

MATH7501 Practical 11 (Week 12), Semester 1-2021

Topic: Probability Distributions

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Pre-Tutorial Activity

- Students must have familiarised themselves with units 9 and 10 contents of the reading materials for MATH7501

Resources

- Chapters 9 and 10: Computation of mean, variance, expectation, gradient decent method
- https://en.wikipedia.org/wiki/Rayleigh_distribution

Section 1: The Rayleigh Distribution

In probability theory and statistics,

the **Rayleigh distribution** is a continuous probability distribution for nonnegative – valued random variables.

The notation $X \sim \text{Rayleigh}(\sigma)$ means that the random variable X has a Rayleigh distribution with shape parameter σ . The probability density function (pdf) is :

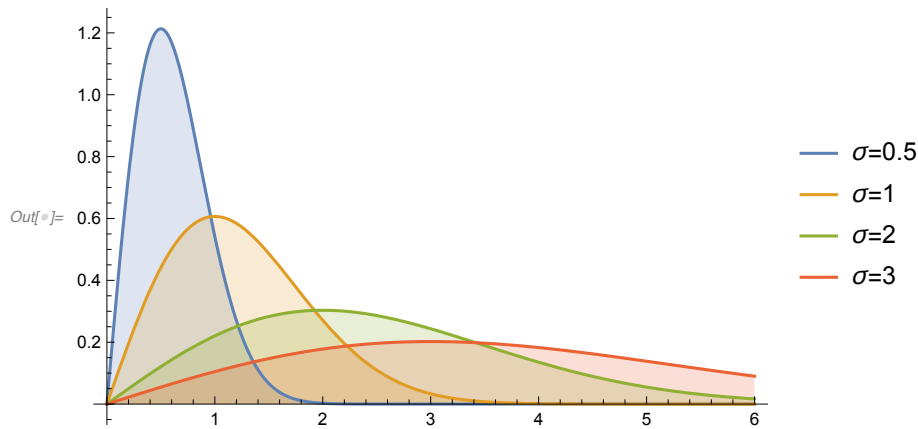
$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0, \sigma > 0 \\ 0, & x < 0 \end{cases}$$

Q1) Plot the pdf of X for $\sigma = 0.5, 1, 2, 3$

```

In[ ]:= Plot[Table[PDF[RayleighDistribution[σ], x], {σ, {.5, 1, 2, 3}}] // Evaluate,
  {x, 0, 6}, Filling -> Axis, PlotRange -> All,
  PlotLegends -> {"σ=0.5", "σ=1", "σ=2", "σ=3"}]

```



Q2) Show that the cumulative distribution function (cdf) of X is

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\begin{aligned} F_X(x) &= P(X \leq x) = \int_0^x f_X(t) dt \\ &= \int_0^x \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} dt \end{aligned}$$

Let $u = \frac{t^2}{2\sigma^2}$. Then $du = \frac{t}{\sigma^2} dt$,

and the limits of the integral are $u = 0$ and $u = \frac{x^2}{2\sigma^2}$. Thus,

substituting $dt = \frac{\sigma^2}{t} du$ in the above integration, we have,

$$\begin{aligned} F_X(x) &= \int_0^{\frac{x^2}{2\sigma^2}} e^{-u} du \\ &= -[e^{-u}]_{u=0}^{u=\frac{x^2}{2\sigma^2}} \\ &= -\left\{e^{-\frac{x^2}{2\sigma^2}} - e^0\right\} \\ &= \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0, \end{cases} \end{aligned}$$

as required.

You can also use the `Integrate[]` function to compute the cdf as shown below.

```

In[ ]:= f[x_] := x/σ^2 E^(-x^2/(2σ^2)) (*Rayleigh probability density function*)

```

```

Integrate[f[x], {x, 0, u}] (*cdf*)

```

```

Out[ ]:= 1 - E^(-u^2/(2σ^2))

```

Q3) Show that $f_X(x)$ is a valid probability density function by showing that the integral over $[0, \infty)$ is 1

Method 1 : Show this using calculus

$$\begin{aligned}
 \int_0^{\infty} f_X(x) dx &= \int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
 &= \lim_{y \rightarrow \infty} \int_0^y \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
 &= \lim_{y \rightarrow \infty} F_X(y), \text{ by the definition of the cdf} \\
 &= \lim_{y \rightarrow \infty} \left(1 - e^{-\frac{y^2}{2\sigma^2}}\right) \\
 &= 1 - 0 \\
 &= 1, \text{ as required.}
 \end{aligned}$$

Method 2 :

Use the `Integrate[]` function to compute this integration exactly in Mathematica.

```
In[ ]:= Integrate[f[x], {x, 0, ∞}, Assumptions → σ > 0]
```

```
Out[ ]:= 1
```

Method 3 : You can also use `NIntegrate[]`

function to derive a numerical approximation to $\int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx$,
for a given value of σ (say for example, $\sigma = 1$).

```
NIntegrate[x E^(-x^2/2), {x, 0, ∞}]
```

```
Out[ ]:= 1.
```

Method 4 :

Approximate the integral by Reiman sum. That is $\int_0^{\infty} f_X(x) dx \approx \sum_{k=0}^{\infty} f(k) \delta$,

where δ is the width of the rectangles.

(*discretisation sum *)

```
In[ ]:= Clear[δ] (*δ is the width of the rectangle*)
```

```
In[13]:= Total[Table[x E^(-x^2/2) δ, {x, 0, 1000, δ = 0.001}]]
```

```
Out[13]:= 1.
```

Q4) Show that the mean of $X \sim \text{Rayleigh}(\sigma) = \sigma \sqrt{\frac{\pi}{2}}$

$$\begin{aligned}
 \mu_X = E(X) &= \int_0^{\infty} x f_X(x) dx \\
 &= \int_0^{\infty} x \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx
 \end{aligned}$$

Let $u = \frac{x^2}{2\sigma^2}$. Then $du = \frac{x}{\sigma^2} dx$,

and the limits of the integral are still $u = 0$ and $u = \infty$. Thus,

substituting $dx = \frac{\sigma^2}{x} du$ and using $x = \sqrt{2\sigma^2 u}$ in the above integration we have,

$$\begin{aligned}
\mu_X &= \int_0^{\infty} \sqrt{2\sigma^2 u} e^{-u} du \\
&= \sigma \sqrt{2} \int_0^{\infty} \sqrt{u} e^{-u} du \\
&= \sigma \sqrt{2} \int_0^{\infty} u^{3/2-1} e^{-u} du \\
&= \sigma \sqrt{2} \Gamma\left(\frac{3}{2}\right), \text{ } \Gamma(x) \text{ is the Gamma function defined as : } \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \\
&= \sigma \sqrt{2} \Gamma\left(\frac{1}{2} + 1\right) \\
&= \sigma \sqrt{2} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right), \text{ using the property of Gamma function, } \Gamma(x+1) = x\Gamma(x) \\
&= \sigma \sqrt{2} \times \frac{1}{2} \sqrt{\pi}, \text{ using the property of Gamma function, } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\
&= \sigma \sqrt{\left(\frac{\pi}{2}\right)}
\end{aligned}$$

You can use the `Integrate[]` function to check your derivation of the mean as follows.

```
In[*]:= Integrate[x f[x], {x, 0, ∞}, Assumptions → σ > 0] (*mean (μ) *)
```

```
Out[*]:=  $\sqrt{\frac{\pi}{2}} \sigma$ 
```

```
In[*]:= (*using numerical approximation to compute the mean when σ = 1*)
```

```
In[*]:= NIntegrate[x x E- $\frac{x^2}{2}$ , {x, 0, ∞}]
```

```
Out[*]:= 1.25331
```

```
δ = 0.01
```

```
Out[*]:= 0.01
```

```
In[*]:= Total[Table[x x E- $\frac{x^2}{2}$  δ, {x, 0, 10, δ}]]
```

```
Out[*]:= 1.25331
```

Q5) Show that the variance of $X \sim \text{Rayleigh}(\sigma) = \sigma^2 \left(\frac{4 - \pi}{2} \right)$

From Week 11 practical, we know that $\sigma^2_X = \text{var}(X) = E[X^2] - \mu_X^2$,

where $E[X^2]$ is the second moment of X and μ_X is the mean of X ,

which we computed in Q4. Thus,

it remains to find an expression for $E[X^2]$. We will require the integration by -

parts formula $\int u dv = uv - \int v du$ for this calculation.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^{\infty} x^2 \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \left[x^2 \int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right]_{x=0}^{x=\infty} - \int_0^{\infty} 2x \left(\int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right) dx, \end{aligned}$$

by using $u = x^2$ and $dv = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$,

$$= \left[-x^2 e^{-\frac{x^2}{2\sigma^2}} \right]_{x=0}^{x=\infty} + \int_0^{\infty} 2x e^{-\frac{x^2}{2\sigma^2}} dx$$

To evaluate the integral, let $u = \frac{x^2}{2\sigma^2}$. Then $du = \frac{x}{\sigma^2} dx$,

and the limits of the integral are still $u = 0$ and $u = \infty$. Thus,

substituting $dx = \frac{\sigma^2}{x} du$ we have

$$\begin{aligned} E[X^2] &= 0 + 2\sigma^2 \left[\int_0^{\infty} e^{-u} du \right] \\ &= -2\sigma^2 \left[e^{-u} \right]_{u=0}^{u=\infty} \\ &= -2\sigma^2 [0 - 1] \\ &= 2\sigma^2 \end{aligned}$$

Thus, $\sigma^2_X = \text{var}(X) = E[X^2] - \mu_X^2$

$$\begin{aligned} &= 2\sigma^2 - \left(\sigma \sqrt{\frac{\pi}{2}} \right)^2 \\ &= \sigma^2 \left(\frac{4 - \pi}{2} \right) \text{ as required.} \end{aligned}$$

You can use the `Integrate[]` function to check your work.

`Integrate[x^2 f[x], {x, 0, ∞}, Assumptions → σ > 0]`

(*compute second moment ($E[X^2]$ *))

`Out[]:= 2 σ^2`

(*using numerical approximation to compute the second moment when $\sigma = 1$ *)

`In[]:= NIntegrate[x^2 x E^(-x^2/2), {x, 0, ∞}]`

`Out[]:= 2.`

`In[]:= Total[Table[x^2 x E^(-x^2/2) δ, {x, 0, 10, δ}]]`

`Out[]:= 2.`

In[*]:= (*variance*)

$$E[X^2] - \mu^2 = 2\sigma^2 - \left(\sqrt{\frac{\pi}{2}}\sigma\right)^2$$

$$\text{Out[*]} = 2\sigma^2 - \frac{\pi\sigma^2}{2}$$

Q6) Find the median of X .

Note that the median of X is the number M such that,

$$\int_0^M f_X(x) dx = \frac{1}{2}$$

Note that the left hand side is the cdf of X evaluated at M . Thus we have

$$\begin{aligned} F_X(M) &= \frac{1}{2} \\ 1 - e^{-\frac{M^2}{2\sigma^2}} &= \frac{1}{2} \\ e^{-\frac{M^2}{2\sigma^2}} &= \frac{1}{2} \\ -\frac{M^2}{2\sigma^2} &= \ln\left(\frac{1}{2}\right) \\ M^2 &= -2\sigma^2 \ln\left(\frac{1}{2}\right) \\ M &= \sigma \sqrt{-2 \ln\left(\frac{1}{2}\right)} \text{ as the median.} \end{aligned}$$

Q7) The quantile function of the distribution, $q(u)$ for $u \in [0, 1]$, is defined as follows : For each u , we should have,

$$\int_0^{q(u)} f_X(x) dx = u$$

a) Find an expression for $q(u)$

By noting that the left -

hand side of the equation is the cdf of X evaluated at $q(u)$, we have

$$\begin{aligned} F_X(q(u)) &= u \\ 1 - e^{-\frac{q(u)^2}{2\sigma^2}} &= u \\ e^{-\frac{q(u)^2}{2\sigma^2}} &= 1 - u \\ -\frac{q(u)^2}{2\sigma^2} &= \ln(1 - u) \\ q(u)^2 &= -2\sigma^2 \ln(1 - u) \\ q(u) &= \sigma \sqrt{-2 \ln(1 - u)} \end{aligned}$$

b) Say that $X = q(U)$ with U as Uniformly distributed on $[0, 1]$. Then, X has a Rayleigh distribution. Show this empirically for $\sigma = 1$, by generating 10^4 uniform random variables on $[0, 1]$

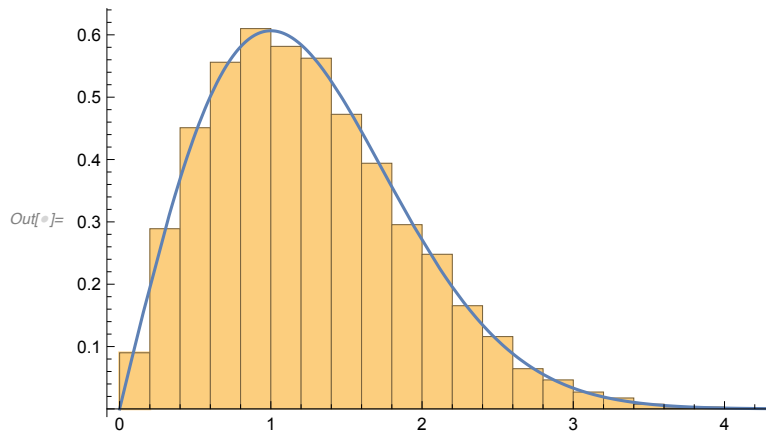
(*Generate a Rayleigh random Variable from Uniform[0,1] random variable*)

```

In[ ]:= U = RandomReal[1, 10 000];
        X = Sqrt[-2 Log[1 - U]];

In[ ]:= Show[Histogram[X, Automatic, "PDF"],
             Plot[PDF[RayleighDistribution[1], x],
                 {x, 0, 8}, PlotRange -> All]]

```



The plot shows histogram of the data X along with the pdf of $X \sim \text{RayleighDistribution}(1)$.

Section 2: Simple Linear Regression Problem

Consider the simple linear regression problem with data points $(x_1, y_1), \dots, (x_n, y_n)$. The aim is to seek β_0 and β_1 to fit the line,

$$y = \beta_0 + \beta_1 x,$$

by minimizing the loss function

$$L(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2.$$

Here, β_0 and β_1 are the slope and the intercept, respectively, of the line of best fit.

Q1) Compute an expression for the gradient of $L(\beta_0, \beta_1)$

Since the loss function has two variables, we need to use partial derivatives here.

$$\begin{aligned}
 \frac{\partial L(\beta_0, \beta_1)}{\partial \beta_1} &= \frac{\partial}{\partial \beta_1} \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2 \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \beta_1} (y_i - (\beta_0 + \beta_1 x_i))^2 \\
 &= -2 \sum_{i=1}^n x_i (y_i - (\beta_0 + \beta_1 x_i))
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial L(\beta_0, \beta_1)}{\partial \beta_0} &= \frac{\partial}{\partial \beta_0} \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2 \\
 &= \sum_{i=1}^n \frac{\partial}{\partial \beta_0} (y_i - (\beta_0 + \beta_1 x_i))^2 \\
 &= -2 \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))
 \end{aligned}$$

Q2) Use Gradient Descent algorithm to find the line of best fit for the following data

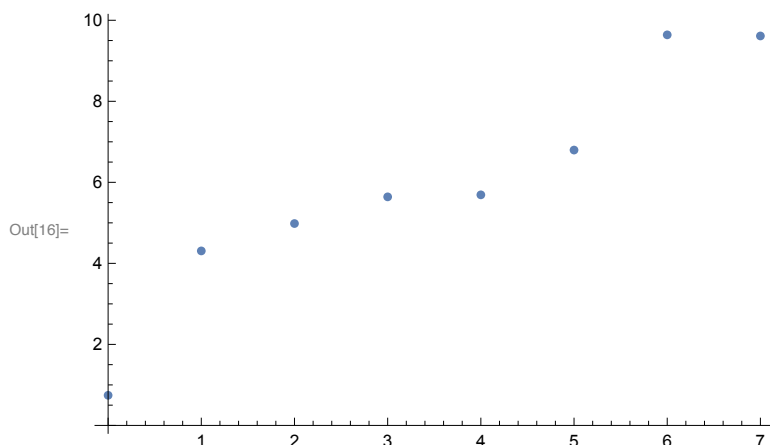
Note : You can watch Prac 12 video for
the explanation of the Gradient Descent algorithm

(*First generate data from a given simple linear model and plot the data*)

```
ClearAll[data, x];
```

```
data = Table[{x, 1 + 2 x + RandomReal[]}, {x, 0, 7}];
```

```
In[16]:= ListPlot[data]
```



(*get the x and y values from the data*)

```
In[33]:= X = data[[All, 1]]
```

```
Out[33]= {0, 1, 2, 3, 4, 5, 6, 7}
```

```
In[34]:= Y = data[[All, 2]];
```

(*perform the Gradient Descent*)

```
In[181]:= ClearAll[Ypred, Db0, Db1, b0, b1]
```

```
L = 0.0001; (* The learning Rate or step size*)
```

```
In[182]:= b0 = 0; (*initial value of  $\beta_0$ *)
```

```
b1 = 0; (*initial value of  $\beta_1$ *)
```

```
For[i = 1, i < 1000, i++,
```

```
Ypred = b0 + b1 * X; (* The current predicted value of Y*)
```

```
Db0 = -2 * Total[Y - Ypred]; (* Derivative wrt  $\beta_0$ *)
```

```
Db1 = -2 * Total[X (Y - Ypred)]; (* Derivative wrt  $\beta_1$ *)
```

```
b0 = b0 - L * Db0; (* Update  $\beta_0$ *)
```

```
b1 = b1 - L * Db1 (* Update  $\beta_1$ *)
```

```
(*Print[{b0,b1}]*)
```

```
]
```

```
Print[{b0, b1}]
```

```
{0.831304, 2.116}
```