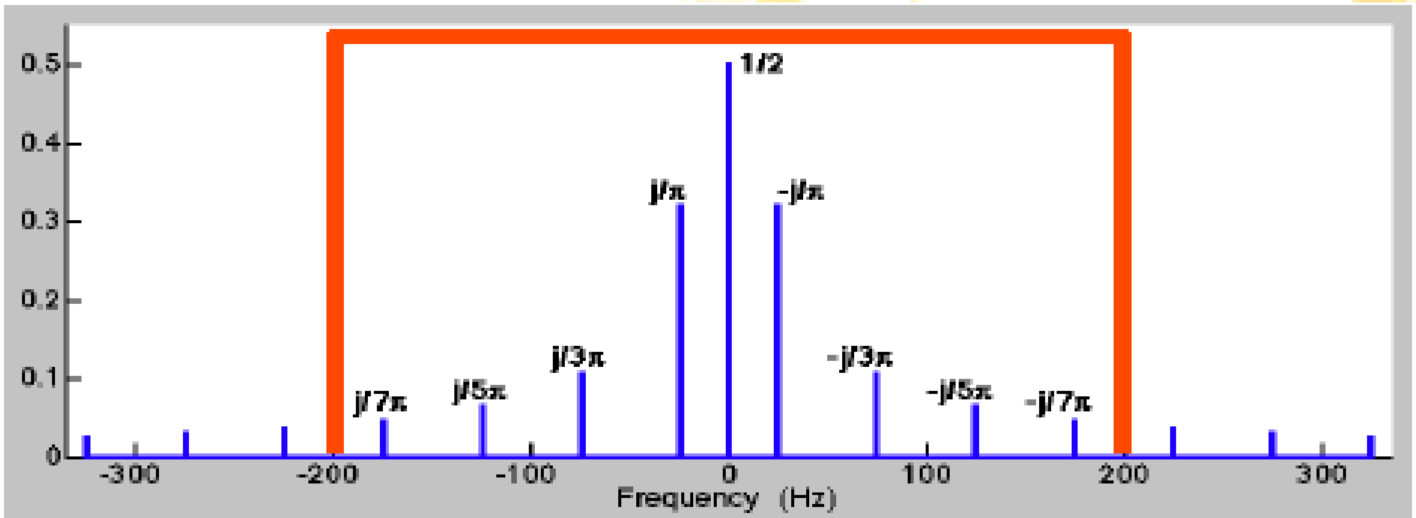


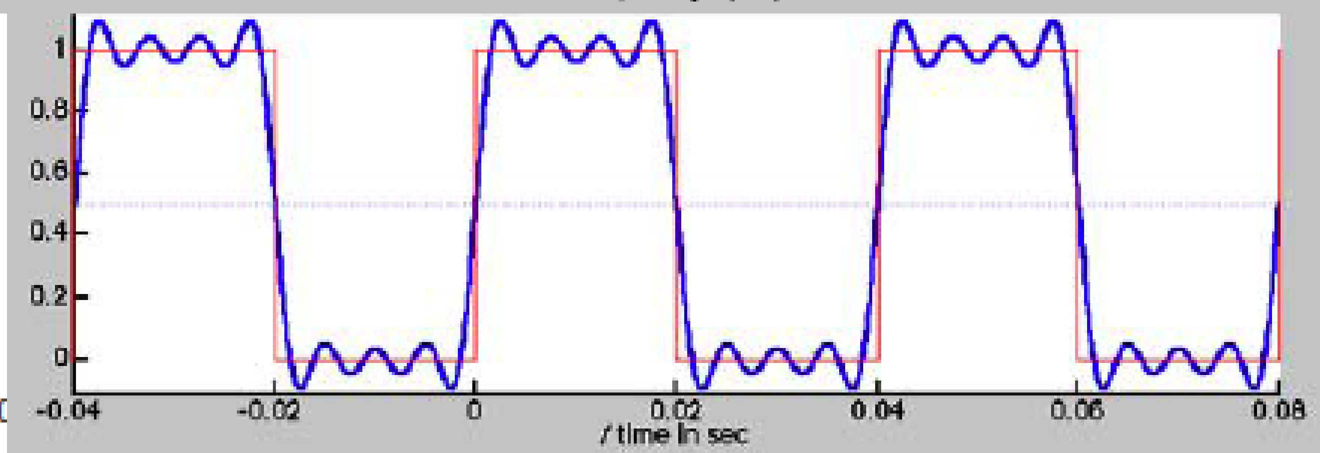


Synthesis: up to 7th Harmonic

Synthesis: up to 7th Harmonic

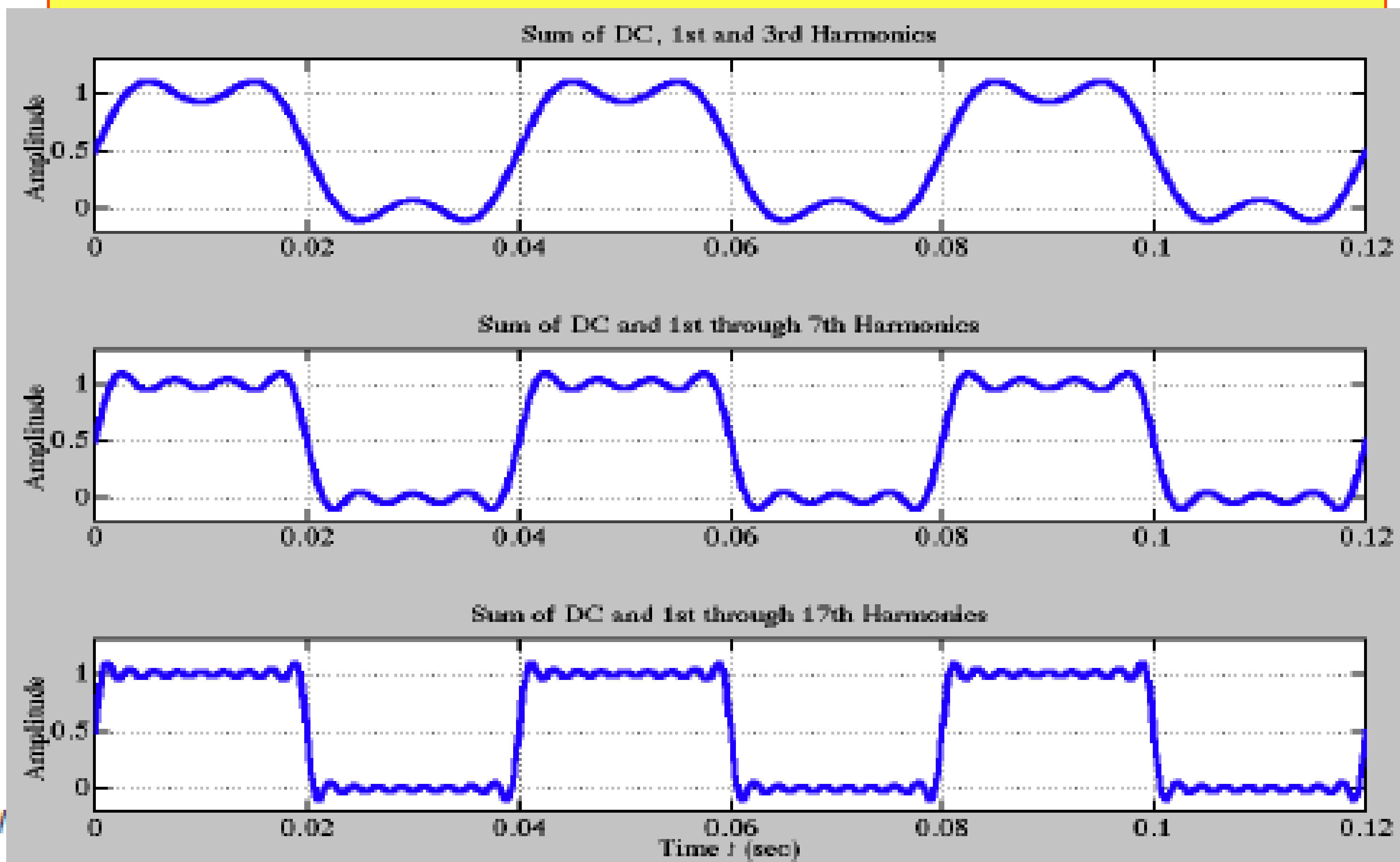
$$t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





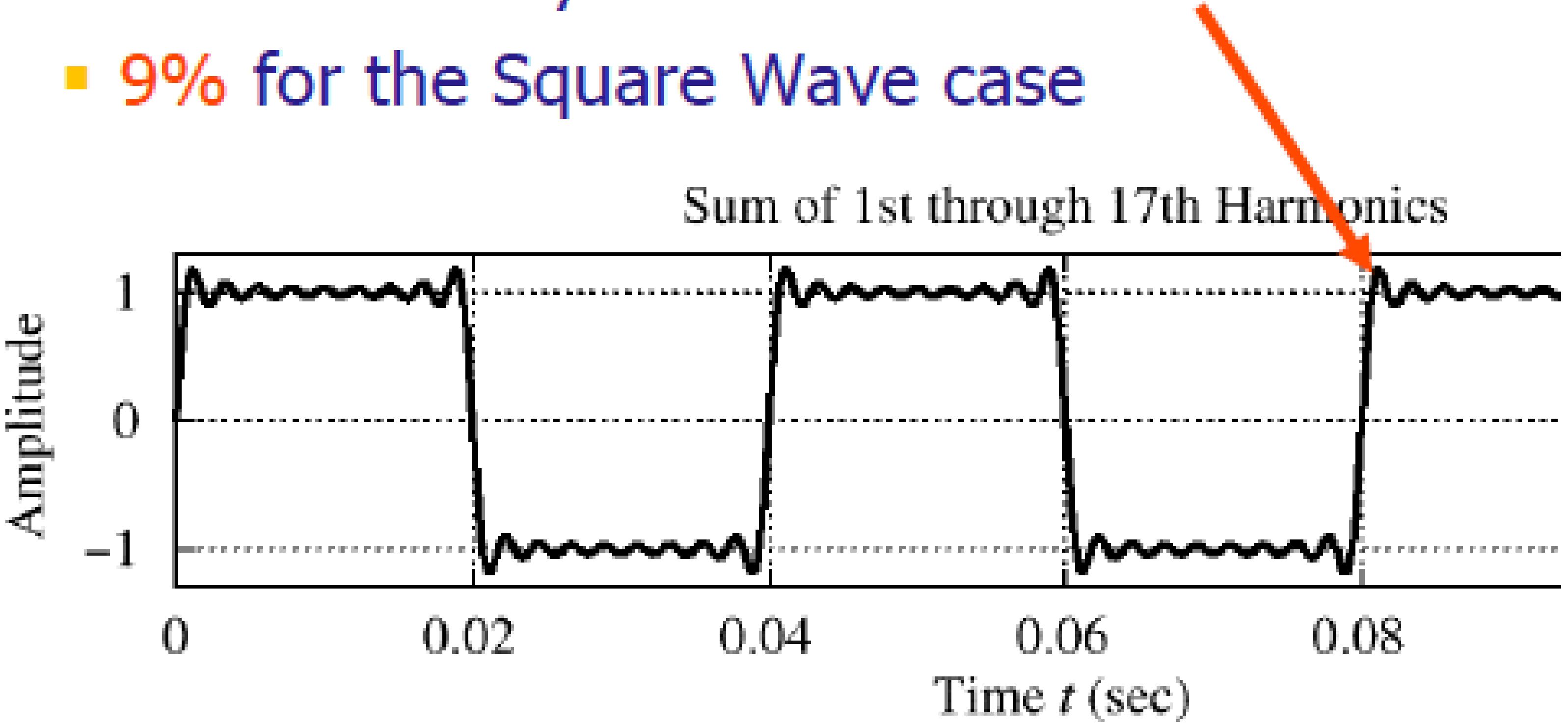
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \mathrm{K}$$



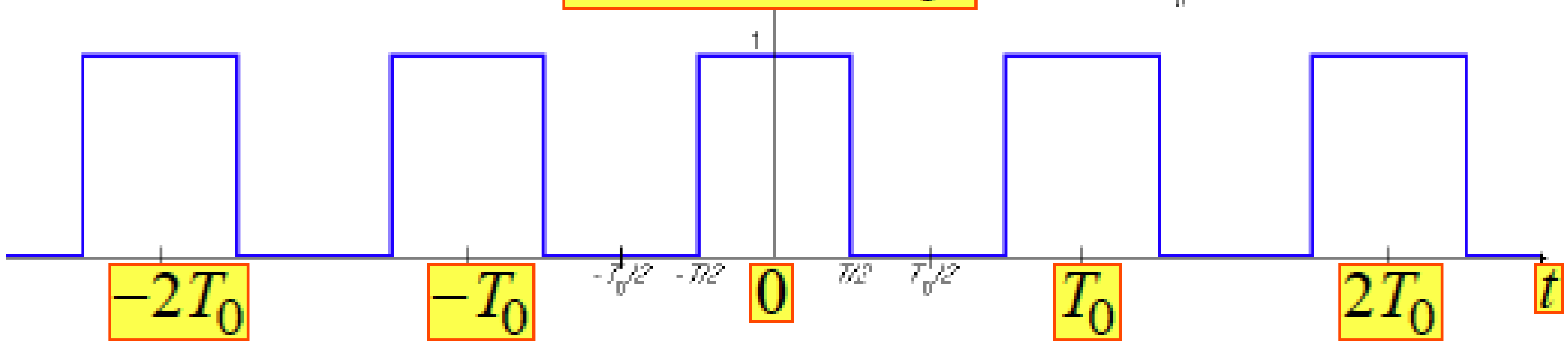
Gibb's Phenomenon

- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



General Periodic Signals

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

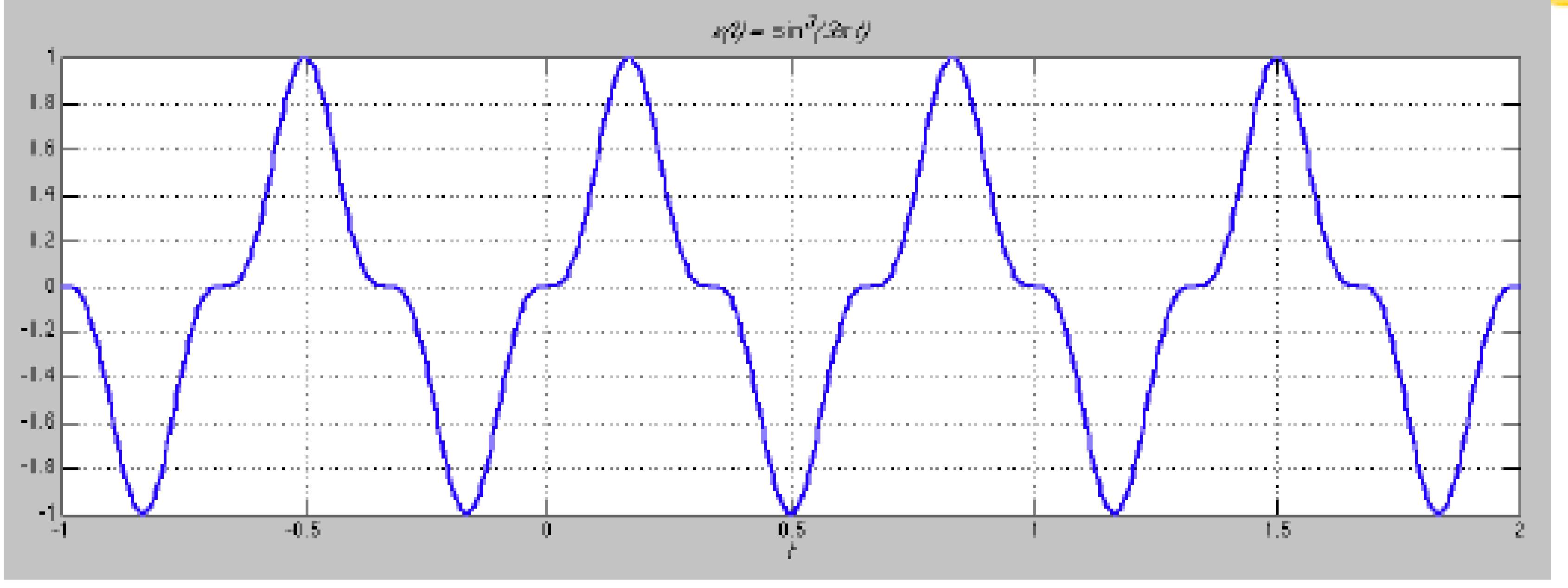
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Example

$$x(t) = \sin^3(3\pi t)$$



$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

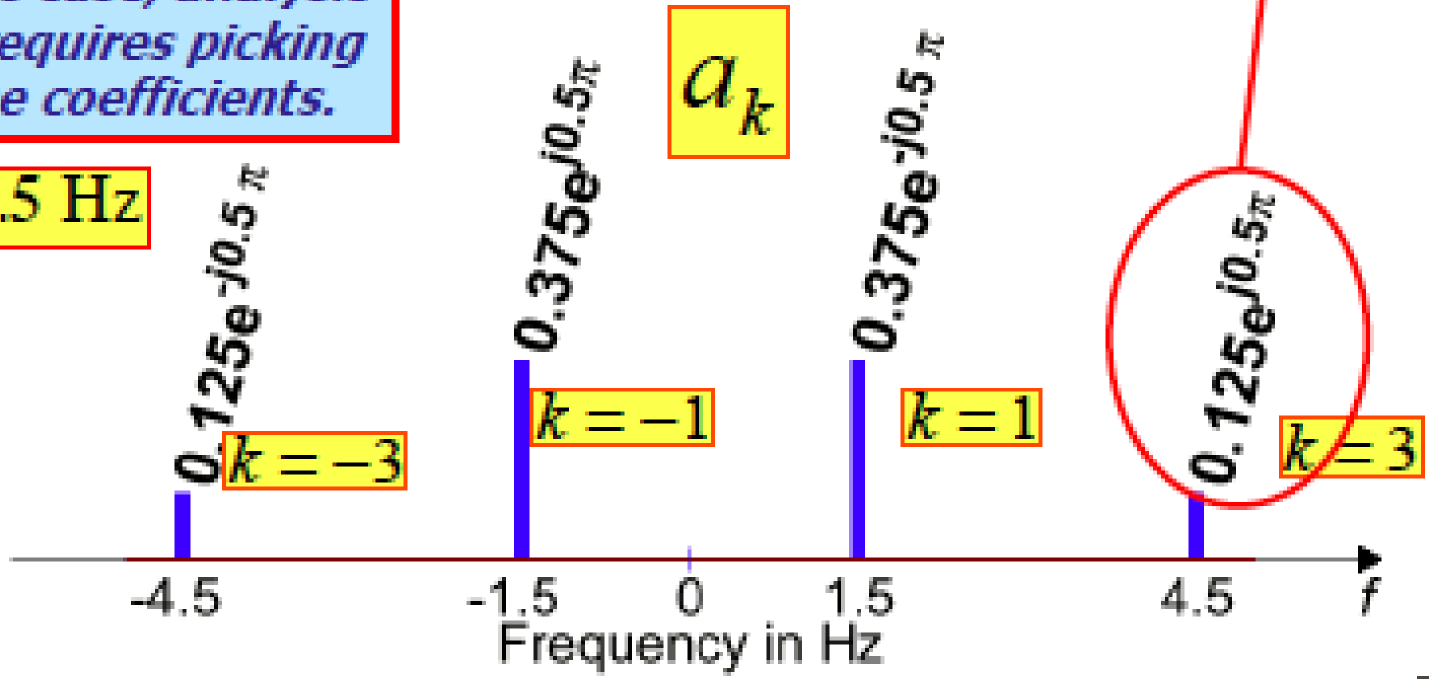
Example

$$x(t)=\sin^3(3\pi t)$$

$$x(t)=\left(\frac{-j}{8}\right)e^{-j9\pi t}+\left(\frac{3j}{8}\right)e^{-j3\pi t}+\left(\frac{-3j}{8}\right)e^{j3\pi t}+\left(\frac{j}{8}\right)e^{j9\pi t}$$

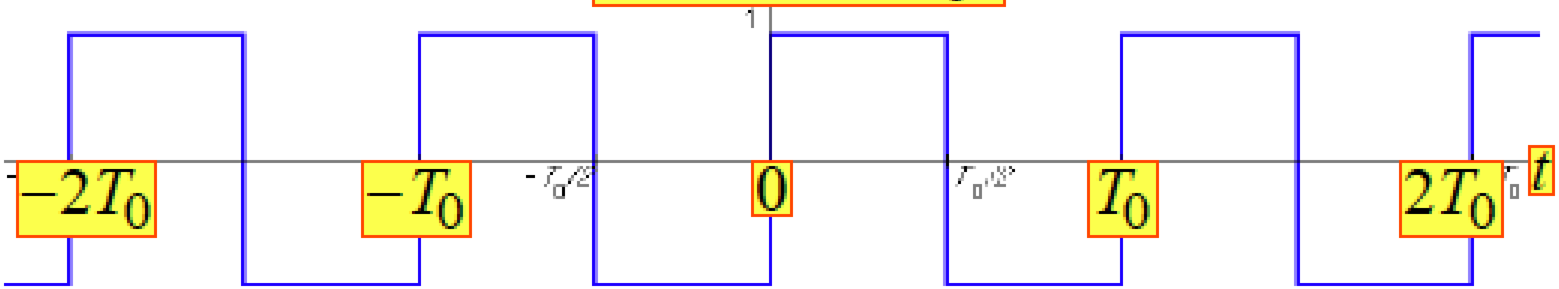
In this case, analysis just requires picking off the coefficients.

$f_0 = 1.5$ Hz



Square Wave Signal

$$x(t) = x(t + T_0)$$



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\,\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$X_0 = A_0 e^{j0}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{ X_k e^{j2\pi f_k t} \right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t} \right\}$$

Bandlimited Signals

- A bandlimited signal has all its frequencies below a certain limit ω_N.
  - A square wave is *not* a bandlimited signal since its non-zero spectrum components go all the way up to infinity.
  - Bandlimited signals are very smooth.
  - Bandlimited signals can be sampled and then reconstructed exactly. This is the basis for all of modern communications and signal processing.

Fourier Series and other Demos