May 13

$$P(X=k) = \sum_{i=0}^{k} P(X_1=i)\,P(X_2=k-i) = \frac{i^{-(\lambda_1+\lambda_2)}}{k!} \sum k! \frac{\lambda_1^{i}}{i!}\frac{\lambda_2^{k-i}}{(k-i)!}$$

$$= \frac{i^{-(\lambda_1+\lambda_2)}}{k!} \cancel{(\lambda_1+\lambda_2)^k} \sum C_i^k \frac{\lambda_1^i \lambda_2^{k-i}}{(\lambda_1+\lambda_2)^k}$$

Ex: ② Prisson distribution

$X \sim \text{prisson}(\lambda)$

$$m_X(t) = \sum_{k=0}^{+\infty} e^{tk}\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{+\infty}\frac{(\lambda e^t)^k}{k!}$$

→ Recall that $\sum_{k=0}^{+\infty} \frac{x^k}{k!} = e^x \quad \forall x \in \mathbb{R}$

$$m_X(t) = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)} \quad \forall t \in \mathbb{R}$$

$$m_X(t) = \exp(\lambda(\exp(t)-1))$$

$$\frac{d}{dt} m_X(t) = \lambda e^t e^{\lambda(e^t-1)}$$

$$\mathbb{E}(X) = \frac{d}{dt}(m_X(t))\Big|_{t=0} = \lambda$$

$$\frac{d^2 m_X(t)}{dt^2} = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}$$

$$\frac{d^2 m_X(t)}{dt^2}\Big|_{t=0} = \lambda + \lambda^2 = \mathbb{E}(X^2)$$

③ Geometric distribution

$X \sim \text{Geometric}(p)$

$$m_X(t) = \mathbb{E}(e^{tX}) = \sum_{k=1}^{+\infty} e^{tk}(1-p)^{k-1}p = pe^t \sum_{k=1}^{+\infty}[e^t(1-p)]^{k-1}$$

→ Recall that $\sum_{j=0}^{+\infty} x^j = \frac{1}{1-x}$ for $|x|<1$

$(l = k-1)$

$$pe^t \sum_{l=0}^{+\infty}[e^t(1-p)]^l$$

$$= \frac{pe^t}{1-(1-p)e^t} \quad \text{provided that } (1-p)e^t < 1$$

Conclusion:

$$m_X(t) = \frac{pe^t}{1-(1-p)e^t} \quad \text{for } t \in \left(-\infty, \overbrace{\ln\left(\frac{1}{1-p}\right)}^{>0}\right)$$

Properties:

① If $Y = X + a$ where a is a constant, then $m_Y(t) = e^{at} m_X(t)$

② If $Y = \beta X$ then $m_Y(t) = m_X(\beta t)$

Proof: Assume that $Y = \beta X + a$.

$$m_Y(t) = \mathbb{E}(e^{tY}) = \mathbb{E}(e^{t(\beta X + a)}) = \mathbb{E}(e^{\beta t X} e^{at}) = e^{at}\,\mathbb{E}(e^{\beta t X}) = e^{at} m_X(\beta t)$$

Ex: Describe the r.v. X such that $m_X(t) = \boxed{\frac{\frac{1}{3}e^t}{1-\frac{2}{3}e^t} \cdot \frac{1}{e^t}} = \frac{\frac{1}{3}}{1-\frac{2}{3}e^t} = \frac{1}{3-2e^t}$

$$= e^{-t} \cdot \underbrace{\frac{\frac{1}{3}e^t}{1-\frac{2}{3}e^t}}_{m_Y(t)}$$

$X = Y - 1$ where $Y \sim \text{Geometric}\ (p = 1/3)$.

$m_Y(t) e^{-t}$. $\quad X = \#$ of failure before first success.

$m_X(t) = e^{-t} m_Y(t)$

$X \sim Y - 1$

What is the mgf of $Y-1$? $\rightarrow m_Y(t)\, e^{-t}$

Ex: Suppose that the numbers of claims received by an insurance company within a year is $X \sim \text{Poisson}(\lambda = 4)$.

It is estimated that the cost to the company is $\left(\frac{3}{2}\right)^X = Y$.

What is the expected cost to the company?

$$\rightarrow \mathbb{E}(Y) = \mathbb{E}\left(\left(\tfrac{3}{2}\right)^X\right) = \mathbb{E}\left(\left(e^{\ln(\frac{3}{2})}\right)^X\right) = \mathbb{E}\left(e^{\ln(\frac{3}{2})X}\right) = m_X\left(t = \ln\left(\tfrac{3}{2}\right)\right)$$

$$= e^{\lambda(e^t - 1)}\Big|_{t = \ln(3/2)} = e^2$$

$\therefore m_X(t) = \mathbb{E}(e^{tX})$

Ex: You start w/ a fortune $P = 100$. $P(H) = 1/3$. Flip the coin 10 times. Whenever the coin turns up H, your fortune is halved. What is the expected fortune at the end?

$H \to 300$

$HT \to 150$

$\vdots$

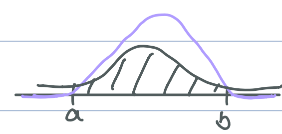
→ Ch. 2 & 3

Ch. 3 (Quiz 2)

CHAPTER 4

Continuous R.V. ⟷ as opposed to discrete r.v.

discrete r.v.	continuous r.v.
pf ⟷	pdf (probability density func.)
① $P_X(x) \geq 0$	$p(x) \geq 0 \quad \forall x$
② $\sum_x P_X(x) = 1$ ⟷	$\int_{-\infty \, a}^{+\infty \, b} p(x)\,dx = 1$

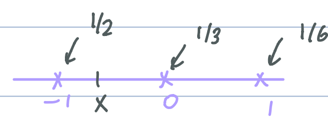


① Cumulative distribution function (cdf)

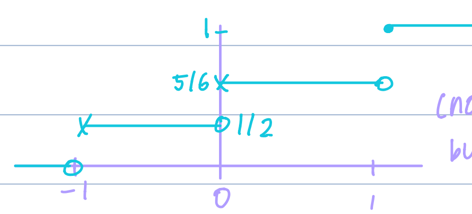
Def: Given a r.v. X, the cdf of X the function: $F_X(x) = P(X \leq x)$.

Ex: (discrete case)

x	-1	0	1
$P_X(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$



$$F_X(x) = \begin{cases} 0 & \text{if } x < n \leq -1 \\ \frac{1}{2} & \text{if } -1 \leq x < 0 \\ \frac{5}{6} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



(not left continuous, but right continuous)

$F_X \overset{?}{\leftrightarrows} P_X$

Properties of the cdf of a discrete r.v.

If F is the cdf of a discrete r.v X, then :

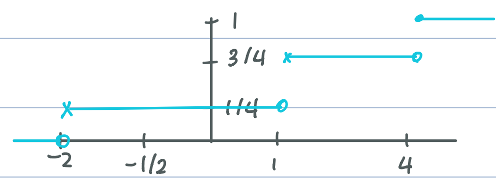
① F is a right continuous step function

② $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

③ If p is the pf of X, $p(x_0) \neq 0 \iff F$ is discontinuous at x_0

and $F(x_0) - \lim_{x \to x_0^-} F(x) =$ size of the jump at x_0

Ex:



$\rightarrow$ What is P_X ?

$\rightarrow$

x	-2	1	4
$p_X(x)$	$1/4$	$1/2$	$1/4$

Def: If a function F is s.t.

① F continuous and nondecreasing (i.e. $F(x_2) \geq F(x_1)$ whenever $x_2 \geq x_1$)

② $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

Then F is the cdf of some random variable X. (i.e. $F(x) = P(X = x)$).

X is said to be a continuous r.v.

Facts: If F is the cdf of a continuous r.v. X, then :

① F is differentiable "almost everywhere" (i.e. except at "a few" points)

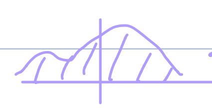
② If $f'(x) = \frac{dF(x)}{dx}$ then $\int_{-\infty}^{x} f(t)\,dt = F(x)$ | $f(x) \geq 0$ (non decreasing)

f is called the probability density function (pdf) of X.

Properties of pdf

① $f(x) \geq 0 \quad \forall x$

② $\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{x \to +\infty} F(x) - \lim_{x \to -\infty} F(x) = 1 - 0 = 1$



$\rightarrow$ Total Area $= 1$

Remark: If a function $f: \mathbb{R} \to \mathbb{R}$ is such that ① $f(x) \geq 0 \quad \forall x \in \mathbb{R}$ and
② $\int_{-\infty}^{\infty} f(x)\,dx = 1$, then f is the pdf of some continuous r.v. X and the cdf of X is $F_X(x) = \int_{-\infty}^{x} f(t)\,dt$.

The set $\{x \mid f(x) \neq 0\}$ is called the support of X.

Ex: Let $f(x) = \frac{c}{1+x^2}$ if $x \in \mathbb{R}$. Find c such that f is a pdf.

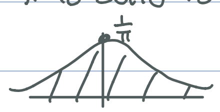
① $f(x) \geq 0 \quad \forall x \iff c \geq 0$

② $\int_{-\infty}^{+\infty} \frac{c}{1+x^2}\,dy = c\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = c\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = c\pi = 1$

$c = \frac{1}{\pi}$

$f(x) = \frac{1}{\pi(1+x^2)}$ is the pdf of a continuous r.v. X.

X is said to have a cauchy distribution (name given to that pdf)



cdf of X: $F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+t^2}\,dt$

$F_X(x) = \frac{1}{\pi}\left(\tan^{-1}(x) + \frac{\pi}{2}\right) \rightarrow$

(graph of F_X)

Computing Probabilities using cdf or pdf

Let X be a continuous r.v. with pdf f

① $P(X=a) = 0$

② $P(a \leq X \leq b) = P(a < X \leq b) = P(a < X < b) = P(a \leq X < b) =$
$F_X(b) - F_X(a) = \int_a^b f(t)\,dt$