$a_n \; b_n$

Properties

For matrices of appropriate size

(1) $(AB)C = A(BC)$, Associativity

(2) $(A+B)C = AC + BC$, $A(B+C) = AB + AC$ Distributive Laws

Another unusual property of matrices is that $AB = 0$ does *not* imply $A = 0$ or $B = 0$. It is possible for the product of two non-zero matrices to be zero:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Also, if $AC = BC$, or $CA = CB$ then it is not true in general that $A = B$. (If $AC = BC$, then $AC - BC = 0$ and $(A-B)C = 0$, but this does **not** imply $A - B = 0$ or $C = 0$.)

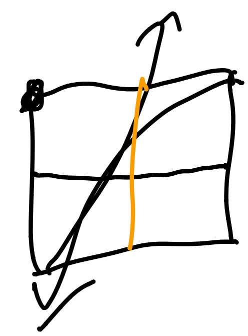
Transposition

Definition 3. *The* transpose *of an $m \times n$ matrix $A = (a_{ij})$ is the $n \times m$ matrix A^T with entries*

$$a_{ji} = a^T_{ij}, \quad \textit{for all } i, j$$

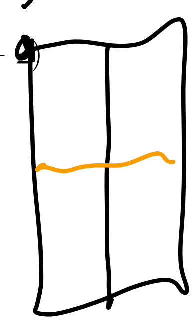
ie.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots\cdots & a_{m1} \\ a_{12} & a_{22} & \cdots\cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots\cdots & a_{mn} \end{pmatrix}$$



So the row vectors of A become column vectors of A^T and vice versa.

Example 7 (transposition)**.** *If $A = \begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$, $C = (7 \;\; 5 \;\; -2)$ then find A^T, B^T, C^T.*



$$A^T = \begin{pmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$C^T = (7 \;\; 5 \;\; -2)^T = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}.$$

Properties

For matrices of appropriate size

$$(\alpha A)^T_{ij} = (\alpha A)_{ji} = \alpha A_{ji}$$

(1) $(\alpha A)^T = \alpha \cdot A^T \quad , \quad \alpha \in \mathbb{R}$

(2) $(A+B)^T = A^T + B^T$

(3) $\boxed{(A^T)^T = A}$

(4) $\boxed{(AB)^T = B^T A^T}$ (*not* $A^T B^T$!).

$AB \neq BA$

Dot product expressed as matrix multiplication

A column vector

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$$

may be interpreted as an $n \times 1$ matrix. Then the dot product (for two column vectors $\boldsymbol{v}$ and $\boldsymbol{w}$) may be expressed using matrix multiplication:

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \underset{1 \times n}{(v_1\ v_2\ \cdots\ v_n)} \underset{n \times 1}{\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}} \\ &= \boxed{\boldsymbol{v}^T \boldsymbol{w}.} \neq \cancel{\boxed{v^T \cdot w}} \end{aligned}$$

Also

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{v} &= v_1^2 + v_2^2 + \cdots + v_n^2 = \|\boldsymbol{v}\|^2 \\ &= \boldsymbol{v}^T \boldsymbol{v}. \end{aligned}$$

$$\|v\|^2 = \sum_{i=1}^{n} v_i^2$$

1.4 Identity matrix, inverses and determinants

The identity matrix

An $n \times n$ matrix ($m = n$) is called a *square matrix* of *order* n. The diagonal containing the entries

$$a_{11},\ a_{22},\ \cdots,\ a_{nn}$$

is called the *main diagonal* (or *principal diagonal*) of A. If the entries above this diagonal are all zero then A is called *lower* triangular. If all the entries below the diagonal are zero, A is called *upper triangular*

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ \vdots & a_{22} & 0 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ a_{n1} & \cdots & & \cdots & a_{nn} \end{pmatrix} \qquad \begin{pmatrix} a_{11} & \cdots & & \cdots & a_{1n} \\ 0 & a_{22} & & & \vdots \\ 0 & 0 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}$$

lower triangular upper triangular

If elements above *and* below the principal diagonal are zero, so

$$a_{ij} = 0\,, \quad i \neq j$$

then A is called a *diagonal matrix*. Note that if A is diagonal then $A = A^T$.

Example 8. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *is a diagonal* 3×3 *matrix.*

Definition 4 (Identity matrix)**.** *The* $n \times n$ identity matrix $I = I_n$*, is the diagonal matrix whose entries are all* 1*:*

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}.$$

$A_{n\times n} I_n = A$

$I_m A_{m\times n} = A$

Remarks on the identity matrix

Recall from linear transformations of $\mathbb{R}^2$: $A\boldsymbol{i}$ is the first column of A and $A\boldsymbol{j}$ is the second column.

For example, for the 2×2 case we have

$$AI = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A = IA.$$

In general the jth column vector of I is the coordinate vector

$$\boldsymbol{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j.$$

e_j

in dim 10

Thus

$$
\begin{aligned}
A\boldsymbol{e}_j \;&=\; \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix} \begin{array}{c} \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array}\right) \end{array} \begin{array}{c} \\ \\ \\ \leftarrow j \\ \\ \\ \\ \end{array} \\
&=\; \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} \;=\; j\text{th column vector of } A.
\end{aligned}
$$

$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$

e_j in $\mathbb{R}^n$

AI

So $AI = A$ for all A.

Replacing A by A^T, $A^T I = A^T$ also. Transpose both sides: $A = (A^T)^T = (A^T I)^T = I^T (A^T)^T = IA$, so $IA = A$ for all A.

We have proved that for all square matrices A

$$IA \;=\; AI \;=\; A.$$

Definition 5 (Inverse)**.** *Let I denote the identity matrix.*

A square matrix A is **invertible** *(or* **non-singular***) if there exists a matrix B such that*

$$AB = BA = I.$$

$A = 5 \quad B = \frac{1}{5} = 5^{-1}$

Then B is called the inverse *of A and is denoted A^{-1}. A matrix that is not invertible is also said to be* singular.

$5 \cdot \frac{1}{5} = 1$

$\frac{1}{5} \cdot 5 = 1$

Inverse for the 2×2 case

Let A be a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Set $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = BA.$$

Let

$$\Delta = ad - bc.$$

If $\Delta \neq 0$, then

$$A^{-1} = \frac{1}{\Delta} B = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} g_{12} \\ g_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We call Δ the *determinant* of A.

Is inverse unique

A and B_1 is inverse so
$$AB_1 = I$$
$$B_1A = I$$

A and B_2 is inverse so
$$AB_2 = I$$
$$B_2A = I$$

$$(\underbrace{B_1 A}_{I})B_2 = B_1(\underbrace{A B_2}_{I})$$
$$\Downarrow$$
$$B_2 = B_1$$

Example 9. *Find the inverse matrix of* $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$*. Check your answer.* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = I$

$\Delta = 1 \times 5 - 2 \times 3 = -1.$

$$\Rightarrow A^{-1} = \frac{1}{-1}\begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Check whether $AA^{-1} = A^{-1}A = I$:
Multiplying the two matrices A and A^{-1} gives

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So A is invertible with inverse

$$\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}.$$

Example 10 (Inverse of a 3×3 matrix)**.** *Let*

$$A = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}, \qquad B = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}.$$

Show that $B = A^{-1}$.

An easy way to do this is to multiply the two matrices together.

$$AB = \begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix}\begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, = I$$

and, similarly

$$BA = \begin{pmatrix} 16 & -17 & 7 \\ 11 & -12 & 5 \\ -2 & 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 & -1 \\ 1 & -2 & -3 \\ -2 & 2 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad I$$

Properties of inverses

(1) A has at most **one** inverse

(2) If A, B are invertible so is AB, and

$$(AB)^{-1} = \boxed{B^{-1}A^{-1}} \quad (\textit{not } A^{-1}B^{-1}!)$$

(3) A is invertible if and only if A^T is invertible

(4) If A is invertible then $(A^T)^{-1} = (A^{-1})^T$. $= A^{-T}$

$$\boxed{A\,B}\,(\boxed{A\,B})^{-1} = I$$

$$A\,\underbrace{B\,B^{-1}}_{I}\,A^{-1}$$

$$A\,A^{-1} = I$$

Determinants

Definition 6. *Associated to each square matrix A is a number called the* determinant *of A and denoted $|A|$ or $\det(A)$. It is defined as follows.*

$|A|$

- *If A is a 1×1 matrix, say $A = (a)$ then $|A|$ is defined to be a.*
- *If $A = (a_{ij})$ is 2×2 then we define*

$$\boxed{|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}}.$$

- *In general if $A = (a_{ij})$ is $n \times n$, first set*

$$C_{ij} = \boxed{(-1)^{i+j}} \left| \begin{array}{cccc|ccc} a_{11} & a_{12} & \dots & a_{1\,j-1} & a_{1\,j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2\,j-1} & a_{2\,j+1} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1\,1} & a_{i-1\,2} & \dots & a_{i-1\,j-1} & a_{i-1\,j+1} & \dots & a_{i-1\,n} \\ \hline a_{i+1\,1} & a_{i+1\,2} & \dots & a_{i+1\,j-1} & a_{i+1\,j+1} & \dots & a_{i+1\,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n\,j-1} & a_{n\,j+1} & \dots & a_{nn} \end{array} \right|$$

called the cofactor *of a_{ij}. The $(n-1) \times (n-1)$ determinant is obtained by* omitting *the ith row and jth column from A (indicated by the horizontal and vertical lines in the matrix).*

$j=3$

$i=2$

4×4

We then define

$$|A| = a_{11}C_{11} + a_{12}C_{12} + \cdots\cdots + a_{1n}C_{1n}$$

which gives a *recursive* definition of the determinant.

Observe that $(-1)^{i+j}$ gives the pattern $\begin{pmatrix} + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ - & + & - & + & \cdot & \cdot & \cdot & \cdot \\ + & - & + & - & \cdot & \cdot & \cdot & \cdot \\ \vdots & & & & & & & \end{pmatrix}$

Thus for a 3×3 matrix A we have

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

where

$$C_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad C_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Properties of determinants

Property (1): $|A| = |A^T|$

Consider for example the 2×2 case. Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Rightarrow \quad A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Thus $|A| = a_{11}a_{22} - a_{12}a_{21} = |A^T|$. It can be shown that this holds for any square matrix, not just in the 2×2 case.

This means that any results about the *rows* in a general determinant is also true about the *columns* (since the rows of A^T are the columns of A). In particular, any statement about the effect of row operations on determinants is also true for column operations.

Warning: We used row operations to simplify systems of equations, because they do not change the solution. Column operations may change the solution of linear systems, so we should not use column operations on such systems.

Property (2): The determinant may by taking cofactors along any row (not just the first) or down any column. Eg. for a 3×3 matrix A

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} && \text{(definition, expansion along 1st row)} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} && \text{(expansion along 2nd row)} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} && \text{(expansion down 3rd column) etc.} \end{aligned}$$

This is useful if one row or column contains a larger number of zeros.

Determinants of triangular matrices

Suppose $A = \begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & \ddots & & \vdots \\ & & & & 0 \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{pmatrix}$

is lower triangular. Then by repeated expansion along $\boldsymbol{r}_1$ we obtain

$$|A| = a_{11} \begin{vmatrix} a_{22} & & \\ \vdots & \ddots & \\ \vdots & & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = \ldots = a_{11}\ a_{22} \cdot \cdots \cdot a_{nn},$$

which is the product of the diagonal entries. The same result holds for upper triangular and diagonal matrices.

In particular

$$|I| = 1.$$

Connection with inverses

An important property of determinants is the following.

Fact (product of determinants)

Let $A,\ B$ be $n \times n$ matrices. Then

$$\boxed{|AB| = |A| \cdot |B|\,.}$$

$$\boxed{|A+B| \neq |A|+|B|}$$

Theorem 1 (Invertible matrices)**.**

$$A \textit{ is invertible } \iff |A| \neq 0.$$

Proof $\quad\Longrightarrow$ If A is invertible, then

$$\begin{aligned} I &= AA^{-1} \\ \Rightarrow \quad 1 = |I| &= |AA^{-1}| = |A| \cdot |A^{-1}|\,, \\ \text{So } |A| &\neq 0. \end{aligned}$$

$\Longleftarrow$ Follows from below.

Remark 1. *It follows immediately from the proof that if A is invertible, then*

$$\boxed{|A^{-1}| = \frac{1}{|A|},}$$

i.e., the inverse of the determinant is the determinant of the inverse.

1.5 Vectors operations (in 2 and n dimensions)

- A *vector* quantity has both a magnitude and a direction. Force and velocity are examples of vector quantities.
- A *scalar* quantity has only a magnitude (it has no direction). Time, area and temperature are examples of scalar quantities.

A vector is represented geometrically in the (x, y) plane (or in (x, y, z) space) by a directed line segment (arrow). The direction of the arrow is the direction of the vector, and the length of the arrow is proportional to the magnitude of the vector. Only the length and direction of the arrow are significant: it can be placed anywhere convenient in the (x, y) plane (or (x, y, z) space).

If P, Q are points, $\overrightarrow{PQ}$ denotes the vector from P to Q.

A vector $\boldsymbol{v} = \overrightarrow{PQ}$ in the (x, y) plane may be represented by a pair of numbers

$$\boldsymbol{v} = \boxed{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}} = \begin{pmatrix} x_Q - x_P \\ y_Q - y_P \end{pmatrix}$$

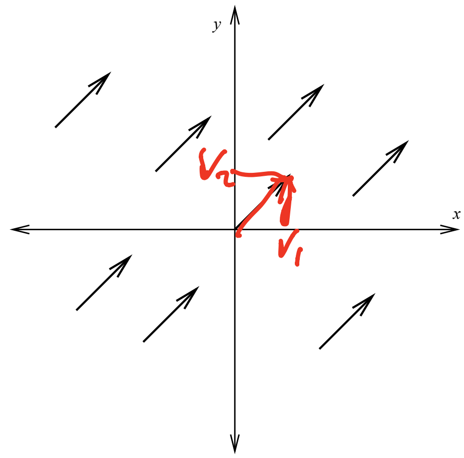


Figure 4: The vector of unit length at $45°$ to the x-axis has many representations.

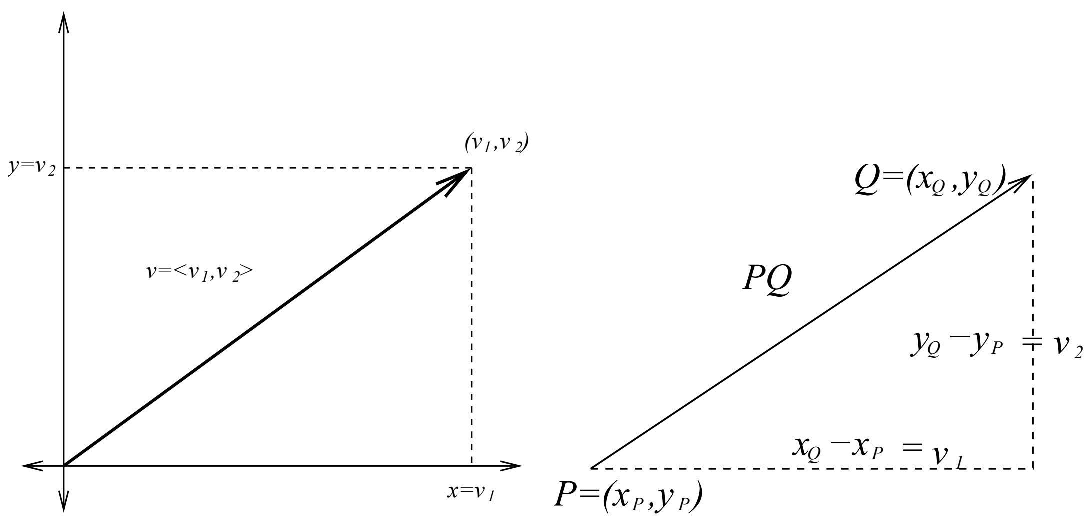


Figure 5: Geometric representation of a vector.

which is the same for all representations $\overrightarrow{PQ}$ of $\boldsymbol{v}$. We call v_1, v_2 the *components* of the vector $\boldsymbol{v}$.

We call the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the *zero vector*. It is denoted by $\mathbf{0}$.

Vectors will be indicated by bold lowercase letters $\boldsymbol{v}$, $\boldsymbol{w}$ etc. Writing by hand you may use $\underline{v}$ or $\vec{v}$ or $\widetilde{v}$.

Position vectors

Let $P = (x_p, y_p)$ be a point in the (x, y) plane. The vector $\overrightarrow{OP}$, where O is the origin, is called the *position vector* of P. Obviously

$$\overrightarrow{OP} = \begin{pmatrix} x_p \\ y_p \end{pmatrix}.$$

Norm

For vector $\boldsymbol{v} = \overrightarrow{PQ}$, the *norm*(or *length* or *magnitude*) of $\boldsymbol{v}$, written $\|\boldsymbol{v}\|$, is the distance between P and Q. Thus for $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}$$

Vector addition

We add vectors by the triangle rule.

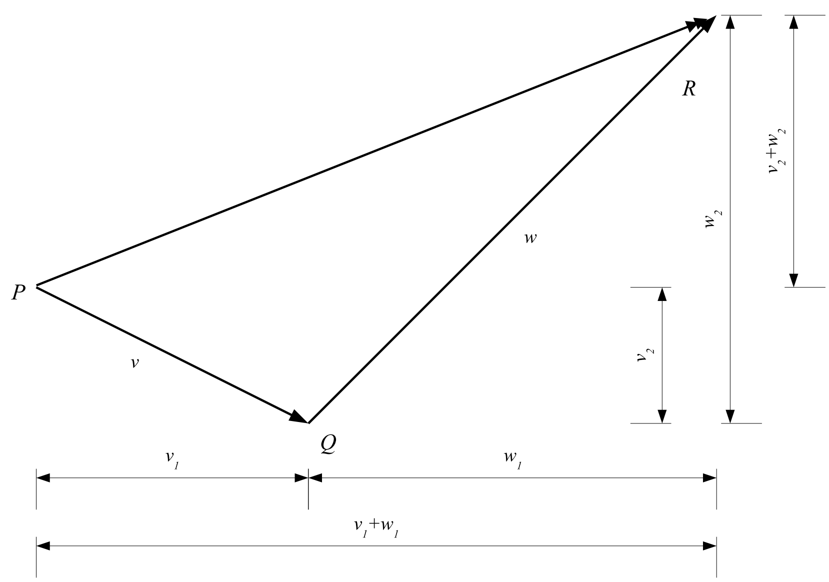


Figure 6: $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$

Consider the triangle PQR with $\boldsymbol{v} = \overrightarrow{PQ}$, $\boldsymbol{w} = \overrightarrow{QR}$. Then $\boldsymbol{v} + \boldsymbol{w} = \overrightarrow{PR}$; see Figure 6. In terms of components, if

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

then

$$\boldsymbol{v} + \boldsymbol{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}.$$

It follows from the component description that vector addition satisfies the following properties:

$$\begin{aligned} \boldsymbol{v} + \boldsymbol{w} &= \boldsymbol{w} + \boldsymbol{v} && \text{(commutative law)} \\ \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) &= (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} && \text{(associative law)} \\ \boldsymbol{v} + \boldsymbol{0} &= \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v} \end{aligned}$$

Scalar multiplication

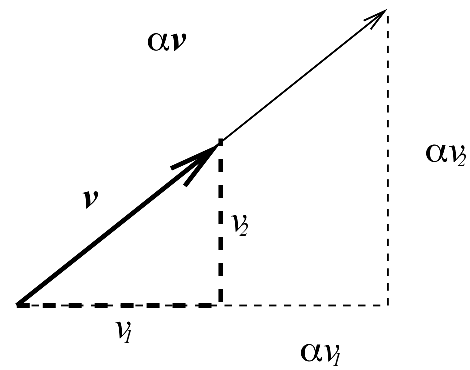


Figure 7: We can multiply the vector $\boldsymbol{v}$ by a number α (scalar).

If α is a real number (called a *scalar*), we define $\alpha\boldsymbol{v}$ to be the vector of norm

$$\|\alpha\boldsymbol{v}\| = |\alpha| \cdot \|\boldsymbol{v}\|$$

in the same direction as $\boldsymbol{v}$ if $\alpha > 0$, and opposite direction if $\alpha < 0$.

Using similar triangles it follows that if $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ then $\alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$.

If we multiply any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by zero we obtain the zero vector:

$$0 \cdot \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boldsymbol{0}$$

Unit vectors

A *unit vector* is a vector of norm 1. If $\boldsymbol{v} \neq \boldsymbol{0}$ is a vector, then

$$\frac{1}{\|\boldsymbol{v}\|}\,\boldsymbol{v}$$

is a unit vector in the direction of $\boldsymbol{v}$.

In particular

$$\boldsymbol{i} = \overset{e_1}{\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}}\ , \quad \boldsymbol{j} = \overset{e}{\boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

determine unit vectors along the x and y axes respectively.

For any vector $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$v_1 e_1 + v_2 e_2$

$$\boldsymbol{v} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j},$$

Hence we can decompose $\boldsymbol{v}$ into a vector $v_1\boldsymbol{i}$ along the x-axis and $v_2\boldsymbol{j}$ along the y-axis. The numbers v_1 and v_2 are called the *components* of $\boldsymbol{v}$ with respect to $\boldsymbol{i}$ and $\boldsymbol{j}$.

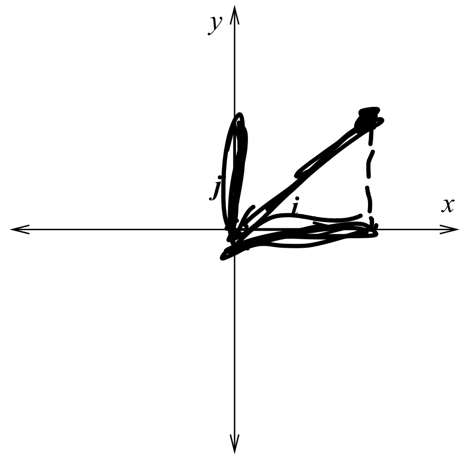


Figure 8: Unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ in the x and y directions respectively.

Row and column vectors

We may write vectors using columns e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, or as row vectors $\begin{pmatrix} a & b \end{pmatrix}$. We usually use column vectors in this course.

Example 11. *An albatross is flying NE at 20 km/h into a 10 km/h wind in direction E60°S. Find the speed and direction of the bird relative to the ground.*

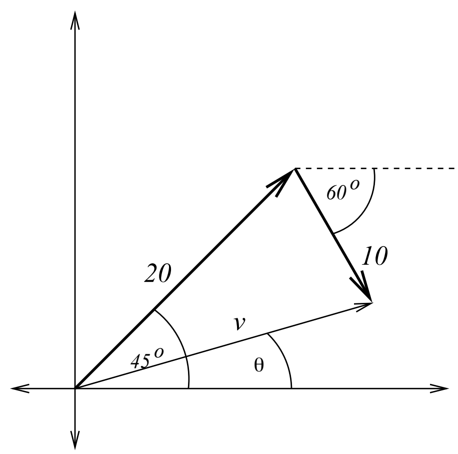


Figure 9: $\boldsymbol{v}$ gives the velocity vector of the albatross.

We have:

$$\begin{aligned}
\mathbf{z} &= 20\cos 45^\circ \boldsymbol{i} + 20\sin 45^\circ \boldsymbol{j} \\
&= 10\sqrt{2}\boldsymbol{i} + 10\sqrt{2}\boldsymbol{j} \\
\mathbf{w} &= 10\cos 60^\circ \boldsymbol{i} - 10\sin 60^\circ \boldsymbol{j} \\
&= 5\boldsymbol{i} - 5\sqrt{3}\boldsymbol{j}.
\end{aligned}$$

So $\mathbf{v} = \mathbf{z} + \mathbf{w} = (10\sqrt{2}+5)\boldsymbol{i} + (10\sqrt{2} - 5\sqrt{3})\boldsymbol{j}$
Therefore the magnitude of $\mathbf{v}$ *is*

$$
\begin{aligned}
\|\boldsymbol{v}\| &= \sqrt{(10\sqrt{2}+5)^2 + (10\sqrt{2}-5\sqrt{3})^2} \\
&= \sqrt{500 + 100\sqrt{2}(1-\sqrt{3})} \\
&\approx 19.91.
\end{aligned}
$$

To calculate the angle θ we use

$$
\begin{aligned}
\tan\theta &= \frac{10\sqrt{2}-5\sqrt{3}}{10\sqrt{2}+5} \\
\Rightarrow \theta &\approx 16^\circ
\end{aligned}
$$

So $\boldsymbol{v}$ = $19.9kmh^{-1}$ at E16°N.

Vectors in $\mathbb{R}^n$ n-vector

We are familiar with vectors in two and three dimensional space, $\mathbb{R}^2$ and $\mathbb{R}^3$. These generalize to n dimensional space, denoted $\mathbb{R}^n$. A vector $\boldsymbol{v}$ in $\mathbb{R}^n$ is specified by n components:

$$
\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \qquad v_i \in \mathbb{R}.
$$

We define addition of vectors and multiplication by a scalar component-wise. Thus if $\alpha \in \mathbb{R}$ is a scalar and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$ is another vector then

$$
\boldsymbol{v}+\boldsymbol{w} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_n+w_n \end{pmatrix} = \boldsymbol{w}+\boldsymbol{v}, \qquad \alpha\boldsymbol{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{pmatrix} \quad \alpha \in \mathbb{R},
$$

We define the *dot* (or *scalar*) *product* between $\boldsymbol{v}$ and $\boldsymbol{w}$ as

$$
\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{w} &= v_1w_1 + v_2w_2 + \cdots\cdots + v_nw_n \\
&= \sum_{k=1}^{n} v_kw_k.
\end{aligned}
$$

The *length* or *norm* of the vector $\boldsymbol{v}$ is

$$
\begin{aligned}
\|\boldsymbol{v}\| &= \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} \\
&= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\end{aligned}
$$

Theorem 2. *Every vector* $\boldsymbol{v}$ *in* $\mathbb{R}^n$ *is expressible in terms of* n *special vectors* $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ *called* coordinate vectors, *where*

$$\boldsymbol{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th position.}$$

Proof An arbitrary vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written as

$$\boxed{\boldsymbol{v}} = \boxed{\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}} = \boxed{\begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots\cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ v_n \end{pmatrix}}$$

$$= \boxed{v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots\cdots + v_n\boldsymbol{e}_n.}$$

Notation

(1) E.g. in $\mathbb{R}^3$ we have

$$\boldsymbol{i} = \boldsymbol{e}_1\,, \quad \boldsymbol{j} = \boldsymbol{e}_2\,, \quad \boldsymbol{k} = \boldsymbol{e}_3.$$

(2) We have also the zero vector

$$\boldsymbol{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{satisfying}$$

$$\boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}, \qquad 0 \cdot \boldsymbol{v} = \boldsymbol{0}, \qquad \text{for every vector } \boldsymbol{v}.$$

(3) If $\boldsymbol{v} \neq \boldsymbol{0}$, $\boldsymbol{w} \neq \boldsymbol{0}$, we say that vectors $\boldsymbol{v}$, $\boldsymbol{w}$ are *perpendicular* (or *orthogonal*) if

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0.$$

1.6 Dot product, norm and distance

Dot product

For non zero vectors $\boldsymbol{v} = \overrightarrow{OP}$, $\boldsymbol{w} = \overrightarrow{OQ}$ the *angle between* $\boldsymbol{v}$ and $\boldsymbol{w}$ is the angle θ with $0 \leq \theta \leq \pi$ radians between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ at the origin O; see Figure 10.

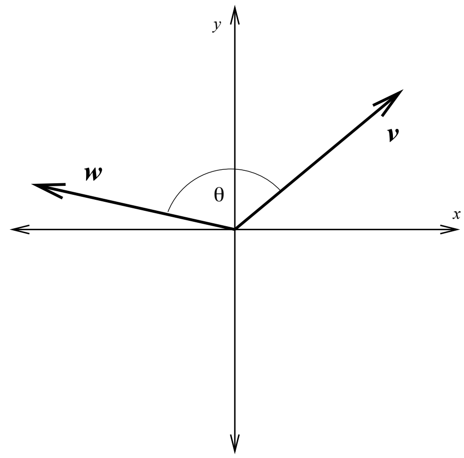


Figure 10: θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

The *dot* (or *scalar* or *inner*) product of vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, denoted by $\boldsymbol{v} \cdot \boldsymbol{w}$, is the *number* given by

$$\boldsymbol{v} \cdot \boldsymbol{w} = \begin{cases} 0\,, & \text{if } \boldsymbol{v} \text{ or } \boldsymbol{w} = \boldsymbol{0} \\ \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cos\theta\,, & \text{otherwise} \end{cases}$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

If $\boldsymbol{v}$, $\boldsymbol{w} \neq \boldsymbol{0}$ and $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are said to be *orthogonal* or *perpendicular*.

If $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ and $\boldsymbol{w} = w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}$ are two vectors, then $\boldsymbol{v} \cdot \boldsymbol{w}$ is given by:

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3,$$

In particular, for $\boldsymbol{v} \in \mathbb{R}^3$,

$$\|\boldsymbol{v}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + v_3^2.$$

Example 12. *Find the angle θ between the vectors:*

$$\boldsymbol{v} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{\|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|}.$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 3 - 5 \cdot 3 + 4 \cdot 3 = 0.$$

So $\cos\theta = 0$*, so the vectors are perpendicular, i.e.* $\theta = \dfrac{\pi}{2}$.

Example 13. *If $P = (2,4,-1)$, $Q = (1,1,1)$, $R = (-2,2,3)$, find the angle $\theta = PQR$.*

Find vectors joining Q to P, and Q to R:

$$\overrightarrow{QP} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix},$$

$$\overrightarrow{QR} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}.$$

$$\therefore \quad \left\|\overrightarrow{QP}\right\| = \left\|\overrightarrow{QR}\right\| = \sqrt{1+9+4} = \sqrt{14}.$$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 3 - 4 = -4$$

$$\therefore \quad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left\|\overrightarrow{QP}\right\| \cdot \left\|\overrightarrow{QR}\right\|} = \frac{-4}{14} \Rightarrow \theta \simeq 107^\circ.$$

The projection formula

Fix a vector $\boldsymbol{v}$. Given another vector $\boldsymbol{w}$ we can write $\boldsymbol{w}$ as

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

where:

- $\boldsymbol{w}_1$ is in the direction of $\boldsymbol{v}$
- $\boldsymbol{w}_2$ is perpendicular to $\boldsymbol{v}$.

See Figure 11. We want to find $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$.

Since $\boldsymbol{w}_1$ is in the direction of $\boldsymbol{v}$, let

$$\boldsymbol{w}_1 = \alpha\boldsymbol{v} \qquad \text{for some } \alpha \in \mathbb{R}.$$

Then $\boldsymbol{w}_2 = \boldsymbol{w} - \alpha\boldsymbol{v}$.

We need to choose α to make $\boldsymbol{v}$ and $\boldsymbol{w}_2$ orthogonal.

$$0 = \boldsymbol{w}_2 \cdot \boldsymbol{v} = (\boldsymbol{w} - \alpha\boldsymbol{v}) \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{v} - \alpha\boldsymbol{v} \cdot \boldsymbol{v} = \boldsymbol{w} \cdot \boldsymbol{v} - \alpha\|\boldsymbol{v}\|^2.$$

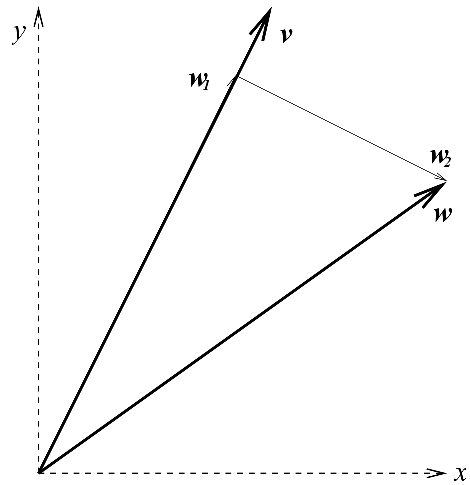


Figure 11: $\boldsymbol{w}$ can be decomposed into a component $\boldsymbol{w}_1$ in the direction of $\boldsymbol{v}$ and a component $\boldsymbol{w}_2$ perpendicular to $\boldsymbol{v}$.

So we need

$$\alpha = \frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}.$$

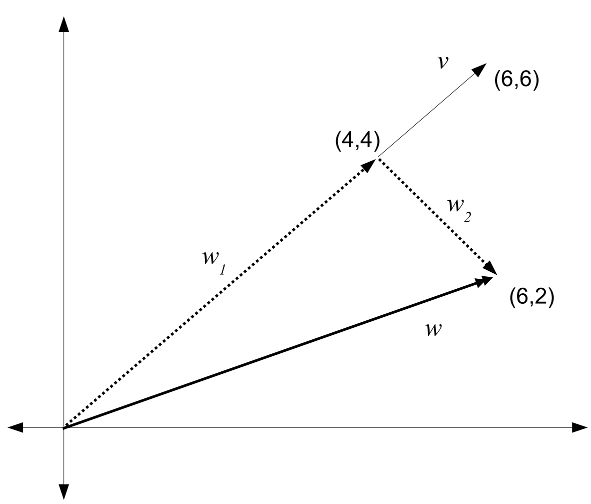
We have derived the *projection formula:*

$$\boldsymbol{w}_1 = \frac{(\boldsymbol{w}\cdot\boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\ , \qquad \boldsymbol{w}_2 = \boldsymbol{w} - \frac{(\boldsymbol{w}\cdot\boldsymbol{v})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}. \tag{1}$$

Example 14. *Find the projection of* $\boldsymbol{w} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ *onto* $\boldsymbol{v} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$.

$$\boldsymbol{w}_1 = \frac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = \frac{36+12}{6^2+6^2}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}.$$

$$\boldsymbol{w}_2 = \boldsymbol{w} - \boldsymbol{w}_1 = \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2. \end{pmatrix}$$



$(AB)^T = B^T A^T$

Properties of dot ~~product~~

If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then

(1) $\boxed{\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}}$.

(2) $\boxed{\boldsymbol{v} \cdot (\boldsymbol{w} + \boldsymbol{u})} = \boxed{\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{u}}$.

(3) $(\alpha \boldsymbol{v}) \cdot \boldsymbol{w} = \alpha(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{v} \cdot (\alpha \boldsymbol{w})$.

(4) $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$.

$u \cdot v = u^T v = (u^T v)^T = v^T u^{T^T} = v^T u$

$17^T = 17 = v \cdot u$

1.7 Application: k-means clustering

$0^T 0 = 0$

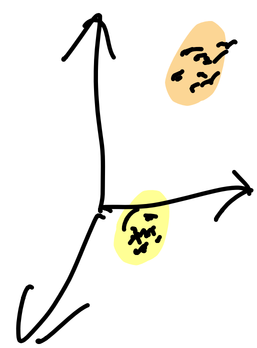
$v^T v = 0_{scalar} \Rightarrow v = 0_{vec}$

$v^T(w+u) = v^T w + v^T u$

$\sqrt{}$ $\quad ||x||$

$v \cdot v = \sum_{i=1}^{n} \boxed{v_i^2} = 0$

$d(x, y) = ||x - y||$



Inner product (dot product): $u, v \in \mathbb{R}^n$

$$u^T v = u \cdot v = \sum_{i=1}^{n} u_i v_i = \langle u, v \rangle$$

Bra Ket

Scalar

Outer product

Rank-1 Matrix

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}_{1\times n} = \begin{bmatrix} \boxed{u_1 v_1} & & \vdots & \\ & & \vdots & \\ i \cdots & \cdots & \boxed{u_j v_j} & \\ & & & \end{bmatrix}$$

n×n Matrix

Four ways to think of matrix multiplication

$$\underset{m\times n}{A}\ \underset{n\times p}{B}$$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & \alpha_1 & - \\ - & \alpha_2 & - \\ & \vdots & \\ - & \alpha_m & - \end{bmatrix}$$

$$B = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} - & b_1 & - \\ & \vdots & \\ - & b_n & - \end{bmatrix}$$

Way 1 (inner product):

$$(AB)_{ij} = \tilde{a}_i^T b_j$$

Way 2 (linear combination of cols)

$$A\begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boxed{A}\boxed{b_1} & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}$$

Way 3 (linear on rows

$$AB = \begin{bmatrix} — & \tilde{a}_1 \boxed{B} & — \\ — & \tilde{a}_2 B & — \\ & \vdots & \\ — & \tilde{a}_m B & — \end{bmatrix}$$

Way 4: Sum of outer products

$$AB = \boxed{a_1 b_1^T} + a_2 b_2^T + \cdots + a_n b_n^T$$

2 Sets, counting and cardinality

The terms *set* and *element* won't be formally defined here. Roughly speaking, a **set** will be a collection of objects called **elements**, and given any element and any set, we should be able to say whether the element belongs to the set or not.

2.1 Sets

// a is an element of S

- Recall that the notation $\boxed{a \in S}$ means that a is an element of the set S, or a belongs to S.
- We can list elements in a set using braces or curly brackets: $\{x_1, x_2, x_3\}$. The order in which we list the elements of a set is irrelevant, so $\{x_1, x_2, x_3\} = \{x_3, x_1, x_2\}$, etc.
  The number of times that each element is listed is also irrelevant; $\{a, b, a\} = \{a, b\}$ for example.
- The **empty** set is the set containing NO elements, denoted by $\boxed{\emptyset}$. $\emptyset$
  (We'll see soon that "the" empty set is the right terminology, because we'll show that there is only one empty set.)
- For any set A, the **cardinality** of A is the number of elements in the set A; we shall denote this as $\boxed{|A|}$. $|A|$

Note that in *Mathematica*, we can calculate with sets provided we use the Union function.

```
Union[{a,b,a}] = = Union[{a, b}]
MemberQ[{2,{2}},2]
```

Otherwise these will behave like vectors.

Example 15. **(a)** *How many elements does the set $\{2, 2, \{2\}\}$ have?* $|\{2,2,\{2\}\}| = 2$ $\{2, \boxed{\{2\}}\}$

(b) *Is it true that $\{1, 1, 2\} = \{1, 2\}$?* True

Singleton **(c)** *Is it true that $1 \in \{1\}$?* True

(d) *Is it true that $1 \in \{\{1\}\}$?* False

$\{2\} \neq 2$

Example: *Describe the following sets in words.*

(a) $\{1, 2, \ldots, 100\}$ ✓

(b) $\{x \in \mathbb{R} : x > 0\}$ $= W$ $\mathbb{R}$

−20 0 +10

(c) $\{y \in \boxed{\mathbb{Z}^+} : -3 \leqslant y \leqslant 3\}$

OEW //

- *If A and B are any sets, A is called a* **subset** *of B, written $A \subseteq B$, if and only if every element of A is also an element of B.* $\leq$

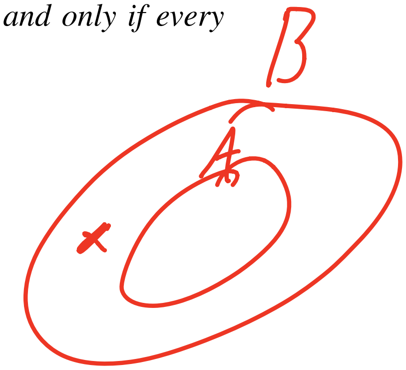
$A \subseteq B$ $<$

Venn Diagram

- *For sets A and B, we say A is a* **proper subset** *of B if and only if $A \subseteq B$ and $A \neq B$. So A is a proper subset of B if and only if every element of A is also an element of B, and there is some element of B which is not in A.* $\subsetneq$

- *For sets A and B, we say sets A and B are* **equal**, *$A = B$, if and only if every element of A is in B, and also every element of B is in A.*

  So $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.



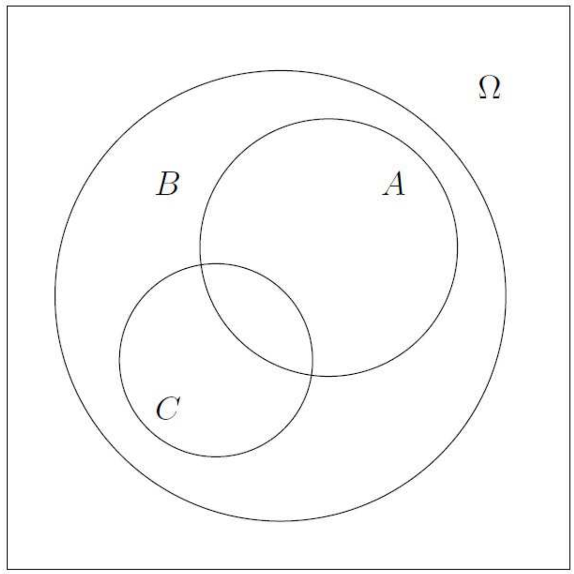


Figure 12: A Venn diagram with universe Ω.

Example 16. *Suppose $A = \{a, b, c, d\}$, $B = \{a, b, e\}$ and $C = \{a, b, c, d, e\}$. Answer the following, and give reasons.*

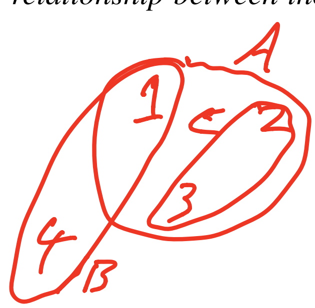
(a) *Is it true that $B \subseteq A$?* F

(b) *Is it true that $A \subseteq C$?* T (b') $A \in C$ False

(c) *Is A a proper subset of C?* T

(d) *Is it true that $B \subseteq B$?* T

Example 17. *Draw a Venn diagram to represent the relationship between the following sets: $A = \{1, 2, 3\}$, $B = \{1, 4\}$, $C = \{2, 3\}$.*



Example 18. *True or false?*

(a) $\{4\} \in \{1, \{3\}, 4\}$ F

(b) $\{4\} \subseteq \{1, \{3\}, 4\}$ T

(c) $\{3\} \in \{1, \{3\}, 4\}$ T

(d) $1 \subseteq \{1, \{3\}, 4\}$ F!!!

Note the set notation used here; verbally read ":" as "such that".

Example 19. *Let* $A = \{x \in \mathbb{Z} : x = 4p - 1 \text{ for some } p \in \mathbb{Z}\}$, $B = \{y \in \mathbb{Z} : y = 4q - 5 \text{ for some } q \in \mathbb{Z}\}$. *Prove that* $A = B$.

We investigate some **operations** on sets now, obtaining new sets from existing sets. Let U be some **universal set**, depending on the context.
(So U could perhaps be $\mathbb{R}$ in some contexts.)

In the following, suppose A and B are some subsets of a universal set U.

$A, B \subseteq U$

- The **union** of sets A and B, denoted $A \cup B$, is the set of all elements x in U such that $x \in A$ *or* $x \in B$ *(or both)*.
- The **intersection** of sets A and B, denoted $A \cap B$, is the set of all elements x in U such that $x \in A$ *and* $x \in B$.
- The **set difference** of B minus A, denoted $B - A$, and sometimes also called the relative complement of A in B, is the set of all x in U such that $x \in B$ and $x \notin A$. Some texts write $B \setminus A$ instead of $B - A$.
- The **complement** of A, denoted A^c, is the set of all x in U such that $x \notin A$.

Summarising the above:

$$\begin{aligned}
A \cup B &= \{x \in U : x \in A \text{ or } x \in B\}; \\
A \cap B &= \{x \in U : x \in A \text{ and } x \in B\}; \\
B \setminus A &= \{x \in U : x \in B \text{ and } x \notin A\}; \\
A^c &= \{x \in U : x \notin A\}
\end{aligned}$$

$= U$

Example 20. *Let the universal set be* $\{1, 2, \ldots, 10\}$, *and let* $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8, 10\}$ *and* $C = \{1, 3, 5, 7, 9\}$.
Write down the following sets:

B and C are a partion of U

(a) $A \cap B =$ $\{2, 4\}$

(b) $A \cup B =$ $\{1, 2, 3, 4, 6, 8, 10\}$

(c) $B \cup C =$ U

(d) $B \setminus A =$ $\{6, 8, 10\}$

(e) $A \setminus C =$ $\{2, 4\}$

(f) $B^c =$ C

(g) $A^c =$ $\{x \in U : x \geq 5\}$

(h) $A^c \cup B =$

$B \setminus A = B \cap A^c$

$\boxed{B \setminus A = B \cap A^c}$

Definition 7. *Let n be a positive integer (so $n \in \mathbb{Z}^+$), and let $x_1, x_2, \ldots, x_n$ be n not necessarily distinct elements. The* **ordered n-tuple**, *denoted $(x_1, x_2, \ldots, x_n)$, consists of the n elements with their ordering: first x_1, then x_2, and so on up to x_n. (Note round brackets, not braces)*

An ordered 2-tuple is an **ordered pair**.

An ordered 3-tuple is an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if and only if $x_i = y_i$ for all i with $1 \leqslant i \leqslant n$.

T

Example 21. *Is it true that $(3, 1) = (1, 3)$?* F

Example 22. *If $((-2)^2, y, \sqrt{9}) = (4, 3, z)$, find y and z.* $y = 3$, $z = 3$

Definition 8. *Given two sets A and B, the* **Cartesian product** *of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. (Note the word "all") So*

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Similarly

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i,\ 1 \leqslant i \leqslant n\}.$$

A Cartesian product with which you are probably already familiar is the xy-plane. The points in the real xy-plane are the elements of the set $\mathbb{R} \times \mathbb{R}$.

If you plot $(3, 1)$ and $(1, 3)$ you'll see these ordered pairs are not equal.

Example 23. *Let $A = \{-1, 0, 1\}$ and $B = \{x, y\}$. Write out the set $A \times B$. How many elements are in $A \times B$?*

$\boxed{\emptyset \subseteq S}$

Recall that we denote a set with NO elements in it by $\emptyset$.

$\cancel{\emptyset \in S}$

- The empty set is a subset of every set. So if S is any set, we have $\emptyset \subseteq S$.
- Every set is a subset of itself. So if S is any set, we have $S \subseteq S$.

Theorem 3. *The empty set $\emptyset$ is unique.*

Proof We use a contradiction argument. So suppose that $\emptyset_1$ and $\emptyset_2$ are each sets with no elements. Since $\emptyset_1$ has no elements, it is a subset of $\emptyset_2$, that is, $\emptyset_1 \subseteq \emptyset_2$. Also since $\emptyset_2$ has no elements, we have $\emptyset_2 \subseteq \emptyset_1$. Thus $\emptyset_1 = \emptyset_2$, by definition of set equality.

Partitions of sets

- Two sets are called **disjoint** if and only if they have no elements in common.

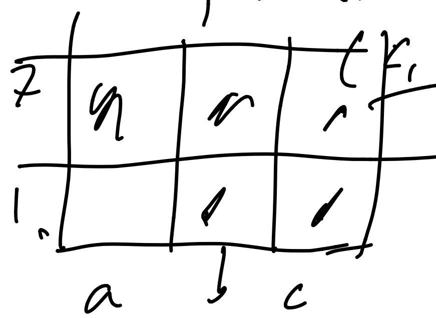
  So A and B are disjoint if and only if $A \cap B = \emptyset$.

$A = \{a, b, c\}$

$B = \{1, 7\}$

$\underline{A \times B} = \{(u,v) : u \in A,\ v \in B\}$

$= \{(a,1), (b,1), (c,1), (a,7), (b,7), (c,7)\}$



$A = \{1,2,3\}$

$A^2 = A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

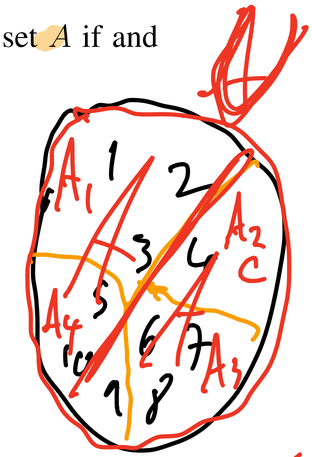
$\boxed{\mathbb{R}^n}$

- Sets $A_1, A_2, \ldots, A_n$ are **mutually disjoint** (or **pairwise disjoint**) if and only if for all pairs of sets A_i and A_j with $i \neq j$, their intersection is empty; that is, if and only if $A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$.

- A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set A if and only if

  (1) $A = \boxed{A_1 \cup A_2 \cup \cdots \cup A_n}$ and $\bigcup_{i=1}^{n} A_i$

  (2) the sets $A_1, A_2, \ldots, A_n$ are mutually disjoint.



$A_i \cap A_j = \emptyset$

$i \neq j$

Example 24. *Determine whether the following statements are true or false.*

(a) $\emptyset = \{\emptyset\}$ F

(b) $A \cup \emptyset = A$ T

(c) $\boxed{A \cap A^c} = \emptyset$ T

(d) $A \cup A^c = \emptyset$ F

(e) $\boxed{A \cap \emptyset} = \emptyset$ T

$A \cap (B^c \cap B)$

(f) $(A - B) \cap B = \emptyset$ T

$\emptyset$

(g) $\{a, b, c\}$ *and* $\{d, e\}$ *are disjoint sets.* T

(h) $\{1, 2\}$, $\{5, 7, 9\}$ *and* $\{3, 4, 5\}$ *are mutually disjoint sets.* F

$\boxed{\bigcup A_i = A}$

Example 25. *Let*

$$A_1 = \{n \in \mathbb{Z} : n < 0\}, \qquad A_2 = \{n \in \mathbb{Z} : n > 0\}.$$

Is $\{A_1, A_2\}$ a partition of $\mathbb{Z}$? If so, explain why; if not, see if you can turn it into a partition with a small change.

Example 26. *Find a partition of $\mathbb{Z}$ into four parts such that none of the four parts is finite in size.*

Definition 9. *Given a set X, the* **power set** *of X is the set of all subsets of X.*
It is denoted by $\mathcal{P}(X)$.

Example 27. *If $B = \{1, 2, 3\}$, write down the set $\mathcal{P}(B)$.*

Example 28. *Let $X = \emptyset$. Write down $\mathcal{P}(X)$, and $\mathcal{P}(\mathcal{P}(X))$.*

If $|S| = n$, how many elements does the power set $\mathcal{P}(S)$ have?

Subset relations

- For all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

- For all sets A and B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.