MATH7501 Practical 9 (Week 10), Semester 1-2021

Topic: Sequences & Series, Limits and Derivatives

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Pre-Tutorial Activity

- Students must have familiarised themselves with units 5 to 8 contents of the reading materials for MATH7501

Resources

- Chapters 5 to 8 of course reader

Q1 Limit

Calculate $\text{limit}_{x \to \text{Infinity}} \sin(e^{-x})$ and explain why this is true

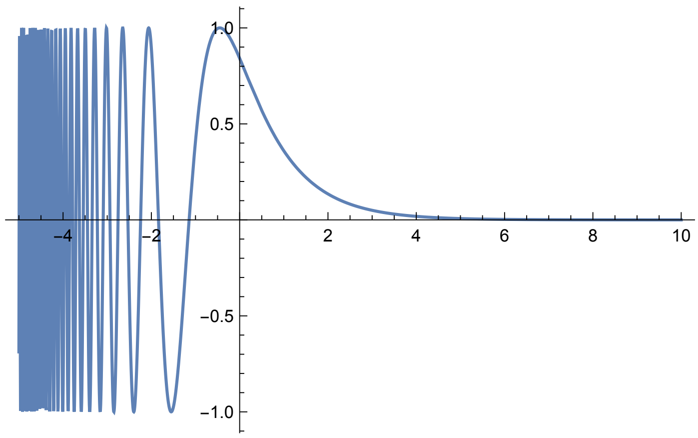
$\lim_{x \to \infty} e^{-x} = 0$. As $\sin(0)$ is continuous,

$\lim_{x \to \infty} \sin(e^{-x}) = 0$ (see page 105 of course reader for the concept of limit of composite functions).

An illustration of this limit is given below.

In[33]:=
```
Plot[Sin[Exp[-x]], {x, -5, 10}]
```

Out[33]=



Q2 Limit

Calculate $\text{limit}_{x \to \text{Infinity}} \dfrac{1}{\cos(x) + x}$

Note that $-1 \leq \cos(x) \leq$
1. So adding x to each term of the inequality gives $-1+x \leq$
$\cos(x)+x \leq 1+x$.

For $x > 1$, $\dfrac{1}{1+x} \leq \dfrac{1}{\cos(x)+x} \leq \dfrac{1}{x-1}$

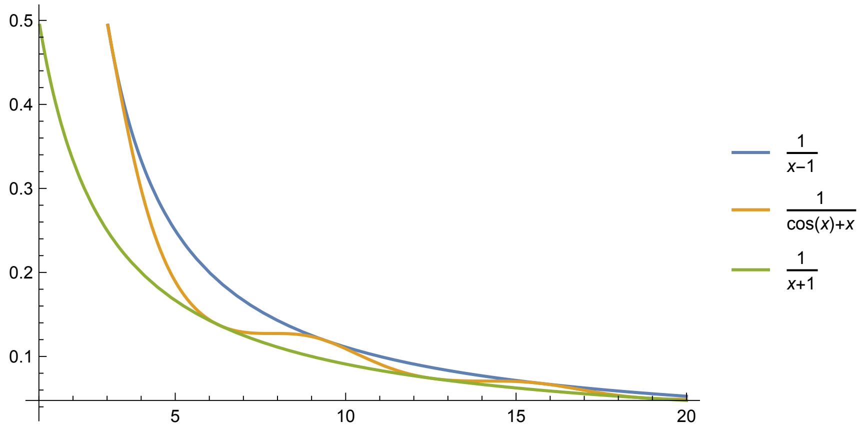
As $\text{limit}_{x\to\infty} \dfrac{1}{-1+x} = 0$ and $\text{limit}_{x\to\infty} \dfrac{1}{1+x} = 0$,

by the sqeez theorem, $\text{limit}_{x\to\text{Infinity}} \dfrac{1}{\cos(x)+x} = 0$.

An illustration of this concept is given below.

In[46]:=
```
Plot[{1/(x - 1), 1/(Cos[x] + x), 1/(x + 1)}, {x, 1, 20}, PlotLegends → "Expressions"]
```

Out[46]=



Q3 Continuous function

Consider the function

$$f(x) = \begin{cases} \frac{\sin(x)}{x}, & x > 0 \\ 1, & x \leq 0 \end{cases}$$

Show that this function is continuous for all x.

Hint : Continuous at a point (see Section 6.4, Definition 12)
Continuous on interval (see pg 104)

f (x) is continuous everywhere except, possibly,
at $x = 0$. So check continuity at $x = 0$, using Definition 12.

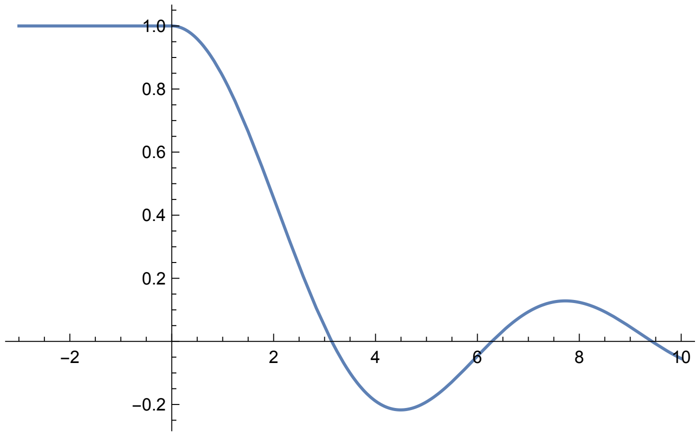
Check : (1) Is f (0) defined?
This condition is satisfied since $f(0) = 1$.
(2) Does $\text{limit}_{x\to 0} f(0)$ exist?
$\text{limit}_{x\to 0^-} f(x) =$
$f(0) = 1$ and $\text{limit}_{x\to 0^+} f(x) = \sin(x)/x = 1$. So $\text{limit}_{x\to 0} f(x) = 1$
(3) Is $\text{limit}_{x\to 0} f(x) = f(0)$?
$\text{limit}_{x\to 0} f(x) = 1 = f(0)$, so this condition is satisfied.

Thus f (x) is continuous at x = 0, and hence is continuous at all x.

The plot below shows graph of f (x) for x between - 3 and 10, for illustration. It is clear from the graph that f (x) is continuous within the chosen interval

In[36]:= `Plot[Piecewise[{{1, x ≤ 0}, {Sin[x]/x, x > 0}}], {x, -3, 10}]`

Out[36]=



Q4 IVT and MVT

a) Consider the function $f(x) = x^4 + x + 1$. Show that this function does not cross the x – axis.

Step 1 : Show that there is only one critical point by solving $f'(x) = 0$ for x

$4x^3 + 1 = 0$ gives $x = \sqrt[3]{\frac{-1}{4}}$ as the only critical point.

Step 2 : find $f''(x)$ and show that the critical point is global minimum

$f''(x) = 12x^2 > 0$ for $x = \sqrt[3]{\frac{-1}{4}}$. Thus the critical point is a global minimum.

step 3 : find $f(x)$ at $x = \sqrt[3]{\frac{-1}{4}}$.

$$f\left(\sqrt[3]{\frac{-1}{4}}\right) = \left(\sqrt[3]{\frac{-1}{4}}\right)^4 + \sqrt[3]{\frac{-1}{4}} + 1 > 0$$

Therefore, f (x) does not cross the x - axis.

b) Consider the new function $f(x) = x^4 + x - 1$. Show that there are exactly two solutions to $f(x) = 0$, using Intermediate Value Theorem (IVT) and Mean Value Theorem (MVT).

IVT (look at pg 105)

MVT (look at pg 115)

Step 1 : Use IVT to get an idea of how many roots

f (0) = −1, f (1) = 1, f (−2) = 13.

So by IVT there exists at least one root in the interval (−2, 0) and at least one root in the interal (0, 1). This means there exists at least two roots of f (x).

Step 2 : Use the MVT to show that there are exactly 2 roots

Suppose there are at least three distinct roots (say x1 < x2 < x3). Then by the MVT,

there exists a number c in (x1, x2) such that f ' (c) = 0 and

there exists a number d in (x2, x3) such that f ' (d) = 0

But $f'(x) = 4x^3 + 1$ and $f'(x) = 0$ gives $4x^3 + 1 = 0$,

$x = \sqrt[3]{\frac{-1}{4}}$ (only one root).

This contradicts to the assumption of having at last 3 roots. Thus there must be only 2 roots.

Q5 Series

Determine if the following sums converge or not. If they do, work out their limit

a) $\sum_{n=1}^{\infty} n^{2/e}$

$\sum_{n=1}^{\infty} n^{2/e} = \sum_{n=1}^{\infty} \frac{1}{n^{-2/e}}$ diverges by the p − series test with p = −2 / e

(b) $\sum_{n=0}^{\infty} (\pi - e)^n$

$\sum_{n=0}^{\infty} (\pi - e)^n$ converges to $\frac{1}{1 + e - \pi}$ by the Geometric series test with a = 1 and r = $\pi - e$