On Necessary and Sufficient Conditions for Wedging in Two Contact Node System

Sangkyu Kim and Yong Hoon Jang*

School of Mechanical Engineering, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul 120-749, Republic of Korea

jyh@yonsei.ac.kr

Abstract. A necessary and sufficient condition for wedging in two node system is explored. When external loading is zero, wedging is possible if and only if the *constraint vectors* consisting of contact stiffness and coefficient of friction, directing either *admissible* or *inadmissible* region, are positive linear dependent. This condition is validated by comparing with the conventional necessary condition.

Keywords: Wedging, Necessary and Sufficient Condition, Positively Linear Dependence.

1 Introduction

A wedge with inclined planes can divide an object in two, lift up an object, or hold an object in place. The friction between bodies plays an important role to this frictional behavior. The wedging on the friction surface is known as a state in which the object is maintained by itself even after all the external force is removed. In the process of assembling structures, a wedge state can be established if a small change in alignment occurs.

In order to better understand this frictional phenomenon, the Coulomb frictional law is usually applied. Recently a series of researchers have studied friction phenomenon, such as frictional shakedown and wedging, using a newly developed method to investigate the conditions of relative displacements including contact stiffness and frictional coefficient under the Coulomb frictional law [1-5]. It is reported that the Klarbring's model with a single node in the rate loading can predict the wedge state when the frictional coefficient is beyond the critical value [6]. For two node systems and beyond, a critical coefficient of friction at which the wedge occurs is related to Klarbring's P-matrix criterion [7]. Specifically, Ahn has pointed that if the two node frictional system has a critical coefficient of friction, the discontinuous separation (or "jumps") can occur according to a specific loading scenario, indicating that the critical coefficient of friction can be obtained from the Klarbring's P-matrix criterion [3]. Ahn et. al.[2] have stated that if the two node system encounters an open admissible region obtained

T. Uhl (ed.), *Advances in Mechanism and Machine Science*,
Mechanisms and Machine Science 73,
https://doi.org/10.1007/978-3-030-20131-9_396

through the slip equality conditions of the Coulomb's frictional law at each node, wedging can occur. At this instant, the system always evolves into a state of discontinuous abrupt change of displacement. However, they state the con-verse relation is not always true, meaning that if the discontinuity happens, wedging may not occur.

Through the outcomes obtained previously regarding wedging and the discontinuity of frictional displacement, the following statement can be deduced: If the friction-al system is in a state of wedge, it will violate the Klarbring's P-matrix condition, and on the contrary, if it violates the Klarbring's P-matrix condition, it is possible that this system will not be wedged [3]. Thus, the necessary condition of the statement is valid but the sufficient condition is not satisfied. To explain this discrepancy, Ahn [3] uses a specific condition, "trap", which a slip state on the slip equality conditions consisting of contact stiffness and frictional coefficient cannot evolve further, satisfying the frictional law. This "trap" state is equivalent to the two-node discontinuity explained above. Under a circumstance in which the system satisfies the "trap" condition, it is not possible for wedging to occur. Since the statement regarding the wedging condition does not satisfy the necessary and sufficient conditions, one should estimate every situation in which wedging occurs regardless of the usage of Klarbring's P-matrix condition or trap.

In this study, we shall present a condition for wedging which satisfies the necessary and sufficient condition in a two-node system. Firstly, we will explain the Klarbring's P-matrix condition. Secondly, the relation of positive linear dependence of linear inequality. Lastly, we will compare the current statement with a new condition with the previous statement for a two-node frictional system.

2 Klarbring's P-matrix Criterion and Wedging

A discrete two dimensional frictional system with two contact nodes is considered, in which the tangential and normal forces on the surface is $[q\ p]^T$ and the corresponding displacement is denoted as $[v\ w]^T$. If the system is linearly elastic, the equation of motion is

$$\begin{Bmatrix} q \\ p \end{Bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix} \begin{Bmatrix} v \\ w \end{Bmatrix} + \begin{Bmatrix} q^* \\ p^* \end{Bmatrix} \tag{1}$$

where the matrix $[\mathrm{A\ B^T}; B\ C]$ is the contact stiffness and the vector $\{\mathrm{q}^*\ p^*\}$ is the contact forces when there is no-relative motion on the contact surface.

According to the Klarbring's P-matrix criterion [6], the statement, the matrix $(A + f\Lambda B)$ satisfies P-matrix condition, meaning that the matrix is positive definite and all principal minors are also positive, is equivalent to the statement, frictional rate problem in the discrete system is well-posed. The matrix Λ has a diagonal element of ± 1.

The wedging on the friction surface is known as a state in which the object is maintained by itself even after all the external forces have been removed. Klarbring had shown that a single node is in a wedge state when the frictional coefficient is greater than a critical value [6]. However, in the system containing more than two contact nodes, it is suggested that the critical value defining wedge state is related to Klarbring's P-matrix condition [2] and the wedging can happen if an open admissible region in $\mathrm{v_i}$ -space obtained through the slip equality conditions at each node exists, resulting in the state of discontinuous slip displacements. The results for wedging is then summarized that if the frictional system has a state of wedge, it will violate the Klarbring's P-matrix condition, and on the contrary, if it violates the Klarbring's P-matrix condition, it is possible that this system will not be wedged [2]. Thus, the necessary condition of the statement is valid but the sufficient condition is not satisfied.

3 Condition of Positive Linear Dependency

In order to establish a new condition for the wedging, we would need to explain positive linear dependency of the vectors which determines the direction of the inequality obtained from the Coulomb frictional law. We will explain this relation in a two node system to effectively transfer the meaning.

When the external force is zero in a two-node discrete frictional system, the four inequalities generated by the Coulomb frictional law are:

$$\begin{aligned} &(A_{11} - fB_{11})v_1 + (A_{12} - fB_{12})v_2 \leq 0 \quad (\mathrm{I}) \\ &(A_{11} + fB_{11})v_1 + (A_{12} + fB_{12})v_2 \geq 0 \quad (\mathrm{II}) \\ &(A_{21} - fB_{21})v_1 + (A_{22} - fB_{22})v_2 \leq 0 \quad (\mathrm{III}) \\ &(A_{21} + fB_{21})v_1 + (A_{22} + fB_{22})v_2 \geq 0 \quad (\mathrm{IV}) \end{aligned} \tag{2}$$

The four constraints are represented by four straight lines on the $\mathrm{v_i}$ -space, and the intersection represented by these four inequalities is the admissible region. In general, the only possible admissible region is the origin since all four straight lines pass through the origin. However, when wedging occurs, the admissible region is open to one direction. If the slip displacements $\mathrm{v_1, v_2}$ exists in this region, the displacements which do not pass the origin can be solutions even when the external forces are zero.

Since the inner product of the vector $(A_{11} - fB_{11},\ A_{12} - fB_{12})$ and the vector (v_1, v_2) which is located on the line $(A_{11} - fB_{11})v_1 + (A_{12} - fB_{12})v_2 = 0$ is zero, the vector $(A_{11} - fB_{11},\ A_{12} - fB_{12})$ is perpendicular to the line (I). In addition, the direction of the vector $(A_{11} - fB_{11},\ A_{12} - fB_{12})$ always directs to an inadmissible region represented by the inequality (I). Let the vector $(A_{11} - fB_{11},\ A_{12} - fB_{12})$ be a constraint vector regarding the backward slip motion of node 1. For the inequality (II), the constraint vector $(A_{11} + fB_{11},\ A_{12} + fB_{12})$ directs to an admissible region since the inequality is reverse, compared with the inequality (I). In order to indicate the direction of constraint vector to the inadmissible region for the inequality (II), a negative sign is multiplied to the constraint vector, such as $(-(A_{11} + fB_{11}),\ -(A_{12} + fB_{12}))$. Likewise, the constraint vector for the inequality (III) is $(A_{21} - fB_{21},\ A_{22} - fB_{22})$ and the constraint vector for inequality (IV) is $(-(A_{21} + fB_{21}),\ -(A_{22} + fB_{22}))$.

Throughout this text, all the constraint vectors are chosen to direct inward the inadmissible region, which is depicted as shaded region in Fig. 1.

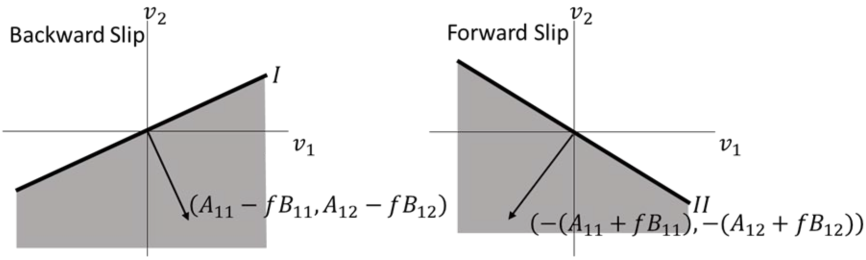


Fig. 1. Constraint vectors which direct to an inadmissible region for the inequality (I,II)

The range of constraint vectors for the lines (I,II,III,IV) can be identified. The slip motions at each line is possible when $A_{11} - fB_{11} > 0$, $A_{11} + fB_{11} > 0$, $A_{22} - fB_{22} > 0$, $A_{22} + fB_{22} > 0$. If we denote the constraint vector of the lines (I,II,III,IV) as ①, ②, ③, ④, respectively, the existence range of constraint vectors can be visualized in Figure 2. For the line (I), the constraint vector can be located in the first and fourth quadrant, which is $v_1 > 0$. The constraint vector for the line (II) is located in the second and third quadrant, that is $v_1 < 0$. The constraint vectors for the lines (III,IV) are also located in the first and second quadrant ($v_2 > 0$) and in the third and fourth quadrant, ($v_2 < 0$) respectively.

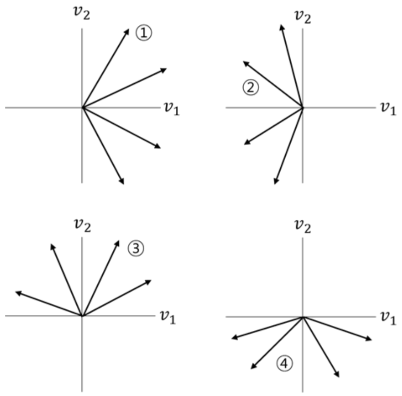
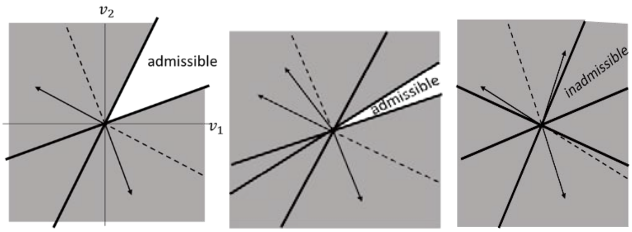


Fig. 2. Existence range of constraint vectors for the lines (I,II, III,IV)

Since the wedging occurs when the admissible region exists and it is open, it is important to find the existence of an admissible region under the zero loading state. Under given conditions, the states which the inequalities (I,II,III,IV) can generate are only two different states; one has both admissible and inadmissible regions and the other has just inadmissible regions. Thus, we will compare two conditions to confirm for the existence of admissible region.

Suppose that there are two arbitrary constraint vectors which form inadmissible and admissible regions in v-space, as shown in Fig. 3 (a). It is noted that in general, two arbitrary vectors cannot make the entire region inadmissible, except that two constraint vectors are collinear and in opposite directions. For the general case where two vectors are not collinear and in opposite directions, if one vector is additionally placed between the vector extension lines (the dashed lines in Fig. 3 (b)), the entire region becomes an inadmissible region, as shown in Fig. 3 (c). Then, it is evident that the positive combination of two vectors do not become the other vectors, meaning that $\alpha C_1 + \beta C_2 \neq C_3$ where C_i (i=1,3) are the constraint vectors and α, β>0. On the contrary, if we select two constraint vectors among three constraint vectors which are placed as shown in Figure 3(a), the positive combination of two vectors can be the other vector, meaning that $\alpha C_1 + \beta C_2 = C_3$. This relation is generally stated as positive linear dependent [8]. Thus, three constraint vectors which satisfy the condition that one vector is not placed between the extensions of the other vectors, should be positive linear dependent. Therefore, when an admissible region exists, three arbitrarily selected constraint vectors among four are positive linear dependent. The reverse relation, that is, if three arbitrary selected constraint vectors among four are positive linear dependent, then an admissible region exists, is also true.



(a) (b) (c)

Fig. 3. (a) Two arbitrary constraint vectors forming inadmissible and admissible regions. The arrow lines are constraint vectors and the dashed lines are the extended line of constraint vector lines. (b) When the third constraint vector is not placed between the vector extension lines, an admissible region exists (c) When the third constraint vector is placed between the vector extension lines, all inadmissible regions are existed.

As above, the condition for wedging is summarized as follows; when the wedging exists, any three constraint vectors selected from four vectors in two node system are positive linear dependent. This is a necessary and sufficient condition. The constraint vectors consist of the row vectors of the Klarbring's matrix $\boldsymbol{A} + \boldsymbol{f\Lambda B}$ and the vector directions should be rearranged so that they can point toward one direction, either admissible or inadmissible region for each constraint. In the above explanation, we choose the vector directions turning to inadmissible region.

A practical condition derived from the wedging condition can be obtained. Suppose that there are four constraint vectors, $C_i, (i = 1,4)$. Four subsets which selects three constraint vectors among four can be determined as $\{C_1, C_2, C_3\}$, $\{C_1, C_2, C_4\}$,

$\{C_1, C_3, C_4\}, \{C_2, C_3, C_4\}$ and all subsets satisfy positive linear dependency. For example, in order to show the positive linear dependency of the subset $\{C_1, C_2, C_3\}$, represent each constraint vector as $C_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$, $C_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$, $C_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}$. The positive linear dependency of three constraint vector can be represented that two positive quantities $\alpha, \beta > 0$ exist, satisfying $\alpha C_1 + \beta C_2 = C_3$. This equation are represented as

$$\alpha \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \beta \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3}$$

This last equation can be interpreted as the column vector $[\alpha, \ \beta, -1]^T$ as a vector perpendicular to two vectors $[a_1 \ \ a_2 \ \ a_3]$ and $[b_1 \ \ b_2 \ \ b_3]$, meaning that the cross product of the two vectors $[a_1 \ \ a_2 \ \ a_3]$ and $[b_1 \ \ b_2 \ \ b_3]$ are parallel to the vector $(\alpha, \ \beta, -1)$. Then,

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2\, b_3 - a_3\, b_2,\ a_3\, b_1 - a_1\, b_3,\ a_1\, b_2 - a_2\, b_1) \tag{4}$$

Note that the vector obtained from equation (4) is collinear to the column vector of equation (3) such as $(\alpha, \beta, -1)$. Thus, the sign of one component of the vector in equation (4) should be different from the other of the cross products in equation (4). Therefore, if three constraint vectors selected from four constraint vectors are positive linear dependent, the signs of components in (4) are mutually different, for example, [(+,+,-),(+,-,+),(-,-,+),(-,+,-)]. In the next section, we will determine the wedging using the conditions of positive linear dependency for two cases where wedging is suspected in violation of Klarbring's P-matrix criterion.

4 Comparison of Wedging Conditions

According to the Klarbring's P-matrix criterion, if the matrix $(\boldsymbol{A} + f\boldsymbol{\Lambda B})$ does not satisfy P-matrix condition, wedging may happen but it does not necessarily happen. To show an example, we investigate a two-node system with a specific contact stiffness.

Suppose that the contact stiffness of $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0.4 \\ 1.5 & 0.4 \end{bmatrix}$. The coefficient of friction is 0.6. The matrix $(\boldsymbol{A} + f\boldsymbol{\Lambda B})$ gives four different matrixes depending on the matrix of Λ. The determinants of the matrix $(\boldsymbol{A} + f\boldsymbol{\Lambda B})$ are -2.0520, 0.3720, -0.2280, and 1.9080. According to the Klarbring's P-matrix criterion, the cases where the determinants are negative such as -2.0520 and -0.2280 may have the wedging.

To validate the positive linear dependency of four constraint vectors from the previous contact stiffness matrix, we firstly calculate the results of equation (4) as follows, (0.228, 2.052, -1.68), (2.28, -1.908, 0.228), (2.28, 0.372, -2.052), (1.908, 0.372, -1.68).

Since the components of cross product such as equation (4) are not the same sign like (+,+,-) or (+,-,+), the four constraint vectors are positive linear dependent, meaning that wedging can happen. If there is no separation at any node, wedging occurs. We can confirm that this case has an open admissible region in v-space, as shown in Fig. 4.

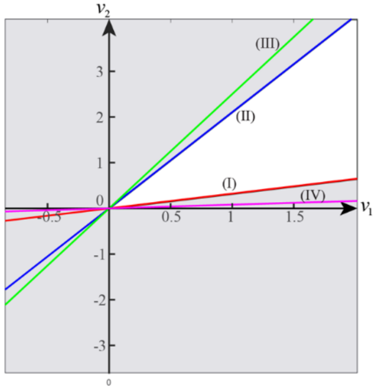


Fig. 4. An example of wedging with open admissible region in v-space

The second example tells that the matrix of $(\boldsymbol{A} + \boldsymbol{f\Lambda B})$ do not satisfy the P-matrix criterion, thus the wedging may happen. but the system cannot be wedged. The contact stiffness are $\boldsymbol{A} = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0.35 \\ 0.3 & 0.2 \end{bmatrix}$. The determinants of the matrix $(\boldsymbol{A} + \boldsymbol{f\Lambda B})$ are 0.0284, 0.0116, 0.0296, and -0.1096. Thus, the wedging may happen for the case where the determinants is negative, -0.1096. Next, the results of equation (4) are (-0.0296 -0.0284 -0.0360), (0.0960 0.1096 -0.0296), (0.0960 0.0116 0.0284), (-0.1096 0.0116 -0.0360). The first and third results have all the same sign, which are positive linear independent, resulting in no wedging. This is also confirmed by the result of v-space which shows all inadmissible region.

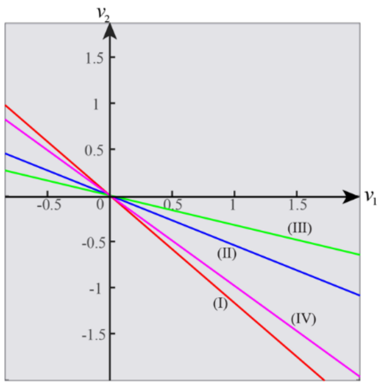


Fig. 5. An example of no-wedging with inadmissible region in v-space

5 Conclusions

The main objectives of this study is to find a wedging condition that satisfies both necessary and sufficient conditions. For two node frictional system, there are four constraint vectors, consisting of the components of Klarbring's matrix $(\boldsymbol{A} + \boldsymbol{f\Lambda B})$ and directing either admissible or inadmissible region. Wedging can occur if and only if three constraint vectors selected from four are positive linear dependent.

Acknowledgements

We are pleased to acknowledge support from the National Research Foundation of Korea (NRF) funded by the Korea government (MSIP) (Y.H. Jang and S.K Kim, Grant No. 2018R1A2B6008891).

References

1. Klarbring, A.: Examples of non-uniqueness and non-existence of solutions to quasi-static contact problems with friction. Ingenieur-Archiv, 60, 529–541, (1990).
2. Ahn, Y. J., Bertocchi, E., and Barber, J. R.: Shakedown of coupled two dimensional discrete frictional systems. Journal of the Mechanics and Physics of Solids, 56, 3433–3440, (2008).
3. Ahn, Y. J.: Discontinuity of quasi-static solution in the two-node coulomb frictional system. International Journal of Solids and Structures, 47, 2866–2871, (2010).
4. Andersson, L-E. Barber, J. R., and Ahn, Y-J.: Attractors in frictional systems subjected to periodic loads. SIAM Journal of Applied Mathematics, 73, 1097–1116, (2013).
5. Andersson, L-E. Barber, J. R., and Ponter, A. R. S.: Existence and uniqueness of attractors in frictional systems with uncoupled tangential displacements and normal tractions. International Journal of Solids and Structures, 51, 3710–3714, (2014).
6. Barber, J.R., Hild, P., 2006. On wedged configurations with Coulomb friction. In: Wriggers, Peter, Nackenhorst, Udo (eds.), ANALYSIS AND SIMULATION OF CONTACT PROBLEMS, pp. 205–213. Springer-Verlag, Berlin. (2006).
7. Klarbring, A. Contact, friction, discrete mechanical structures and discrete frictional systems and mathematical programming. In P. Wriggers and P. Panagiotopoulos (eds), NEW DEVELOPMENTS IN CONTACT PROBLEMS, pp. 55–100. Springer, Wien, (1999)..
8. Davis. C.: Theory of positive linear dependence, American Journal of Mathematics, 76:733-746, (1954).