Image Processing - 22913

Maman 12

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Question 1 (25%)

Write a program that reads a gray-scale image. Sharpen the image using a Laplacian filtering. keep the source and the result.

Solution The source code can be found here: `https://godbolt.org/z/4x5a5MnPj`.

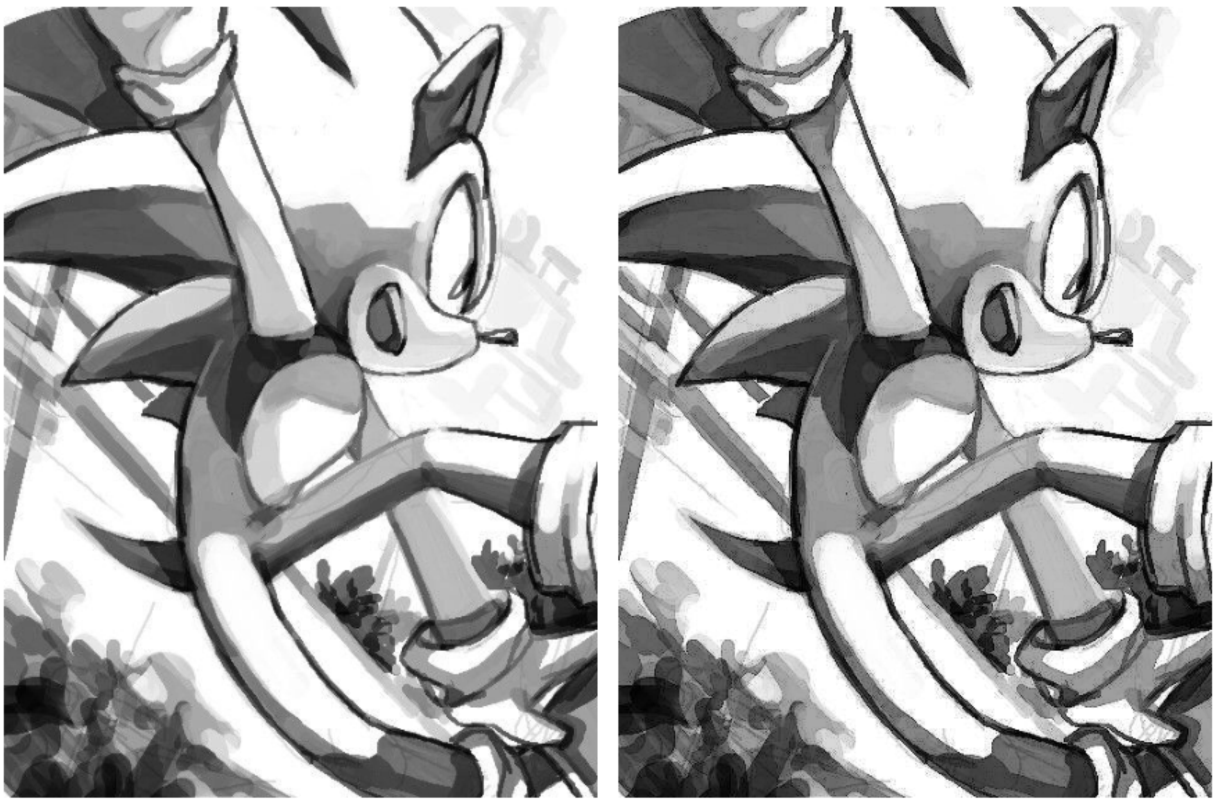

Figure 1: Original (left) compared to sharpened image (right)

Question 2 (25%)

Section 1

Given $x = \{1, 2, 4, 3\}$ and $h = \{-1, 2, -1\}$, calculate $y = h \star x$.

Solution The formula of discrete convolution is

$$f(x) \star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m) \quad x = 0, 1, 2, \ldots, M-1 \tag{4-48}$$

Let's apply that formula to the given series:

$$y(0) = \sum_{m=0}^{6} h(m)x(0-m) = -1 \cdot 1 = -1$$
$$y(1) = \sum_{m=0}^{6} h(m)x(1-m) = -1 \cdot 2 + 2 \cdot 1 = 0$$
$$y(2) = \sum_{m=0}^{6} h(m)x(2-m) = -1 \cdot 4 + 2 \cdot 2 + -1 \cdot 1 = -1$$
$$y(3) = \sum_{m=0}^{6} h(m)x(3-m) = -1 \cdot 3 + 2 \cdot 4 + -1 \cdot 2 = 3$$
$$y(4) = \sum_{m=0}^{6} h(m)x(4-m) = 2 \cdot 3 + -1 \cdot 4 = 2$$
$$y(5) = \sum_{m=0}^{6} h(m)x(5-m) = -1 \cdot 3 = -3$$

To summarize, the convolution of h and x is $y = \{-1, 0, -1, 3, 2, -3\}$ ■

Question 2 continued on next page. . .

Section 2

Prove that the (inverse) Fourier transform of a Gaussian filter is a Gaussian function:

$$\hat{f}(u,v) = \exp\left(-(u^2+v^2)/2\sigma^2\right) \Rightarrow f(x,y) = 2\pi\sigma^2 \exp\left(-2\pi^2\sigma^2(x^2+y^2)\right)$$

Proof

$$
\begin{aligned}
\hat{f}(u,v) &= e^{-(u^2+v^2)/2\sigma^2} \\
\Rightarrow f(x,y) &= \mathfrak{I}^{-1}\left\{e^{-(u^2+v^2)/2\sigma^2}\right\} \\
\Rightarrow f(x,y) &= \iint_{-\infty<u,v<\infty} e^{-(u^2+v^2)/2\sigma^2} \cdot e^{i2\pi(xu+yv)} du\, dv \\
\Rightarrow f(x,y) &= \iint_{-\infty<u,v<\infty} e^{i2\pi(xu+yv)-(u^2+v^2)/2\sigma^2} du\, dv \\
\Rightarrow f(x,y) &= \iint_{-\infty<u,v<\infty} e^{-\frac{1}{2\sigma^2}\left(i2\pi\sigma^2(xu+yv)-(u^2+v^2)\right)} du\, dv \\
\Rightarrow f(x,y) &= \iint_{-\infty<u,v<\infty} e^{-\frac{1}{2\sigma^2}\left(\left(u-i2\pi\sigma^2 x\right)^2+\left(v-i2\pi\sigma^2 y\right)^2+\left(2\sigma^2\pi\right)^2+\left(x^2+y^2\right)\right)} du\, dv \\
\Rightarrow f(x,y) &= \iint_{-\infty<u,v<\infty} e^{-2\pi^2\sigma^2(x^2+y^2)-\frac{1}{2\sigma^2}\left(\left(u-\sigma^2\cdot i2\pi x\right)^2+\left(v-\sigma^2\cdot i2\pi y\right)^2\right)} du\, dv \\
\Rightarrow f(x,y) &= e^{-2\pi^2\sigma^2(x^2+y^2)} \iint_{-\infty<u,v<\infty} e^{-\frac{1}{2\sigma^2}\left(\left(u-\sigma^2\cdot i2\pi x\right)^2+\left(v-\sigma^2\cdot i2\pi y\right)^2\right)} du\, dv \\
\Rightarrow f(x,y) &= e^{-2\pi^2\sigma^2(x^2+y^2)} \cdot 2\pi\sigma^2 \cdot \underbrace{\frac{1}{2\pi\sigma^2} \cdot \iint_{-\infty<u,v<\infty} e^{-\frac{1}{2\sigma^2}\left(\left(u-\sigma^2\cdot i2\pi x\right)^2+\left(v-\sigma^2\cdot i2\pi y\right)^2\right)} du\, dv}_{=1 \text{ (2D Gaussian distribution)}} \\
\Rightarrow f(x,y) &= e^{-2\pi^2\sigma^2(x^2+y^2)} \cdot 2\pi\sigma^2
\end{aligned}
$$

■

Question 3 (25%)

Show that the 1-D convolution theorem given in Eqs. (4-25) and (4-26) also holds for discrete variables, but with the right side of Eq. (4-26) multiplied by $1/M$. That is, show that:

(a) $(f \star h)(x) \iff (F \bullet H)(u)$

(b) $(f \bullet h)(x) \iff \frac{1}{M}(F \star H)(u)$

Section 1

$$
\begin{aligned}
\Im\left[(f \star h)(x)\right] &= \Im\left[\sum_{x=0}^{M-1} f(m)h(x-m) \quad x = 0, 1, 2, \ldots, M-1\right] && \text{Eq 4-48} \\
&= \Im\left[\sum_{x=0}^{M-1}\sum_{m=0}^{M-1} f(m)h(x-m)\right] \\
&= \sum_{x=0}^{M-1}\sum_{m=0}^{M-1} f(m)h(x-m)e^{-i2\pi ux/M} \quad u = 0, 1, 2, \ldots, M-1 && \text{Eq 4-44} \\
&= \sum_{m=0}^{M-1} f(m)\left[\sum_{x=0}^{M-1} h(x-m)e^{-i2\pi ux/M}\right] \\
&= \sum_{m=0}^{M-1} f(m)H(u)e^{-i2\pi um/M} && \text{Ex 4.17} \\
&= H(u)\sum_{m=0}^{M-1} f(m)e^{-i2\pi um/M} \\
&= H(u)F(u) \\
&= (F \bullet H)(u) && \blacksquare
\end{aligned}
$$

Question 3 continued on next page. . .

Section 2

$$
\begin{aligned}
\Im\left[(f \bullet h)(x)\right] &= \Im\left[h(x)\cdot\frac{1}{M}\sum_{u=0}^{M-1}F(u)e^{i2\pi ux/M}\right] && \text{Eq 4-45}\\
&= \Im\left[\sum_{u=0}^{M-1}F(u)\cdot\frac{1}{M}\sum_{u=0}^{M-1}h(x)e^{i2\pi ux/M}\right]\\
&= \Im\left[\frac{1}{M}\sum_{u=0}^{M-1}F(u)\sum_{u=0}^{M-1}H(x-u)\right] && \text{Ex 4.17}\\
&= \frac{1}{M^2}\sum_{x=0}^{M-1}\sum_{u=0}^{M-1}F(u)\sum_{u=0}^{M-1}H(x-u)e^{i2\pi ux/M} && \text{Eq 4-44}\\
&= \frac{1}{M}\sum_{x=0}^{M-1}\sum_{u=0}^{M-1}F(u)H(x-u) && \text{Eq 4-45}\\
&= \frac{1}{M}\left(F\star H\right)(u) && \text{Eq 4-48} \qquad \blacksquare
\end{aligned}
$$

Question 4 (25%)

Section 1

$$\hat{f}\left(k_x, k_y\right) = \iint_{-\infty}^{\infty} f(x, y) e^{-i2\pi(k_x x + k_y y)} dx\, dy \tag{2}$$

The equation above describes the 2D Fourier transform of the object whose density is described using the function $f(x, y)$. Note that k_x, k_y are the components of the frequency domain.

Section 2

$$s(k_x) = \hat{f}\left(k_x, 0\right) = \iint_{-\infty}^{\infty} f(x, y) e^{-i2\pi(k_x x)} dx\, dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x, y) dy\right) e^{-i2\pi k_x x} dx \tag{3}$$

The equation above describes the 1D Fourier transform of *one slice* of the object whose density is described using the function $f(x, y)$. Note that the *one slice* described in this equation is the very one where $k_y = 0$, i.e. top view of the frequency domain.

Section 3

$$s(k_x) = \int_{-\infty}^{\infty} p(x) e^{-i2\pi k_x x} dx = \hat{p}(k_x) \tag{4}$$

The transition from equation (3) to equation (4) is done using equation (1):

$$P(x) = \int_{-\infty}^{\infty} f(x, y) dy \tag{1}$$

which describes the sum of the density of the points through which the ray passed that hit the array of receivers at point x. Note that equation (3) was already re-arranged with parentheses that indicate the location of the very specific integral that is the integral of equation (1).

Section 4

The image $f(x, y)$ can be reconstructed using $s(k_x)$ by using the Fourier-slice theorem, i.e. **taking many slices at different angles** (that can be calculated using polar coordinates of k_x), and then reconstruct the image $f(x, y)$ using **interpolation on those slices** - which is used to estimate the values between these slices. Various interpolation methods can be used, such as nearest-neighbor, bi-linear, cubic spline interpolation and more.

After interpolating the Fourier-transformed projection data and filling in the gaps, we should perform the **inverse 2D Fourier transform** on the data to bring it back from the frequency domain to the spatial domain. This results in a reconstructed image that represents the attenuation properties of the object.