

MATH7501: Mathematics for Data Science I

Unit 7: Derivatives, Optimisation and basic ODEs

Slides by Timothy Buttsworth (2021)

Motivation: The Need for Derivatives

Derivatives are “instantaneous rates of change.”

Fundamentally, these are important because everything in life is changing, and we need rigorous ways of describing and quantifying these changes.

Motivation: The Need for Derivatives

There is often a practical need for derivatives. For example, when we are driving, our speedometer gives us the instantaneous rate of change of our location.

Why does the speedometer give us this information instead of, for example, our average speed over the last minute?

Possible answer: the instantaneous car speed at the precise moment of a crash is much more significant than the average speed over the last minute.

7.1 Derivatives

The idea of 'instantaneous rate of change'

The rate of change of a function f at a point a is the slope of a line that lies tangential to the graph of f .

Such a tangent line can be approximated with a straight line joining $(a, f(a))$ to $(b, f(b))$, where b is close to a . Using the 'rise over run' formula, the slope of this line is

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a + h) - f(a)}{h},$$

where $b = a + h$. This slope should be a good approximation to the actual tangent line by making b close to a (equivalently, having h close to 0).

7.1 Derivatives

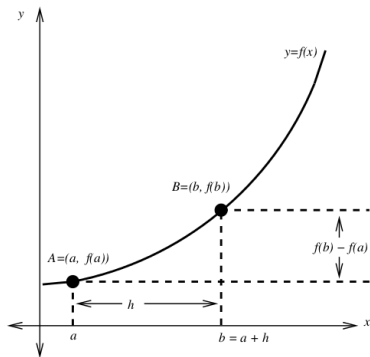


Figure 47: We want to determine the tangent line at the point A .

7.1 Derivatives

Definition (Derivative at a Point)

The *derivative of a function f at a point a* is defined to be

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h},$$

provided this limit exists. If this limit exists, we say that f is *differentiable at the point a* . Otherwise, we say that f is *not differentiable at the point a* .

Definition (Differentiability)

If f is differentiable at every point on an open interval (a, b) , then f is said to be *differentiable on (a, b)* . In this case, we quite often write $\frac{df}{dx}$ to denote the function which assigns to each x the derivative at that point.

7.1 Derivatives

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = x^2 + x$.

Solution

For each x , we have

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 + (x+h) - (x^2 + x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + h}{h} \\ &= 2x + 1.\end{aligned}$$

Therefore, f is differentiable on \mathbb{R} and $f'(x) = 2x + 1$.

7.1 Derivatives

Question

Using the definition of derivative, calculate $f'(x)$, where $f(x) = e^x$.

Solution

For each x , we have

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h}.\end{aligned}$$

Recall that $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$, so $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = e^x$.
Therefore, f is differentiable on \mathbb{R} and $f'(x) = e^x$.

7.2 Working with Derivatives

Mathematica can compute derivatives:

```
In[7]:= D[x^2 + x^3, x]
```

```
Out[7]= 2 x + 3 x^2
```

```
In[8]:= f[x_] = x^2 + x^3  
D[f[x], x]
```

```
Out[8]= x^2 + x^3
```

```
Out[9]= 2 x + 3 x^2
```

```
In[10]:= f'[x]
```

```
Out[10]= 2 x + 3 x^2
```

7.2 Working with Derivatives

Some Useful Derivatives

- The derivative of a constant is 0 (because it does not change);
- $\frac{d(x)}{dx} = 1$, i.e., the function $f(x) = x$ has a slope of 1 everywhere;
- $\frac{d(x^\alpha)}{dx} = \alpha x^{\alpha-1}$ whenever these expressions are well-defined (be careful with expressions like $(-1)^{\frac{1}{2}}$);
- $\frac{d(\sin(x))}{dx} = \cos(x)$;
- $\frac{d(\cos(x))}{dx} = -\sin(x)$.

Higher Derivatives

If f' is the derivative of f , then f'' is the derivative of f' . Repeating this gives the n th order derivative of f , denoted $f^{(n)}(x)$ or $\frac{d^n f}{dx^n}$. For example, $\frac{d^2 \sin(x)}{dx^2} = -\sin(x)$. If $f(t)$ is the displacement of an object after time t , then $f'(t)$ is the object's velocity, and $f''(t)$ is its acceleration.

7.2 Working with Derivatives

Mathematica can compute higher derivatives:

```
In[13]:= D[x^3, {x, 2}]
```

```
Out[13]= 6 x
```

```
In[14]:= D[x^3, {x, n}]
```

```
Out[14]= x3-n FactorialPower[3, n]
```

```
In[15]:= D[x^k, {x, n}]
```

```
Out[15]= xk-n FactorialPower[k, n]
```

```
In[16]:= f[x_] = x^k
```

```
D[f[x], {x, n}]
```

```
Out[16]= xk
```

```
Out[17]= xk-n FactorialPower[k, n]
```

7.2 Working with Derivatives

Theorem (Rules for Differentiation)

If f and g are differentiable functions and c is a constant, then:

- $(cf)' = cf'$;
- $(f \pm g)' = f' \pm g'$;
- $(fg)' = f'g + fg'$ (this is called the product rule, or the Leibniz rule);
- $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$, if $g \neq 0$ (this is called the quotient rule).

In Leibniz notation, these rules are:

- $\frac{d(cf)}{dx} = c \frac{df}{dx}$;
- $\frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$;
- $\frac{d(fg)}{dx} = \frac{df}{dx}g + f \frac{dg}{dx}$;
- $\frac{d(f/g)}{dx} = \frac{\frac{df}{dx}g - \frac{dg}{dx}f}{g^2}$.

7.2 Working with Derivatives

Question (Derivative of the tangent function)

Consider the function $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$. Calculate $f'(x)$, whenever it exists.

Solution

If $x = (n + \frac{1}{2})\pi$ for some integer n , then $\cos(x) = 0$ and the function is not defined. Otherwise, we can use the quotient rule:

$$\begin{aligned}(\tan(x))' &= \frac{(\sin(x))' \cos(x) - (\cos(x))' \sin(x)}{\cos(x)^2} \\&= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} \\&= \frac{1}{\cos(x)^2} \\&= \sec(x)^2.\end{aligned}$$

7.2 Working with Derivatives

Theorem (The Chain Rule)

Suppose g and h are differentiable functions. Then the function $f = g \circ h$ (i.e., $f(x) = g(h(x))$) is differentiable, and

$$f'(x) = g'(h(x))h'(x).$$

In Leibniz notation, we write $u = h(x)$ and $y = g(u)$ so that

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

7.2 Working with Derivatives

Question

Let $y = \sqrt{\sin(x)}$. Calculate $\frac{dy}{dx}$.

Solution

Let $u = \sin(x)$ so that $y = \sqrt{u}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2\sqrt{u}} \frac{du}{dx} \\ &= \frac{\cos(x)}{2\sqrt{\sin(x)}},\end{aligned}$$

as long as $\sin(x) > 0$.

7.2 Working with Derivatives

Inverse Function Derivatives

Suppose f is a function, and $y = f^{-1}(x)$ is the inverse function. Then, for all x on an appropriate domain,

$$x = f(f^{-1}(x)).$$

Then by differentiating, we obtain

$$\begin{aligned} 1 &= \frac{df(y)}{dx} \\ &= \frac{df}{dy} \frac{dy}{dx}. \end{aligned}$$

This implies that $\frac{dy}{dx} = \frac{1}{\frac{df}{dy}}$. In Newton's notation this is

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

7.2 Working with Derivatives

Question (Derivative of Log)

Let $g(x) = \ln(x)$, with domain $(0, \infty)$ and range \mathbb{R} . Find $g'(x)$.

Solution

Let $f(x) = e^x$, so that $g(x) = f^{-1}(x)$. Then

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{e^{\ln(x)}} \\ &= \frac{1}{x}, \end{aligned}$$

whenever $x > 0$.

7.2 Working with Derivatives

Question (Derivative of arcsin)

Let $g(x) = \arcsin(x)$, with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Find $g'(x)$.

Solution

Let $f(x) = \sin(x)$ so that $g(x) = f^{-1}(x)$. Then

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\cos(\arcsin(x))} \\ &= \frac{1}{\sqrt{1 - \sin(\arcsin(x))^2}} \\ &= \frac{1}{\sqrt{1 - x^2}}. \end{aligned}$$

Similar techniques show that

$$(\arccos)'(x) = -\frac{1}{\sqrt{1 - x^2}}, \quad (\arctan)'(x) = \frac{1}{1 + x^2}.$$

7.2 Working with Derivatives

Theorem (L'Hôpital's Rule)

Suppose that f and g are defined and differentiable, and $g'(x) \neq 0$ near a (except possibly at a itself). Suppose that

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0,$$

or

$$\lim_{x \rightarrow a} f(x) = \pm\infty, \quad \lim_{x \rightarrow a} g(x) = \pm\infty.$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

if the right hand side exists.

7.2 Working with Derivatives

Question

Evaluate $\lim_{x \rightarrow 1} \frac{\ln(x)}{x-1}$.

Solution

Note that $\lim_{x \rightarrow 1} \ln(x) = \lim_{x \rightarrow 1} (x - 1) = 0$, so the quotient rule for limits is not applicable. However, l'Hôpital's rule implies that

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{x-1} = \lim_{x \rightarrow 1} \frac{1/x}{1},$$

provided the right hand limit exists. But this limit does exist, and is equal to 1.

7.2 Working with Derivatives

Question

Evaluate $\lim_{x \rightarrow 0^+} x \ln(x)$.

Solution

Write $x \ln(x) = \frac{\ln(x)}{1/x}$. Then by L'Hôpital's rule,

$$\lim_{x \rightarrow 0^+} x \ln(x) = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2},$$

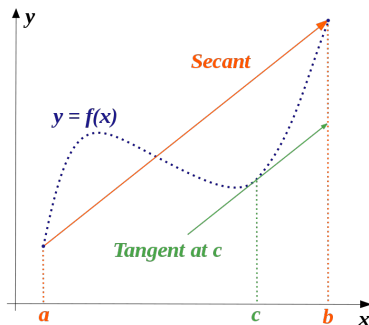
provided this right-hand-side limit exists. This limit does exist, and is equal to 0.

7.2 Working with Derivatives

Theorem (The Mean Value Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and is differentiable on (a, b) . Then there exists a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



7.2 Working with Derivatives

Question

Let $f(x) = x^4 + 4x + 1$. Show that $f(x) = 0$ has exactly two real solutions, without finding the solutions.

Solution

- Step One: Show there are at least two solutions.
 - Note that $f(-2) = 9$, $f(-1) = -2$ and $f(1) = 6$.
 - Since f is continuous, the IVT implies that f has a root in $(-2, -1)$, and has another root in $(-1, 1)$.
- Step Two: Show there are no more than two solutions.
 - We compute $f'(x) = 4x^3 + 4$. Therefore, $f'(x) < 0$ if $x \in (-\infty, -1)$ and $f'(x) > 0$ if $x \in (-1, \infty)$.
 - If a and b were two distinct roots for f in $(-\infty, -1]$, the MVT would imply the existence of $c \in (-\infty, -1)$ with $f'(c) = 0$, which is a contradiction. Thus f can have no more than one root in $(-\infty, -1]$.
 - Similarly, there is at most one root of f on $[-1, \infty)$, so f can have no more than two real roots.

7.3 Smoothness

Theorem (Differentiability implies Continuity)

If f is differentiable at a , then f is continuous at a .

To see this, note that differentiability implies that the limit

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. But then

$$\lim_{x \rightarrow a} (f(x) - f(a)) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) = f'(a) \cdot 0 = 0,$$

so f is continuous at a .

7.3 Smoothness

Question

Show that the function $f(x) = |x|$ is continuous, but not differentiable at $x = 0$.

Solution

Note that

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

It is clear that f is continuous at $x = 0$. To see that f is not differentiable at $x = 0$, note that

$$\lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1, \quad \text{while} \quad \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1.$$

These one-sided limits do not agree, so the overall limit, i.e., the derivative, does not exist.

7.3 Smoothness

We can also find multi-variable functions that do not have tangent planes everywhere:

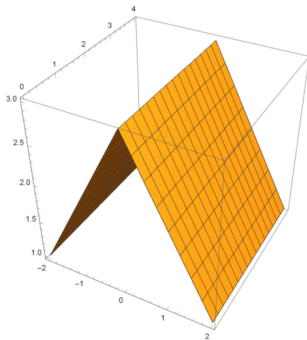
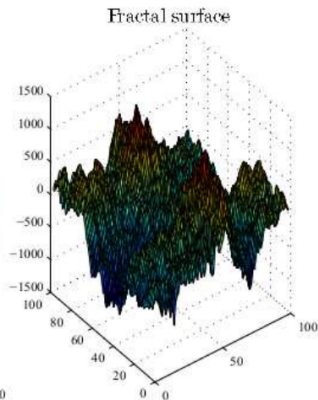
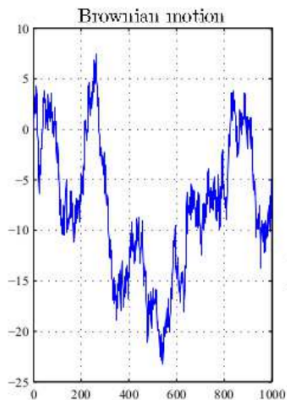


Figure 49: The function $z = f(x) = 3 - |x|$. $f_x(0)$ is undefined.

In general, we say that a surface $z = f(x, y)$ is smooth if f , f_x , f_y all exist and are continuous.

7.3 Smoothness

Many fractals (like Brownian motion) are continuous, but not smooth.



7.4 Optimisation

Definition (Increasing)

A function defined on a closed interval I is said to be *increasing* on I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I .

Definition (Decreasing)

A function defined on an closed interval I is said to be *decreasing* on I if $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I .

7.4 Optimisation

Theorem (Derivatives of Increasing/Decreasing Functions)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and differentiable on (a, b) .

- 1 If $f'(x) > 0$ on (a, b) , then f is increasing on $[a, b]$.
- 2 If $f'(x) < 0$ on (a, b) , then f is decreasing on $[a, b]$.
- 3 If $f'(x) = 0$ on (a, b) , then f is constant on $[a, b]$.

Proof

Choose any $x_1 < x_2$ in $[a, b]$. Then by applying the MVT, we can find $c \in (x_1, x_2) \subseteq (a, b)$ so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

If $f' > 0$, then $f(x_2) - f(x_1) > 0$, so item 1 is true. The other claims follow similarly.

7.4 Optimisation

Question

Find the intervals on which $f(x) = x^3 + x$ is increasing or decreasing.

Solution

We compute $f'(x) = x^2 + 1 > 0$. Therefore, f is increasing on \mathbb{R} .

7.4 Optimisation

Definition (Local Extrema)

A function f is said to have a *local maximum* (*local minimum*) at a point a if there is an open interval I containing a so that $f(a) \geq f(x)$ ($f(a) \leq f(x)$) for all $x \in I \cap \text{dom}(f)$.

Definition (Global Extrema)

A function f is said to have a *global maximum* (*global minimum*) at a point a if $f(a) \geq f(x)$ ($f(a) \leq f(x)$) for all $x \in \text{dom}(f)$.

All global maximums (minimums) are local maximums (minimums).

7.4 Optimisation

Definition (Critical Point)

A function f is said to have a *critical point* at $a \in \text{dom}(f)$ if $f'(a)$ vanishes, or does not exist.

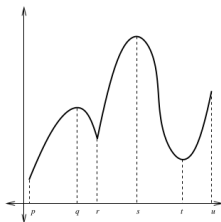


Figure 50: What do the points $x = p, q, r, s, t, u$ represent?

7.4 Optimisation

Theorem (Local Extrema are Critical Points)

If $a \in \text{dom}(f)$ is a local minimum or maximum of f , then a is a critical point of f .

Proof

If a is a local maximum, then $f(x) \leq f(a)$ for all x close to a . If $f'(a)$ does not exist, then a is a CP. If $f'(a)$ does exist, then

$$f'(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \leq 0$$

and

$$f'(a) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} \geq 0,$$

so $f'(a) = 0$ as required. The case that a is a local minimum is almost identical.

7.4 Optimisation

Theorem (First Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and differentiable on (a, b) .

7.4 Optimisation

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Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and differentiable on (a, b) .
Choose a point $c \in (a, b)$.

- 1 If $f'(x) > 0$ for all $x \in (a, c)$ and $f'(x) < 0$ for all $x \in (c, b)$,
then f has a local maximum at c .

7.4 Optimisation

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7.4 Optimisation

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- 2 If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c .

Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c) , so $f(x) \leq f(c)$ for all $a < x < c$.



7.4 Optimisation

Theorem (First Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and differentiable on (a, b) . Choose a point $c \in (a, b)$.

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- 2 If $f'(x) < 0$ for all $x \in (a, c)$ and $f'(x) > 0$ for all $x \in (c, b)$, then f has a local minimum at c .

Proof of 1 (proof of 2 is similar)

We know that f is increasing on (a, c) , so $f(x) \leq f(c)$ for all $a < x < c$. Similarly, f is decreasing on (c, b) , so $f(x) \leq f(c)$ for all $x \in (c, b)$, as required.

7.4 Optimisation

Theorem (Second Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

we need f'' to exist at c ,
AND in a small nbhd of
 c , and f'' must be continuous
here.

7.4 Optimisation

Theorem (Second Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

- 1 If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f .

7.4 Optimisation

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7.4 Optimisation

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7.4 Optimisation

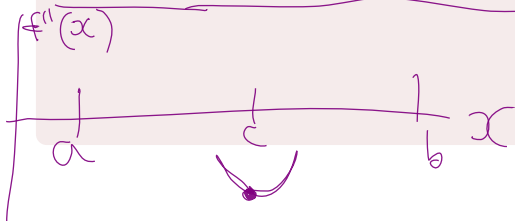
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Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

- 1 If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f .
- 2 If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f .

Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c , by continuity of f'' .



7.4 Optimisation

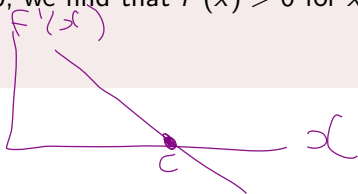
Theorem (Second Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

- 1 If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f .
- 2 If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f .

Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c , by continuity of f'' . Then $f'(x)$ is a decreasing function for x close to c . Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$.



7.4 Optimisation

Theorem (Second Derivative Test)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Suppose that f'' exists and is continuous at a point $c \in (a, b)$.

- 1 If $f'(c) = 0$ and $f''(c) < 0$, then c is a local maximum of f .
- 2 If $f'(c) = 0$ and $f''(c) > 0$, then c is a local minimum of f .

Proof of 1 (proof of 2 is similar)

If $f''(c) < 0$, then $f''(x) < 0$ for x close to c , by continuity of f'' . Then $f'(x)$ is a decreasing function for x close to c . Since $f'(c) = 0$, we find that $f'(x) > 0$ for $x < c$, and $f'(x) < 0$ for $x > c$. The first derivative test implies that c is a local maximum of f .

7.4 Optimisation

$$f(x) = \frac{1}{1+x^2}$$

$x=0$
is the unique
maximum.

Mathematica can sort of deal with optimisation.

```
In[4]:= FindMaximum[1/(1+x^2), {x, 5}]
```

```
Out[4]:= {1., {x -> -1.40789*10^15}}
```

```
In[5]:= FindMinimum[Cos[x], {x, 0}]
```

FindMinimum: Encountered a gradient that is effectively zero. The result returned may not be a minimum; it may be a maximum or a saddle point.

```
Out[5]:= {1., {x -> 0.}}
```

```
In[6]:= FindMinimum[Cos[x], {x, 0.5}]
```

```
Out[6]:= {-1., {x -> 3.14159}}
```

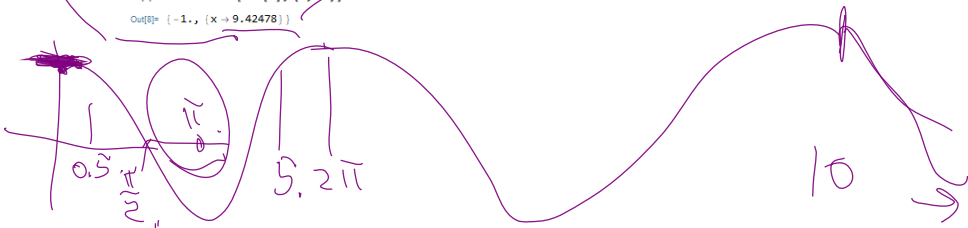
```
In[7]:= FindMinimum[Cos[x], {x, 5}]
```

```
Out[7]:= {-1., {x -> 3.14159}}
```

```
In[8]:= FindMinimum[Cos[x], {x, 10}]
```

```
Out[8]:= {-1., {x -> 9.42478}}
```

$$\approx 3\pi$$



7.4 Optimisation

Question

Find all local maximums and minimums of $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(x) = 2x^3 + 3x^2 - 12x + 4$.

Solution

- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.

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Solution

- We differentiate to find $f'(x) = 6x^2 + 6x - 12$ and $f''(x) = 12x + 6$; these functions exist and are continuous everywhere.
- Therefore the critical points are $x = -2$ and $x = 1$.

$$6x^2 + 6x - 12 = 0$$
$$\Rightarrow \underline{x = -2}, \underline{x = 1}.$$

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- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.

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- We have $f''(1) > 0$, so the $x = 1$ critical point is a local minimum. We have $f(1) = -3$.

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- Therefore the critical points are $x = -2$ and $x = 1$.
- We have $f''(-2) < 0$, so the $x = -2$ critical point is a local maximum. We have $f(-2) = 24$.
- We have $f''(1) > 0$, so the $x = 1$ critical point is a local minimum. We have $f(1) = -3$.
- All local extrema are critical points, so we have found all local extrema.

7.4 Optimisation

Theorem (Extreme Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f achieves a global maximum and a global minimum. Each of these values is achieved at an interior critical point, or at an end point.

$$\begin{aligned} a &\neq -\infty, \\ b &\neq \infty \end{aligned}$$



7.4 Optimisation

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Bonus Theorem

Suppose $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, and has only one critical point. If the critical point is a local minimum, then it is the global minimum. If the critical point is a local maximum then it is a global maximum.

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Suppose $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, and has only one critical point. If the critical point is a local minimum, then it is the global minimum. If the critical point is a local maximum then it is a global maximum.

This still works if a is $-\infty$, or if b is ∞ .

7.4 Optimisation

Question

Find the global maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-\frac{1}{2}, 4]$.

Solution

The derivative is

interior

$$f'(x) = 3x^2 - 6x,$$

so the only critical points are $x = 0, 2$.

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so the only critical points are $x = 0, 2$. Therefore, the global maximums and minimums have to occur at $0, 2, -\frac{1}{2}$ or 4 (being the end points of the interval). We compute:

$$\underline{f(0) = 1}, \quad \underline{f(2) = -3}, \quad \underline{f(\frac{1}{2}) = \frac{1}{8}}, \quad \underline{f(4) = 17},$$

so $(4, 17)$ is the global maximum, and $(2, -3)$ is the global minimum.

7.5 Basic Ordinary Differential Equations

Functions of interest often arise as solutions to differential equations which specify how a function is changing with respect to its input variables.

$$f(x) = x^2 + 1 \quad \text{— explicit,}$$

$$f'(x) = \dots \quad \text{something — implicit}$$

7.5 Basic Ordinary Differential Equations

Functions of interest often arise as solutions to *differential equations* which specify how a function is changing with respect to its input variables. There are two main types:

- *ordinary differential equations*, where the unknown functions depend on only one variable (like time, or a single spatial variable);



A handwritten diagram in purple ink. It features a wavy line that starts from the underlined phrase "single spatial variable" in the list above and points towards the expression $f(x)$. The expression $f(x)$ is written in a cursive style, with the x enclosed in parentheses. There is a small dot to the right of the closing parenthesis.

$$f(x)$$

7.5 Basic Ordinary Differential Equations

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- *partial differential equations*, where the unknown functions depend on several variables.

$$f(t^{\text{time}}, x, y) \dots$$

→ 2

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- *ordinary differential equations*, where the unknown functions depend on only one variable (like time, or a single spatial variable);
- *partial differential equations*, where the unknown functions depend on several variables.

The order of a differential equation is the order of the highest derivative that appears in the equation.

$$f''(x) = \pi \quad \text{2nd order ODE.}$$
$$f'(x) = f(x)^2 \quad \text{1st order ODE}$$

7.5 Basic Ordinary Differential Equations

Example (Population Modelling)

Let $P(t)$ be some population at some time t .

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Let $P(t)$ be some population at some time t . Assuming that the rate of change of the population is proportional to the population itself at any time, we find that there is a constant k so that

$$\frac{dP}{dt} = kP.$$

k is
the constant
proportionality.

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Example (Population Modelling)

Let $P(t)$ be some population at some time t . Assuming that the rate of change of the population is proportional to the population itself at any time, we find that there is a constant k so that

$$\frac{dP}{dt} = kP.$$

exponential
growth

If $k > 0$, the population is growing; if $k < 0$, the population is shrinking; if $k = 0$, the population is constant.

$$P(t) = P(0)e^{kt}$$

7.5 Basic Ordinary Differential Equations

Question

Show that $y(x) = A \exp(\frac{x^2}{2})$ is a solution to the ODE $y' = xy$ for any real constant A .

function $y(x)$.

$$y'(x) = x y(x).$$

7.5 Basic Ordinary Differential Equations

Question

Show that $y(x) = A \exp(\frac{x^2}{2})$ is a solution to the ODE $y' = xy$ for any real constant A .

Solution

For any A , $y'(x) = x A e^{\frac{x^2}{2}}$, and $xy = x A e^{\frac{x^2}{2}}$ as required.

$$\begin{aligned} A e^{\frac{x^2}{2}} &= A e^{\frac{x^2}{2}} \times \frac{d(\frac{x^2}{2})}{dx} \\ &= A x e^{\frac{x^2}{2}} \end{aligned}$$

7.5 Basic Ordinary Differential Equations

Solutions of differential equations

When asked to solve an ordinary differential equation (ODE), you must find *all* possible solutions. ODEs involving only one function and its first derivative (first order ODEs of one function) typically have one arbitrary constant that must be accounted for.

$$f''(x) = \dots$$

usually
has two arbitrary
constants

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Question

Solve the ODE $y'(x) = x^2$. $\frac{x^3}{3}$

If y_1, y_2 were both solⁿs, then
 $(y_1(x) - y_2(x))' = y_1'(x) - y_2'(x) = x^2 - x^2 = 0$.
 $\Rightarrow y_1(x) - y_2(x) = C$, constant.

7.5 Basic Ordinary Differential Equations

Solutions of differential equations

When asked to solve an ordinary differential equation (ODE), you must find *all* possible solutions. ODEs involving only one function and its first derivative (first order ODEs of one function) typically have one arbitrary constant that must be accounted for.

Question

Solve the ODE $y'(x) = x^2$.

Solution

By integrating, we find that $y(x) = \frac{x^3}{3} + C$ for some arbitrary real constant C .

→ how do you know this is all of them?

7.5 Basic Ordinary Differential Equations

Question

Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

(initial conditions
are typically used
to specify your arbitrary
constants?)

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Question

Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

Solution

The ODEs arise by specifying acceleration:

$$\frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2y}{dt^2} = -g \approx -9.81.$$

gravity
(constant)

physics.

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Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

Solution

The ODEs arise by specifying acceleration:

(undoing one derivative)

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -g \approx -9.81.$$

Integrating once and using the initial velocity gives

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v - gt.$$

($t \in \mathbb{R}$, $C = v$)

(constant)

7.5 Basic Ordinary Differential Equations

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Suppose you are throwing apples. Find an expression for the position $(x(t), y(t))$ of the apple if you assume the initial position is $x(0) = y(0) = 0$, and the initial velocity is $x'(0) = u$, $y'(0) = v$.

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Integrating once and using the initial velocity gives

$$\frac{dx}{dt} = u,$$

$$\frac{dy}{dt} = v - gt.$$

Integrating once more and using the initial position gives

$$x(t) = ut, \quad y(t) = vt - \frac{gt^2}{2}.$$



7.5 Basic Ordinary Differential Equations

Numerical v Analytic Solutions of ODEs

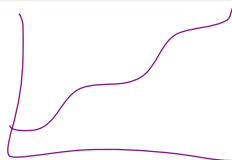
To solve an ODE analytically means to explicitly find the solution in terms well-understood continuous functions (like polynomials, trigonometric functions, exponentials, as well as sums, products, quotients and inverses of these functions).

7.5 Basic Ordinary Differential Equations

Numerical v Analytic Solutions of ODEs

To solve an ODE analytically means to explicitly find the solution in terms well-understood continuous functions (like polynomials, trigonometric functions, exponentials, as well as sums, products, quotients and inverses of these functions).

This is not always possible, so sometimes, it is useful to solve an ODE numerically, which means to use an algorithm to generate a function which is *almost* solution.



7.6 Application: Maximising Profits

Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a rectangular paddock for grazing. What is the maximal area of the paddock?

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Suppose a farmer has 400m of fencing with which to fence off a rectangular paddock for grazing. What is the maximal area of the paddock?

Solution

- Let x, y be the side lengths of the rectangle in meters.
- Then $2(x + y) = 400$ and we wish to maximise $A = xy$.

$$400\text{m} = \text{perimeter}$$

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Solution

- Let x, y be the side lengths of the rectangle in meters.
- Then $2(x + y) = 400$ and we wish to maximise $A = xy$.

- We find

$$y = 200 - x$$

$$A = x(200 - x);$$

we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.

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Solution

- Let x, y be the side lengths of the rectangle in meters.
- Then $2(x + y) = 400$ and we wish to maximise $A = xy$.

- We find

$(0, 200)$

what is the domain of??

$$A = x(200 - x);$$


we have $A'(x) = 200 - 2x$ and $A''(x) = -2$, so $x = 100$ is the only critical point, which is a global maximum.

- The answer is $A = 10000$ in m^2 (achieved by the square of side length 100m).

7.6 Application: Maximising Profits

Bonus Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?



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Bonus Question (Maximising Farming Land)

Suppose a farmer has 400m of fencing with which to fence off a paddock for grazing (any shape, not necessarily rectangular). What is the maximal area of the paddock?

The best shape is the circle, but why?

Techniques
are well beyond
this subject.