Differential Hoare Logics and Refinement Calculi for Hybrid Systems with Isabelle/HOL

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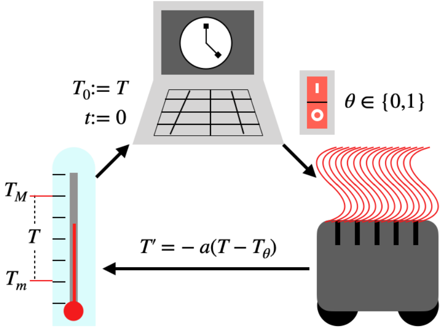
Verification of Hybrid Systems



$$
\begin{aligned}
\text{dynamics} &= T' = -a(T - T_\theta) \\
\text{pre} &= T_m \le T \le T_M \\
\text{pos} &= T_m \le T \le T_M \\
\text{control} &= t := 0\,;\, T_0 := T\,;\, \ldots \\
\text{therm} &= (\text{control}\,;\,\text{dynamics})^* && \text{hybrid program} \\
&\{\text{pre}\}\ \text{therm}\ \{\text{pos}\} && \text{correctness spec}
\end{aligned}
$$

Previous Work

- Isabelle/HOL verification components for hybrid programs that
  - benefit from huge libraries of topology, analysis, and ODEs;
  - based on MKA;
  - work with weakest liberal preconditions;
  - support various verification procedures for systems of ODEs, and
  - are correct by construction.

- Yet, program verification requires simpler methods:
  - Hoare logic serves for program verification, and
  - Morgan's refinement calculus is closely related.

Can we obtain simpler Hoare logic and refinement calculus to do hybrid program verification?

Main Contributions

Development of minimal proof systems for verification of hybrid systems:

1. rules of differential Hoare logic $\mathsf{d}\mathcal{H}$ based on KAT;
2. laws of differential refinement calculus $\mathsf{d}\mathcal{R}$ based on rKAT;
3. integration of lenses as the store model;
4. support invariant reasoning in the style of differential dynamic logic $\mathsf{d}\mathcal{L}$, and
5. tactics for automated verification condition generation.

https://github.com/yonoteam/CPSVerification

Kleene Algebras with Tests

Kleene Algebra

sequential composition

nondeterministic choice

finite iteration

$$(K, +, ;, 0, 1, ^*)$$

actions of programs

abort

skip

Tests

- $(B, +, ;, 0, 1, \neg)$ is a boolean algebra,
- use $\alpha, \beta \in K$ and $p, q \in B$ where $B \subseteq K$,
- $\textbf{if } p \textbf{ then } \alpha \textbf{ else } \beta = p \,;\, \alpha + \neg p \,;\, \beta$, and
- $\{p\}\,\alpha\,\{q\} \leftrightarrow p \,;\, \alpha \leq \beta \,;\, q$

State Transformer Model

Programs are functions $S \to \mathcal{P}\, S$:

$$
\begin{aligned}
(\alpha + \beta)\, s &= \alpha\, s \cup \beta\, s, \\
(\alpha\,;\,\beta)\, s = (\alpha \circ_K \beta)\, s &= \bigcup \{\beta\, s' \mid s' \in \alpha\, s\}, \\
0\, s &= \emptyset, \\
1\, s &= \{s\}, \\
(\neg p)\, s &= \begin{cases} \{s\}, & \text{if } p\, s = \emptyset, \\ \emptyset, & \text{otherwise}, \end{cases} \\
\alpha^*\, s &= \bigcup_{n \geq 0} \alpha^n\, s,
\end{aligned}
$$

where $\alpha^0\, s = 1\, s$ and $\alpha^{n+1} = \alpha^n \circ_K \alpha$.

$$\{p\}\, \alpha\, \{q\} \leftrightarrow (\forall s_1.\ p\, s_1 \to (\forall s_2.\ s_2 \in \alpha\, s_1 \to q\, s))$$

What about Assignments?

Lenses

- Variables are lenses $x = (A, S, get_x, put_x)$ where

$$get_x : S \to A \text{ and } put_x : S \to A \to S$$

- A provides values of variable while S is the state space
- They satisfy the axioms

$$\begin{aligned} get_x\ (put_x\ s\ v) &= v \\ put_x\ (put_x\ s\ u)\ v &= put_x\ s\ v, \\ put_x\ s\ (get_x\ s) &= s. \end{aligned}$$

Semantics $S \to \mathcal{P}\, S$ for assignments are

$$(x := e)\, s = \{put_x\, s\, (e\, s)\}$$

Verification Rules

- Traditional Hoare logic:

$$
\begin{aligned}
p_1 \le p_2 \wedge \{p_2\}\,\alpha\,\{q_2\} \wedge q_2 \le q_1 &\rightarrow \{p_1\}\,\alpha\,\{q_1\}\\
\{p\}\,\alpha\,\{r\} \wedge \{r\}\,\beta\,\{q\} &\rightarrow \{p\}\,\alpha\,;\beta\,\{q\},\\
\{r\,;p\}\,\alpha\,\{q\} \wedge \{\neg r\,;p\}\,\beta\,\{q\} &\rightarrow \{p\}\,\mathbf{if}\ r\ \mathbf{then}\ \alpha\ \mathbf{else}\ \beta\,\{q\},\\
\{r\,;p\}\,\alpha\,\{p\} &\rightarrow \{p\}\,\mathbf{while}\ r\ \mathbf{do}\ \alpha\,\{\neg r\,;p\},\\
&\quad\ \{\lambda s.\ q\,(put_x\,s\,(e\,s))\}\,x := e\,\{q\}.
\end{aligned}
$$

- Extended to regular programs

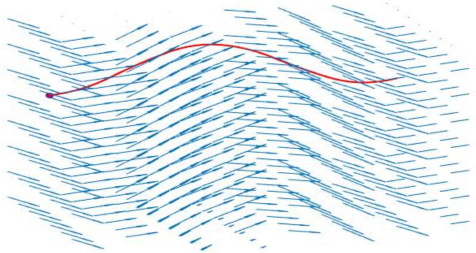
$$
\begin{aligned}
&\{p\}\,\mathbf{skip}\,\{p\},\\
&\{p\}\,\mathbf{abort}\,\{q\},\\
\{p\}\,\alpha\,\{q\} \wedge \{p\}\,\beta\,\{q\} \rightarrow\ &\{p\}\,\alpha+\beta\,\{q\},\\
\{p\}\,\alpha\,\{p\} \rightarrow\ &\{p\}\,\mathbf{loop}\ \alpha\,\{p\},
\end{aligned}
$$

where $\mathbf{loop}\ \alpha = \alpha^*$, $\mathbf{skip} = 1$, and $\mathbf{abort} = 0$.

What about ODEs?

Vector Field

$$X' \ t = f \ t \, (X \ t)$$



where

$$X : T \subseteq \mathbb{R} \to S \qquad f : T \to S \to S \qquad X \, 0 = s$$

$$\text{orbit} : s \mapsto \{X \ t \mid t \in T\}$$

Semantics for ODEs

- Solutions to initial value problems (IVPs)

$$\mathsf{Sols}\, f\; T\; s = \{X : T \to S \mid (\forall t \in T.\; X'\, t = f\; t\,(X\; t)) \wedge X\, 0 = s\}$$

- Guarded orbit

$$\mathsf{orbit}_G^X\, s = \{X\; t \mid t \in T \wedge (\forall \tau \in [0, t].\; G\,(X\; \tau))\}$$

- Semantics $S \to \mathcal{P}\, S$ for assignments are

$$(x' = f\; \&\; G)\, s = \bigcup \{\mathsf{orbit}_G^X\, s \mid X \in \mathsf{Sols}\, f\; T\; s\}$$

- The corresponding rule of inference is

$$\{\lambda\, s.\; \forall t \in T.\; (\forall \tau \in [0, t].\; G\,(X\; t)) \to Q\,(X\; t)\}\; (x'{=}f\; \&\; G)\; \{Q\}$$

easy to obtain if there is a unique solution $X : T \to S$ to the IVPs associated to each s and the vector field f

Invariants in d$\mathcal{H}$

- I is an invariant for f iff $\{I\}\, x'{=}f \,\&\, G\, \{I\}$, or equivalently

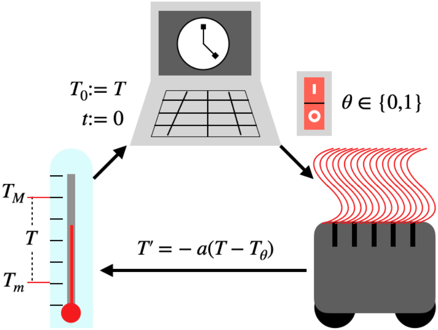
$$\bigcup(\mathcal{P}\,(x'{=}f \,\&\, G)\, I) \subseteq I,$$

- We obtain the following rules

$$\begin{aligned}
p \leq i \wedge \{i\}\, \alpha\, \{i\} \wedge i \leq q \;&\rightarrow \{p\}\, \alpha \;\textbf{inv}\; i\, \{q\},\\
\{i\}\, \alpha\, \{i\} \wedge \{j\}\, \alpha\, \{j\} &\rightarrow \{i\,;j\}\, \alpha\, \{i\,;j\},\\
\{i\}\, \alpha\, \{i\} \wedge \{j\}\, \alpha\, \{j\} &\rightarrow \{i+j\}\, \alpha\, \{i+j\},\\
p \leq i \wedge \{i\,;t\}\, \alpha\, \{i\} \wedge \neg r\,;i \leq q \;&\rightarrow \{p\}\, \textbf{while}\; r \;\textbf{do}\; \alpha \;\textbf{inv}\; i\, \{q\},\\
p \leq i \wedge \{i\}\, \alpha\, \{i\} \wedge i \leq q \;&\rightarrow \{p\}\, \textbf{loop}\; \alpha \;\textbf{inv}\; i\, \{q\},\\
p \leq i \wedge i \text{ is inv. for } f \wedge (G\,;i) \leq q \;&\rightarrow \{p\}\, x' = f \,\&\, G \;\textbf{inv}\; i\, \{q\}.
\end{aligned}$$

where operationally α **inv** $i = \alpha$.

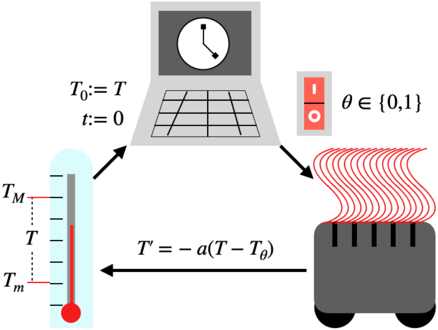
Formalisation of the Thermostat



Lenses $\Pi[n] = (\mathbb{R}, \mathbb{R}^{\{0,1,2,3\}}, \lambda s.\ s\ n, \lambda s\ t.\ s[n \mapsto t])$ give us variables

abbreviation T :: *real* $\Longrightarrow$ *real^4* **where** $T \equiv \Pi[0]$
abbreviation t :: *real* $\Longrightarrow$ *real^4* **where** $t \equiv \Pi[1]$
abbreviation T_0 :: *real* $\Longrightarrow$ *real^4* **where** $T_0 \equiv \Pi[2]$
abbreviation ϑ :: *real* $\Longrightarrow$ *real^4* **where** $\vartheta \equiv \Pi[3]$

Formalisation of the Thermostat



Provide vector field and unique solution

abbreviation $f\ a\ c \equiv [T \mapsto_s -(a * (T - c)),\ T_0 \mapsto_s 0,\ \vartheta \mapsto_s 0,\ t \mapsto_s 1]$

abbreviation $\varphi\ a\ c\ \tau \equiv [T \mapsto_s -exp(-a * \tau) * (c - T) + c,$
$T_0 \mapsto_s T_0,\ \vartheta \mapsto_s \vartheta,\ t \mapsto_s \tau + t]$

Verification of the Thermostat

abbreviation $G\ T_m\ T_M\ a\ L \equiv$
$\mathbf{U}(t \leq -\ (\mathit{ln}\ ((L-(\mathit{if}\ L{=}0\ \mathit{then}\ T_m\ \mathit{else}\ T_M))/(L-T_0)))/a)$

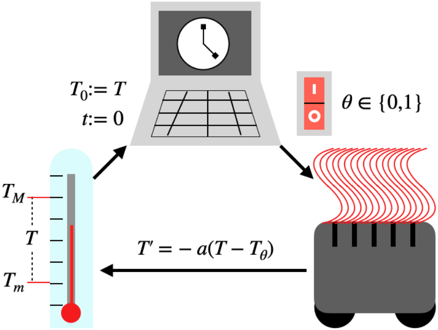
abbreviation $I\ T_m\ T_M \equiv \mathbf{U}(T_m \leq T \wedge T \leq T_M \wedge (\vartheta = 0 \vee \vartheta = 1))$

abbreviation $\mathit{ctrl}\ T_m\ T_M \equiv$
$(t ::= 0); (T_0 ::= T);$
$(\mathit{IF}\ (\vartheta = 0 \wedge T_0 \leq T_m + 1)\ \mathit{THEN}\ (\vartheta ::= 1)\ \mathit{ELSE}$
$\mathit{IF}\ (\vartheta = 1 \wedge T_0 \geq T_h - 1)\ \mathit{THEN}\ (\vartheta ::= 0)\ \mathit{ELSE\ skip})$

abbreviation $\mathit{dyn}\ T_m\ T_M\ a\ T_u\ \tau \equiv$
$\mathit{IF}\ (\vartheta = 0)\ \mathit{THEN}\ x' = f\ a\ 0\ \&\ G\ T_m\ T_M\ a\ 0\ \mathit{on}\ \{0..\tau\}\ \mathit{UNIV}\ @\ 0$
$\mathit{ELSE}\ x' = f\ a\ T_u\ \&\ G\ T_m\ T_M\ a\ T_u\ \mathit{on}\ \{0..\tau\}\ \mathit{UNIV}\ @\ 0$

abbreviation $\mathit{therm}\ T_m\ T_M\ a\ L\ \tau \equiv$
$\mathit{LOOP}\ (\mathit{ctrl}\ T_m\ T_M\ ;\ \mathit{dyn}\ T_m\ T_M\ a\ L\ \tau)\ \mathit{INV}\ (I\ T_m\ T_M)$

Verification of the Thermostat



lemma *thermostat-flow*:
  assumes $0 < a$ **and** $0 \leq \tau$ **and** $0 < T_m$ **and** $T_M < T_u$
  shows $\{I\ T_m\ T_M\}$ *therm* $T_m\ T_M\ a\ T_u\ \tau\ \{I\ T_m\ T_M\}$
  apply (*hyb-hoare* **U**$(I\ T_m\ T_M \wedge t{=}0 \wedge T_0 = T)$)
  prefer *4* **prefer** *8* **using** *local-flow-therm assms* **apply** *force+*
  using *assms therm-dyn-up therm-dyn-down* **by** *rel-auto′*

Differential Refinement Calculus d$\mathcal{R}$

Extend KAT with refinement operation $[-,-] : B \times B \to K$ such that

$$\{p\}\,\alpha\,\{q\} \leftrightarrow \alpha \leq [p,q].$$

Obtain traditional Morgan Style Refinement laws:

$$\begin{aligned}
\mathbf{skip} &\leq [p,p], \\
\mathbf{abort} &\leq [p,q], \\
[p',q'] &\leq [p,q], && \text{if } p \leq p' \text{ and } q' \leq q, \\
[p,r]\,;[r,q] &\leq [p,q], \\
[p,q] + [p,q] &\leq [p,q], \\
\mathbf{if}\ t\ \mathbf{then}\ [t\,;p,q]\ \mathbf{else}\ [\neg t\,;p,q] &\leq [p,q], \\
\mathbf{while}\ t\ \mathbf{do}\ [t\,;p,p] &\leq [p,\neg t\,;p], \\
\mathbf{loop}\ [p,p] &\leq [p,p].
\end{aligned}$$

More Refinement Laws

- Laws for assignments $(x := e) \leq [\lambda s.\ Q\,(put_x\, s\,(e\, s)), Q]$.
- Laws for evolution commands where $X'\, t = f\,(X\, t)$ and $X\, 0 = s$

$$(x' = f\ \&\ G) \leq [\lambda s \in S. \forall t \in T.\ (\forall \tau \in [0, t].\ G\,(X\, \tau)) \rightarrow Q\,(X\, t), Q].$$

- Monotonoic laws and laws with invariants

$$\begin{aligned}
\textbf{if}\ t\ \textbf{then}\ \alpha_1\ \textbf{else}\ \beta_1 \leq \textbf{if}\ t\ \textbf{then}\ \alpha_2\ \textbf{else}\ \beta_2, &\quad \text{if } \alpha_1 \leq \alpha_2 \text{ and } \beta_1 \leq \beta_2;\\
\textbf{while}\ t\ \textbf{do}\ \alpha_1 \leq \textbf{while}\ t\ \textbf{do}\ \alpha_2, &\quad \text{if } \alpha_1 \leq \alpha_2;\\
\textbf{loop}\ \alpha_1 \leq \textbf{loop}\ \alpha_2, &\quad \text{if } \alpha_1 \leq \alpha_2;\\
\textbf{while}\ t\ \textbf{do}\ \alpha\ \textbf{inv}\ i \leq [p, q] &\quad \text{if } p \leq i\,;\, t \text{ and } \alpha \leq [i, i] \text{ and } \neg t\,;\, i \leq q;\\
\textbf{loop}\ \alpha\ \textbf{inv}\ i \leq [p, q] &\quad \text{if } p \leq i \text{ and } \alpha \leq [i, i] \text{ and } i \leq q.
\end{aligned}$$

Refinement of the Thermostat

abbreviation $dyn\ T_m\ T_M\ a\ T_u\ \tau \equiv$
$IF\ (\vartheta = 0)\ THEN\ x' = f\ a\ 0\ \&\ G\ T_m\ T_M\ a\ 0\ on\ \{0..\tau\}\ UNIV\ @\ 0$
$ELSE\ x' = f\ a\ T_u\ \&\ G\ T_m\ T_M\ a\ T_u\ on\ \{0..\tau\}\ UNIV\ @\ 0$

lemma *R-therm-down*:
assumes $a > 0$ **and** $0 \le \tau$ **and** $0 < T_m$ **and** $T_M < T_u$
shows $[\vartheta = 0 \wedge I\ T_m\ T_M \wedge t = 0 \wedge T_0 = T,\ I\ T_m\ T_M] \ge$
$(x' = f\ a\ 0\ \&\ G\ T_m\ T_M\ a\ 0\ on\ \{0..\tau\}\ UNIV\ @\ 0)$
apply(*rule local-flow.R-g-ode-ivl*[*OF local-flow-therm*])
using *therm-dyn-down*[*OF assms*(1,3), *of* - T_M] *assms* **by** *rel-auto'*

lemma *R-therm-up*:
assumes $a > 0$ **and** $0 \le \tau$ **and** $0 < T_m$ **and** $T_M < T_u$
shows $[\neg\ \vartheta = 0 \wedge I\ T_m\ T_M \wedge t = 0 \wedge T_0 = T,\ I\ T_m\ T_M] \ge$
$(x' = f\ a\ T_u\ \&\ G\ T_m\ T_M\ a\ T_u\ on\ \{0..\tau\}\ UNIV\ @\ 0)$
apply(*rule local-flow.R-g-ode-ivl*[*OF local-flow-therm*])
using *therm-dyn-up*[*OF assms*(1) - - *assms*(4), *of* T_m] *assms* **by** *rel-auto'*

Refinement of the Thermostat

abbreviation $ctrl\ T_m\ T_M \equiv$
$(t ::= 0); (T_0 ::= T);$
$(IF\ (\vartheta = 0 \wedge T_0 \le T_m + 1)\ THEN\ (\vartheta ::= 1)\ ELSE$
$IF\ (\vartheta = 1 \wedge T_0 \ge T_h - 1)\ THEN\ (\vartheta ::= 0)\ ELSE\ skip)$

lemma *R-therm-time*: $[I\ T_m\ T_M, I\ T_m\ T_M \wedge t = 0] \ge (t ::= 0)$
by (*rule R-assign-law*, *pred-simp*)

lemma *R-therm-temp*:
$[I\ T_m\ T_M \wedge t = 0, I\ T_m\ T_M \wedge t = 0 \wedge T_0 = T] \ge (T_0 ::= T)$
by (*rule R-assign-law*, *pred-simp*)

lemma *R-thermostat-flow*:
assumes $a > 0$ **and** $0 \le \tau$ **and** $0 < T_m$ **and** $T_M < T_u$
shows $[I\ T_m\ T_M, I\ T_m\ T_M] \ge therm\ T_m\ T_M\ a\ T_u\ \tau$
by (*refinement*;(*rule R-therm-time*)?,(*rule R-therm-temp*)?,
(*rule R-assign-law*)?, (*rule R-therm-up*[*OF assms*])?,
(*rule R-therm-down*[*OF assms*])?) *rel-auto'*

Conclusions

- Used modular semantic framework in Isabelle/HOL to
  - derive a minimal logic $\mathrm{d}\mathcal{H}$ for verification of hybrid programs,
  - obtain refinement components via the laws of $\mathrm{d}\mathcal{R}$,
- Added lenses for better parsing and alternative program stores
- Future work:
  - Explore total correctness,
  - Adversarial dynamics like in differential game logic,
  - Code generation of verified executable code,
  - Integrate with a CAS that supplies solutions and invariants, leaving the certification work to Isabelle.

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