Linear Algebra II - 20229

Maman 13

Yehonatan Simian
206584021

Question 1 (20%)

Let $A = \begin{pmatrix} 0 & a & 1 \\ a & 0 & -1 \\ 0 & 0 & a \end{pmatrix}, a \in \mathbf{R}$.

Section 1

For which values of a does the matrix A diagonalizable?

Solution Let's find A's characteristic polynomial: $p_A(t) = \begin{vmatrix} t & -a & -1 \\ -a & t & 1 \\ 0 & 0 & t-a \end{vmatrix} = (t-a)(t^2-a^2) = (t-a)^2(t+a)$. Note that I started with the last row to ease the calculation. We've got two options:

- $\mathbf{a = 0}$: the matrix A has a $\mathbf{0}$ row, hence not diagonalizable.
- $\mathbf{a \neq 0}$: the matrix A has two eigenvalues: $\lambda_1 = a, \lambda_2 = -a$ let's check the algebraic and geometric multiplicities of each eigenvalue.
  - $\lambda_1 = a$: algebraic multiplicity is 2. Let's find the geometric multiplicity:

$$aI - A = \begin{pmatrix} a & -a & -1 \\ -a & a & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} a & -a & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

    Hence $\rho(aI - A) = 1$ and the geometric multiplicity is 2.
  - $\lambda_2 = -a$: algebraic multiplicity is 1. From Algebra I we know that the geometric multiplicity is always less or equal to the algebraic multiplicity, and also different than zero, hence the geometric multiplicity is also 1.

  We found that the characteristic polynomial can be completely factored into linear factors, and the algebraic multiplicity of each eigenvalue equals its geometric multiplicity, therefore the matrix is diagonalizable.

To summarize, the matrix A is diagonalizable for $a \neq 0$. ■

Question 1 continued on next page...

Section 2

Let $a = 1$, i.e. $A\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ 0 & 0 & 1\end{pmatrix}$. Find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$. Use D to calculate A^{2021}.

Solution In the last section we found that the eigenvalues of A are $\lambda_1 = 1$ with algebraic multiplicity of 2 and $\lambda_2 = -1$ with algebraic multiplicity of 1, hence the matrix A is similar to the matrix $D = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$. Let's find the matrix P, which is composed by the eigenvectors of A. Let V_1, V_{-1} the eigenspaces of the eigenvalues $1, -1$ respectively:

- V_1 is the solution space of $(I - A)\mathbf{x} = \mathbf{0}$:
$$I - A = \begin{pmatrix}1 & -1 & -1\\ -1 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \to \begin{pmatrix}1 & -1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$
Hence $B_1 = \{(1,1,0),(1,0,1)\}$ is a base of V_1.

- V_{-1} is the solution space of $(-I - A)\mathbf{x} = \mathbf{0}$:
$$-I - A = \begin{pmatrix}-1 & -1 & -1\\ -1 & -1 & 1\\ 0 & 0 & -2\end{pmatrix} \to \begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$
Hence $B_{-1} = \{(1,-1,0)\}$ is a base of V_{-1}.

Thus, $B = \{(1,1,0),(1,0,1),(1,-1,0)\}$ is a base of $\mathbb{R}^3$ that is composed by the eigenvectors of A, hence $P = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$ is an invertible matrix that diagonalizes A, i.e. $D = P^{-1}AP$. Let's calculate P^{-1}: $\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1\end{array}\right) \to \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 0.5 & -0.5 & -0.5\\ 0 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0.5 & -0.5 & -0.5\end{array}\right)$. Now let's use D to calculate A^{2021}: $D = P^{-1}AP \Rightarrow A = PDP^{-1} \Rightarrow A^{2021} = (PDP^{-1})^{2021} = PD^{2021}P^{-1}$, hence

$$A = \begin{pmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}^{2021}\begin{pmatrix}0.5 & -0.5 & -0.5\\ 0 & 0 & 1\\ 0.5 & -0.5 & -0.5\end{pmatrix} = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & -1\\ 0 & 0 & 1\end{pmatrix} \quad \blacksquare$$

NOTE $D^{2021} = D$. *In addition, I skipped the last calculation in order to make it fit in one page :)*

Question 2 (20%)

Section 1

Prove that no matrix of rank 3 exists with the characteristic polynomial $p(x) = x^7 - x^5 + x^3$.

Proof Let's assume that such matrix exists. That matrix would be a 7×7 matrix, with eigenvalue 0 that has an algebraic multiplicity of 3 (since $p(x) = x^3(x^4 - x^2 + 1)$). The geometric multiplicity of 0 must equal the dimension of the solution space of that matrix, which is 4 since the rank of that matrix is 3. We found out that the geometric multiplicity of 0 is greater than its algebraic multiplicity, in contradiction to the theorem from Algebra I which states that the the geometric multiplicity must be lesser or equal to the algebraic multiplicity. We found a contradiction, hence the assumption is false, i.e. no matrix of rank 3 exists with the characteristic polynomial $p(x) = x^7 - x^5 + x^3$. ■

Section 2

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ a linear transformation with characteristic polynomial $p(x) = x^2 + 2x - 3$.

1. Prove that the linear transformation $S = 2T + I$ is an isomorphism.

   Proof $p(x)$ can be factored into $(x + 3)(x - 1)$, therefore the eigenvalues of T are $\lambda_1 = -3$ and $\lambda_2 = 1$. S can be re-written as $-2(-0.5I - T)$, and λ is an eigenvalue of T if and only if $\exists_{0 \neq v \in \mathbb{R}^2}$ s.t. $T(v) = \lambda v$, i.e. $(\lambda I - T)v = 0$, i.e. T is not an isomorphism. However, $\lambda = -0.5 \notin \{\lambda_1, \lambda_2\}$ is *not* an eigenvalue of T, therefore (and by the properties of "if and only if") $-0.5I - T = -0.5S$ *is* an isomorphism, thus making S an isomorphism as well. ■

2. What is the characteristic polynomial of T^3?

   Solution $\lambda_1 = -3$ and $\lambda_2 = 1$ are the eigenvalues of T, hence $\lambda_1^3 = -27$ and $\lambda_2^3 = 1$ are the eigenvalues of T^3, therefore the characteristic polynomial of T^3 is $(x + 27)(x - 1) = x^2 + 26x - 27$. ■

Question 2 continued on next page. . .

Section 3

Let A a singular 4×4 matrix. Given $\rho(A + 2I) = 2$ and $\det(A - 2I) = 0$, what is the characteristic polynomial of A? Is it diagonalizable?

Solution A is singular, thus making $\lambda_1 = 0$ its eigenvalue. In addition, $\det(2I - A) = (-1)^4 \det(A - 2I) = 0$, thus making $\lambda_2 = 2$ another eigenvalue. In additioner, $\rho(A + 2I) = \rho(-2I - A) = 2$, thus making $\lambda_3 = -2$ another eigenvalue, with a geometric multiplicity of $4 - 2 = 2$, hence its algebraic multiplicity must be at least 2. We found 3 eigenvalues of A, one of them with algebraic multiplicity of 2, hence the algebraic multiplicity of the other eigenvalues must be 1, thus making the characteristic polynomial not other than the monic polynomial of rank 4: $x(x - 2)(x + 2)^2$.

Note that the geometric multiplicity of each eigenvalue must be at least one and no greater than the algebraic multiplicity, thus making the geometric multiplicity of each eigenvalue of A equal to is algebraic multiplicity. In additionest, that characteristic polynomial can be factored into linear factors, thus making the matrix A diagonalizable. ■

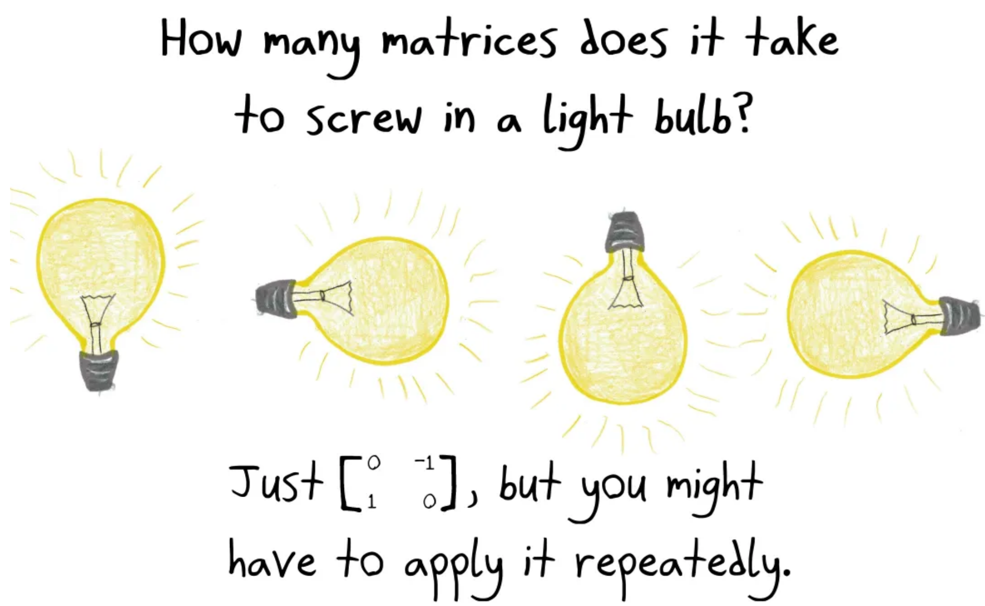

Figure 1: Nerds' Meme

Question 3 (15%)

Let A a diagonalizable $n \times n$ matrix. Let $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ its characteristic polynomial. Let $p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$ an $n \times n$ matrix. Prove: $p(A) = 0$.

Proof A is diagonalizable, therefore there exist a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Hence, $\forall_{k \in \mathbb{N}} : A^k = (PDP^{-1})^k = PD^k P^{-1}$. Now, we can re-write $p(A)$ the following way:

$$p(A) = \sum_{k=0}^{n} a_k A^k = \sum_{k=0}^{n} a_k P D^k P^{-1} = P \left(\sum_{k=0}^{n} a_k D^k \right) P^{-1} = P p(D) P^{-1}$$

Let $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A. $\forall_{0 \le k \le n} : a_k D^k = a_k \begin{pmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} = \begin{pmatrix} a_k \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_k \lambda_n^k \end{pmatrix}$.

Now, we can re-write $p(D)$ the following way:

$$p(D) = \sum_{k=0}^{n} a_k D^k = \sum_{k=0}^{n} \begin{pmatrix} a_k \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_k \lambda_n^k \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{n} a_k \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{k=0}^{n} a_k \lambda_n^k \end{pmatrix} = \begin{pmatrix} p(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p(\lambda_k) \end{pmatrix}$$

and since $\forall_{1 \le i \le n} : \lambda_i$ is an eigenvalue of A, $p(\lambda_i) = 0$, i.e. $p(D)$ is the zero matrix. ■

Question 4 (20%)

Prove or disprove each of the following statements:

Section 1

If the matrices A and B have the same characteristic polynomial, then they have the same rank.

Solution Let $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; both have the same characteristic polynomial $p(x) = 0$, however the former has a rank of 0 whilst the latter's rank is 1, hence the statement is *not* true. ∎

Section 2

The matrices $A = \begin{pmatrix} 2 & -\sqrt{3} \\ \sqrt{3} & -2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are similar.

Proof B's eigenvalues are easy to find, $\lambda_1 = 1$ and $\lambda_2 = -1$. Let's find A's eigenvalues to see if they match:

$$\det(A - \lambda I) = (2 - \lambda)(-2 - \lambda) + 3 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

We found that A and B have the same eigenvalues (with geometric multiplicity of 1 each of course), hence they are similar. ∎

Section 3

The matrices $A = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}$ and $\begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -4 \end{pmatrix}$ are similar.

Proof B is a triangular matrix, hence its characteristic polynomial is $p(x) = (x + 4)^2(x + 1)$, i.e. its eigenvalues are $\lambda_1 = -4$ with algebraic multiplicity of 2 and $\lambda_2 = -1$ with algebraic multiplicity of 1. Let's find the geometric multiplicity of λ_1:

$$-4I - B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & -5 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $B_1 = \{(1, 0, 0), (0, 0, 1)\}$ is a basis of the solution space of $(-4I - A)\mathbf{x} = \mathbf{0}$, i.e. the geometric multiplicity of λ_1 is 2. We found that the geometric multiplicity of each eigenvalue equals its algebraic multiplicity, and the characteristic polynomial can be factored into linear factors, hence B is diagonalizable. We'll continue on the next page, don't go anywhere!

Question 4 continued on next page. . .

Alright I'm glad you came back yayyyy. Let's find A's eigenvalues:

$$\begin{aligned}
p(x) &= \begin{vmatrix} t+3 & -1 & -1 \\ -1 & t+3 & -1 \\ -1 & -1 & t+3 \end{vmatrix} \\
&= \begin{vmatrix} t+1 & t+1 & t+1 \\ -1 & t+3 & -1 \\ -1 & -1 & t+3 \end{vmatrix} \\
&= (t+1)\begin{vmatrix} 1 & 1 & 1 \\ -1 & t+3 & -1 \\ -1 & -1 & t+3 \end{vmatrix} \\
&= (t+1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & t+4 & 0 \\ 0 & 0 & t+4 \end{vmatrix} \\
&= (t+1)(t+4)^2
\end{aligned}$$

So... we've already found out that A and B have the same eigenvalues. Hooray! What's left? What's... left. What is it that is left. Left is not right. Not right is not alright. I think I'd better go to sleep soon. Soon. Noon. Moon. Okay enough nonsense, let's find the geometric multiplicity of -4 and end this proof already:

$$-4I - A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence the geometric multiplicity of this eigenvalue is also 2. We've found out the both A and B have the same eigenvalues with the same algebraic and geometric multiplicities, hence both are similar to $\begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, therefore by transitivity both are similar as well. ∎

Question 5 (15%)

Let $T : M^{\mathbb{R}}_{2\times2} \to M^{\mathbb{R}}_{2\times2}$ the linear transform defined by $T(A) = A - A^t$ for all $A \in M^{\mathbb{R}}_{2\times2}$.

Section 1

Find a basis for $\operatorname{Ker}(T)$. What is the dimension of $\operatorname{Im}(T)$? Find a basis of $\operatorname{Im}(T)$.

Solution The kernel of T is all matrices $A \in M^{\mathbb{R}}_{2\times2}$ that satisfy $T(A) = 0 \Rightarrow A - A^t = 0 \Rightarrow A = A^t$. Those matrices have the form of $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$, hence are spanned by $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$. Note that those three matrices are linearly independent, hence they form a basis of $\operatorname{Ker}(T)$. Since the dimension of the kernel of T is 3, and the dimension of $M^{\mathbb{R}}_{2\times2}$ is 4, necessarily the dimension of the image of T must be 1.

To find a basis for $\operatorname{Im}(T)$ we need to determine the form of the matrices that lie that image. $\forall_{A\in M^{\mathbb{R}}_{2\times2}} : \left(A - A^t\right)^t = A^t - A = -\left(A - A^t\right)$, hence $T(A)$ is always a skew-symmetric matrix, i.e. has the form of $\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, hence is it spanned by $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$ and we found a basis for $\operatorname{Im}(T)$.

Section 2

Prove that T is diagonalizable and write a diagonal matrix that represents T, as well as a matrix P that diagonalizes $[T]_E$.

Proof To diagonalize T, we must first find its eigenvalues and eigenvectors. The standard basis $E = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $M^{\mathbb{R}}_{2\times2}$ is utilized to compute the action of T on each basis matrix:

$$T(E_{11}) = 0, \quad T(E_{12}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T(E_{21}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T(E_{22}) = 0.$$

Hence $[T]_E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $|tI - A| = \begin{vmatrix} t & 0 & 0 & 0 \\ 0 & t-1 & 1 & 0 \\ 0 & 1 & t-1 & 0 \\ 0 & 0 & 0 & t \end{vmatrix} = t \begin{vmatrix} t-1 & 1 & 0 \\ 1 & t-1 & 0 \\ 0 & 0 & t \end{vmatrix} = t^3(t-2)$

According to the last section, the eigenvalue of 0 has a geometric multiplication of 3, thus making the geometric multiplication of each eigenvalue of T equal its algebraic multiplication. We also found that the characteristic polynomial of T can be factored into linear factors, thus making T diagonalizable. ■

Question 6 (12%)

Let A a singular 3×3 matrix. Given $\rho(A + 3I) = 2$ and $\det(A - I) = 0$:

Section 1

What is the characteristic polynomial of A? What is it's trace? Is it diagonalizable?

Solution A is singular, thus making $\lambda_1 = 0$ its eigenvalue. In addition, $\det(I - A) = -\det(A - I) = 0$, thus making $\lambda_2 = 1$ another eigenvalue. In additioner, $\rho(A + 3I) = \rho(-3I - A) = 2$, thus making $\lambda_3 = -3$ another eigenvalue, with a geometric multiplicity of $3-2 = 1$, hence its algebraic multiplicity must be at least 1. We found 3 eigenvalues of A, each of them with algebraic multiplicity of 1, hence the geometric multiplicities of all eigenvalues must be also 1, thus making the characteristic polynomial not other than the monic polynomial of rank 3: $x(x - 1)(x + 3)$, and trace of -2.

Note that the geometric multiplicity of each eigenvalue must be at least one and no greater than the algebraic multiplicity, thus making the geometric multiplicity of each eigenvalue of A equal to is algebraic multiplicity. In additionest, that characteristic polynomial can be factored into linear factors, thus making the matrix A diagonalizable. ∎

NOTE Did anyone else feel an uncanny, strange Déjà-vu?

Section 2

Is the matrix $A - 3I$ invertible?

Solution If λ is an eigenvalue of A, then $\lambda - 3$ is an eigenvalue of $A - 3I$, thus making the eigenvalues of A $\{-3, -2, 0\}$. We found out that $A - 3I$ has 0 as its eigenvalue, hence it is similar to the zero matrix, therefore it is *not* invertible. ∎