

# MATH7501 Practical 11 (Week 12), Semester 1-2021

Topic: Probability Distributions

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## Pre-Tutorial Activity

- Students must have familiarised themselves with units 9 and 10 contents of the reading materials for MATH7501

## Resources

- Chapters 9 and 10: Computation of mean, variance, expectation, gradient decent method
- [https://en.wikipedia.org/wiki/Rayleigh\\_distribution](https://en.wikipedia.org/wiki/Rayleigh_distribution)

## Section 1: The Rayleigh Distribution

In probability theory and statistics, the **Rayleigh distribution** is a continuous probability distribution for nonnegative – valued random variables.

The notation  $X \sim \text{Rayleigh}(\sigma)$  means that the random variable  $X$  has a Rayleigh distribution with shape parameter  $\sigma$ . The probability density function (pdf) is :

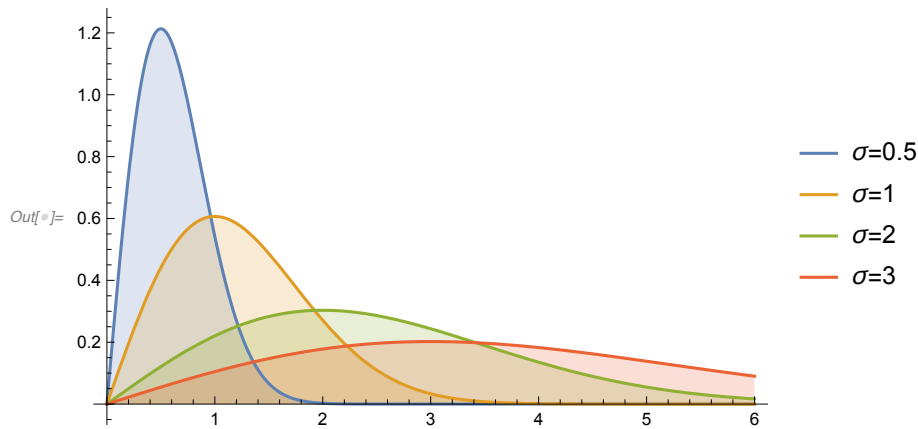
$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0, \sigma > 0 \\ 0, & x < 0 \end{cases}$$

Q1) Plot the pdf of  $X$  for  $\sigma = 0.5, 1, 2, 3$

```

In[8]:= Plot[Table[PDF[RayleighDistribution[σ], x], {σ, {.5, 1, 2, 3}}] // Evaluate,
  {x, 0, 6}, Filling -> Axis, PlotRange -> All,
  PlotLegends -> {"σ=0.5", "σ=1", "σ=2", "σ=3"}]

```



Q2) Show that the cumulative distribution function (cdf) of  $X$  is

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\begin{aligned} F_X(x) &= P(X \leq x) = \int_0^x f_X(t) dt \\ &= \int_0^x \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} dt \end{aligned}$$

Let  $u = \frac{t^2}{2\sigma^2}$ . Then  $du = \frac{t}{\sigma^2} dt$ ,

and the limits of the integral are  $u = 0$  and  $u = \frac{x^2}{2\sigma^2}$ . Thus,

substituting  $dt = \frac{\sigma^2}{t} du$  in the above integration, we have,

$$\begin{aligned} F_X(x) &= \int_0^{\frac{x^2}{2\sigma^2}} e^{-u} du \\ &= -[e^{-u}]_{u=0}^{u=\frac{x^2}{2\sigma^2}} \\ &= -\left\{e^{-\frac{x^2}{2\sigma^2}} - e^0\right\} \\ &= \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0, \end{cases} \end{aligned}$$

as required.

You can also use the `Integrate[]` function to compute the cdf as shown below.

```

In[39]:= f[x_] := x/σ^2 E^(-x^2/(2σ^2)) (*Rayleigh probability density function*)

```

```

Integrate[f[x], {x, 0, u}] (*cdf*)

```

```

Out[39]= 1 - E^(-u^2/(2σ^2))

```

Q3) Show that  $f_X(x)$  is a valid probability density function by showing that the integral over  $[0, \infty)$  is 1

$$\begin{aligned}
\int_0^{\infty} f_X(t) dt &= \int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \lim_{y \rightarrow \infty} \int_0^y \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \lim_{y \rightarrow \infty} F_X(y), \text{ by the definition of the cdf} \\
&= \lim_{y \rightarrow \infty} \left(1 - e^{-\frac{y^2}{2\sigma^2}}\right) \\
&= 1 - 0 \\
&= 1, \text{ as required.}
\end{aligned}$$

You can use the `Integrate[]` function to compute this integration exactly.

```
In[65]:= Integrate[f[x], {x, 0, ∞}, Assumptions → σ > 0]
```

```
Out[65]= 1
```

You can also use `NIntegrate[]` function to derive a numerical approximation

to  $\int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx$ , for a given value of  $\sigma$  (say for example,  $\sigma = 1$ ).

```
NIntegrate[x E^(-x^2/2), {x, 0, ∞}]
```

```
Out[66]= 1.
```

**Q4)** Show that the mean of  $X \sim \text{Rayleigh}(\sigma) = \sigma \sqrt{\frac{\pi}{2}}$

$$\begin{aligned}
\mu_X = E(X) &= \int_0^{\infty} x f_X(x) dx \\
&= \int_0^{\infty} x \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx
\end{aligned}$$

Let  $u = \frac{x^2}{2\sigma^2}$ . Then  $du = \frac{x}{\sigma^2} dx$ ,

and the limits of the integral are still  $u = 0$  and  $u = \infty$ . Thus, substituting  $dx = \frac{\sigma^2}{x} du$  and using  $x = \sqrt{2\sigma^2 u}$  in the above integration we have,

$$\begin{aligned}
\mu_X &= \int_0^{\infty} \sqrt{2\sigma^2 u} e^{-u} du \\
&= \sigma \sqrt{2} \int_0^{\infty} \sqrt{u} e^{-u} du \\
&= \sigma \sqrt{2} \int_0^{\infty} u^{3/2-1} e^{-u} du \\
&= \sigma \sqrt{2} \Gamma\left(\frac{3}{2}\right), \Gamma(x) \text{ is the Gamma function defined as: } \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \\
&= \sigma \sqrt{2} \Gamma\left(\frac{1}{2} + 1\right) \\
&= \sigma \sqrt{2} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right), \text{ using the property of Gamma function, } \Gamma(x+1) = x\Gamma(x) \\
&= \sigma \sqrt{2} \times \frac{1}{2} \sqrt{\pi}, \text{ using the property of Gamma function, } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\
&= \sigma \sqrt{\left(\frac{\pi}{2}\right)}
\end{aligned}$$

You can use the `Integrate[]` function to check your derivation of the mean as follows.

```
In[*]:= Integrate[x f[x], {x, 0, ∞}, Assumptions → σ > 0] (*mean (μ) *)
```

$$\text{Out[*]} = \sqrt{\frac{\pi}{2}} \sigma$$

```
In[*]:= (*using numerical approximation to compute the mean when σ = 1*)
```

```
In[*]:= NIntegrate[x x E^(-x^2/2), {x, 0, ∞}]
```

```
Out[*] = 1.25331
```

$$\delta = 0.01$$

```
Out[*] = 0.01
```

```
In[*]:= Total[Table[x x E^(-x^2/2) δ, {x, 0, 10, δ}]]
```

```
Out[*] = 1.25331
```

**Q5)** Show that the variance of  $X \sim \text{Rayleigh}(\sigma) = \sigma^2 \left( \frac{4 - \pi}{2} \right)$

From Week 11 practical, we know that  $\sigma^2_X = \text{var}(X) = E[X^2] - \mu_X^2$ ,

where  $E[X^2]$  is the second moment of  $X$  and  $\mu_X$  is the mean of  $X$ ,

which we computed in Q4. Thus,

it remains to find an expression for  $E[X^2]$ . We will require the integration by -

parts formula  $\int u dv = uv - \int v du$  for this calculation.

$$\begin{aligned} E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^{\infty} x^2 \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \\ &= \left[ x^2 \int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right]_{x=0}^{x=\infty} - \int_0^{\infty} 2x \left( \int_0^{\infty} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \right) dx, \end{aligned}$$

by using  $u = x^2$  and  $dv = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$ ,

$$= \left[ -x^2 e^{-\frac{x^2}{2\sigma^2}} \right]_{x=0}^{x=\infty} + \int_0^{\infty} 2x e^{-\frac{x^2}{2\sigma^2}} dx$$

To evaluate the integral, let  $u = \frac{x^2}{2\sigma^2}$ . Then  $du = \frac{x}{\sigma^2} dx$ ,

and the limits of the integral are still  $u = 0$  and  $u = \infty$ . Thus,

substituting  $dx = \frac{\sigma^2}{x} du$  we have

$$\begin{aligned} E[X^2] &= 0 + 2\sigma^2 \left[ \int_0^{\infty} e^{-u} du \right] \\ &= -2\sigma^2 \left[ e^{-u} \right]_{u=0}^{u=\infty} \\ &= -2\sigma^2 [0 - 1] \\ &= 2\sigma^2 \end{aligned}$$

Thus,  $\sigma^2_X = \text{var}(X) = E[X^2] - \mu_X^2$

$$\begin{aligned} &= 2\sigma^2 - \left( \sigma \sqrt{\frac{\pi}{2}} \right)^2 \\ &= \sigma^2 \left( \frac{4 - \pi}{2} \right) \text{ as required.} \end{aligned}$$

You can use the `Integrate[]` function to check your work.

```
Integrate[x^2 f[x], {x, 0, ∞}, Assumptions → σ > 0]
(*compute second moment (E[X^2]*)
```

```
Out[ ]:= 2 σ^2
```

```
(*using numerical approximation to compute the second moment when σ = 1*)
```

```
In[ ]:= NIntegrate[x^2 x E^(-x^2/2), {x, 0, ∞}]
```

```
Out[ ]:= 2.
```

```
In[ ]:= Total[Table[x^2 x E^(-x^2/2) δ, {x, 0, 10, δ}]]
```

```
Out[ ]:= 2.
```

```
In[ ]:= (*variance*)
```

$$E[X^2] - \mu^2 = 2\sigma^2 - \left(\sqrt{\frac{\pi}{2}}\sigma\right)^2$$

```
Out[ ]:= 2 σ^2 - \frac{\pi \sigma^2}{2}
```

## Section 2: Simple Linear Regression Problem

Consider the simple linear regression problem with data points  $(x_1, y_1), \dots, (x_n, y_n)$ . The aim is to seek  $\beta_0$  and  $\beta_1$  to fit the line,

$$y = \beta_0 + \beta_1 x,$$

by minimizing the loss function

$$L(\beta_0, \beta_1) = \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2.$$

Here,  $\beta_0$  and  $\beta_1$  are the slope and the intercept, respectively, of the line of best fit.

Q6) Compute an expression for the gradient of  $L(\beta_0, \beta_1)$

Since the loss function has two variables, we need to use partial derivatives here.

$$\begin{aligned} \frac{\partial L(\beta_0, \beta_1)}{\partial \beta_1} &= \frac{\partial}{\partial \beta_1} \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2 \\ &= \sum_{i=1}^n \frac{\partial}{\partial \beta_1} (y_i - (\beta_0 + \beta_1 x_i))^2 \\ &= -2 \sum_{i=1}^n x_i (y_i - (\beta_0 + \beta_1 x_i)) \end{aligned}$$

$$\begin{aligned} \frac{\partial L(\beta_0, \beta_1)}{\partial \beta_0} &= \frac{\partial}{\partial \beta_0} \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i))^2 \\ &= \sum_{i=1}^n \frac{\partial}{\partial \beta_0} (y_i - (\beta_0 + \beta_1 x_i))^2 \\ &= -2 \sum_{i=1}^n (y_i - (\beta_0 + \beta_1 x_i)) \end{aligned}$$