MATH7501: Mathematics for Data Science I

Unit 9: Integration

Motivation: The need to integrate

The integral is, in some sense, the opposite of a derivative. When you take derivatives, you seek to study a function by looking at its 'infinitesimal' changes, whereas when you integrate, you seek to combine all of these infinitesimal changes to recover information about the original function.

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Integration is **essential** in probability theory.

9.1 Area under a curve

Consider a positive and continuous function $f : [a, b] \to \mathbb{R}^+$.

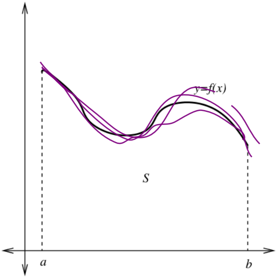

Figure 56: We need to define the area of the region $S = \{(x, y) | a \leq x \leq b, 0 \leq y \leq f(x)\}$.

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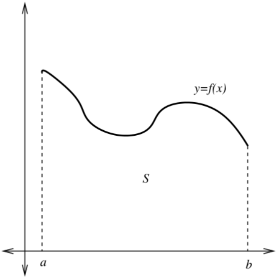
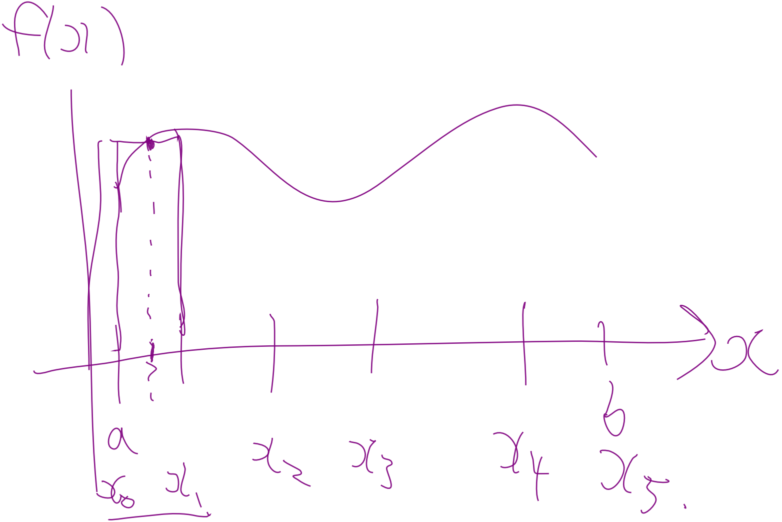


Figure 56: We need to define the area of the region $S = \{(x, y)|a \leq x \leq b, 0 \leq y \leq f(x)\}$.

How does one rigorously define the area underneath the curve $y = f(x)$?

9.1 Area under a curve

Divide the interval $[a, b]$ into subintervals $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$, with $a = x_0$, $b = x_n$ and $x_0 < x_1 < \cdots < x_n$.



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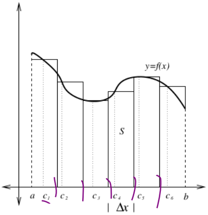

Figure 57: A method of approximating the area of the region S.

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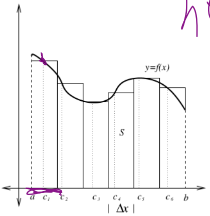
height X length



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Then the total area will be approximately $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$.

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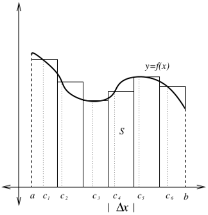

Figure 57: A method of approximating the area of the region S.

Then the total area will be approximately $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$. This approximation is called a *Riemann sum*.

9.1 Area under a curve

Definition (Area under a Curve)

The *area* under a positive and continuous function $f : [a, b] \to \mathbb{R}^+$ is defined to be

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}),$$

(could replace this with any $\xi \in [x_{i-1}, x_i]$)

as long as the size of the biggest sub-interval $[x_{i-1}, x_i]$ goes to 0 as n goes to ∞.

Choice of partition does not matter!

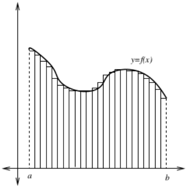

Figure 58: The smaller the width of the rectangle, the better the approximation, in general. Notice that in this diagram we chose $c_i = a + i\Delta x$.

Definition (Riemann Integral)

For *any* continuous function $f : [a, b] \to \mathbb{R}$, the expression

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

from before is known as the *Riemann integral of f over* $[a, b]$, and is denoted $\int_a^b f(x)dx$

still need the length of maximal sub-interval to vanish.

9.1 Area under a curve

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$$\int_a^b f(u)\,du = \int_a^b f(x)\,dx$$

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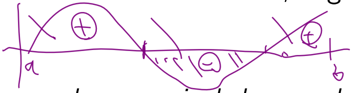
If f is not positive on $[a, b]$, then the integral cannot be interpreted as the area under the curve. Rather, it gives a 'signed area', meaning



$$\int_a^b f(x)dx = \textit{area above x-axis, below graph} - \textit{area below x-axis, above graph}$$

9.1 Area under a curve

Definition (Anti-Derivative)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. An *antiderivative* for f is a function $F : [a, b] \to \mathbb{R}$ so that $F'(x) = f(x)$ for all $x \in (a, b)$.

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Theorem (Fundamental Theorem of Calculus)

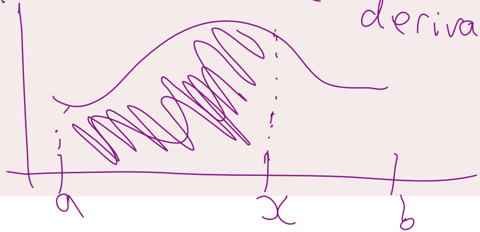
For a continuous function $f : [a, b] \to \mathbb{R}$, the function

$$A(x) = \int_a^x f(t)dt$$

is an anti-derivative for f.

How unique are anti-derivatives?

$A(x)$:



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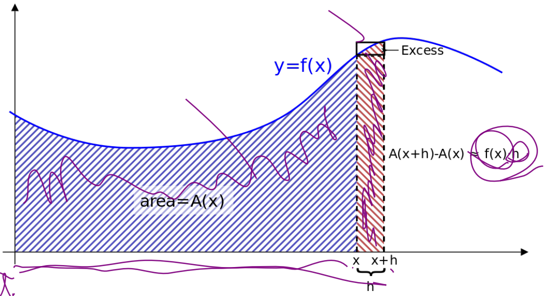
As notation, we often use $F(b) - F(a) = F(x)|_a^b = [F(x)]_a^b$.

$A(x) = \int_a^x f(t)\,dt$ is an anti-derivative of f



https://en.wikipedia.org/wiki/Fundamental_theorem_of_calculus

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \frac{\int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt}{h}$$

$$\approx \frac{f(x) \cdot h}{h} \to f(x)$$

Theorem (Properties of Integrals)

Let c be a real number, and consider continuous functions $f, g : [a, b] \to \mathbb{R}$. Then

- $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (this is more of a definition);

$$\int_b^a f(x)\,dx \;??$$

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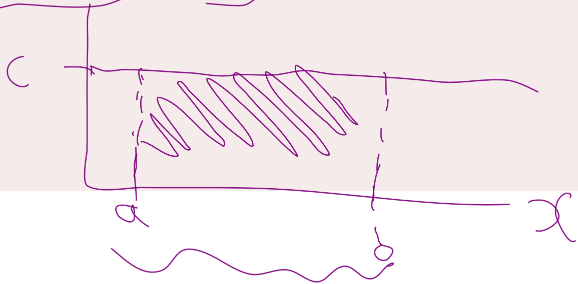

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- $\int_a^b c\,dx = c(b-a)$;
- $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;

"integral of sums is the sum of integrals",

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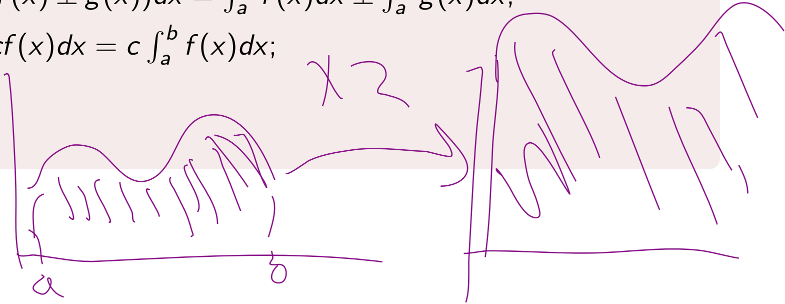
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;



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- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;

true if $c \notin [a,b]$??

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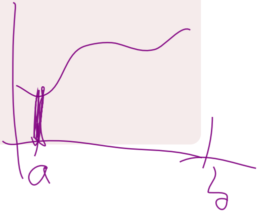
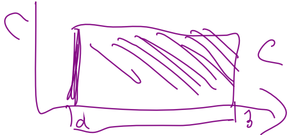
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^a f(x)dx = 0$ (again, sort of a definition).



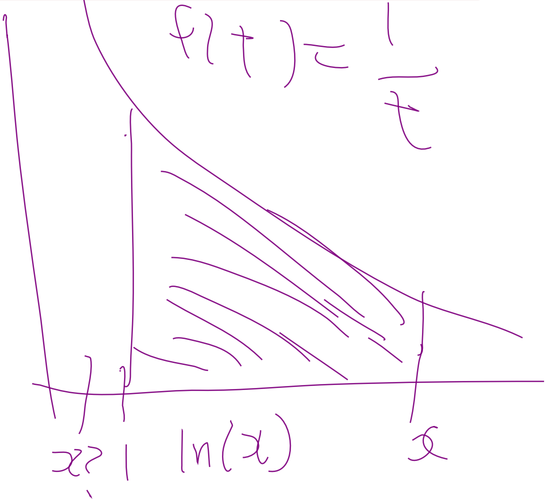


$(b-a)$

Question

Show that $\ln(x) = \int_1^x \frac{1}{t}\,dt$ for all $x > 0$.

$x < 1$



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Solution

Consider the function $g(x) = \ln(x) - \int_1^x \frac{1}{t}dt$.

want to show

$g(x) = 0$ for all $x > 0$.

9.1 Area under a curve

Question

Show that $\ln(x) = \int_1^x \frac{1}{t}\,dt$ for all $x > 0$.

Solution

Consider the function $g(x) = \ln(x) - \int_1^x \frac{1}{t}\,dt$. Using the properties of derivatives and the FTC, we find

$$g'(x) = \frac{1}{x} - \frac{1}{x} = 0.$$

(for all $x > 0$).

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$\Rightarrow g$ is constant
in x.

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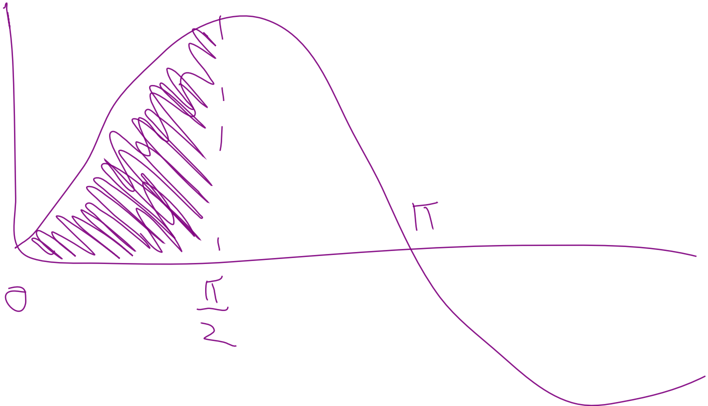
The only functions with derivative 0 are constant (MVT). We therefore find that for all $x > 0$,

$$g(x) = g(1) = \ln(1) - \int_1^1 \frac{1}{t} dt = 0.$$

9.1 Area under a curve

Question

Evaluate $\int_0^{\frac{\pi}{2}} \sin(x)dx$.



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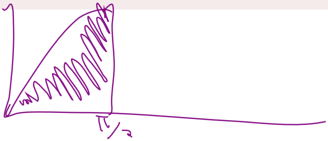
Solution

The function $-\cos(x)$ is an antiderivative for $\sin(x)$, so

$F(\frac{\pi}{2}) - F(0) =$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(x)dx &= [-\cos(x)]|_0^{\frac{\pi}{2}} \\ &= \underbrace{-\cos(\frac{\pi}{2})}_{0} - (\underbrace{-\cos(0)}_{-1}) \\ &= 1.\end{aligned}$$



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Solution

By the FTC,

$$F'(u) = \frac{1}{\sqrt{1+u^2}}.$$

Question

Let $G(x) = \int_2^{\sin(x)} \frac{dt}{\sqrt{1+t^2}}$. Find $G'(x)$.

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Solution

We can write $G(x) = F(\sin(x))$, where F is from the previous question.

$$F(u) = \int_2^u \frac{dt}{\sqrt{1+t^2}}$$

$$F(\sin(x)) = \int_2^{\sin(x)} \frac{dt}{\sqrt{1+t^2}} = G(x).$$

Question

Let $G(x) = \int_2^{\sin(x)} \frac{dt}{\sqrt{1+t^2}}$. Find $G'(x)$.

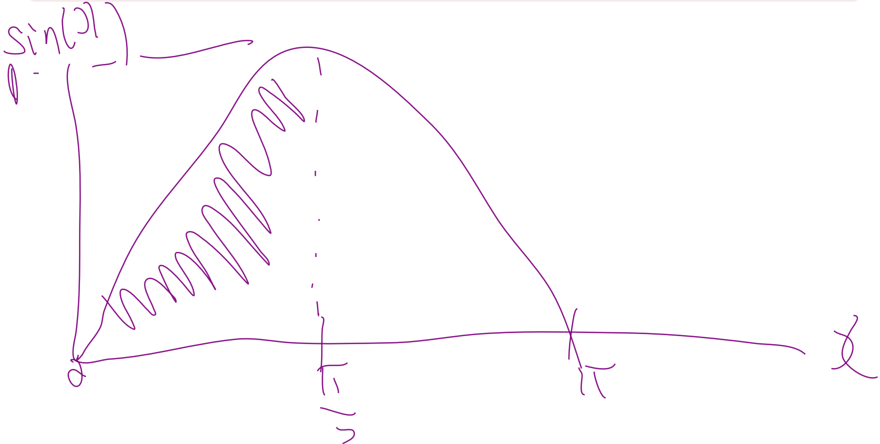
Solution

We can write $G(x) = F(\sin(x))$, where F is from the previous question. We already know F', so by the chain rule,

$$\begin{aligned} G'(x) &= F'(\sin(x))\cos(x) \\ &= \frac{\cos(x)}{\sqrt{1+\sin^2(x)}} \end{aligned}$$

Question

Find the area between the x-axis and the curve $y = \sin(x)$ for $0 \leq x \leq \pi$.



Question

Find the area between the x-axis and the curve $y = \sin(x)$ for $0 \leq x \leq \pi$.

Solution

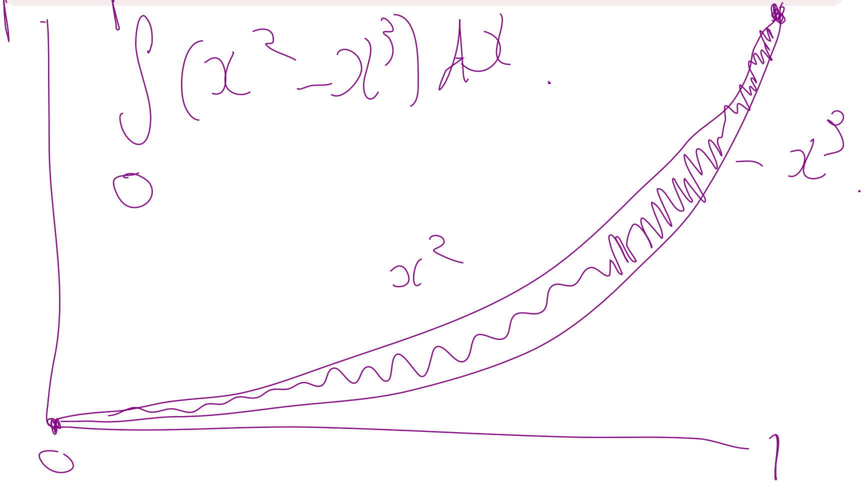
The function $\sin(x)$ is non-negative on $[0, \pi]$, so the area is

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(x)dx &= [-\cos(x)]|_0^{\pi} \\ &= -\cos(\pi) - (-\cos(0)) \\ &= 2.\end{aligned}$$

Question

Find the area bounded by the two curves $y = x^2$ and $y = x^3$ on the interval $[0, 1]$.

$$\int_0^1 (x^2 - x^3)\,dx.$$



9.1 Area under a curve

Question

Find the area bounded by the two curves $y = x^2$ and $y = x^3$ on the interval $[0, 1]$.

Solution

On the interval $[0, 1]$, we have $x^2 \geq x^3$, so the area between the two is the area below x^2, subtract the area below x^3.

Question

Find the area bounded by the two curves $y = x^2$ and $y = x^3$ on the interval $[0, 1]$.

Solution

On the interval $[0, 1]$, we have $x^2 \geq x^3$, so the area between the two is the area below x^2, subtract the area below x^3. We compute

$\int_0^1 (x^2 - x^3)\, dx =$

$$\begin{aligned}
\text{area} &= \int_0^1 x^2 dx - \int_0^1 x^3 dx \\
&= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 \\
&= \frac{1}{3} - \frac{1}{4} \\
&= \frac{1}{12}.
\end{aligned}$$

Corollary (of the FTC)

If f is non-negative and continuous on $[a, b]$, then $\int_a^b f(x)dx \geq 0$.

because $A(x) = \int_a^x f(t)dt$,

then $A'(x) \geq 0$.

so since $A(a) = 0 \Rightarrow A(b) \geq 0$.

9.2 Approximate Integration

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and choose $a = x_0 < x_1 < \cdots < x_n = b$.

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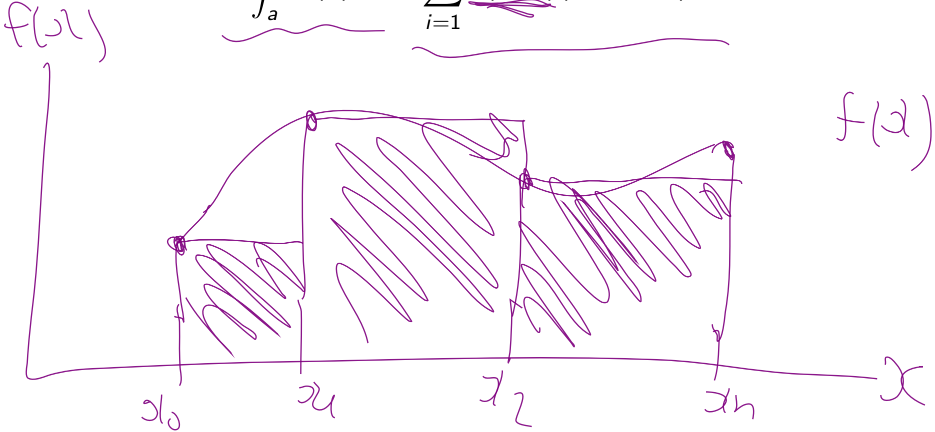
$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$



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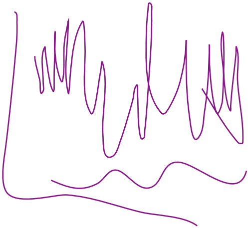

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Another way is the right endpoint approximation:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}).$$

Another way is the *trapezoidal rule*, which averages the previous two:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \boxed{\frac{f(x_i) + f(x_{i-1})}{2}}(x_i - x_{i-1}).$$

9.3 Indefinite Integrals

Definition (Indefinite Integrals)

Let f be some function. The *indefinite integral* of f, denoted $\int f(x)dx$, is the set of all anti-derivatives of f.

Definition (Indefinite Integrals)

Let f be some function. The *indefinite integral* of f, denoted $\int f(x)dx$, is the set of all anti-derivatives of f.

By the fundamental theorem of calculus, anti-derivatives only differ by a constant, so if F is *some* anti-derivative for f, it is common to write

$$\int f(x) = F(x) + c$$

for $c \in \mathbb{R}$.

9.3 Indefinite Integrals

Recall that if f is defined on an interval I, then an anti-derivative for f is a function F so that $F'(x) = f(x)$.

9.3 Indefinite Integrals

Recall that if f is defined on an interval I, then an anti-derivative for f is a function F so that $F'(x) = f(x)$. Let c be some constant, f, g be some functions, and F, G some anti-derivatives for f, g. Some common anti-derivatives are given below:

Function	Antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^\alpha, (\alpha \neq -1)$	$\boxed{\frac{x^{\alpha+1}}{\alpha+1}}$
$\sin x$	$-\cos x$

Function	Antiderivative
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln x$
e^x	e^x

$\sec = \frac{1}{\cos}$

$x > 0.$

9.3 Indefinite Integrals

Recall that if f is defined on an interval I, then an anti-derivative for f is a function F so that $F'(x) = f(x)$. Let c be some constant, f, g be some functions, and F, G some anti-derivatives for f, g. Some common anti-derivatives are given below:

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Function	Antiderivative
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln x$
e^x	e^x

But recall, the set of *all* anti-derivatives is found by adding constants to those in the table.

Question

Find $\int(x^2 + 3x)dx$, meaning, find all antiderivatives of the function $f(x) = x^2 + 3x$.

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Solution

Using the table, we find that

$$\int x^2 + 3x dx = \frac{x^3}{3} + \frac{3x^2}{2} + C$$

for $C \in \mathbb{R}$.

Question

Suppose $f''(x) = x - \sqrt{x}$. Find $f(x)$.

Question

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Solution

Using the table, we find that $f'(x) = \frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3} + C$ for some $C \in \mathbb{R}$.

Question

Suppose $f''(x) = x - \sqrt{x}$. Find $f(x)$.

Solution

Using the table, we find that $f'(x) = \frac{x^2}{2} - \frac{2x^{\frac{3}{2}}}{3} + C$ for some $C \in \mathbb{R}$. Then using the table again,

$$f(x) = \frac{x^3}{6} - \frac{4x^{\frac{5}{2}}}{15} + Cx + D$$

for some $C, D \in \mathbb{R}$.

9.4 Explicit Integration Techniques

If possible, it is often handy to solve integrals explicitly. There are a number of standard tricks you should know.

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Integration by Substitution

If $u = g(x)$ is a differentiable function with range in an interval I and f is continuous on I, then

Bad manners

$$\int f(g(x))g'(x)dx = \int f(u)\frac{du}{dt}dt = \int f(u).$$

If g' is continuous on $[a, b]$, then

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.}$$

Question

Find $\int x^3 \cos(x^4 + 2)dx$.

Question

Find $\int x^3 \cos(x^4 + 2)dx$.

Solution

Let $u = x^4 + 2$, so that $x^3 = \frac{1}{4}\frac{du}{dx}$, and

$$\int x^3 \cos(x^4+2)dx = \int \frac{\cos(u)}{4}du$$
$$= \frac{\sin(u)}{4} + C$$
$$= \frac{\sin(x^4+2)}{4} + C.$$

$\frac{du}{dx} = 4x^3$

C any real number

Question

Evaluate $\int_1^2 \frac{dx}{(5x-3)^2}$.

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Evaluate $\int_1^2 \frac{dx}{(5x-3)^2}$.

Solution

Substitute $u = g(x) = 5x - 3$, so that $\frac{du}{dx} = 5$, $= g'(x)$

Question

Evaluate $\int_1^2 \frac{dx}{(5x-3)^2}$.

Solution

Substitute $u = g(x) = 5x - 3$, so that $\frac{du}{dx} = 5$, and

$$\begin{aligned}\int_1^2 \frac{dx}{(5x-3)^2} &= \int_{g(1)}^{g(2)} \frac{du}{5u^2} \\ &= \frac{1}{5}\int_2^7 \frac{1}{u^2}\,du \\ &= \frac{1}{5}\Big[-u^{-1}\Big]_2^7 \\ &= \frac{1}{5}\left(-\frac{1}{7}+\frac{1}{2}\right) = \frac{1}{14}.\end{aligned}$$

$g(1)$
$= 5 \cdot 1 - 3$
$= 2$

$g(2) = 5 \cdot 2 - 3$
$= 7.$

9.4 Explicit Integration Techniques

Question

Find $\int \frac{dx}{\sqrt{a^2-x^2}}$ for any $a > 0$.

Question

Find $\int \frac{dx}{\sqrt{a^2-x^2}}$ for any $a > 0$.

Solution

Substitute $x = a\sin(\theta)$, so that $dx = a\cos(\theta)d\theta$.

$$\frac{dx}{d\theta} = a\cos\theta$$

9.4 Explicit Integration Techniques

Question

Find $\int \frac{dx}{\sqrt{a^2-x^2}}$ for any $a>0$.

Solution

Substitute $x = a\sin(\theta)$, so that $dx = a\cos(\theta)d\theta$. Then

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{a\cos(\theta)d\theta}{\sqrt{a^2 - a^2\sin^2(\theta)}}$$

$$= \int \frac{\cancel{a}\cos(\theta)d\theta}{\cancel{a}\sqrt{1-\sin^2(\theta)}}$$

$$= \int \frac{\cancel{\cos(\theta)}d\theta}{\cancel{\cos(\theta)}}$$

$$= \theta + c$$

$$= \arcsin\left(\frac{x}{a}\right) + c.$$

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int d\theta$$

Question

Find $\int \cos(x) \sin(x)$.

Question

Find $\int \cos(x) \sin(x)$. dx

Solution

Use $u = \sin(x)$ so that $\frac{du}{dx} = \cos(x)$.

Question

Find $\int \cos(x)\sin(x)$.

Solution

Use $u = \sin(x)$ so that $\frac{du}{dx} = \cos(x)$. Then

$$\begin{aligned}\int \cos(x)\sin(x)dx &= \int u du \\ &= \frac{u^2}{2} + c \\ &= \frac{\sin^2(x)}{2} + c.\end{aligned}$$

Question

Find $\int (3x+1)(3x^2+2x)^3 dx$.

$u = 3x^2 + 2x$

$\frac{du}{dx} = 6x + 2$

Question

Find $\int (3x+1)(3x^2+2x)^3 dx$.

Solution

Use $u = 3x^2 + 2x$ so that $\frac{du}{dx} = 6x + 2$ and $\boxed{3x+1} = \frac{1}{2}\frac{du}{dx}$.

9.4 Explicit Integration Techniques

Question

Find $\int (3x+1)(3x^2+2x)^3 dx$.

Solution

Use $u = 3x^2 + 2x$ so that $\frac{du}{dx} = 6x + 2$ and $3x + 1 = \frac{1}{2}\frac{du}{dx}$. Then

$$\int (3x+1)(3x^2+2x)^3 dx = \int \frac{u^3}{2} du$$

$$= \frac{u^4}{8} + c$$

$$= \frac{1}{8}(3x^2+2x)^4 + c.$$

$\frac{1}{2}\frac{du}{dx}$ u^3

Our next integral technique is essentially the opposite of the product rule.

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Integration by Parts

Recall that if $u(x)$ and $v(x)$ are two differentiable functions, then $(uv)' = u'v + uv'$.

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Integration by Parts

Recall that if $u(x)$ and $v(x)$ are two differentiable functions, then $\int (uv)' = \int u'v + uv'$. Therefore

$$\int uv'dx = uv - \int u'vdx.$$

$uv = \int u'v + \int uv'$

9.4 Explicit Integration Techniques

Our next integral technique is essentially the opposite of the product rule.

Integration by Parts

Recall that if $u(x)$ and $v(x)$ are two differentiable functions, then $(uv)' = u'v + uv'$. Therefore

$$\int uv' dx = uv - \int u' v dx.$$

This is useful if, for whatever reason, $\int u'v$ happens to be easier to solve than $\int uv'$.

Question

Find $\int x^3 \ln(x) dx$.

$u = \ln(x), \quad v' = x^3$

$\int uv' = uv - \int u'v$

easier to deal with.

Question

Find $\int x^3 \ln(x) dx$.

Solution

Use $u = \ln(x)$ and $v = \frac{x^4}{4}$, so that $u' = \frac{1}{x}$ and $v' = x^3$.

Question

Find $\int x^3 \ln(x) dx$.

Solution

Use $u = \ln(x)$ and $v = \frac{x^4}{4}$, so that $u' = \frac{1}{x}$ and $v' = x^3$. Then

$\int uv' = uv - \int u'v$

$$
\begin{aligned}
\int x^3 \ln(x) &= \frac{x^4}{4}\ln(x) - \int \frac{x^4}{4} \cdot \frac{1}{x} dx \\
&= \frac{x^4}{4}\ln(x) - \frac{1}{4}\int x^3 dx \\
&= \frac{x^4}{4}\ln(x) - \frac{x^4}{16} + c
\end{aligned}
$$

Question

Find $\int xe^x \, dx$.

Question

Find $\int xe^x\,dx$.

Solution

Use $u = x$ and $v = e^x$, so that $u' = 1$ and $v' = e^x$.

$$= \int u v'$$

$$= uv - \int u' v$$

9.4 Explicit Integration Techniques

Question

Find $\int xe^x dx$.

Solution

Use $u = x$ and $v = e^x$, so that $u' = 1$ and $v' = e^x$. Then

du v

$$\int xe^x dx = xe^x - \int e^x \qquad \int u'v$$
$$= (x-1)e^x + c.$$

$\int uv'$

What is $\int \frac{1}{x} dx$? We know that $\ln(x)' = \frac{1}{x}$, but the logarithm function is only defined for positive x.

$\ln'(x) = \frac{1}{x}$ but only for $x > 0$.

$\int \frac{1}{x} dx$ should make sense for all $x \neq 0$.

What is $\int \frac{1}{x} dx$? We know that $\ln(x)' = \frac{1}{x}$, but the logarithm function is only defined for positive x. Therefore, we are allowed to say that

$$\int \frac{1}{x} = \ln(x), \qquad x > 0 .$$

but only for positive x.

9.4 Explicit Integration Techniques

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but only for positive x. On the other hand, if $x < 0$, then by the chain rule, $\ln(-x)' = \frac{-1}{-x} = \frac{1}{x}$.

$$\ln'(-x) \times \frac{d(-x)}{dx}$$

$$= \frac{1}{(-x)} \times -1 = \frac{-1}{-x}.$$

What is $\int \frac{1}{x} dx$? We know that $\ln(x)' = \frac{1}{x}$, but the logarithm function is only defined for positive x. Therefore, we are allowed to say that

$$\int \frac{1}{x} = \ln(x),$$

but only for positive x. On the other hand, if $x < 0$, then by the chain rule, $\ln(-x)' = \frac{-1}{-x} = \frac{1}{x}$. Therefore, the general antiderivative for $\frac{1}{x}$ is given by

$$\int \frac{1}{x} = \ln(|x|) + c.$$

9.4 Explicit Integration Techniques

Our next integral technique is useful for integrating rational functions, that is, ratios of polynomials.

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Integration by Partial Fractions

Suppose that $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials.

9.4 Explicit Integration Techniques

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Integration by Partial Fractions

Suppose that $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials. To integrate, you should factorise Q into powers of first or second order polynomials.

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Integration by Partial Fractions

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For example, if you are required to integrate $\frac{cx+d}{(x-a)(x-b)}$, you should find A and B so that

$$\frac{cx+d}{(x-a)(x-b)} = \int \frac{A}{x-a} + \int \frac{B}{x-b},$$

and integrate that instead.

Question

Find $\int \frac{(x+2)dx}{x^2+x}$.

Question

Find $\int \frac{(x+2)dx}{x^2+x}$.

$x^2+x = x(x+1)$

Solution

The denominator factors as $x(x+1)$, so we aim to find A, B so that

$$\frac{(x+1)\,A}{(x+1)\,x} + \frac{B\,x}{x+1\,x} = \frac{x+2}{x^2+x}.$$

$$= \frac{A(x+1)+Bx}{(x+1)x} = \frac{x(A+B)+A}{x^2+x}$$

9.4 Explicit Integration Techniques

Question

Find $\int \frac{(x+2)dx}{x^2+x}$.

Solution

The denominator factors as $x(x+1)$, so we aim to find A, B ~~so~~ that

$$\frac{A}{x} + \frac{B}{x+1} = \frac{x+2}{x^2+x}. \quad = \frac{2}{x} + \frac{(-1)}{x+1}.$$

Then $\frac{A(x+1)}{x(x+1)} + \frac{Bx}{(x+1)x} = \frac{x^{\cancel{2}}+2}{x^2+x}$ so that $A = 2$ and $A + B = 1$, so $B = -1$.

$$= \frac{x(A+B)+A}{x^2+x}$$

$$\Rightarrow x+2 = x(A+B) + A.$$

Question

Find $\int \frac{(x+2)dx}{x^2+x}$.

Solution

The denominator factors as $x(x+1)$, so we aim to find A, B so that

$$\frac{A}{x} + \frac{B}{x+1} = \frac{x+2}{x^2+x}.$$

Then $\frac{A(x+1)}{x(x+1)} + \frac{Bx}{(x+1)x} = \frac{x^2+2}{x^2+x}$ so that $A = 2$ and $A + B = 1$, so $B = -1$. Therefore

$$\int \frac{(x+2)dx}{x^2+x} = \int \frac{2dx}{x} - \int \frac{dx}{x+1} = 2\ln|x| - \ln|x+1| + C.$$

Question

Find $\int \frac{dx}{x^2-a^2}$ for $a \neq 0$.

$$x^2 - a^2$$

$$= (x-a)(x+a).$$

$$\frac{1}{x^2-a^2} = \frac{A}{(x+a)} + \frac{B}{(x-a)}.$$

9.4 Explicit Integration Techniques

Question

Find $\int \frac{dx}{x^2-a^2}$ for $a \neq 0$.

Solution

The denominator factors as $(x-a)(x+a)$, so we aim to find A, B so that

$(x-a)$ $(x+a)$

$$\frac{1}{x^2-a^2} = \frac{A}{x+a} + \frac{B}{x-a} = \frac{(A+B)x - a(A-B)}{x^2-a^2}.$$

$(x-a)$ $(x+a)$

Question

Find $\int \frac{dx}{x^2-a^2}$ for $a \neq 0$.

Solution

The denominator factors as $(x-a)(x+a)$, so we aim to find A, B so that

$$\frac{1}{x^2-a^2} = \frac{A}{x+a} + \frac{B}{x-a} = \frac{(A+B)x - a(A-B)}{x^2-a^2}.$$

Then $A+B=0$ and $A-B=-\frac{1}{a}$; solving gives $A=-\frac{1}{2a}$ and $B=\frac{1}{2a}$.

$$\frac{1}{x^2-a^2} = \frac{-1}{2a(x+a)} + \frac{1}{2a(x-a)}.$$

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Solution

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Then $A+B=0$ and $A-B=-\frac{1}{a}$; solving gives $A=-\frac{1}{2a}$ and $B=\frac{1}{2a}$. Therefore

$$\begin{aligned}\int \frac{dx}{x^2-a^2} &= -\int \frac{dx}{2a(x+a)} + \int \frac{dx}{2a(x-a)} \\ &= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C\end{aligned}$$

9.5 Volume Integrals

Recall that the area under the curve $y = f(x)$ between is found with

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum f(x_i^*)(x_i - x_{i-1})$$

where n describes the maximal length of some partition of $[a, b]$.

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where n describes the maximal length of some partition of $[a, b]$.

What about finding volume under a surface $z = f(x, y)$, above a region R?

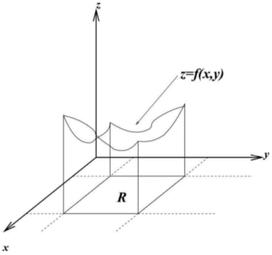

Figure 59: What is the volume V under the surface?

9.5 Volume Integrals

If our region R is the rectangle $R = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq x \leq d\}$, we can approximate the volume by dividing $[a, b]$ into m subintervals of length $\Delta x = \frac{b-a}{m}$, and $[c, d]$ into n subintervals of length $\Delta y = \frac{d-c}{n}$.

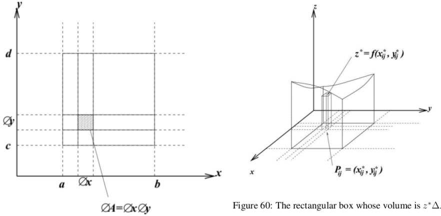

Figure 60: The rectangular box whose volume is $z^* \Delta A$.

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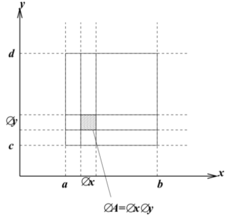

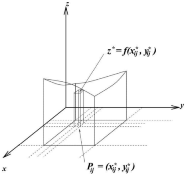

Figure 60: The rectangular box whose volume is $z^*\Delta A$.

Then the volume can be approximated by picking a point (x^*_{ij}, y^*_{ij}) in each smaller rectangle $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$ and having

$$V \approx \sum_{i,j} f(x^*_{ij}, y^*_{ij})\Delta x \Delta y.$$

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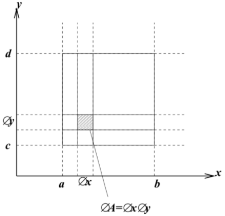

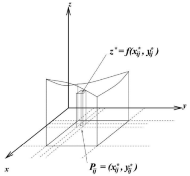

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$$V \approx \sum_{i,j} f(x^*_{ij}, y^*_{ij}) \Delta x \Delta y.$$

Similar ideas can be used for non-rectangular regions.

9.5 Volume Integrals

Definition (Volume Integrals)

Let f be a continuous function in the region $R \subset \mathbb{R}^2$. Then the associated volume is

$$V = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{i,j} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Whenever this limit exists, it is denoted $\int \int_R f(x, y) dA$.

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Theorem (Properties of Volume Integrals)

- $\int\int_R (f \pm g)dA = \int\int_R f dA + \int\int_R g dA$;

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Theorem (Properties of Volume Integrals)

- $\int \int_R (f \pm g) dA = \int \int_R f dA + \int \int_R g dA$;
- $\int \int c f dA = c \int \int f dA$;
- If R_1, R_2 is a partition of R, then $\int \int_R f dA = \int \int_{R_1} f dA + \int \int_{R_2} f dA$;

9.5 Volume Integrals

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Let f be a continuous function in the region $R \subset \mathbb{R}^2$. Then the associated volume is

$$V = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i,j} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Whenever this limit exists, it is denoted $\int\int_R f(x,y)dA$.

Theorem (Properties of Volume Integrals)

- $\int\int_R (f \pm g)dA = \int\int_R fdA + \int\int_R gdA$;
- $\int\int cfdA = c\int\int fdA$;
- If R_1, R_2 is a partition of R, then $\int\int_R fdA = \int\int_{R_1} fdA + \int\int_{R_2} fdA$;
- If $f \geq g$ in R, then $\int\int_R fdA \geq \int\int_R gdA$.

9.5 Volume Integrals

The idea of *iterated integrals* gives us a concrete way to actually evaluate volume integrals using single-variable integrals we are already familiar with.

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The idea of *iterated integrals* gives us a concrete way to actually evaluate volume integrals using single-variable integrals we are already familiar with.
Indeed, if $R = [a, b] \times [c, d] \in \mathbb{R}^2$ is a rectangle, then

$$\int_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy,$$

where $\int_c^d f(x,y)dy$ means integrating f with respect to y (keeping x fixed), and $\int_a^b f(x,y)dx$ means integrate f with respect to x (keeping y fixed).

Question

Evaluate $\int_0^2 \int_1^3 x^2 y dy dx$.

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$$\begin{aligned} \int_0^2 \int_1^3 x^2 y dy dx &= \int_0^2 4x^2 dx \\ &= \frac{4x^3}{3}|_0^2 \\ &= \frac{32}{3}. \end{aligned}$$

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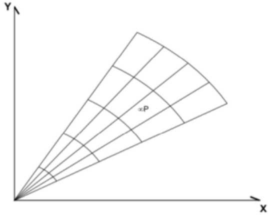

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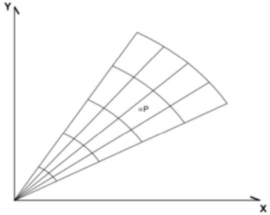

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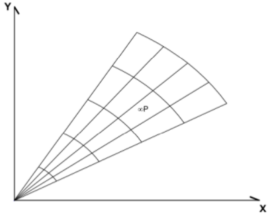

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The area of a component containing the point $P = (r^*, \theta^*)$ is approximately $r^*\Delta r\Delta\theta$. So the integral over this patch is approximately $f(r^*, \theta^*)r^*\Delta r\Delta\theta$. Therefore the volume integral is

$$\int\int_D f(r\cos(\theta), r\cos(\theta))rdrd\theta.$$

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Evaluate $\int\int_D e^{-x^2-y^2} dxdy$, where D is the region bounded by the circle $x^2 + y^2 = R^2$.

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$$\begin{aligned}\int\int_D e^{-x^2-y^2}dxdy &= \int_0^{2\theta}\left(\int_0^R e^{-r^2}rdr\right)d\theta \\ &= 2\pi\left(\frac{-e^{-r^2}}{2}\right)_0^R \\ &= \pi(1-e^{-R^2}).\end{aligned}$$

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Question

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Using the polar co-ordinates, this is $0 \le \theta \le 2\pi$ and $0 \le r < \infty$, so like before, we find

$$\begin{aligned}\int\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr d\theta \\ &= 2\pi\left(\frac{-e^{-r^2}}{2}\right)_0^{\infty} = \pi.\end{aligned}$$

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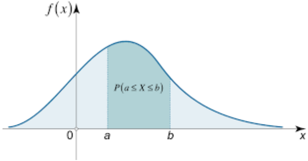

https://www.math24.net/probability-density-function

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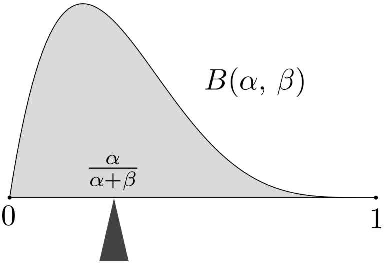
The expected value is the 'mean' value of X, weighted according to the pdf.



https://en.wikipedia.org/wiki/Expected_value

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Definition (Variance)

The *variance* of the random variable X with pdf f is

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \int_{-\infty}^{\infty} (x - \mathbb{E}[X])^2 f(x) dx.$$

The variance describes how much the data from X tends to be spread out from the expected value $\mathbb{E}[X]$.