MATH7501 Practical 10 (Week 11), Semester 1-2021

Topic: Sequences & Series, Limits and Derivatives

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Pre-Tutorial Activity

- Students must have familiarised themselves with units 5 to 8 contents of the reading materials for MATH7501

Resources

- Chapters 5 to 8 of
- https://en.wikipedia.org/wiki/Law_of_the_unconscious_statistician

Section 1: Quiz 2

Do Quiz 2 of Sem1, 2021

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See Quiz 2 solution s
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Section 2: Exponential Distribution

In probability theory and statistics, the exponential distribution is a continuous distribution that is commonly used to measure the expected time for an event to occur. For example,

- in **physics** it is often used to measure radioactive decay,
- in **engineering** it is used to measure the time associated with receiving a defective part on an assembly line,
- in **finance** it is often used to measure the likelihood of the next default for a portfolio of financial assets.I
- it can also be used to measure the likelihood of incurring a specified number of defaults within a specified time period.

If a random variable has the exponential distribution with parameter λ,
we write it as : $X \sim \text{Exp}(\lambda)$.

Here $\lambda > 0$ is called the **rate parameter** and X takes values in the interval $[0, \infty)$.

The **probability density function** (pdf), f_X, of X is given by :

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

The **cumulative distribution function** (cdf), F_X, of X is given by :

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Relationship between pdf and cdf of a continuous random variable

In general, suppose X is a continuous random variable defined on the entire real line, then we compute :

- the probability of X falling within a given interval [a,b] is computed by integrating its pdf, f_X.

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

- The **cumulative distribution function** (cdf), F_X, of X is defined as

$$F_X(x) = P(-\infty \leq X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

- The pdf of a continuous random variable X can be computed by differentiating its cdf, as long as the derivative exists :

$$f_X(x) = \frac{d}{dx} F_X(x).$$

- Intuitively this means $f_X(x)\,dx$ is the probability of X falling within the infinitesimal interval [x, x + dx]

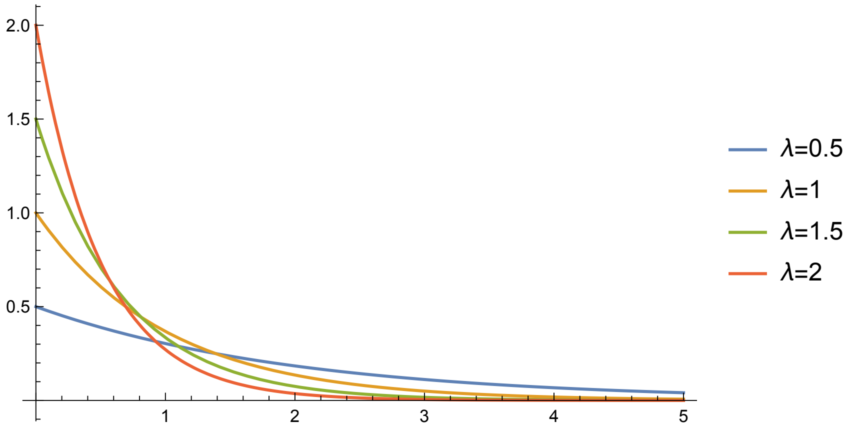
Q1) Use the above general definition of cdf to show that the cdf of X~Exp (λ) is given as in Eq (1)

$$\begin{aligned} F_X(x) &= P(0 \leq X \leq x) \\ &= \int_0^x f_X(t)\,dt \\ &= \int_0^x \lambda e^{-\lambda t}\,dt \\ &= \lambda \int_0^x e^{-\lambda t}\,dt \\ &= \lambda \left[\frac{-1}{\lambda} e^{-\lambda t}\right]_{t=0}^{t=x} \\ &= \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \end{aligned}$$

Q2) Plot the pdf and cdf of X~Exp (λ) for λ = 0.5, 1, 1.5, 2

```
(* plot the pdf*)
Plot[
 Table[PDF[ExponentialDistribution[λ], x], {λ, {1/2, 1, 1.5, 2}}] // Evaluate,
 {x, 0, 5}, PlotRange → All, PlotLegends → {"λ=0.5", "λ=1", "λ=1.5", "λ=2"}]
```

Out[52]=

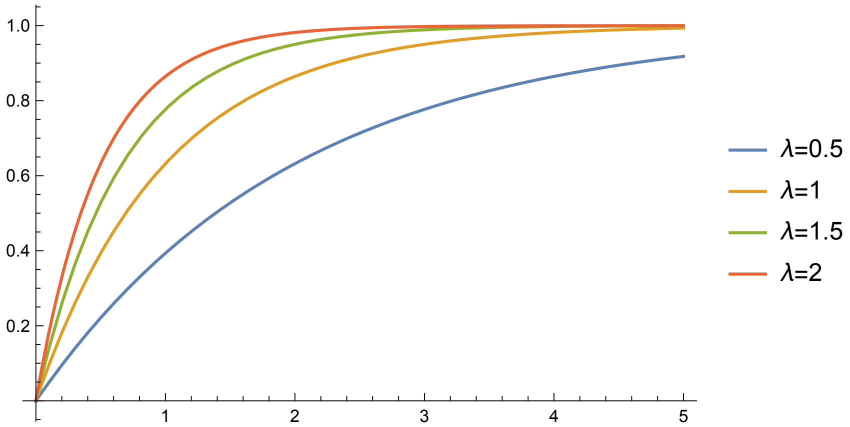



```
(*plot the cdf*)
```

In[55]:=

```
Plot[
 Table[CDF[ExponentialDistribution[λ], x], {λ, {1/2, 1, 1.5, 2}}] // Evaluate,
 {x, 0, 5}, PlotRange → All, PlotLegends → {"λ=0.5", "λ=1", "λ=1.5", "λ=2"}]
```

Out[55]=



Mean and variance of a continuous random variable

suppose X is a continuous random variable defined on the entire real line with pdf $f_X(x)$, then

- its **mean** , μ_X, is the expected value of X which is defined as follows

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- and **variance** of X, $\sigma^2{}_X$, is the expected variability of X form its mean, μ_X. Variance of X is computed as

$$\sigma^2{}_X = \mathrm{var}(X) = E\left[(X - \mu_X)^2\right]$$
$$= E\left[X^2 - 2\,\mu_X\, X + \mu_X{}^2\right],\ \text{after expanding } (X - \text{mu})^2$$
$$= E\left[X^2\right] - 2\,\mu_X\, E[X] + \mu_X{}^2\ ,$$

after using the peoperty : $E(aX + bY) = aE(X) + bE(Y)$ for two random variables X and Y

$$= E\left[X^2\right] - 2\,\mu_X{}^2 + \mu_X{}^2,\ \text{using the fact that } E[X] = \mu_X$$
$$= E\left[X^2\right] - \mu_X{}^2 \text{ ----------------------------- Eq (2)}$$

Note that X^2 is a function of X $\left(\text{say } g(X) = X^2\right)$. Then by the law of the unconscious statistician for continuous random variable,

we have $E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$. Thus Eq (2) is computed as :

$$\sigma^2{}_X = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \mu_X{}^2.$$

Q3) Show that the mean of $X \sim \mathrm{Exp}(\lambda) = \frac{1}{\lambda}$

$$\mu_X = E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx$$
$$= \lambda \int_0^{\infty} x e^{-\lambda x}\,dx$$
$$= \frac{\lambda}{\lambda}\left\{\left[-x e^{-\lambda x}\right]_{x=0}^{x=\infty} + \int_0^{\infty} e^{-\lambda x}\,dx\right\},$$

using integration bi parts : $\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx,$

where $f(x) = x$ and $g'(x) = \int_0^{\infty} e^{-\lambda x}$

$$= \left\{0 - \frac{1}{\lambda}\left[e^{-\lambda x}\right]_{x=0}^{x=\infty}\right\}$$
$$= \frac{1}{\lambda}$$

Q4) Show that the variance of $X \sim \mathrm{Exp}(\lambda) = \frac{1}{\lambda^2}$

To compute $\sigma^2{}_X = \int_{-\infty}^{\infty} x^2\ f_X(x)\ dx - \mu_X{}^2$, first we need to find

$$\int_{-\infty}^{\infty} x^2 f_X(x) = \int_0^{\infty} x^2\,\lambda e^{-\lambda x}\,dx$$
$$= \lambda \int_0^{\infty} x^2\, e^{-\lambda x}\,dx$$
$$= \frac{\lambda}{\lambda}\left\{\left[-x^2\, e^{-\lambda x}\right]_{x=0}^{x=\infty} + 2\int_0^{\infty} x e^{-\lambda x}\,dx\right\},$$

using integration bi parts with $f(x) = x^2$ and $g'(x) = \int_0^{\infty} e^{-\lambda x}$

$$= \left\{0 + \frac{2}{\lambda}\int_0^{\infty} x\lambda e^{-\lambda x}\, dx\right\}, \text{ multiplying by } \frac{\lambda}{\lambda}$$

$$= \frac{2}{\lambda} E[X]$$

$$= \frac{2}{\lambda^2}$$

Thus

$$\sigma^2{}_X = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$