ECES-352
Homework #5
Solutions

Reading: Chapter 4 on Sampling

PROBLEM 1

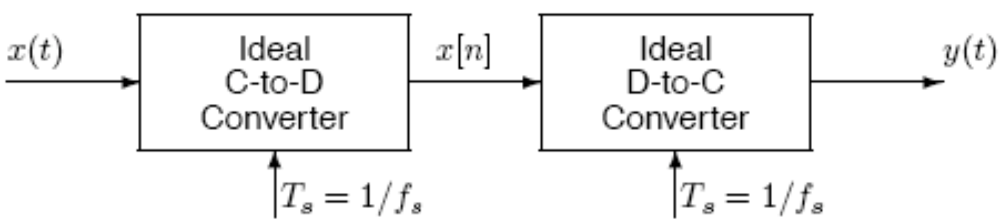


Figure 1: Ideal sampling and reconstruction system.

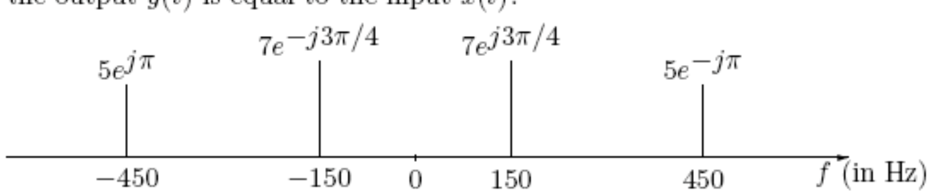
Again consider the ideal sampling and reconstruction system shown in Figure 1 of the previous problem.

(a) Suppose that the discrete-time signal $x[n]$ in Figure 1 is given by the formula

$$x[n] = 3\cos(0.25\pi n + \pi/5)$$

If the sampling rate of the C-to-D converter is $f_s = 11000$ samples/second, many *different* continuous-time signals $x(t) = x_\ell(t)$ could have been inputs to the above system. Determine two such inputs with frequency less than 11000 Hz; i.e., find $x_1(t) = A_1\cos(\omega_1 t + \phi_1)$ and $x_2(t) = A_2\cos(\omega_2 t + \phi_2)$ such that $x[n] = x_1(nT_s) = x_2(nT_s)$ if $T_s = 1/10000$ secs.

(b) Now if the input $x(t)$ to the system in Figure 1 of Problem 5.1 has the two-sided spectrum representation shown below, what is the *minimum* sampling rate f_s such that the output $y(t)$ is equal to the input $x(t)$?



(c) Determine the spectrum for $x[n]$ when $f_s = 450$ samples/sec. Make a $\boxed{\text{plot}}$ for your answer, but label the frequency, amplitude and phase of each spectral component.

(a) $x[n] = 3\cos(0.25\pi n + \pi/5)$

$f_s = 11000$ Hz

$\hat{\omega} = 0.25\pi = \frac{\omega}{f_s}$

$\omega_1 = (11000)(0.25\pi) = 2750\pi$ r/s, $\boxed{f_1 = 1375 \text{ Hz}}$

$\hat{\omega} = (2\pi - 0.25\pi) = 1.75\pi = \frac{\omega}{f_s}$

$\omega_2 = (11000)(1.75\pi) = 19250\pi$ r/s, $\boxed{f_2 = 9625 \text{ Hz} \left(\begin{matrix}\text{folded}\\ \text{alias}\end{matrix}\right)}$

(b) $f_{max} = 450$ Hz

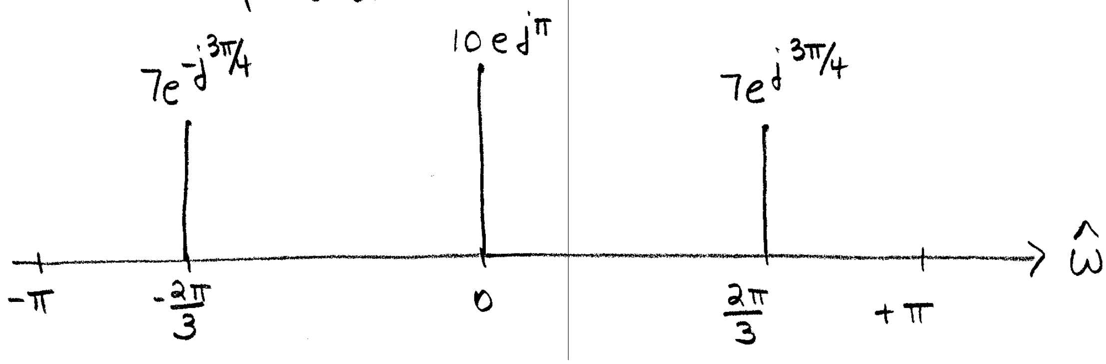
minimum $f_s = 2 \cdot f_{max} = \boxed{900 \text{ Hz}}$

(c) $f_s = 450$ Hz

$\hat{\omega}_1 = \frac{150 \cdot 2\pi}{450} = \frac{2\pi}{3}$

$\hat{\omega}_2 = \frac{450 \cdot 2\pi}{450} = 2\pi$

Note that the plus & minus components at $f = 450$ Hz both map to $\hat{\omega} = 0$ and thus reinforce.



PROBLEM 2

A non-ideal D-to-C converter takes a sequence $y[n]$ as input and produces a continuous-time output $y(t)$ according to the relation

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

where $T_s = 0.1$ second. The input sequence is given by the formula

$$y[n] = \begin{cases} 32 & 0 \leq n \leq 4 \\ 32(.5)^{n-4} & 5 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot $y[n]$ versus n.

(b) For the pulse shape

$$p(t) = \begin{cases} 1 & -0.05 \leq t \leq 0.05 \\ 0 & \text{otherwise} \end{cases}$$

carefully sketch the output waveform $y(t)$ over its non-zero region.

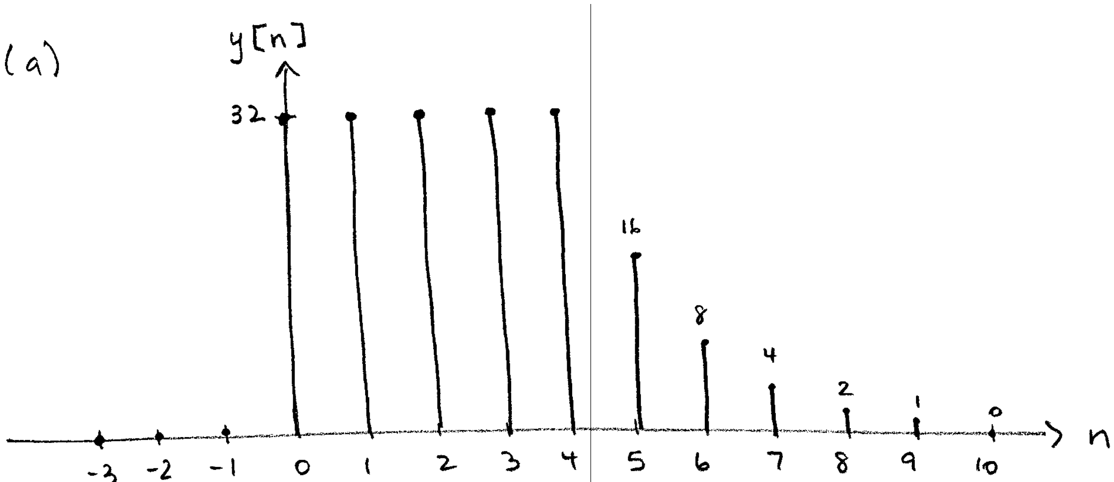
(c) For the pulse shape

$$p(t) = \begin{cases} 1 - 10|t| & -0.1 \leq t \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

carefully sketch the output waveform $y(t)$ over its non-zero region.

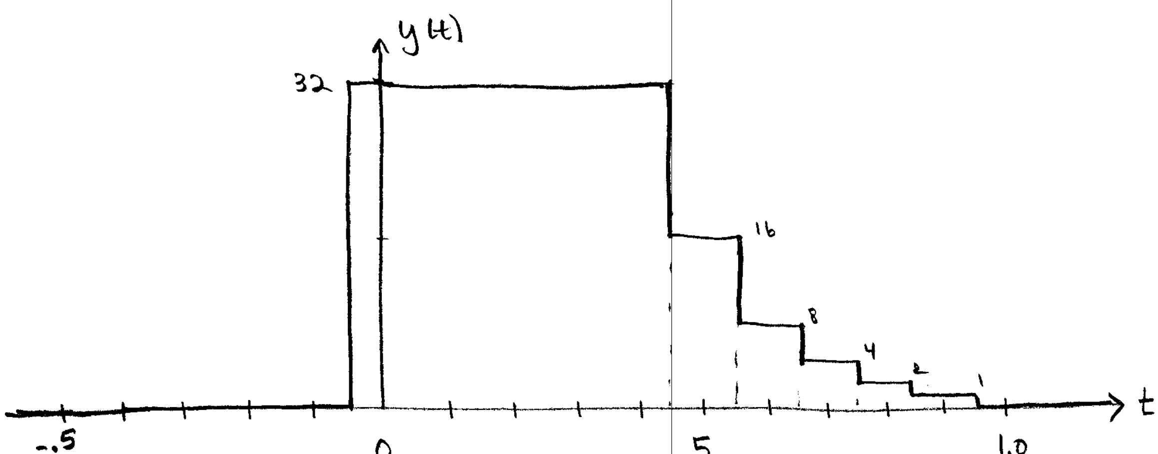
$$y[n] = \begin{cases} 32 & 0 \leq n \leq 4 \\ 32(.5)^{n-4} & 5 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

(a)

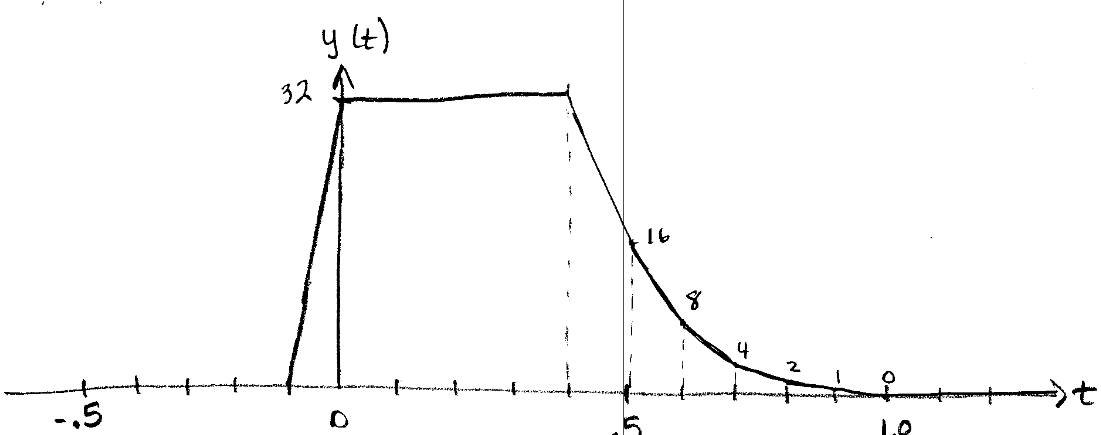


(b)

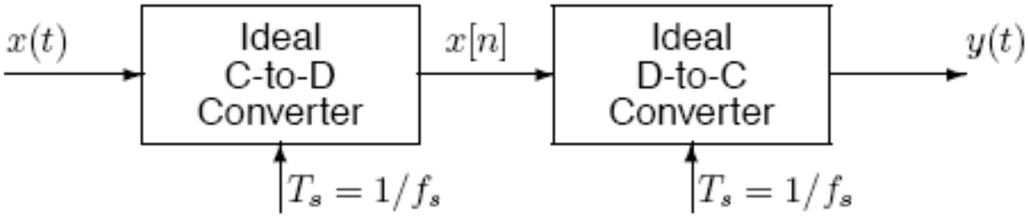
$$p(t) = \begin{cases} 1 & -0.05 \le t \le 0.05 \\ 0 & \text{otherwise} \end{cases}$$



(c) $p(t) = \begin{cases} 1 - 10|t| & ; -.1 \le t \le .1 \\ 0 & ; \text{otherwise} \end{cases}$



PROBLEM 3



Chirps are very useful signals for probing the behavior of sampling operations and illustrating the "folding" type of aliasing (see Fig. 4.4 in the book).

(a) If the input to the ideal C/D converter is $x(t) = 7\cos(1800\pi t + \pi/4)$, and the sampling frequency is 1000 Hz, then the output $y(t)$ is a sinusoid. Determine the formula for the output signal.

(b) Suppose that the input signal is a chirp signal defined as follows:

$$x(t) = \cos(2000\pi t - 400\pi t^2) \qquad \text{for } 0 \le t \le 5 \text{ sec.}$$

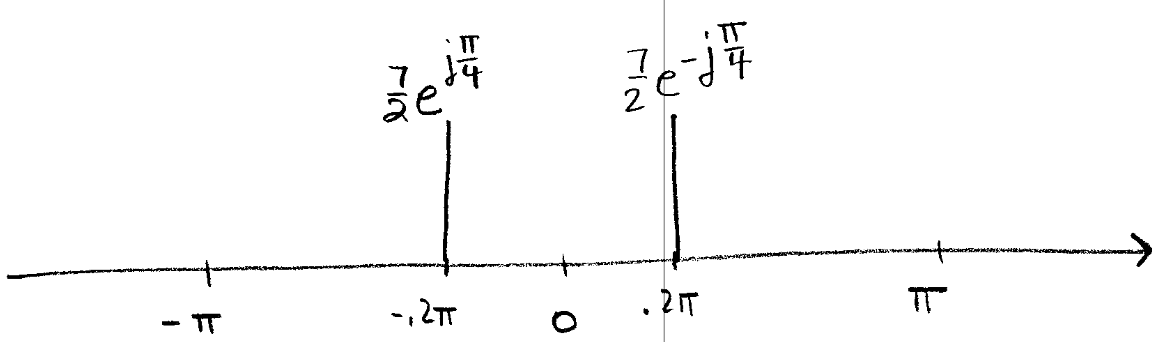
If the sampling rate is $f_s = 1000$ Hz, then the output signal $y(t)$ will have time-varying frequency content. Draw a graph of the resulting analog *instantaneous* frequency (in Hz) versus time of the signal $y(t)$ **after reconstruction.** Hint: this could be done in MATLAB by putting a sampled chirp signal into the MATLAB function `specgram()`, or the DSP-First function `plotspec()`.

(a) $x(t) = 7\cos(1800\pi t + \pi/4)$

$f_s = 1000$ Hz

$$\hat{\omega} = \frac{\omega}{f_s} = \frac{1800\pi}{1000} = 1.8\pi$$

$$\hat{\omega} - 2\pi = 1.8\pi - 2\pi = -0.2\pi = \text{folded alias}$$



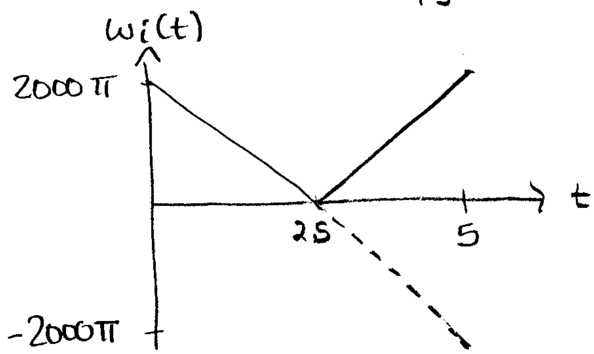
Reconstruction: $\omega = \hat{\omega} \cdot f_s = 0.2\pi(1000) = 200\pi$

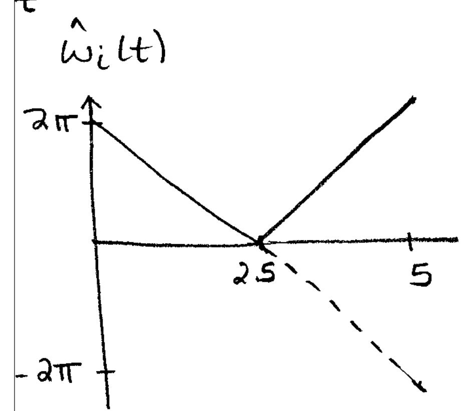
$$\boxed{y(t) = 7\cos(200\pi t - \pi/4)}$$

(b) $x(t) = \cos(2000\pi t - 400\pi t^2)$ $\qquad 0 \le t \le 5$

$\omega_i(t) = 2000\pi - 800\pi t$

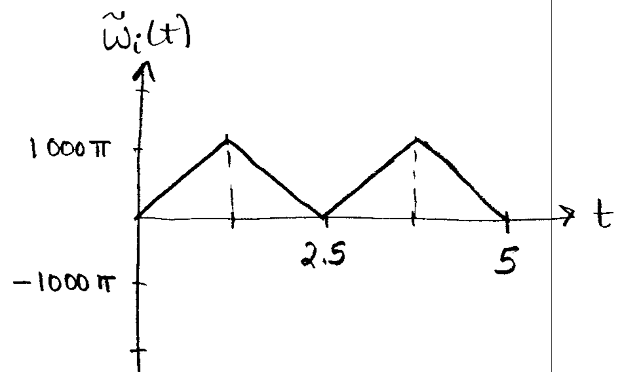
$\hat{\omega}_i(t) = \dfrac{\omega_i(t)}{f_s} = 2\pi - 0.8\pi t$



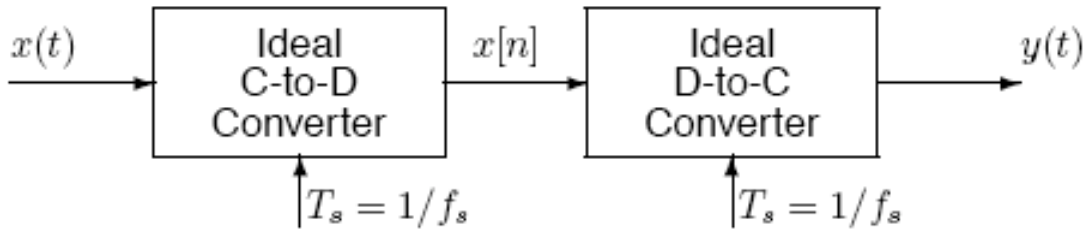


Since $|\hat{\omega}_i| > \pi$, aliasing occurs.

After reconstruction:



PROBLEM 4



In all parts below, the sampling rates of the C/D and D/C converters are **equal,** and the input to the Ideal C/D converter is

$$x(t) = 3\cos(2\pi(50)t - \pi/2) + 2\cos(2\pi(300)t).$$

(a) If the output of the ideal D-to-C Converter is

$$y(t) = x(t) = 3\cos(2\pi(50)t - \pi/2) + 2\cos(2\pi(300)t),$$

what general statement can you make about the sampling frequency f_s in this case?

(b) If the sampling rate is $f_s = 200$ samples/sec., determine the discrete-time signal $x[n]$, and give an expression for $x[n]$ as a sum of cosines. *Make sure that all frequencies in your answer are positive and less than π radians.* Plot the spectrum of this signal over the range of frequencies $-\pi \le \hat{\omega} \le \pi$. Make a $\boxed{\text{plot}}$ for your answer, but label the frequency, amplitude and phase of each spectral component.

(c) If the output of the Ideal D-to-C Converter is

$$y(t) = 3\cos(2\pi(50)t - \pi/2) + 2,$$

determine the value of the sampling frequency f_s. (Remember that the input $x(t)$ is as defined above.)

$$x(t) = 3\cos(2\pi(50)t - \pi/2) + 2\cos(2\pi(300)t)$$

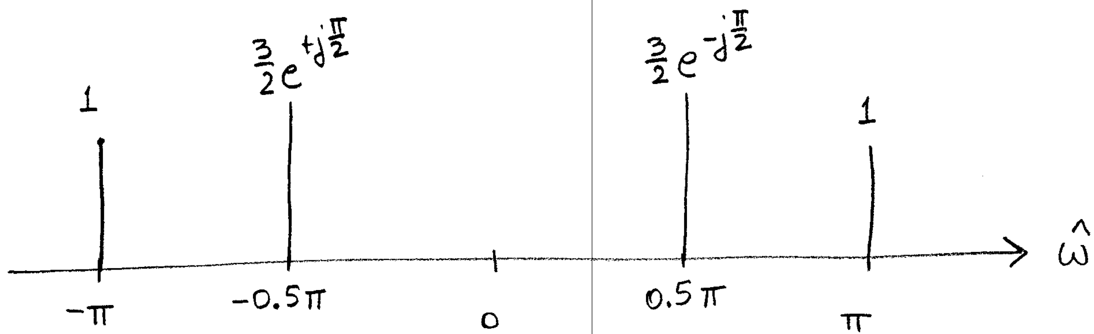
(a) output is identical to input; therefore $f_s > 600$ Hz

(b) $f_s = 200$ Hz

$$x[n] = 3\cos\left(2\pi \cdot 50 \cdot \frac{n}{200} - \pi/2\right) + 2\cos\left(2\pi \cdot 300 \cdot \frac{n}{200}\right)$$

$$= 3\cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) + 2\cos(3\pi n)$$

$$\boxed{x[n] = 3\cos\left(\frac{\pi}{2}n - \frac{\pi}{2}\right) + 2\cos(\pi n)}$$



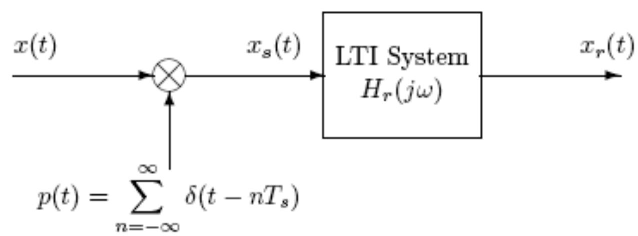
(c) The DC term is an alias of the original 300 Hz term.

$\hat{\omega} = 2\pi$ for $\omega = 300$ Hz

$\hat{\omega} = 2\pi = \dfrac{2\pi \cdot 300}{f_s}$; $\boxed{f_s = 300 \text{ Hz}}$

PROBLEM 5

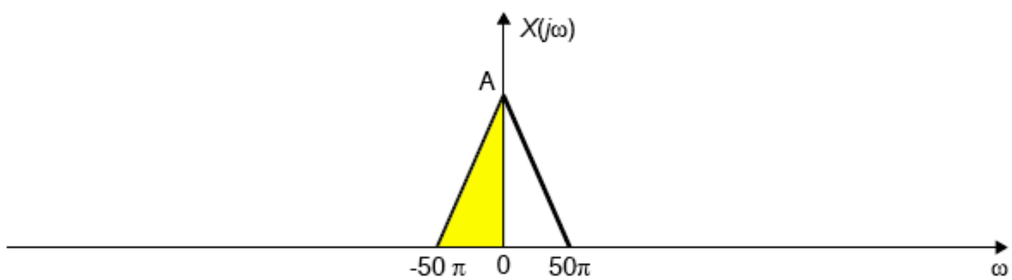
The derivation of the Sampling Theorem involves the operations of impulse train sampling and reconstruction as shown in the following system:



The LTI system is the ideal bandlimited reconstruction filter with frequency response given by

$$H_r(j\omega) = \begin{cases} T_s & |\omega| \le \pi/T_s \\ 0 & |\omega| > \pi/T_s. \end{cases}$$

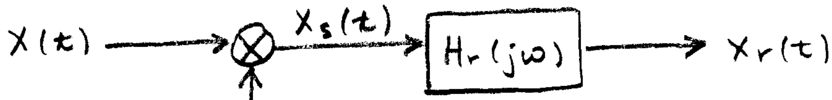
The "typical" bandlimited Fourier transform of the input is depicted below:



(a) For the input with Fourier transform depicted above, use the Sampling Theorem to choose the sampling rate $\omega_s = 2\pi/T_s$ so that $x_r(t) = x(t)$. Plot $X_s(j\omega)$ for the value of $\omega_s = 2\pi/T_s$ that is equal to the *Nyquist* rate.[1]

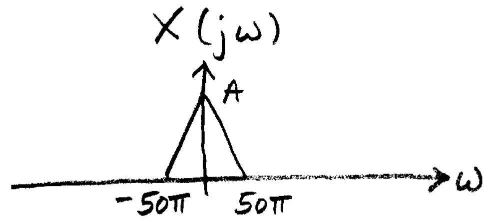
(b) If $\omega_s = 2\pi/T_s = 80\pi$ in the above system and $X(j\omega)$ is as depicted above, plot the Fourier transform $X_s(j\omega)$ and show that aliasing occurs. There will be an infinite number of shifted copies of $X(j\omega)$, so indicate what the pattern is versus ω.

(c) For the conditions of part (b), determine and sketch the Fourier transform of the output $X_r(j\omega)$.

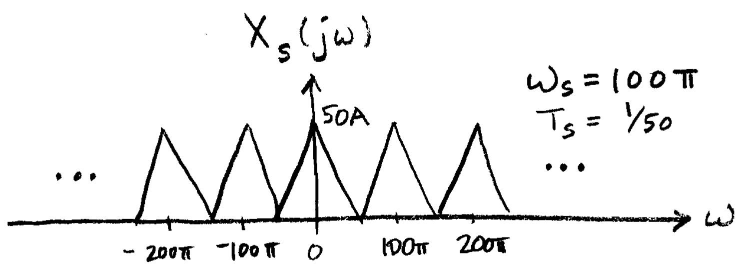


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$H_r(j\omega) = \begin{cases} T_s, & |\omega| \leq \pi/T_s \\ 0, & \text{else} \end{cases}$$

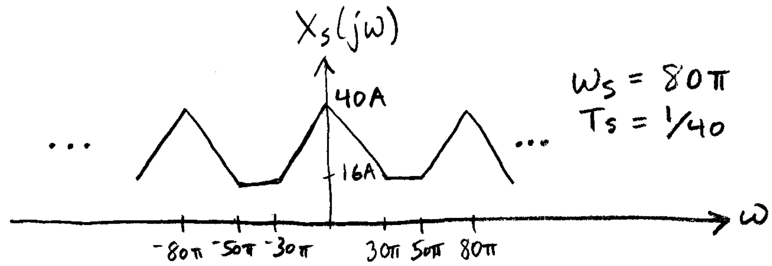


(a) To obtain $x_r(t) = x(t)$, choose $\omega_s \geq 100\pi$.



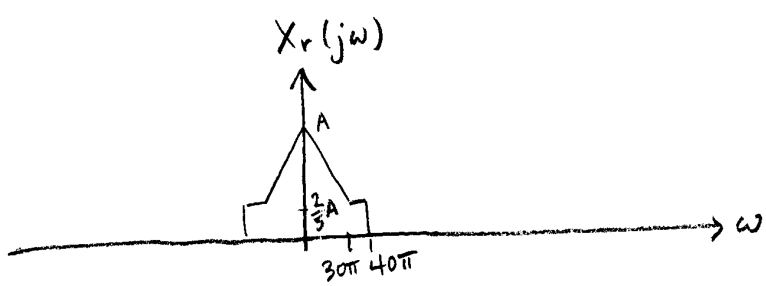
$\omega_s = 100\pi$
$T_s = 1/50$
(Nyquist rate)

(b)



$\omega_s = 80\pi$
$T_s = 1/40$
(aliasing occurs)

(c)



PROBLEM 6

a) Determine the DTFT of the following signals:
   i) $x[n] = \delta[n-3]$
   ii) $x[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$
   iii) $x[n] = (1/4)^{n-3}u[n-3]$
   iv) $x[n] = u[n+3] - u[n-4]$

b) Consider a radix-2 16-point FFT. Please answer the following questions assuming a butterfly-type decimation-in-time implementation discussed in class.
   i) How many butterfly stages are required?
   ii) How many butterflies per stage are required?
   iii) How many 2-pt butterflies are required in total?
   iv) Assuming a decimation-in-time FFT, what is the sequential order of the input samples to compute the fft?
   v) Extra credit: What is the sequential order of input samples for decimation-infrequency?