# Derivation of OLS: Simple and Multiple Linear Regression

## Simple Linear Regression

The simple linear regression model is defined as:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i,$$

where:

- $y_i$: Dependent variable (response variable).
- $x_i$: Independent variable (predictor variable).
- $\beta_0, \beta_1$: Regression coefficients (intercept and slope).
- $\epsilon_i$: Error term (unobservable).

The objective of Ordinary Least Squares (OLS) is to minimize the **Sum of Squared Errors (SSE)**, which is given by:

$$SSE = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right)^2 .$$

Step 1: Partial derivatives of SSE To find the values of $\beta_0$ and $\beta_1$ that minimize $SSE$, we compute the partial derivatives with respect to $\beta_0$ and $\beta_1$ and set them equal to 0.

Partial derivative with respect to $\beta_0$:

$$\frac{\partial SSE}{\partial \beta_0} = -2 \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 x_i)\right) = 0.$$

Partial derivative with respect to $\beta_1$:

$$\frac{\partial SSE}{\partial \beta_1} = -2 \sum_{i=1}^{n} x_i \left(y_i - (\beta_0 + \beta_1 x_i)\right) = 0.$$

Step 2: Solve for $\beta_0$ and $\beta_1$ Expanding the first equation:

$$\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0.$$

This simplifies to:

$$\sum_{i=1}^{n} y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} x_i = 0,$$

which gives:

$$\beta_0 = \bar{y} - \beta_1 \bar{x},$$

where $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

Expanding the second equation:

$$\sum_{i=1}^{n} x_i \left(y_i - \beta_0 - \beta_1 x_i\right) = 0.$$

Substitute $\beta_0 = \bar{y} - \beta_1 \bar{x}$:

$$\sum_{i=1}^{n} x_i y_i - \beta_0 \sum_{i=1}^{n} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 = 0.$$

This simplifies to:

$$\beta_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

Thus, the OLS estimators for simple linear regression are:

$$\beta_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}, \quad \beta_0 = \bar{y} - \beta_1 \bar{x}.$$

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# Multiple Linear Regression

The multiple linear regression model is defined as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where:

- $\mathbf{y} \in \mathbb{R}^n$: Dependent variable vector.
- $\mathbf{X} \in \mathbb{R}^{n \times p}$: Design matrix containing $p$ predictors (including a column of ones for the intercept).
- $\boldsymbol{\beta} \in \mathbb{R}^p$: Coefficient vector.
- $\boldsymbol{\epsilon} \in \mathbb{R}^n$: Error term vector.

The objective of OLS is to minimize the **Sum of Squared Errors (SSE)**, which is given in matrix form as:

$$J(\boldsymbol{\beta}) = \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2.$$

Expanding the norm:

$$J(\boldsymbol{\beta}) = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Step 1: Partial derivative of $J(\boldsymbol{\beta})$ Take the derivative with respect to $\boldsymbol{\beta}$:

$$\frac{\partial J(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} = -2\mathbf{X}^\top(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}).$$

Set the derivative to 0:

$$\mathbf{X}^\top(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) = 0.$$

Step 2: Solve for $\boldsymbol{\beta}$ Rearranging:

$$\mathbf{X}^\top\mathbf{y} = \mathbf{X}^\top\mathbf{X}\boldsymbol{\beta}.$$

Assuming $\mathbf{X}^\top\mathbf{X}$ is invertible, multiply by its inverse:

$$\boldsymbol{\beta} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}.$$

Thus, the OLS estimator for multiple linear regression is:

$$\boldsymbol{\beta} = (\mathbf{X}^\top\mathbf{X})^{-1}\mathbf{X}^\top\mathbf{y}.$$

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# Conclusion

In this document, we derived the OLS estimators for both simple and multiple linear regression. The simple linear regression derivation focused on solving for $\beta_0$ and $\beta_1$ analytically, while the multiple linear regression derivation utilized matrix operations to generalize the solution for multiple predictors. These derivations form the foundation for linear regression in statistics and machine learning.