MATH7501: Mathematics for Data Science I

Final Lecture: The Normal Distribution

The Normal Distribution

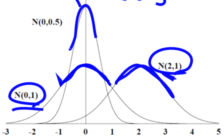
Definition (Gaussian Distribution)

We say that X has a *Gaussian/normal distribution with parameters* μ *and* σ^2 on $\mathbb{R}$ (denoted $X \sim \mathrm{N}(\mu, \sigma^2)$) if its pdf is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.

$\sigma^2 > 0$.

X is a random variable.

$\mu = 0 = E(X)$, $\sigma^2 = 0.5$



Properties of the Gaussian Distribution

If $X \sim \mathrm{N}(\mu, \sigma^2)$ then

- $\mathbb{E}[X] = \mu$.
- $\mathrm{Var}[X] = \sigma^2$.

$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \mu$

$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$.

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 1 of 3 – Changing Variables)

Let $u = \frac{x-\mu}{\sigma}$, so that $\frac{du}{dx} = \frac{1}{\sigma}$.

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 1 of 3 – Changing Variables)

Let $u = \frac{x-\mu}{\sigma}$, so that $\frac{du}{dx} = \frac{1}{\sigma}$. Then

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u)^2} du,$$

$dx = du\,\sigma$

μ, σ ?

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 1 of 3 – Changing Variables)

Let $u = \frac{x-\mu}{\sigma}$, so that $\frac{du}{dx} = \frac{1}{\sigma}$. Then

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u)^2} du,$$

but $g(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u)^2}$ is precisely the pdf of the normal distribution with $\mu = 0$ and $\sigma = 1$.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1.$$

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 1 of 3 – Changing Variables)

Let $u = \frac{x-\mu}{\sigma}$, so that $\frac{du}{dx} = \frac{1}{\sigma}$. Then

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u)^2} du,$$

but $g(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u)^2}$ is precisely the pdf of the normal distribution with $\mu = 0$ and $\sigma = 1$. It therefore suffices to prove the result in this case.

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 2 of 3 – Introducing a Volume Integral)

Recall we are trying to show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$.

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 2 of 3 – Introducing a Volume Integral)

Recall we are trying to show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = 1.$ This is equivalent to showing that $\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)^2 = 2\pi.$

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 2 of 3 – Introducing a Volume Integral)

Recall we are trying to show that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$. This is equivalent to showing that $\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)^2 = 2\pi$. But $\in \mathbb{R}$.

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) \\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dydx. \quad = 2\pi
\end{aligned}$$

want to show

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 3 of 3 – Evaluating Volume Integral via Polar)

Using polar co-ordinates, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dydx = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\,\mathbf{r}drd\theta.$$

θ r

$r^2 = x^2 + y^2$

$dydx = rdrd\theta$

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 3 of 3 – Evaluating Volume Integral via Polar)

Using polar co-ordinates, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dydx = \int_0^{2\pi}\left(\int_0^{\infty} e^{-\frac{r^2}{2}}\,\mathbf{r}\,dr\right)d\theta.$$

Now using integration by substitution, we have $\int_0^\infty r\,e^{-\frac{r^2}{2}}\,dr = 1$.

$$u = r^2, \quad \frac{du}{dr} = 2r \rightarrow \int_0^\infty e^{-\frac{u}{2}}\frac{du}{2}$$

The Normal Distribution

Important Observation

The function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is actually a pdf, i.e., it is continuous, positive, and integrates to 1.

Proof (Part 3 of 3 – Evaluating Volume Integral via Polar)

Using polar co-ordinates, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dydx = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\,\mathbf{r}\,drd\theta.$$

Now using integration by substitution, we have $\int_0^{\infty} e^{-\frac{r^2}{2}}\,dr = 1$.

Therefore $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dydx = \int_0^{2\pi}\left(\int_0^{\infty} e^{-\frac{r^2}{2}}\,\mathbf{r}\,dr\right)d\theta = 2\pi$, as required.

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}.$$

The Normal Distribution

Question

Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that $\mathbb{E}[X] = \mu$.

Solution

Since the pdf is $\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$,

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x+\mu}{\sigma}\right)^2} dx$$

want to show $= \mu$.

The Normal Distribution

Question

Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that $\mathbb{E}[X] = \mu$.

Solution

Since the pdf is $\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2}$, we have

definition.

substitute $y = x - \mu$, $dy = dx$

linearity of integral

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} \frac{x}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \int_{-\infty}^{\infty} \frac{y+\mu}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y}{\sigma}\right)^2} dy \\
&= \int_{-\infty}^{\infty} \frac{\mu}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y}{\sigma}\right)^2} dy + \int_{-\infty}^{\infty} \frac{y}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y}{\sigma}\right)^2} dy \\
&= \mu + \int_{-\infty}^{0} \frac{y}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y}{\sigma}\right)^2} dy + \int_{0}^{\infty} \frac{y}{\sigma\sqrt{2\pi}} e^{-\left(\frac{y}{\sigma}\right)^2} dy \\
&= \mu
\end{aligned}
$$

is a pdf

$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_{0}^{\infty}$

$y = -z$

The Normal Distribution

Question

Suppose that $X \sim N(\mu, \sigma^2)$. Show that $Var[X] = \sigma^2$.

Solution (Sketch)

We have to show that $\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$.

$$Var(X) = \int_{-\infty}^{\infty} f(x)(x-\mu)^2 \, dx$$

The Normal Distribution

Question

Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that $Var[X] = \sigma^2$.

Solution (Sketch)

We have to show that $\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$. After substituting $u = \frac{x-\mu}{2\sigma}$, we get

$$\frac{du}{dx} = \frac{1}{2\sigma}$$

$$\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du.$$

? ?

The Normal Distribution

Question

Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that $Var[X] = \sigma^2$.

Solution (Sketch)

We have to show that $\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$. After substituting $u = \frac{x-\mu}{2\sigma}$, we get

$$\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du.$$

Using integration by parts, we obtain the answer.

what is the anti-deri

erf – error function.

The Normal Distribution

Question

Suppose that $X \sim \mathrm{N}(\mu, \sigma^2)$. Show that $Var[X] = \sigma^2$.

Solution (Sketch)

We have to show that $\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sigma^2$. After substituting $u = \frac{x-\mu}{2\sigma}$, we get

$$\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{\sigma^2}{2\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 e^{-u^2} du.$$

Using integration by parts, we obtain the answer.

The Normal Distribution

The Gaussian distribution comes up in physics.

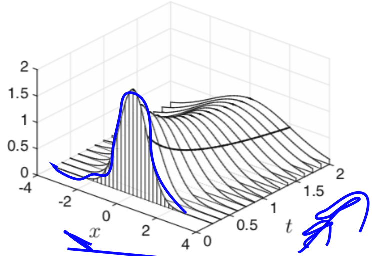
The Normal Distribution

The Gaussian distribution comes up in physics. Indeed, if we think of time t as the same as σ^2 and set $\mu = 0$, then we get a time evolving function that describes how heat spreads out.

$\sigma^2 = t$ time,

The Normal Distribution

The Gaussian distribution comes up in physics. Indeed, if we think of time t as the same as σ^2 and set $\mu = 0$, then we get a time evolving function that describes how heat spreads out.

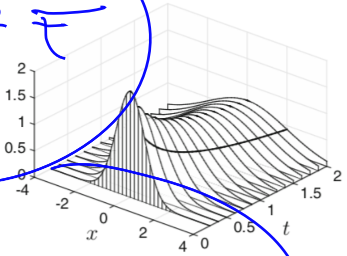


https://math.stackexchange.com/questions/2871506/convexity-and-concavity-in-a-textbook-figure-of-the-fundamental-solution-of-the

The Normal Distribution

The Gaussian distribution comes up in physics. Indeed, if we think of time t as the same as σ^2 and set $\mu = 0$, then we get a time evolving function that describes how heat spreads out.

$f(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{x^2}{t}}$



https://math.stackexchange.com/questions/2871506/convexity-and-concavity-in-a-textbook-figure-of-the-fundamental-solution-of-the

This function $f(t,x)$ satisfies the so-called *heat equation*:

partial differential equation.

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.}$$

The Normal Distribution

Example (Brownian Motion)

Let P be a particle travelling according to Brownian motion in $\mathbb{R}$, with "speed" equal to 1.

The Normal Distribution

Example (Brownian Motion)

Let P be a particle travelling according to Brownian motion in $\mathbb{R}$, with "speed" equal to 1. Suppose the particle starts at position $\mu \in \mathbb{R}$.

$$\mu = \mathbb{E}[X].$$

The Normal Distribution

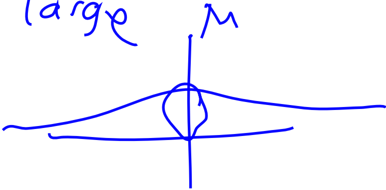
Example (Brownian Motion)

Let P be a particle travelling according to Brownian motion in $\mathbb{R}$, with "speed" equal to 1. Suppose the particle starts at position $\mu \in \mathbb{R}$.

Then the random variable X_t which describes the location of the particle P after time t is distributed as $X_t \sim \mathrm{N}(\mu, t)$.

t small

t large

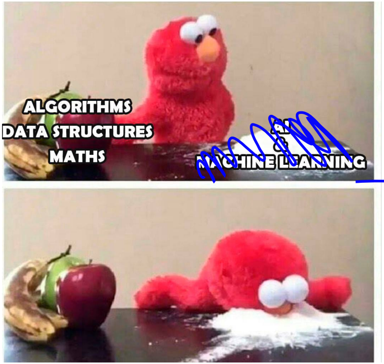


Conclusion

Thankyou! I hope this course has prepared you for your future in data science!

Conclusion

Thankyou! I hope this course has prepared you for your future in data science!



AI & ML without mathematics