MATH7501 Practical 7 (Week 8), Semester 1-2021

Topic: Sequences, Limits and Series

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Pre-Tutorial Activity

- Students must have familiarised themselves with unit 5 contents of the reading materials for MATH7501

Resources

- Chapter 5 and 6 of course reader

Q1 Limit of sum of two sequences

Suppose $lim_{n \longrightarrow \infty}\ a_n = a$ and $lim_{n \longrightarrow \infty}\ b_n = b.$

Use the (ε, N) definition of the limit of a sequence to show that

$$lim_{n \longrightarrow \infty}\ (a_n + b_n) = a + b$$

Solution :

Since $lim_{n \longrightarrow \infty}\ a_n = a,$ *there exists an* N_1 such that, for every $\epsilon_1 > 0$, if $\mathrm{n} \geq \mathrm{N}_1$ then $|\mathrm{a_n} - \mathrm{a}| < \epsilon_1$. Similarly, as $lim_{n \longrightarrow \infty}\ b_n = b,$ *there exists an* N_2 such that, for every $\epsilon_2 > 0$, if $\mathrm{n} \geq \mathrm{N}_2$ then $|\mathrm{b_n} - \mathrm{b}| < \epsilon_2$.

Now,
choose $\epsilon > 0$ such that $\epsilon \geq \epsilon_1 + \epsilon_2$ and and integer N such that $\mathrm{N} = \max(\mathrm{N}_1, \mathrm{N}_2)$. Then we have

$$\begin{aligned} |(\mathrm{a_n} + \mathrm{b_n}) - (\mathrm{a} + \mathrm{b})| &\leq |\mathrm{a_n} - \mathrm{a}| + |\mathrm{b_n} - \mathrm{b}|, \text{ by the triangle inequality} \\ &< \epsilon_1 + \epsilon_2 \\ &\leq \epsilon \end{aligned}$$

This implis that $lim_{n \longrightarrow \infty}\ (a_n + b_n) = a + b.$

The last inequality holds for any ϵ_1 and ϵ_2 such that $\epsilon_1 + \epsilon_2 \leq \epsilon$. Thus leting $\epsilon_1 = \epsilon_2 = \dfrac{\epsilon}{2}$ would satisfy this condition. In the above proof,
we choose $\mathrm{N} = \max(\mathrm{N}_1, \mathrm{N}_2)$ to avoid the case that if $\mathrm{N}_1 < \mathrm{N}_2$,
then for all $\mathrm{n} \geq \mathrm{N}_1$ we have $|\mathrm{a_n} - \mathrm{a}| < \epsilon_1$. However, if $\mathrm{n} < \mathrm{N}_2$, we have $|\mathrm{b_n} - \mathrm{b}| > \epsilon_2$.

Q2 Derivative of sin(x)

(a) Give a geometric explanation of why $\frac{d}{dx}\sin(x) = \cos(x)$

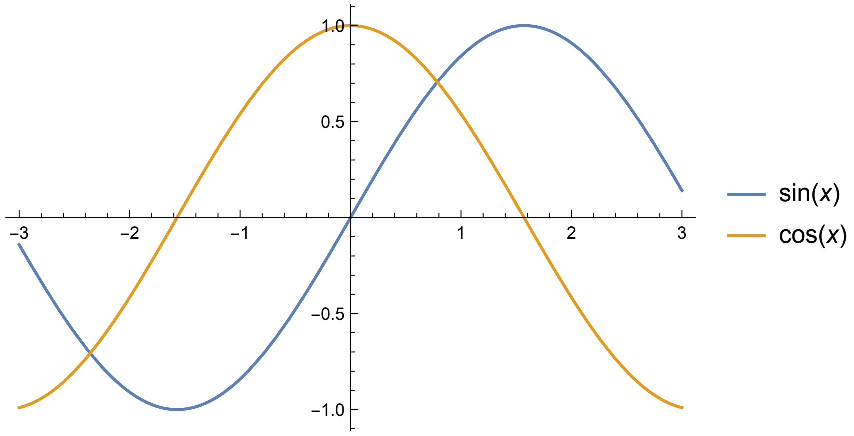
Solution :

Note that by definition, derivative of a function at a given point is given by the slope of the tangent line at that point. Below is the graph of sin (x) and cos (x) on the same plot for x in the range [-3, 3]. Consider the sin (x) curve, and imagine drawing tangent lines at points on the curve. For example, if x = 0, the slope of the tangent line on sin (x) curve is 1, which coincides with cos (0) = 1. At the two turning points, on sin (x), the slope of the tangent line is 0, which coincides with cos (x) = 0. Repeating this process, it looks like that the derivative of sin (x) is cos (x).

The above explanation is only a visualisation of the proof, but **not a formal proof**. A formal proof is explained using a unit circle and right - triangle identities for sin (x) and cos (x). You may refer to this link (which is an MIT OpenCourseWare handout) for a detailed explanation of such a proof.

In[∘]:=
```
Plot[{Sin[x], Cos[x]}, {x, -3, 3}, PlotLegends → "Expressions"]
```

Out[∘]=



(b) prove the result in part (a) using using the fact that

T: = sin (x + h) = sin (x) cos (h) + cos (x) sin (h),

l_1 := sin (h)/h tends to 1 as h tends to 0

$l_{2:=}$ cos (h) - 1)/h tends to 0 as h tends to 0.

Solution :

Using the definition of the derivative, we have that

$$\frac{d}{dx}\sin(x) = \text{limit}_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h}$$

$$= \text{limit}_{h\to 0}\frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}, \text{ by T}$$

$$= \text{limit}_{h\to 0}\frac{\cos(x)\sin(h) - \sin(x) + \sin(x)\cos(h)}{h}$$

$$= \text{limit}_{h\to 0}\frac{\cos(x)\sin(h) + \sin(x)[\cos(h) - 1]}{h}$$

$$= \cos(x)\left[\text{limit}_{h\to 0}\frac{\sin(h)}{h}\right] + \sin(x)\left[\text{limit}_{h\to 0}\frac{(\cos(h) - 1)}{h}\right]$$

$$= \cos(x)\times 1 + \sin(x)\times 0, \quad \text{by } l_1 \text{ and } l_2$$

$$= \cos(x)$$

Below are visual plot to show l_1 and l_2. The first plot shows that

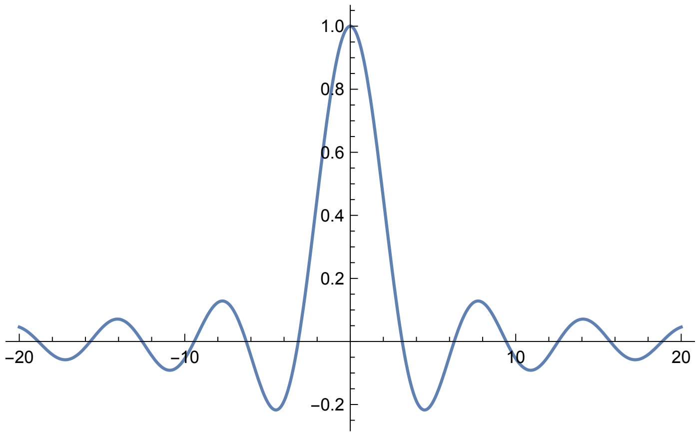
$$\text{limit}_{x \to 0} \frac{\sin(x)}{x} = 1$$

and the second plot shows that

$$\text{limit}_{x \to 0} \frac{(\cos(x) - 1)}{x} = 0$$

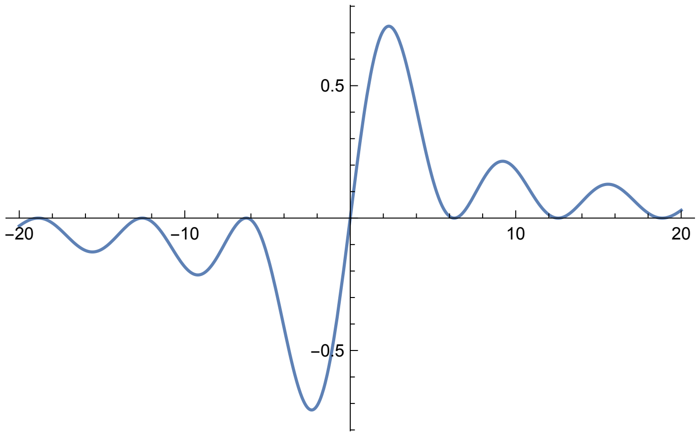
In[∘]:= **Plot[$\frac{\text{Sin[x]}}{\text{x}}$, {x, −20, 20}, PlotRange → All]**

Out[∘]=



In[1]:= `Plot[(1 - Cos[x])/x, {x, -20, 20}, PlotRange → All]`

Out[1]=



Q3 Product rule, chain rule and quotient rule

(a) Using only the product rule and the fact that $\frac{dx}{dx} = 1$, calculate $\frac{d}{dx}\left(x^3\right)$

Solution :

$$\frac{d}{dx}\left(x^3\right) = \frac{d}{dx}\left(x x^2\right), \text{ writing } x^3 = x\, x^2$$

$$= \frac{d}{dx}(x)\, x^2 + x\, \frac{d}{dx}\left(x^2\right), \text{ by the product rule}$$

$$= 1\ x^2 + x\left[\frac{d}{dx}(x\, x)\right], \text{ writing } x^2 = x\, x$$

$$= x^2 + x\left[\frac{d}{dx}(x)\,x + x\,\frac{d}{dx}(x)\right], \text{ by the product rule}$$
$$= x^2 + x[1\,x + 1\,x]$$
$$= x^2 + 2\,x^2$$
$$= 3\,x^2$$

(b) Prove the quotient rule for derivatives using the chain rule, the product rule and the power rule

Solution :

Let $h(x) = \frac{f(x)}{g(x)}$. Then $h(x) = f(x)[g(x)]^{-1}$.

$$\frac{d}{dx}h(x) = \frac{d}{dx}\left(f(x)[g(x)]^{-1}\right)$$
$$= \frac{d}{dx}(f(x))[g(x)]^{-1} + f(x)\frac{d}{dx}\left([g(x)]^{-1}\right), \text{ by the product rule}$$
$$= \frac{d}{dx}(f(x))[g(x)]^{-1} + f(x)\left\{-1[g(x)]^{-2}\frac{d}{dx}[g(x)]\right\},$$

by the power rule and chain rule

$$= \frac{\frac{d}{dx}(f(x))}{[g(x)]} - \frac{f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$
$$= \frac{\frac{d}{dx}(f(x))\,g(x) - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Q4 Evaluation of a series and approximations

Consider $S = \sum_{n=1}^{\infty} n^{-2}$, for $n = 1, 2, 3 \ldots$

(a) Use Mathematica to analytically evaluate S.

In[2]:=
```
Sum[1/n^2, {n, 1, Infinity}]
```
Out[2]= $\frac{\pi^2}{6}$

(b) Use this result to suggest an algorithm for numerically approximating the constant π and implement it in Mathematica.

Solution :

From part (a),

we have $S = \frac{\pi^2}{6}$. Rearanging this for π gives $\pi = \sqrt{(6\,S)}$. Using this formular,

we can obtain a numerical approximation for π

as : $\text{limit}_{k \to \infty} \sqrt{\left[6 \sum_{n=1}^{k} n^{-2}\right]}$. The following program computes

this approximation and plots the approximated value for each k value.

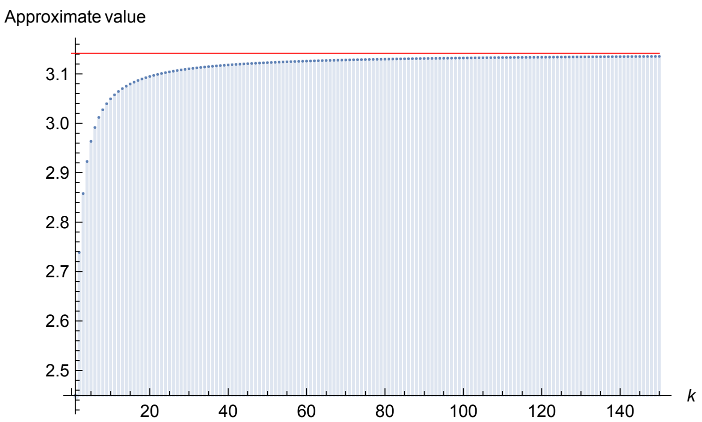
In[14]:=
```
Clear[f]
```

In[15]:=
```
(* first create a function to compute the value
  of √[6 ∑_{n=1}^k n^-2] for each k and check if its output is
  close to the numerical value of π = 3.141592653589793`*)
f[k_] := Sqrt[6 Sum[1/n^2, {n,  1,  k}]]

N[f[100]]
```

Out[10]= 3.13208

In[17]:=
```
(* write a program to compute the approximation for
 k in [1 to nmax]. You can choose the value for nmax *)
With[{nmax = 150},
 (*nmax is the upper limit of the sum*)
  Show[ DiscretePlot[f[k], {k, 1, nmax},
   Epilog → {Red, Line[{{0, Pi}, {nmax, Pi}}]},
   (*This plots a hrizontal red line showing the actual value of π*)
   PlotRange → All,
   AxesLabel → {k, Approximate value}
  ]]
]
```

Out[17]=


As seen from the plot, the approximation is getting close to the actual value of π as k increases. If you wish to know which k value gives an absolute difference between the approximated value and the actual value for a given tolerance value (error), the following code may be used. Note that in this example, I chose the error to be 0.0001 in this example.

In[18]:=
```
Clear[k]
FindRoot[Abs[f[k] - Pi] == 0.0001, {k, 1000}]
```

Out[19]= {k → 9548.95}

Q5 Harmonic series

Consider the harmonic series $S = \sum_{n=1}^{\infty} n^{-1}$, for n = 1, 2, 3 .... The partial sum of S is given by

$$\sum_{n=1}^{k} n^{-1} = \log(k) + \gamma + \epsilon_k,$$

where γ is Euler' s gamma constant and ϵ_k is an $o(1)$ sequence. Use Mathematica to numerically approximate γ.

Solution :

As ϵ_k is an $o(1)$ sequence, it converges to zero in probability as k approaches to an appropriate limit. Then γ can be approximated by

$$\gamma = \text{limit}_{k \to \infty} \left[\sum_{n=1}^{k} n^{-1} - \log(k)\right].$$ The approximated value,

up to fifteen decimal places, is $\gamma = 0.577215664901532$ (see here). We can compute this constant numerically by following a similar algorithm as we constructed for Q4.

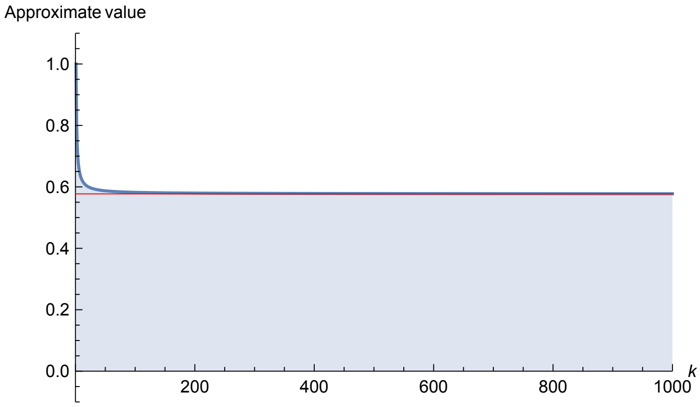
```
In[22]:= Clear[g]
(* first create a function to compute the value
  of [∑_{n=1}^k n^-1 - log(k)] for each k and check if its output is
  close to the numerical value of γ= 0.57721 56649 01532*)
g[k_] := Sum[1 / n, {n, 1, k}] - Log[k]

N[g[1000]]
Out[25]= 0.577716

(* write a program to compute the approximation
 for k in [1 to nmax]. You can choose the value for nmax *)

In[30]:= With[{nmax = 1000},
 DiscretePlot[g[k], {k, 1, nmax},
  Epilog → {Red, Line[{{0, 0.5772}, {nmax, 0.5772}}]},
  PlotRange → {{0, nmax}, {-.1, 1.1}},
  AxesLabel → {k, Approximate value}
 ]]
```

Out[30]=



It is seen from the plot that the approximate value converges to the actual value (the red line) of γ as k approaches infinity.

Q6 Optimisation

suppose $f(x) = 1/(1+x^2)$. Use derivatives to explain why $x = 0$ is the only maximum

First derivative test can be used to show this. The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \frac{-2x}{(1+x^2)}.$$

Solving $f'(x) = 0$ gives $x = 0$ as the only turning point.

Now, if $x < 0$, then $f'(x) > 0$, imlying that $f(x)$ is increasing for $x < 0$.

If $x > 0$, then $f'(x) < 0$, imlying that $f(x)$ is decreasing for $x < 0$. Thus,

there is a maximum at $x = 0$ and since it is the only turning point, it is a global maximum.

Q7 Limit

Find the limit of $\frac{x}{\sqrt{(1+x^2)}}$ as x tends to infinity

$$\text{limit}_{x\to\infty}\left(\frac{x}{\sqrt{(1+x^2)}}\right) = \text{limit}_{x\to\infty}\left(\frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}}\right), \text{ factrise } x^2$$

$$= \text{limit}_{x\to\infty}\left(\frac{x}{x\sqrt{\left(1+\frac{1}{x^2}\right)}}\right), \sqrt{x^2} = x$$

$$= \text{limit}_{x\to\infty}\left(\frac{1}{\sqrt{\left(1+\frac{1}{x^2}\right)}}\right),$$

$$= \frac{1}{\sqrt{(1+0)}}, \quad \text{limit}_{x\to\infty}\left(\frac{1}{x^2}\right) = 0,$$

$$= 1$$