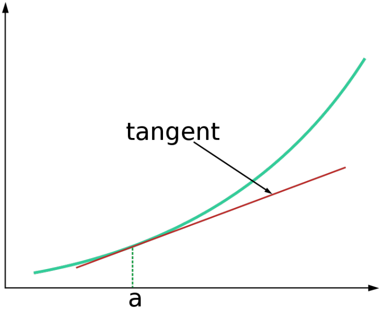
MATH7501: Mathematics for Data Science I

Unit 8: Linear approximations and Taylor series

8.1 Linear Approximations

Provided the function f is 'good enough' close to the point $x = a$, we have the following linear approximation of $f(x)$ for x close to a:

$$f'(x) \approx f(a) + f'(a)(x - a).$$



https://en.wikipedia.org/wiki/Linear_approximation

Example

Consider the population model $P(t) = 14931234e^{0.002t}$, where t is the number of quarter years since 1981. Then for t close to 70, we have the linear approximation

$$P(t) \approx P(70) + P'(70)(t - 70).$$

We compute $P'(t) = 14931234 \times 0.002 \times e^{0.002t}$ so that

$$\begin{aligned} P(t) &\approx 17175007.255 + 34350.014(t - 70) \\ &\approx 14770506.275 + 34350.014t. \end{aligned}$$

8.2 Taylor Series Approximations

Suppose the function $f(x)$ can be expressed as $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some real a, as well as real coefficients c_i.

To find the coefficients, first note that by setting $x = a$, we find that $c_0 = f(a)$. Then by differentiating, and then setting $x = a$, we find $c_1 = f'(a)$. Continuing on in this way gives

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

8.2 Taylor Series Approximations

Definition

If f is a function so that $f^{(n)}(a)$ exists for any n, the *Taylor series of f at a* is the expression

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

If $a = 0$, this is called the *Maclaurin series approximation*.

But when does $f(x)$ actually equal this infinite series?

8.2 Taylor Series Approximations

Theorem

If f is 'sufficiently well behaved' near the point a, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < r,$$

where $r > 0$ is the radius of convergence.

In this context, the phrase 'sufficiently well behaved' means a number of things, including the assumption that $\lim_{n\to\infty} \frac{b_n r^n}{n!} = 0$, where $b_n = \max_{x\in[a-r,a+r]} \left|f^{(n)}(x)\right|$. Unless otherwise stated, we will now assume that all functions we deal with are 'sufficiently well behaved.'

8.2 Taylor Series Approximations

Convergence of Taylor series

If

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \text{ for } |x-a| < r,$$

then the expression $\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is a good approximation of $f(x)$ for $|x-a| < r$, especially for large k.

The value of r can be determined by the *ratio test*, which says that $\sum_{n=0}^{\infty} c_n$ converges if $\lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} < 1$.

8.2 Taylor Series Approximations

Question

Find the Taylor series of $f(x) = \ln(x)$ about $x = 1$; determine its radius of convergence.

Solution

We calculate the derivatives:
$f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ and so on. Therefore,

$$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2,$$
$$\cdots f^{(n)}(1) = (-1)^{n-1}(n-1)!.$$

Then the Taylor series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$; this converges provided $\lim_{n\to\infty} \frac{\left|(x-1)^{n+1}n\right|}{|n+1(x-1)^n|} = |x-1| < 1.$

8.2 Taylor Series Approximations

In the previous question, we found that the Taylor series for $\ln(x)$ about $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$, and that this converges for $0 < x < 2$.

In fact, the following points are also true (even though we did not prove them):

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ is actually convergent for $0 < x \leq 2$;
- $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ for all $0 < x \leq 2$.

8.2 Taylor Series Approximations

Question

Find the Taylor series of $f(x) = e^x$ about $x = a$; determine its radius of convergence.

Solution

Since $f'(x) = e^x$, we find that $f^{(n)}(x) = e^x$ for all n. Therefore $f^{(n)}(a) = e^a$, so the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = e^a \sum_{n=0}^{\infty} \frac{1}{n!}(x-a)^n.$$

By the ratio test, this converges whenever $1 > \lim_{n\to\infty} \frac{\left|(x-a)^{n+1} n!\right|}{\left|(n+1)!(x-a)^n\right|} = \lim_{n\to\infty} \frac{|x-a|}{n+1} = 0$, so this expression converges everywhere. It turns out that this expression is in fact equal to e^x everywhere.

8.2 Taylor Series Approximations

Question

Find the Maclaurin series of $f(x) = \sin(x)$; determine its radius of convergence.

Solution

We have $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f''''(x) = \sin(x)$, and then repeats. Therefore,

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2m \\ (-1)^m & \text{if } n = 2m+1. \end{cases}$$

This implies that the Maclaurin series for f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}.$$

Similarly to e^x, we find that $\lim_{n\to\infty} \frac{|x^2|}{|(2m+3)(2m+2)|} = 0$, so we get convergence everywhere (convergence is to $\sin(x)$).

Taylor series in Mathematica:

```
In[30]:= Series[Cos[x], {x, 0, 10}]
```

Out[30]= $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + O[x]^{11}$

```
In[31]:= Series[Cos[x], {x, 0, Infinity}]
```

Series: Series order specification ∞ is not a machine-sized integer.

Out[31]= Series[Cos[x], {x, 0, ∞}]

```
In[37]:= Series[Cos[x], {x, 0, 10000}]
```

Out[37]= $1 - \frac{x^2}{2} +$ ⋯ 7498 ⋯ $+ O[x]^{10001}$

large output | show less | show more | show all | set size limit...

```
In[35]:= Series[x^2/y, {x, 200, 1}, {y, 400, 1}]
```

Out[35]= $\left(100 - \frac{y-400}{4} + O[y-400]^2\right) + \left(1 - \frac{y-400}{400} + O[y-400]^2\right)(x-200) + O[x-200]^2$

8.3 Linear Approximations for Multi-Variable Functions

Definition (Linear Approximation)

Suppose $f(x, y)$ is a function of two variables. The expression

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the *linear approximation for f at (a, b).* We expect this to be a good approximation if f is 'sufficiently well-behaved' for values of (x, y) close to (a, b).

8.3 Linear Approximations for Multi-Variable Functions

Example

The temperature in a region is given by $T(x,y) = 100 - x^2 - y^2$. Note that $\frac{\partial T}{\partial x} = -2x$, $\frac{\partial T}{\partial y} = -2y$. Therefore, $T(0,5) = 75$, $T_x(0,5) = 0$ and $T_y(0,5) = -10$, so the linear approximation for T at $(0,5)$ is

$$T(x,y) \approx 75 - 10(y-5) = 125 - 10y;$$

we expect this to be a good approximation for (x,y) close to $(0,5)$.

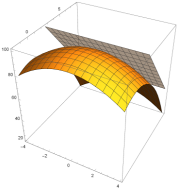


Figure 53: The function $T(x,y) = 100 - x^2 - y^2$ and the tangent plane $T(x,y) = 125 - 10y$.

8.3 Linear Approximations for Multi-Variable Functions

Question

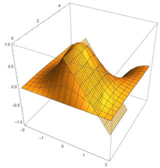
Find the tangent plane to the function $f(x, y) = e^{-x^2}\sin(y)$ at the point $(1, \frac{\pi}{2})$. Hence find an approximate value for $e^{-(0.9)^2}\sin(1.5)$.

Solution

We have $\frac{\partial f}{\partial x} = -2xe^{-x^2}\sin(y)$ and $\frac{\partial f}{\partial y} = e^{-x^2}\cos(y)$. Therefore, the linear approximation is

$$f(x, y) \approx \frac{1}{e} + (-\frac{2}{e})(x-1) + (0)(y - \frac{\pi}{2}) = \frac{1}{e} - \frac{2(x-1)}{e}.$$

Therefore, $e^{-(0.9)^2}\sin(1.5) = f(0.9, 1.5) \approx \frac{1}{e} - \frac{2(0.9-1)}{e} = \frac{1.2}{e}$.



8.3 Linear Approximations for Multi-Variable Functions

The formula for a linear approximation can also be interpreted in terms of small changes to f corresponding to small changes in x and y. Indeed, if we take an approximation to f at the point (a, b), then by rearranging, we get

$$f(x, y) - f(a, b) \approx f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Using Δ to mean 'change in', this becomes

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y;$$

this tells us how f changes with small changes in x and y. 'Infinitesimally', we write this in terms of *differentials*:

$$df = f_x(a, b)dx + f_y(a, b)dy.$$

Question

Electric power is given by $P(E,R) = \frac{E^2}{R}$, where E is the voltage and R is the resistance. Find a linear approximation for $P(E,R)$ for values of E close to 200 (in Volts) and R close to 400 (in Ohms). Use this to find the effect that a change in E and R has on P.

Solution

We calculate $\frac{\partial P}{\partial E} = \frac{2E}{R}$ and $\frac{\partial E}{\partial R} = -\frac{E^2}{R^2}$. Therefore,

$$P(E,R) \approx 100 + (1)(E - 200) + (-\frac{1}{4})(R - 400).$$

Using $\Delta P = P(E,R) - P(200,400)$, $\Delta E = E - 200$ and $\Delta R = R - 400$, we find

$$\Delta P \approx \Delta E - \frac{1}{4}\Delta R.$$

8.3 Linear Approximations for Multi-Variable Functions

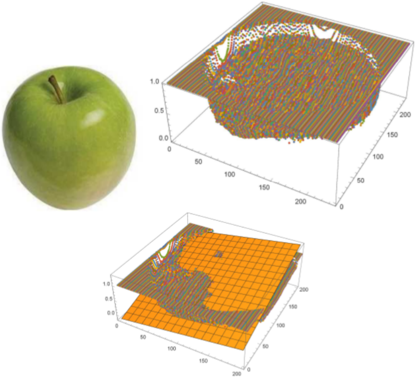


Figure 55: An image of an apple and the pixel values as a function of position in the image, $P(x, y)$, with a tangent plane.

Errors in Linear Approximations

Suppose we want to evaluate $f(x, y)$ at a certain point (x, y), but we do not know (x, y) exactly. Instead, we know that x is within a distance E_1 of a, and y is within a distance E_2 of b. Then the true value of $f(x, y)$ will be within approximately

$$E \approx |f_x(a, b)| \, E_1 + |f_y(a, b)| \, E_2$$

of $f(a, b)$. This gives us a way to treat errors.

Question

Suppose we are making a metal barrel with base radius 1m and height 2m. We allow for a 5% error in the construction of both the base radius and the height. What is the worst-case scenario error for the volume?

Solution

A 5% error in the radius gives an absolute error of $E_1 = 0.05$m in the radius. A 5% error in the height gives an error of $E_2 = 0.1$m. Now the volume is given by $V(r, h) = \pi r^2 h$; computing derivatives gives $V_r(1, 2) = 4\pi$ and $V_h(1, 2) = \pi$. Therefore, the worst error in the volume is

$$E = 4\pi \times 0.05 + \pi \times 0.1 = 0.3\pi.$$

Since $V(1, 2) = 2\pi$, the relative error in the volume is $\frac{0.3\pi}{2\pi} = 15\%$.

8.4 Optimisation via quadratic approximations

Let $f(x)$ be a real function that we wish to maximise. Let x_0 be an initial guess for the location of the maximum.

An approximation for the true location of the maximum proceeds by finding the second order Taylor series approximation $f(x) \approx f(x_i) + f'(x_i)(x - x_i) + \frac{f''(x_i)}{2}(x - x_i)^2$. If $f''(x_i) < 0$, then this approximation has a maximum; let the location of this maximum be x_{i+1}.

Continue in this way until the desired accuracy is achieved.