MATH7501: Mathematics for Data Science I

Unit 10: Partial Derivatives and Gradient Descent

10.1 Partial Derivatives and Tangent Planes

Consider a surface $f(x, y)$ with a particular point $P = (x_0, y_0)$. Then the slope of f at P in the x direction is $\frac{\partial f}{\partial x}$, evaluated at P; this partial derivative is the derivative of the single-variable function $f(x, y_0)$. We can find the slope of f in the y-direction similarly.

For example, if $f(x, y) = 1 - x^2 - y^2$, and $(x_0, y_0) = (1, -1)$, then $f(x, -1) = -x^2$, so $\frac{\partial f}{\partial x}(P) = -2$. Also, $f(1, y) = -y^2$, so $\frac{\partial f}{\partial y}(P) = 2$.

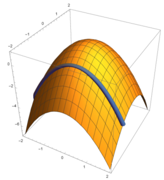


Figure 64: The surface $z = f(x, y) = 1 - x^2 - y^2$ and the intersection of the plane $y = -1$ with the surface.

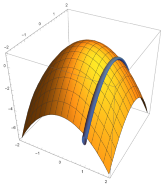


Figure 65: The surface $z = f(x, y) = 1 - x^2 - y^2$ and the intersection of the plane $x = 1$ with the surface.

Question

The volume of a solid with side lengths x, y, z is given by $V(x, y, z) = xyz$. Calculate the partial derivatives of V in all three variables.

Solution

If y, z are held and x changes, then we can see that

$$\frac{\partial V}{\partial x} = yz.$$

Similarly, $\frac{\partial V}{\partial y} = xz$, $\frac{\partial V}{\partial z} = xy$.

10.1 Partial Derivatives and Tangent Planes

For a given function $f(x, y)$, the *second order* partial derivatives are found by differentiating f once with respect to one variable, and then differentiating again with respect to a second variable. We denote this with

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad f_{yy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

Question

Let $f(x, y) = x\sin(y) + y\cos(x)$. Calcualte all of the second order partial derivatives, and show that $f_{xy} = f_{yx}$.

Recall that through linear approximations, we have

$$\Delta f \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y.$$

Therefore, if x and y are both changing with time t, and we know how much x and y are changing by, we can approximate how much f is changing over time.

Theorem (Multi-Variable Chain Rule)

If x, y are differentiable functions of t, and f is a differentiable function of x, y, then

$$\frac{df(x(t), y(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Question

Suppose the radius of a cylinder decreases at a rate of $r'(t) = -2cm/s$, and the height is kept constant at $h(t) = 2cm$. How fast is the volume decreasing when $r(t) = 1cm$?

Solution

We have $V = \pi r^2 h$ so
$V' = \pi 2rhr' + \pi r^2 h' = 2\pi rhr' = -8\pi cm^3/s$.

Question

Suppose now that the height is decreasing at $1cm/s$. What is the rate of change of volume?

Solution

We have $V = \pi r^2 h$ so $V' = \pi 2rhr' + \pi r^2 h' = -9\pi cm^3/s$.

Of course, the chain rule extends to higher dimensions as well.

Question

Suppose a, b, c are the dimensions of a box which are changing over time. From the rate of change of the volume.

Solution

The volume is $V = abc$, so

$$\begin{aligned} \frac{dV}{dt} &= V_a \frac{da}{dt} + V_b \frac{db}{dt} + V_c \frac{dc}{dt} \\ &= bc \frac{da}{dt} + ac \frac{db}{dt} + ab \frac{dc}{dt}. \end{aligned}$$

10.2 Gradient

The partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ tell us the slope of f in the x, y directions, but they can also be used to describe the slope in *any direction*.
Take a point $P \in \mathbb{R}^2$, and say we travel through P with $(x(t), y(t)) = P + \mathbf{u}t$, where $\mathbf{u} = (u_1, u_2)$ is a unit-vector. Then, using the linear approximation,

$$
\begin{aligned}
f(x(t), y(t)) &\approx f(x(0), y(0)) + \frac{\partial f}{\partial x} u_1 t + \frac{\partial f}{\partial y} u_2 t \\
&= f(P) + t\left(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)\right) \cdot \mathbf{u}.
\end{aligned}
$$

The expression $(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \cdot \mathbf{u}$ is therefore referred to as the *slope/directional derivative of f at P in the direction of* u. If $\mathbf{u} \neq 0$ is not a unit vector, then the directional derivative is simply $(\frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P)) \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$.

Question

Find the directional derivative of $f(x, y) = 4 - x^2 - 4y^2$ at the point $(1, 1)$ in the $(1, 1)$ direction.

Solution

The corresponding unit vector is $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. We compute that $\frac{\partial f}{\partial x} = -2x$ and $\frac{\partial f}{\partial y} = -8y$. Therefore the slope at $(1, 1)$ in the $\mathbf{u}$ direction is

$$(-2, -8) \cdot (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = -\frac{10}{\sqrt{2}} = -5\sqrt{2}.$$

Question

If $f(x, y) = x^2 - 3y^2 + 6y$, find the slope at $(1, 0)$ in the direction of $\mathbf{i} - 4\mathbf{j}$.

Solution

The norm of $\mathbf{u} = \mathbf{i} - 4\mathbf{j}$ is $||\mathbf{u}|| = \sqrt{17}$. Also, note that $\frac{\partial f}{\partial x} = 2x$ and $\frac{\partial f}{\partial y} = 6 - 6y$. Therefore

$$f_{\mathbf{u}}(1, 0) = (2, 6) \cdot \frac{(1, -4)}{\sqrt{17}} = -\frac{22}{\sqrt{17}}.$$

10.2 Gradient

It is useful to give the vector (f_x, f_y) a name, since it comes up a lot when computing directional derivatives.

Definition (Gradient Vector)

We define $\nabla f = (f_x, f_y)$ to be the *gradient* of f, (or $\nabla f = (f_x, f_y, f_z)$ for functions of three variables). Then the derivative of f in the direction of $\mathbf{u}$ is $\nabla f \cdot \frac{\mathbf{u}}{||\mathbf{u}||}$

(Question)

Find the gradient of $f(x, y) = x^2 - 3(y-1)^2 + 3$.

Solution

$\nabla f = (2x, -6(y-1)) = 2x\mathbf{i} - 6(y-1)\mathbf{j}$.

Question

Find the directional derivative of $g(x, y) = e^{x^2}\cos(y)$ at $(1, \pi)$ in the direction of $-3\mathbf{i} + 4\mathbf{j}$.

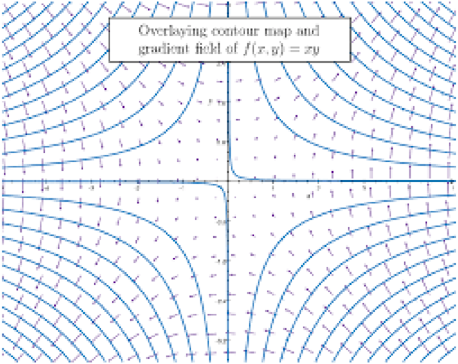
Solution

The norm of $\mathbf{u} = -3\mathbf{i} + 4\mathbf{j}$ is $||\mathbf{u}|| = 5$, and $\nabla g = 2xe^{x^2}\cos(y)\mathbf{i} - \sin(y)e^{x^2}\mathbf{j}$. Therefore

$$\begin{aligned} g_{\mathbf{u}} &= \nabla g(1, \pi) \cdot \frac{\mathbf{u}}{||\mathbf{u}||} \\ &= (2e\cos(\pi), -\sin(\pi)e) \cdot (\frac{-3}{5}, \frac{4}{5}) = \frac{6e}{5}. \end{aligned}$$

10.2 Gradient

In general, the gradient $\nabla f(a,b)$ points in the direction perpendicular to the contour (line of constant f) passing through (a,b). In fact, $\nabla f(a,b)$ points in the direction of the shaprest increase of f.



https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/partial-derivative-and-gradient-articles/a/the-gradient

10.2 Gradient

Question

Consider the function $f(x, y) = mx + ny + c$ for some $m, n, c \in \mathbb{R}$ and $n \neq 0$. Find the gradient of f and compare to the contours.

Solution

The contours are $y = -\frac{mx}{n} + C$. The gradient vector is $\nabla f = (m, n)$, which points in the direction perpendicular to the contours.

Question

Consider the function $f(x, y) = x^2 + y^2$. Describe the gradient and the contours.

Solution

The contours are the concentric circles centered at the origin. The gradient is $(2x, 2y)$, which is radial, always points perpendicular to the circles.

10.2 Gradient

Question

The temperature of a metal plate if $T(x, y) = 20 - 4x^2 - y^2$. Find the gradient at the point $(2, -3)$. Hence describe the direction of maximal increase in temepraturе, and the direction of constant temperature.

Solution

The gradient is $\nabla T = (-8x, -2y)$, which is $(-16, 6)$ at the specified point. To find the angle this makes with the x-axis, write $-16 = r\cos(\theta)$ and $6 = r\sin(\theta)$, so that $r = 2\sqrt{73}$ and $\theta = \pi - \arctan(\frac{6}{16})$.
The direction of constant temperature is then $\frac{\pi}{2} - \arctan(\frac{6}{16})$.

Question

A team is mapping the ocean floor. They produce the model $D(x,y) = 250 - 30x^2 - 50\sin(\frac{\pi y}{2})$ in m, with $(x,y) \in [-2,2] \times [-2,2]$ in km. There is a leaking oil well at $(1, 0.5)$; find the depth at this location, as well as the direction and magnitude of steepest increase here.

Solution

The depth at $(1, 0.5)$ is $D(1, 0.5) = 250 - 30 - 50\sin(\frac{\pi}{4}) \approx 184.6m$. The gradient at this point is

$$\nabla D(1, 0.5) = (-60, -\frac{50\pi}{2}\cos(\frac{\pi}{4})) = (-60, -\frac{25\pi}{\sqrt{2}}),$$

which is measured in m/km.
This makes an angle of $\arctan(\frac{25\pi}{60\sqrt{2}})$ with the positive x-axis, and has a magnitude of $\sqrt{3600 + 625\pi^2}$