MATH7501 Practical 12 (Week 13), Semester 1-2021

Topic: Revision

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Pre-Tutorial Activity

- Students must have familiarised themselves with units 1and 10 contents of the reading materials for MATH7501

Resources

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Q 1: Taylor Series Approximation

a) Find Taylor series approximation for $\ln\left(1+x^2\right)$ about $x = 0$

As the expansion is around 0, this approximation is given by the Maclaurine series :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n.$$

First consider the Maclaurine series approximation for $g(x) = \ln(1+x)$. The first few derivatives of $g(x)$ at $x = 0$ are given below.

$$g'(0) = \frac{1}{1+x}|_{x=0} = 1$$

$$g''(0) = \frac{-1}{(1+x)^2}|_{x=0} = -1$$

$$g^{(3)}(0) = \frac{2}{(1+x)^3}|_{x=0} = 2$$

$$g^{(4)}(0) = \frac{-6}{(1+x)^4}|_{x=0} = -6$$

$$g^{(5)}(0) = \frac{42}{(1+x)^5}|_{x=0} = 24$$

$$g^{(6)}(0) = \frac{-120}{(1+x)^n}|_{x=0} = -120$$

....

$$g^{(n)}(0) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^6}|_{x=0} = (-1)^{n+1}(n-1)!$$

Thus

$$g(x) = \ln(1+x) \simeq \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Replacing x with x^2 in the above we get

$$f(x) = \ln\left(1+x^2\right) \simeq \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(x^2\right)^n$$

$$= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4},$$

up to order 8. You can use the Series[] function to check this.

```
(* this computes Taylor series expansion of f(x) at x=0 upto order 8*)
Series[Log[1 + x^2], {x, 0, 8}]
```

Out[4]= $x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + O[x]^9$

b) Find the values of x for which the series converges.

Let $a_n = \frac{(-1)^{n+1}}{n} x^{2n}$, then by the Ratio rest,

the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \left(x^2\right)^n$ converges if $\text{limit}_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. Now

$$\text{limit}_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \text{limit}_{n\to\infty} \left| \frac{(-1)^{(n+1)+1}}{n+1} x^{2(n+1)} \times \frac{(-1)^{n+1}}{n} x^{2n} \right|$$

$$= \text{limit}_{n\to\infty} \left| x^2 \frac{n}{n+1} \right|$$

$$= \left| x^2 \right|$$

Thus for the above series to converge, it requires that

$\left| x^2 \right| < 1$

$\sqrt{x} < \sqrt{1}$

$| x | < 1.$

Therefore, the series converges for $| x | < 1$.

c) Find Taylor series approximation for $\ln\left(1+x^2\right)$ about x = a

```
(* this computes Taylor series expansion of f(x) at x=a upto order 4*)
```

In[6]:= `Series[Log[1 + x^2], {x, a, 4}]`

Out[6]= $$\text{Log}\left[1+a^2\right] + \frac{2a(x-a)}{1+a^2} + \frac{\left(1-a^2\right)(x-a)^2}{\left(1+a^2\right)^2} + \frac{2a\left(-3+a^2\right)(x-a)^3}{3\left(1+a^2\right)^3} + \frac{\left(-1+6a^2-a^4\right)(x-a)^4}{2\left(1+a^2\right)^4} + O[x-a]^5$$

```
(*Eg. if the expansion is about x=1, then we have*)
```

In[7]:= `Series[Log[1 + x^2], {x, 1, 4}]`

Out[7]= $\text{Log}[2] + (x-1) - \frac{1}{6}(x-1)^3 + \frac{1}{8}(x-1)^4 + O[x-1]^5$

Q 2: Gradient Computation and Taylor Series Approximation

Consider $f(\mathbf{x}) = x_1 + e^{(x_2 - x_1)}$.

a) Find the gradient of $f(\mathbf{x})$: $\nabla f(\mathbf{x})$

$$\nabla f(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} f(\mathbf{x}),\ \frac{\partial}{\partial x_2} f(\mathbf{x})\right)^T$$
$$= \left(1 - e^{(x_2 - x_1)},\ e^{(x_2 - x_1)}\right)^T$$

b) Use $\nabla f(\mathbf{x})$ to find a Taylor series approximation of $f(x)$ around $x = (1, 2)$ up to order 1

$$f(\mathbf{x}) \simeq f(1,2) + \nabla f(1,2)\,(\mathbf{x} - (1,2))^T$$
$$= 3.7183 + (-1.7183,\ 2.7183)\,(x_1 - 1,\ x_2 - 2)^T$$
$$= 3.7183 - 1.783\,(x_1 - 1) + 2.7183\,(x_2 - 2)$$

Q 3. Determinant of a 3 × 3 Matrix

Find the determinant of $A = \begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$

Using cofactors and expanding along the first row, gives:

$$\det(A) = |A|$$
$$= (-1)^{1+1}\,7 \begin{vmatrix} 3 & -1 \\ 4 & -2 \end{vmatrix} + (-1)^{1+2}\,2 \begin{vmatrix} 0 & -1 \\ -3 & -2 \end{vmatrix} + (-1)^{1+3}\,1 \begin{vmatrix} 0 & 3 \\ -3 & 4 \end{vmatrix}$$
$$= 7\,(-6 - (-4)) = 2\,(0 - (3)) + 1\,(0 - (-9))$$
$$= 7\,(-2) - 2\,(-3) + 1\,(9)$$
$$= -14 + 6 + 9$$
$$= 1$$

Q 4. Integration

Find $\int e^x \sin(x)\, dx$

Need to use integration bi - parts : $\int u\, dv = uv - \int v\, du$. Assuming $u = e^x$ and $dv = \sin(x)$ we get :

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

Applying bi - parts for the integral on the right - hand side with u = e^x and dv = cos (x) , we get

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \left\{e^x \sin(x) - \int e^x \sin(x)\,dx\right\}$$

$$2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x)$$

$$\int e^x \sin(x)\,dx = \frac{1}{2} e^x [\sin(x) - \cos(x)]$$

Q 5. The Normal Distribution and Moments of Truncated Normal Distribution

Some notations for standard normal random variable

Note that if a random variable X has the standard normal distribution then its mean is 0 and variance is 1. This is written as X as : $X \sim \text{Normal}(0, 1)$.

- the **pdf** of X~Normal(0,1) is denoted as: $\phi(x) = 1/\left(\sqrt{2\,\pi}\right) e^{-\frac{x^2}{2}}$
- the **cdf** of X~Normal(0,1) is denoted as: $\Phi(x) = \int_{-\infty}^{x} 1/\left(\sqrt{2\,\pi}\right) e^{-\frac{t^2}{2}} dx$

Cdf and pdf of $X \sim \text{Normal}(\mu, \sigma^2)$ in terms of standard normal random variable

- the **pdf** of X~Normal(μ, σ^2) is given as: $f_X = 1/\left(\sigma\sqrt{2\,\pi}\right) e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. This can be rearranged to give $f_X(x) = \frac{1}{\sigma}\left(1/\left(\sqrt{2\,\pi}\right) e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$, where $\phi(\cdot)$ is the pdf of X~Normal(0,1)
- the **pdf** of X~Normal(μ, σ^2) is given as: $F_X(x) = \int_{-\infty}^{x} 1/\left(\sigma\sqrt{2\,\pi}\right) e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$. This can be rearranged to give $F_X(x) = \frac{1}{\sigma}\int_{-\infty}^{x} 1/\left(\sqrt{2\,\pi}\right) e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt = \frac{1}{\sigma}\Phi\left(\frac{x-\mu}{\sigma}\right)$, $\Phi(\cdot)$ is the cdf of X~Normal(0,1)

Truncated Normal Distribution

Let X be a normal random variable with mean μ and variance σ^2 and lies within the interval (a, b) such that $-\infty \le a < b \le \infty$. Then X , conditional on $a < X < b$, has a **truncated normal distribution.** It ' s pdf is given by

$$f_X(x) = \begin{cases} \frac{1}{\sigma}\,\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}, & a \le x \le b,\ \sigma > 0 \\ 0, & \text{Otherwise} \end{cases}$$

Here, $\phi(x) = \frac{1}{\left(\sqrt{2\,\pi}\right)} e^{-\frac{x^2}{2}}$, is the pdf of standard normal random distribution and

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\left(\sqrt{2\,\pi}\right)} e^{-\frac{t^2}{2}} dt,$$

is the cdf of standard normal distribution $(X \sim \text{Normal}(0, 1))$

Consider now the k - th moment of the truncated normal distribution

$m_k := E[X^k]$, for $a \le X \le b$ and a,
$b \in [-\infty, \infty]$, (this implies X is truncated normal)
where $E[X^k]$ denotes the expected value of X^k (the k - th moment of X).

(a) Show that m_k can be given by the recursive formula :

$$m_k = (k-1)\,\sigma^2\, m_{k-2} + \mu\, m_{k-1} - \sigma\,\frac{b^{k-1}\,\phi\left(\frac{b-\mu}{\sigma}\right) - a^{k-1}\,\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}, \text{ for } k = 1, 2, \ldots.$$

with $m_{-1} = 0$ and $m_0 = 1$,
where $\phi(\cdot)$ is the standard normal pdf and $\Phi(\cdot)$ is the standard normal cdf.

ANS : Let $n_0 := \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$. Then

$$m_k\, n_0 = \int_a^b x^k\,\frac{1}{\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right)\,dx$$

$$= \int_a^b x^k\,\frac{1}{\sigma\sqrt{(2\pi)}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_a^b x^{k-1}\left(\frac{x\sigma}{\sigma}\right)\frac{1}{\sigma\sqrt{(2\pi)}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_a^b \frac{\sigma}{\sqrt{(2\pi)}}\,x^{k-1}\left(\frac{x}{\sigma^2}\right)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_a^b \frac{\sigma}{\sqrt{(2\pi)}}\,x^{k-1}\left(\frac{x-\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\right)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \sigma\int_a^b \frac{(x-\mu)}{\sigma^2\sqrt{(2\pi)}}\,x^{k-1}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \int_a^b \frac{\mu}{\sigma\sqrt{(2\pi)}}\,x^{k-1}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \sigma\left[\frac{-x^{k-1}}{\sqrt{(2\pi)}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{x=a}^{x=b} + \int_a^b (k-1)\,\frac{x^{k-2}}{\sqrt{(2\pi)}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \int_a^b \frac{\mu}{\sigma\sqrt{(2\pi)}}\,x^{k-1}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx,$$

using bi - parts

$$= \sigma\left[\frac{-b^{k-1}}{\sqrt{(2\pi)}}\,e^{-\frac{(b-\mu)^2}{2\sigma^2}} + \frac{a^{k-1}}{\sqrt{(2\pi)}}\,e^{-\frac{(a-\mu)^2}{2\sigma^2}}\right] +$$

$$(k-1)\,\sigma^2\int_a^b \frac{x^{k-2}}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx + \mu\int_a^b \frac{1}{\sigma\sqrt{(2\pi)}}\,x^{k-1}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \sigma\left[-b^{k-1}\,\phi\left(\frac{b-\mu}{\sigma}\right) + a^{k-1}\,\phi\left(\frac{a-\mu}{\sigma}\right)\right] + (k-1)\,\sigma^2\int_a^b \frac{x^{k-2}}{\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right)dx +$$

$$\mu\int_a^b \frac{1}{\sigma}\,x^{k-1}\,\phi\left(\frac{x-\mu}{\sigma}\right)dx,$$ using the pdf of standard normal distribution.

Thus

$$m_k = \frac{\sigma\left[-b^{k-1}\,\phi\left(\frac{b-\mu}{\sigma}\right) + a^{k-1}\,\phi\left(\frac{a-\mu}{\sigma}\right)\right]}{n_0} +$$

$$(k-1)\,\sigma^2\int_a^b \frac{x^{k-2}}{n_0\,\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right)dx + \mu\int_a^b \frac{1}{n_0\,\sigma}\,x^{k-1}\,\phi\left(\frac{x-\mu}{\sigma}\right)dx.$$

Observing that the integrals in the second and third terms are lower order moments and reorganising, we obtain the result.

b) Find a recursive formula for the k - th moment of $X \sim \text{Normal}(\mu, \sigma^2)$ where X lies in the interval $(-\infty, \infty)$.

$$m_k = E[X^k] = \int_{-\infty}^{\infty} x^k \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \quad ,$$

$$= \int_{-\infty}^{\infty} x^{k-1} (x-\mu+\mu) \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} x^{k-1} (x-\mu) \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu\, x^{k-1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \left[\frac{-\sigma x^{k-1}}{\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{x=-\infty}^{x=\infty} + \int_{-\infty}^{\infty} (k-1)\, x^{k-2} \frac{\sigma}{\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx +$$

$$\int_{-\infty}^{\infty} \mu\, x^{k-1} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx, \text{ by applying integration bi - parts}$$

$$= 0 + (k-1)\,\sigma^2 \int_{-\infty}^{\infty} x^{k-2} \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu\, m_{k-1}$$

$$= (k-1)\,\sigma^2 m_{k-2} + \mu\, m_{k-1} \;,\; k > 2 .$$