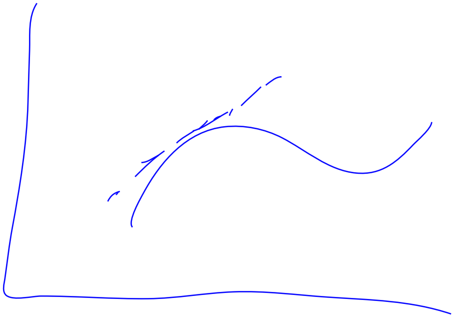
MATH7501: Mathematics for Data Science I

Unit 10: Partial Derivatives and Gradient Descent

Consider a surface $f(x, y)$ with a particular point $P = (x_0, y_0)$.



10.1 Partial Derivatives and Tangent Planes

Consider a surface $f(x, y)$ with a particular point $P = (x_0, y_0)$. Then the slope of f at P in the x direction is $\frac{\partial f}{\partial x}$, evaluated at P; this partial derivative is the derivative of the single-variable function $f(x, y_0)$.

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For example, if $f(x, y) = 1 - x^2 - y^2$, and $(x_0, y_0) = (1, -1)$, then $f(x, -1) = -x^2$, so $\frac{\partial f}{\partial x}(P) = -2$.

$$\frac{\partial f}{\partial x} = -2x \Big|_{x=1}$$

Consider a surface $f(x, y)$ with a particular point $P = (x_0, y_0)$. Then the slope of f at P in the x direction is $\frac{\partial f}{\partial x}$, evaluated at P; this partial derivative is the derivative of the single-variable function $f(x, y_0)$. We can find the slope of f in the y-direction similarly.

For example, if $f(x, y) = 1 - x^2 - y^2$, and $(x_0, y_0) = (1, -1)$, then $f(x, -1) = -x^2$, so $\frac{\partial f}{\partial x}(P) = -2$. Also, $f(1, y) = - y^2$, so $\frac{\partial f}{\partial y}(P) = 2$.

$\frac{\partial f}{\partial y} = -2y$

$y = -1$

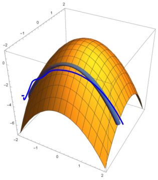

Figure 64: The surface $z = f(x, y) = 1 - x^2 - y^2$ and the intersection of the plane $y = -1$ with the surface.

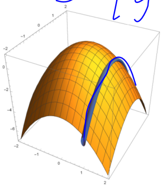

Figure 65: The surface $z = f(x, y) = 1 - x^2 - y^2$ and the intersection of the plane $x = 1$ with the surface.

Question

The volume of a solid with side lengths x, y, z is given by $V(x, y, z) = xyz$. Calculate the partial derivatives of V in all three variables.

10.1 Partial Derivatives and Tangent Planes

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Solution

If y, z are held and x changes, then we can see that

$$\frac{\partial V}{\partial x} = yz.$$

$$\frac{\partial (xyz)}{\partial x} = yz\frac{\partial (x)}{\partial x} = yz$$

Similarly, $\frac{\partial V}{\partial y} = xz$, $\frac{\partial V}{\partial z} = xy$.

For a given function $f(x, y)$, the *second order* partial derivatives are found by differentiating f once with respect to one variable, and then differentiating again with respect to a second variable.

those two variables do not need to be the same.

For a given function $f(x, y)$, the *second order* partial derivatives are found by differentiating f once with respect to one variable, and then differentiating again with respect to a second variable. We denote this with

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad \cancel{f_{xy}} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

f_{yx}

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$\frac{\partial f}{\partial x} = \sin(y) - y\sin(x)$, $\frac{\partial f}{\partial y} = x\cos(y) + \cos(x)$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(y) - \sin(x)$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \cos(y) - \sin(x)$

Question

Let $f(x, y) = x\sin(y) + y\cos(x)$. Calcualte all of the second order partial derivatives, and show that $f_{xy} = f_{yx}$.

— Always true if f is 'sufficiently well-behaved'.

Recall that through linear approximations, we have

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

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Therefore, if x and y are both changing with time t, and we know how much x and y are changing by, we can approximate how much f is changing over time.

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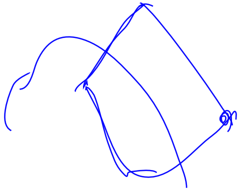

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Theorem (Multi-Variable Chain Rule)

If x, y are differentiable functions of t, and f is a differentiable function of x, y, then

$\frac{d}{dt} f(x(t), y(t)) \approx$

$$\frac{df(x(t),y(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

limit

$a x'(t) + b y'(t)$

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

+ (don't count for derivatives).

Question

Suppose the radius of a cylinder decreases at a rate of $r'(t) = -2cm/s$, and the height is kept constant at $h(t) = 2cm$. How fast is the volume decreasing when $r(t) = 1cm$?

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Solution

We have $V = \pi r^2 h$ so
$V' = \pi 2rhr' + \pi r^2 h' = 2\pi rhr' = -8\pi cm^3/s.$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

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Suppose now that the height is decreasing at $1cm/s$. What is the rate of change of volume?

Solution

We have $V = \pi r^2 h$ so $V' = \boxed{\pi 2rhr' + \pi r^2 h'} = \boxed{-9\pi cm^3/s}$.

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Suppose a, b, c are the dimensions of a box which are changing over time. From the rate of change of the volume.

Solution

The volume is $\boxed{V = abc}$, so

$\frac{\partial V}{\partial a}$

$$\begin{aligned}\frac{dV}{dt} &= V_a\frac{da}{dt} + V_b\frac{db}{dt} + V_c\frac{dc}{dt}\\ &= \boxed{bc\frac{da}{dt} + ac\frac{db}{dt} + ab\frac{dc}{dt}}.\end{aligned}$$

$\frac{\partial V}{\partial a} = bc, \quad \frac{\partial V}{\partial b} = ac, \quad \frac{\partial V}{\partial c} = ab$

10.2 Gradient

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ tell us the slope of f in the x, y directions, but they can also be used to describe the slope in *any direction*.

Dot product: $(a_1, a_2) \cdot (b_1, b_2)$

$= a_1 b_1 + a_2 b_2$.

$= \underbrace{\sqrt{a_1^2 + a_2^2}}_{|a|} \times \underbrace{\sqrt{b_1^2 + b_2^2}}_{|b|} \times \cos\theta$.

θ is the angle between the 2.

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$x(t)$ is a scalar, $y(t)$ is a scalar,

$(x(t), y(t))$ is a vector.

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$$
\begin{aligned}
f(x(t), y(t)) &\approx f(x(0), y(0)) + \frac{\partial f}{\partial x} u_1 t + \frac{\partial f}{\partial y} u_2 t \\
&= f(P) + t\left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right) \cdot \mathbf{u}.
\end{aligned}
$$

$$(a, b) \cdot (c, d)$$

$$= \underline{ac + bd} \rightarrow \text{Scalar}.$$

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The expression $\underline{(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \cdot \mathbf{u}}$ is therefore referred to as the *slope/directional derivative of f at P in the direction of* u.

$$\frac{f(t) - f(0)}{t} \approx \underline{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot u}$$

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The expression $(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \cdot \mathbf{u}$ is therefore referred to as the *slope/directional derivative of f at P in the direction of* u. If $\mathbf{u} \neq 0$ is not a unit vector, then the directional derivative is simply $(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$.

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Question

Find the directional derivative of $f(x, y) = 4 - x^2 - 4y^2$ at the point $(1, 1)$ in the $(1, 1)$ direction.

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Solution

The corresponding unit vector is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$.

$i = (1, 0)$

$j = (0, 1)$.

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Solution

The corresponding unit vector is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. We compute that $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = -8y$.

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$$(-2, -8) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{10}{\sqrt{2}} = -5\sqrt{2}.$$

Question

If $f(x, y) = x^2 - 3y^2 + 6y$, find the slope at $(1, 0)$ in the direction of $\mathbf{i} - 4\mathbf{j}$.

$i = (1, 0)$, $j = (0, 1)$.

('x' direction) ('y' direction.

Question

If $f(x, y) = x^2 - 3y^2 + 6y$, find the slope at $(1, 0)$ in the direction of $\mathbf{i} - 4\mathbf{j}$.

Solution

The norm of $\mathbf{u} = \mathbf{i} - 4\mathbf{j}$ is $||\mathbf{u}|| = \sqrt{17}$.

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Solution

The norm of $\mathbf{u} = \mathbf{i} - 4\mathbf{j}$ is $||\mathbf{u}|| = \sqrt{17}$. Also, note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 6 - 6y$.

$\frac{u}{\sqrt{17}}$ is a unit vector

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Solution

The norm of $\mathbf{u} = \mathbf{i} - 4\mathbf{j}$ is $||\mathbf{u}|| = \sqrt{17}$. Also, note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 6 - 6y$. Therefore

$$f_{\mathbf{u}}(1, 0) = (2, 6) \cdot \frac{(1, -4)}{\sqrt{17}} = -\frac{22}{\sqrt{17}}.$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

10.2 Gradient

It is useful to give the vector (f_x, f_y) a name, since it comes up a lot when computing directional derivatives.

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Definition (Gradient Vector)

We define $\nabla f = (f_x, f_y)$ to be the *gradient* of f, (or $\nabla f = (f_x, f_y, f_z)$ for functions of three variables). Then the derivative of f in the direction of $\mathbf{u}$ is $\nabla f \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$

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(Question)

Find the gradient of $f(x, y) = x^2 - 3(y-1)^2 + 3$.

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(Question)

Find the gradient of $f(x, y) = x^2 - 3(y-1)^2 + 3$.

Solution

$\nabla f = (2x, -6(y-1)) = 2x\mathbf{i} - 6(y-1)\mathbf{j}$.

$\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$

Question

Find the directional derivative of $g(x, y) = e^{x^2}\cos(y)$ at $(1, \pi)$ in the direction of $-3\mathbf{i} + 4\mathbf{j}$.

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Solution

The norm of $\mathbf{u} = -3\mathbf{i} + 4\mathbf{j}$ is $||\mathbf{u}|| = 5$, and
$\nabla g = 2xe^{x^2}\cos(y)\mathbf{i} - \sin(y)e^{x^2}\mathbf{j}$.

$\frac{\partial g}{\partial x}$

$3^2 + 4^2 = 5^2$

$\frac{\partial g}{\partial y}$

Question

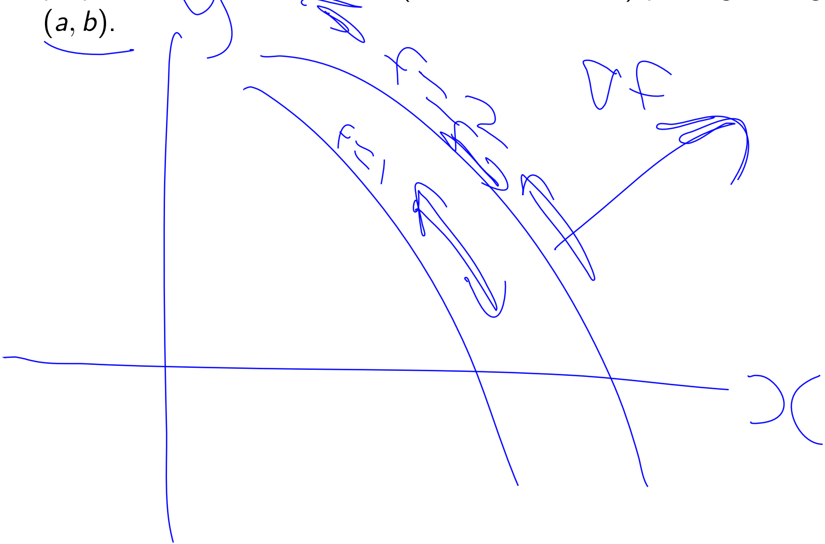
Find the directional derivative of $g(x, y) = e^{x^2}\cos(y)$ at $(1, \pi)$ in the direction of $-3\mathbf{i} + 4\mathbf{j}$.

Solution

The norm of $\mathbf{u} = -3\mathbf{i} + 4\mathbf{j}$ is $||\mathbf{u}|| = 5$, and $\nabla g = 2xe^{x^2}\cos(y)\mathbf{i} - \sin(y)e^{x^2}\mathbf{j}$. Therefore

$$
\begin{aligned}
g_{\mathbf{u}} &= \nabla g(\overset{x=}{1}, \overset{y=}{\pi}) \cdot \frac{\mathbf{u}}{||\mathbf{u}||} \\
&= (2e\cos(\pi), -\sin(\pi)e) \cdot (\frac{-3}{5}, \frac{4}{5}) = \frac{6e}{5}.
\end{aligned}
$$

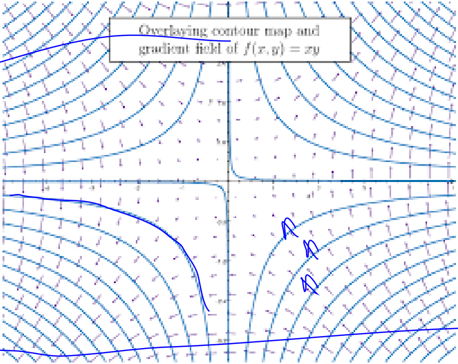
In general, the gradient $\nabla f(a, b)$ points in the direction perpendicular to the contour (line of constant f) passing through (a, b).



10.2 Gradient

In general, the gradient $\nabla f(a, b)$ points in the direction perpendicular to the contour (line of constant f) passing through (a, b). In fact, $\nabla f(a, b)$ points in the direction of the shaprest increase of f.

$f(x,y) = xy$



contours in blue
gradient in purple

https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/partial-derivative-and-gradient-articles/a/the-gradient

10.2 Gradient

Question

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Solution

The contours are $y = -\frac{mx}{n} + C$. The gradient vector is $\nabla f = (m, n)$, which points in the direction perpendicular to the contours.

$$\nabla f \cdot \left(1, -\frac{m}{n}\right)$$

$$= (m, n) \cdot \left(1, -\frac{m}{n}\right) = 0.$$

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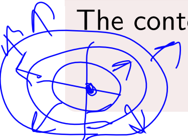
Consider the function $f(x, y) = x^2 + y^2$. Describe the gradient and the contours.

Solution

The contours are the concentric circles centered at the origin.



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Question

Consider the function $f(x, y) = x^2 + y^2$. Describe the gradient and the contours.

Solution

The contours are the concentric circles centered at the origin. The gradient is $(2x, 2y)$, which is radial, always points perpendicular to the circles.

10.2 Gradient

Question

The temperature of a metal plate if $T(x, y) = 20 - 4x^2 - y^2$. Find the gradient at the point $(2, -3)$. Hence describe the direction of maximal increase in temeprature, and the direction of constant temperature.

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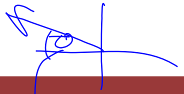
Solution

The gradient is $\nabla T = (-8x, -2y)$, which is $(-16, 6)$ at the specified point.



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Solution

The gradient is $\nabla T = (-8x, -2y)$, which is $(-16, 6)$ at the specified point. To find the angle this makes with the x-axis, write $-16 = r\cos(\theta)$ and $6 = r\sin(\theta)$, so that $r = 2\sqrt{73}$ and $\theta = \pi - \arctan(\frac{6}{16})$.

The direction of constant temperature is perpendicular to ∇T.

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The temperature of a metal plate if $T(x, y) = 20 - 4x^2 - y^2$. Find the gradient at the point $(2, -3)$. Hence describe the direction of maximal increase in temepraturе, and the direction of constant temperature.

Solution

The gradient is $\nabla T = (-8x, -2y)$, which is $(-16, 6)$ at the specified point. To find the angle this makes with the x-axis, write $-16 = r\cos(\theta)$ and $6 = r\sin(\theta)$, so that $r = 2\sqrt{73}$ and $\theta = \pi - \arctan(\frac{6}{16})$. The direction of constant temperature is then $\frac{\pi}{2} - \arctan(\frac{6}{16})$.

Question

A team is mapping the ocean floor. They produce the model $D(x, y) = 1700 - 30x^2 - 50\sin(\frac{\pi y}{2})$ in m, with $(x, y) \in [-2, 2] \times [-2, 2]$ in km.

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Solution

The depth at $(1, 0.5)$ is
$D(1, 0.5) = 250 - 30 - 50\sin(\frac{\pi}{4}) \approx 184.6m.$

1700 $x = 1$, $y = 0.5$ km

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$$\nabla D(1, 0.5) = (-60, -\frac{50\pi}{2}\cos(\frac{\pi}{4})) = (-60, -\frac{25\pi}{\sqrt{2}}),$$

which is measured in m/km.

$\frac{\partial D}{\partial x}$ — $\frac{m}{km}$

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which is measured in m/km. This makes an angle of $\arctan(\frac{25\pi}{60\sqrt{2}})$ with the positive x-axis, and has a magnitude of $\sqrt{3600 + \frac{625\pi^2}{2}}$.