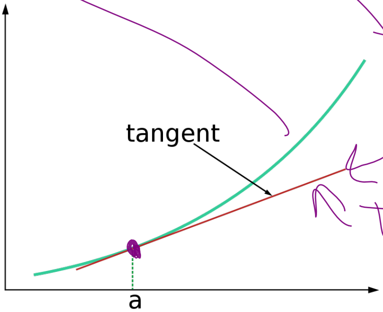
MATH7501: Mathematics for Data Science I

Unit 8: Linear approximations and Taylor series

8.1 Linear Approximations

Provided the function f is 'good enough' close to the point $x = a$, we have the following linear approximation of $f(x)$ for x close to a:

$$f(x) \approx \quad \cancel{f}(x) \approx \underbrace{f(a) + f'(a)(x - a)}.$$



$f(x)$

tangent line

https://en.wikipedia.org/wiki/Linear_approximation

Example

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Example

Consider the population model $P(t) = 14931234e^{0.002t}$, where t is the number of quarter years since 1981. Then for t close to 70, we have the linear approximation

$$P(t) \approx P(70) + P'(70)(t - 70).$$

We compute $P'(t) = 14931234 \times 0.002 \times e^{0.002t}$ so that

$$\begin{aligned} P(t) &\approx 17175007.255 + 34350.014(t - 70) \\ &\approx 14770506.275 + 34350.014t. \end{aligned}$$

Suppose the function $f(x)$ can be expressed as $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some real a, as well as real coefficients c_i.

$a = 0 \Rightarrow$ Maclaurin series.

infinite sum

$f(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

8.2 Taylor Series Approximations

Suppose the function $f(x)$ can be expressed as $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for some real a, as well as real coefficients c_i.

To find the coefficients, first note that by setting $x = a$, we find that $c_0 = f(a)$. Then by differentiating, and then setting $x = a$, we find $c_1 = f'(a)$.

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

set $x = a$

$f(a) = c_0 + c_1(0) + c_2(0)^2 + c_3 \ldots$

$f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2$

$f'(a) = c_1 + 2 \times c_2(0) + 0 + 0 + \ldots 0$

Suppose the function $f(x)$ can be expressed as $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some real a, as well as real coefficients c_i.

$f(x) = \cancel{c_0} + c_1\cancel{(x-a)} + c_2\cancel{(x-a)}^2 + c_3\underline{(x-a)^3}$

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$$c_n = \frac{f^{(n)}(a)}{n!}.$$

$f''(x) = 2c_2 + 3 \times 2 \times c_3 \cdot (x-a) + \ldots$

$f''(a) = 2c_2 \Rightarrow c_2 = \frac{f''(a)}{2}.$

8.2 Taylor Series Approximations

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$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

8.2 Taylor Series Approximations

Definition

If f is a function so that $f^{(n)}(a)$ exists for any n, the *Taylor series of* f *at* a is the expression

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

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If $a = 0$, this is called the *Maclaurin series approximation*.

But when does $f(x)$ actually equal this infinite series?

Theorem

If f is 'sufficiently well behaved' near the point a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < r,$$

where $r > 0$ is the radius of convergence.

Theorem

If f is 'sufficiently well behaved' near the point a, then

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where $r > 0$ is the radius of convergence.

In this context, the phrase 'sufficiently well behaved' means a number of things, including the assumption that $\lim_{n\to\infty} \frac{b_n r^n}{n!} = 0$, where $b_n = \max_{x\in[a-r,a+r]} \left|f^{(n)}(x)\right|$.

" $f^{(n)}(a)$ these decay quite quickly".

Theorem

If f is 'sufficiently well behaved' near the point a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < r,$$

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Convergence of Taylor series

If

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < r,$$

then the expression $\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is a good approximation of $f(x)$ for $|x-a| < r$, especially for large k.

'k' is the (cutoff) point of our approximation.

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then the expression $\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is a good approximation of $f(x)$ for $|x-a| < r$, especially for large k.

The value of r can be determined by the *ratio test*, which says that $\sum_{n=0}^{\infty} c_n$ converges if $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1$.

Question

Find the Taylor series of $f(x) = \ln(x)$ about $x = 1$; determine its radius of convergence.

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Solution

We calculate the derivatives:

$\boxed{f'(x) = \frac{1}{x},}$ $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ and so on.

$(a = 1)$

$$\sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(1)}}{n!}(x-1)^n$$

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$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ and so on. Therefore,

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2,$$
$$\cdots f^{(n)}(1) = (-1)^{n-1}(n-1)!.$$

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Then the Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$;

$\sum_{n=0}^{\infty}$

$n=0$ term

$$\frac{f^{(0)}(1)(x-1)^0}{0!} = 0(x-1)^0 = 0.$$

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Find the Taylor series of $f(x) = \ln(x)$ about $x = 1$; determine its radius of convergence.

1 is radius of convergence.

Solution

We calculate the derivatives:
$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ and so on. Therefore,

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2,$$
$$\cdots f^{(n)}(1) = (-1)^{n-1}(n-1)!.$$

Then the Taylor series is $\sum_{n=1}^{\infty} \frac{\cancel{(-1)^{n-1}}}{n}(x-1)^n$; this converges provided $\lim_{n\to\infty} \frac{|(x-1)^{n+1} n|}{|n+1(x-1)^n|} = |x-1| < 1$.

$(x-1)^{n+1} / n+1$

$(x-1)^n / n$

$= |x-1| \frac{}{n} \times \frac{n}{n+1} \to 1$

8.2 Taylor Series Approximations

In the previous question, we found that the Taylor series for $\ln(x)$ about $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$, and that this converges for $0 < x < 2$.

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In fact, the following points are also true (even though we did not prove them):

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ is actually convergent for $0 < x \leq 2$;

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} \to \text{divergent.}$$

In the previous question, we found that the Taylor series for $\ln(x)$ about $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$, and that this converges for $0 < x < 2$.

In fact, the following points are also true (even though we did not prove them):

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ is actually convergent for $0 < x \leq 2$;
- $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ for all $0 < x \leq 2$.

8.2 Taylor Series Approximations

Question

Find the Taylor series of $f(x) = e^x$ about $x = a$; determine its radius of convergence.

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Solution

Since $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all n.

8.2 Taylor Series Approximations

Question

Find the Taylor series of $f(x) = e^x$ about $x = a$; determine its radius of convergence.

Solution

Since $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = e^a$, so the Taylor series is

$$\sum_{n=0}^{\infty} \frac{\overset{e^a}{f^{(n)}(a)}}{n!}(x-a)^n = e^a \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n.$$

Question

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Since $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = e^a$, so the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = e^a \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n.$$

$$\frac{\frac{(x-a)^{n+1}}{(n+1)!}}{\frac{(x-a)^n}{n!}} = \frac{n!\,(x-a)^{n+1}}{(n+1)!\,(x-a)^n}$$

By the ratio test, this converges whenever $1 > \lim_{n\to\infty} \frac{\left|(x-a)^{n+1} n!\right|}{\left|(n+1)!(x-a)^n\right|} = \boxed{\lim_{n\to\infty} \frac{|x-a|}{n+1}} = 0$, so this expression converges everywhere.

radius of convergence is ∞!

8.2 Taylor Series Approximations

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Since $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = e^a$, so the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = e^a \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n.$$

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Question

Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

Taylor series at $a = 0$.

8.2 Taylor Series Approximations

Question

Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

Solution

We have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f''''(x) = \sin(x)$, and then repeats.

8.2 Taylor Series Approximations

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Solution

We have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f''''(x) = \sin(x)$, and then repeats. Therefore,

$f(0) = 0$

$f'(0) = 1$

$f''(0) = 0$

$f'''(0) = -1$

$f''''(0) = 0$

repeats

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2m \\ (-1)^m & \text{if } n = 2m+1. \end{cases}$$

$m \in \mathbb{N}$.

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Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

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This implies that the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \boxed{\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}}.$$

Question

Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

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$\frac{m+1}{m}$ term

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Similarly to e^x, we find that $\lim_{m\to\infty} \boxed{\frac{|x^2|}{|(2m+3)(2m+2)|}} = 0$, so we get convergence everywhere "radius of convergence is ∞".

8.2 Taylor Series Approximations

Question

Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

Solution

We have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f''''(x) = \sin(x)$, and then repeats. Therefore,

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2m \\ (-1)^m & \text{if } n = 2m+1. \end{cases}$$

This implies that the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}.$$

Similarly to e^x, we find that $\lim_{n\to\infty} \frac{\left|x^2\right|}{|(2m+3)(2m+2)|} = 0$, so we get convergence everywhere (convergence is to $\sin(x)$).

Taylor series in Mathematica:

point about which you take the Taylor Series

$O(x)^{11}$

$\Sigma\infty$

```
In[30]:= Series[Cos[x], {x, 0, 10}]
```

Out[30]= $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + O[x]^{11}$

```
In[31]:= Series[Cos[x], {x, 0, Infinity}]
```

⋯ Series: Series order specification ∞ is not a machine-sized integer.

Out[31]= Series[Cos[x], {x, 0, ∞}]

```
In[37]:= Series[Cos[x], {x, 0, 10000}]
```

Out[37]= $1 - \frac{x^2}{2} + \cdots 7498 \cdots + O[x]^{10001}$

large output | show less | show more | show all | set size limit...

```
In[35]:= Series[x^2/y, {x, 200, 1}, {y, 400, 1}]
```

Out[35]= $\left(100 - \frac{y-400}{4} + O[y-400]^2\right) + \left(1 - \frac{y-400}{400} + O[y-400]^2\right)(x-200) + O[x-200]^2$

Partial — $\frac{\partial f}{\partial x}$

del.

Definition (Linear Approximation)

Suppose $f(x, y)$ is a function of two variables. The expression

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the *linear approximation for f at (a, b)*.

(tangent plane).

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is called the *linear approximation for f at (a, b).* We expect this to be a good approximation if f is 'sufficiently well-behaved' for values of (x, y) close to (a, b).

Example

The temperature in a region is given by $T(x, y) = 100 - x^2 - y^2$.

Example

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Find the linear approximation about $(x,y) = (0,5)$.

$$T(x,y) \approx T(0,5) + \frac{\partial T}{\partial x}(0,5)(x-0) + \frac{\partial T}{\partial y}(0,5)(y-5).$$

Example

The temperature in a region is given by $T(x, y) = 100 - x^2 - y^2$. Note that $\frac{\partial T}{\partial x} = -2x$, $\frac{\partial T}{\partial y} = -2y$. Therefore, $T(0, 5) = 75$, $T_x(0, 5) = 0$ and $T_y(0, 5) = -10$,

Example

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$$T(x,y) \approx 75 - 10(y-5) = 125 - 10y;$$

Tangent plane.

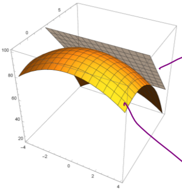
8.3 Linear Approximations for Multi-Variable Functions

Example

The temperature in a region is given by $T(x, y) = 100 - x^2 - y^2$. Note that $\frac{\partial T}{\partial x} = -2x$, $\frac{\partial T}{\partial y} = -2y$. Therefore, $T(0, 5) = 75$, $T_x(0, 5) = 0$ and $T_y(0, 5) = -10$, so the linear approximation for T at $(0, 5)$ is

$$T(x, y) \approx 75 - 10(y - 5) = 125 - 10y;$$

we expect this to be a good approximation for (x, y) close to $(0, 5)$.



tangent plane

$T(x,y)$

Figure 53: The function $T(x, y) = 100 - x^2 - y^2$ and the tangent plane $T(x, y) = 125 - 10y$.

Question

Find the tangent plane to the function $f(x, y) = e^{-x^2}\sin(y)$ at the point $(1, \frac{\pi}{2})$. Hence find an approximate value for $e^{-(0.9)^2}\sin(1.5)$.

8.3 Linear Approximations for Multi-Variable Functions

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Solution

We have $\frac{\partial f}{\partial x} = -2xe^{-x^2}\sin(y)$ and $\frac{\partial f}{\partial y} = e^{-x^2}\cos(y)$.

$$\frac{\partial (e^{-x^2})}{\partial x} = e^{-x^2} \times (-2x)$$

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Find the tangent plane to the function $f(x, y) = e^{-x^2} \sin(y)$ at the point $(1, \frac{\pi}{2})$. Hence find an approximate value for $e^{-(0.9)^2} \sin(1.5)$.

Solution

We have $\frac{\partial f}{\partial x} = -2xe^{-x^2} \sin(y)$ and $\frac{\partial f}{\partial y} = e^{-x^2} \cos(y)$. Therefore, the linear approximation is

$$f(x, y) \approx \frac{1}{e} + (-\frac{2}{e})(x-1) + (0)(y - \frac{\pi}{2}) = \boxed{\frac{1}{e} - \frac{2(x-1)}{e}}.$$

$\frac{\partial f}{\partial y}$

$f(1, \frac{\pi}{2})$

$= e^{-1} \sin(\frac{\pi}{2})$

$\frac{\partial f}{\partial x}(1, \frac{\pi}{2})(x-1).$

8.3 Linear Approximations for Multi-Variable Functions

Question

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Solution

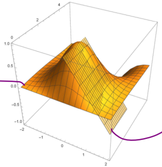
We have $\frac{\partial f}{\partial x} = -2xe^{-x^2}\sin(y)$ and $\frac{\partial f}{\partial y} = e^{-x^2}\cos(y)$. Therefore, the linear approximation is

tangent plane.

$$f(x,y) \approx \frac{1}{e} + \left(-\frac{2}{e}\right)(x-1) + (0)\left(y - \frac{\pi}{2}\right) = \frac{1}{e} - \frac{2(x-1)}{e}.$$

Therefore, $e^{-(0.9)^2}\sin(1.5) = f(0.9, 1.5) \approx \frac{1}{e} - \frac{2(0.9-1)}{e} = \frac{1.2}{e}$.

f(x,y).



tangent plane.

8.3 Linear Approximations for Multi-Variable Functions

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$$f(x, y) - f(a, b) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

'change in f'.

('change in x')

'change in y'.

The formula for a linear approximation can also be interpreted in terms of small changes to f corresponding to small changes in x and y. Indeed, if we take an approximation to f at the point (a, b), then by rearranging, we get

$$f(x, y) - f(a, b) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Using Δ to mean 'change in', this becomes

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y;$$

this tells us how f changes with small changes in x and y.

8.3 Linear Approximations for Multi-Variable Functions

The formula for a linear approximation can also be interpreted in terms of small changes to f corresponding to small changes in x and y. Indeed, if we take an approximation to f at the point (a, b), then by rearranging, we get

$$f(x, y) - f(a, b) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Using Δ to mean 'change in', this becomes

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y;$$

this tells us how f changes with small changes in x and y. 'Infinitesimally', we write this in terms of *differentials*:

$$\boxed{df = f_x(a, b)dx + f_y(a, b)dy.}$$

Question

Electric power is given by $\boxed{P(E, R) = \frac{E^2}{R}}$, where E is the voltage and R is the resistance. Find a linear approximation for $P(E, R)$ for values of E close to 200 (in Volts) and R close to 400 (in Ohms). Use this to find the effect that a change in E and R has on P.

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Solution

We calculate $\frac{\partial P}{\partial E} = \frac{2E}{R}$ and $\frac{\partial E}{\partial R} = -\frac{E^2}{R^2}$.

$$\frac{\partial\left(\frac{1}{R}\right)}{\partial R} = -\frac{1}{R^2}.$$

8.3 Linear Approximations for Multi-Variable Functions

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Solution

We calculate $\frac{\partial P}{\partial E} = \frac{2E}{R}$ and $\frac{\partial E}{\partial R} = -\frac{E^2}{R^2}$. Therefore,

$$\frac{\partial P}{\partial R}(200,400) = \frac{-(200^2)}{(400^2)},$$

$$P(E,R) \approx 100 + (1)(E-200) + \left(-\frac{1}{4}\right)(R-400).$$

$$\frac{\partial P}{\partial E}(200,400) = \frac{2\times 200}{400}$$

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We calculate $\frac{\partial P}{\partial E} = \frac{2E}{R}$ and $\frac{\partial E}{\partial R} = -\frac{E^2}{R^2}$. Therefore,

$$P(E,R) \approx 100 + (1)(E-200) + (-\frac{1}{4})(R-400).$$

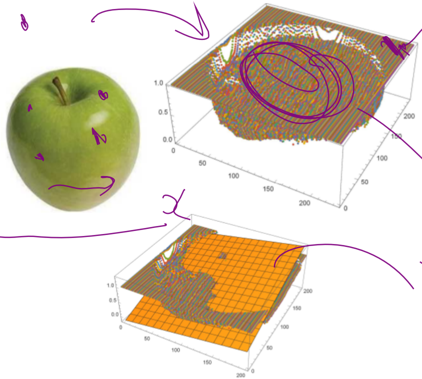
Using $\Delta P = P(E,R) - P(200,400)$, $\Delta E = E - 200$ and $\Delta R = R - 400$, we find

$$\boxed{\Delta P \approx \Delta E - \frac{1}{4}\Delta R.}$$

8.3 Linear Approximations for Multi-Variable Functions

y

white.



apple.

x

tangent plane.

Figure 55: An image of an apple and the pixel values as a function of position in the image, $P(x, y)$, with a tangent plane.

Errors in Linear Approximations

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of $f(a, b)$. This gives us a way to treat errors.

Question

Suppose we are making a metal barrel with base radius 1m and height 2m. We allow for a 5% error in the construction of both the base radius and the height. What is the worst-case scenario error for the volume?

8.3 Linear Approximations for Multi-Variable Functions

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A 5% error in the radius gives an absolute error of $E_1 = 0.05\text{m}$ in the radius.

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Solution

A 5% error in the radius gives an absolute error of $E_1 = 0.05$m in the radius. A 5% error in the height gives an error of $E_2 = 0.1$m. Now the volume is given by $V(r,h) = \pi r^2 h$; computing derivatives gives $V_r(1,2) = 4\pi$ and $V_h(1,2) = \pi$.

$$\frac{\partial V}{\partial r} = 2\pi r h$$

1 2

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$$E = 4\pi \times 0.05 + \pi \times 0.1 = 0.3\pi.$$

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Since $V(1, 2) = 2\pi$, the relative error in the volume is $\frac{0.3\pi}{2\pi} = 15\%$.

8.4 Optimisation via quadratic approximations

Let $f(x)$ be a real function that we wish to maximise. Let x_0 be an initial guess for the location of the maximum.

"quadratics are the simplest function to optimise".

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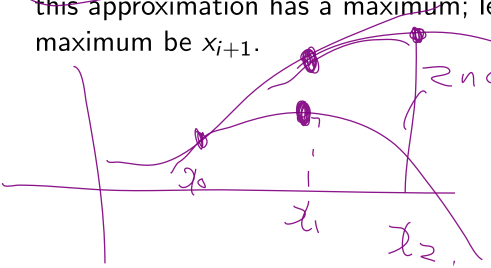
Let $f(x)$ be a real function that we wish to maximise. Let x_0 be an initial guess for the location of the maximum.

An approximation for the true location of the maximum proceeds by finding the second order Taylor series approximation $f(x) \approx f(x_i) + f'(x_i)(x - x_i) + \frac{f''(x_i)}{2}(x - x_i)^2$. If $f''(x_i) < 0$, then this approximation has a maximum; let the location of this maximum be x_{i+1}.



2nd order Taylor approx

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Continue in this way until the desired accuracy is achieved.