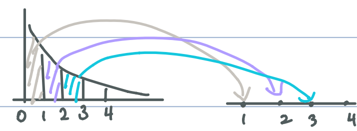
May 20

Exponential Distribution

Ex: (refer to 4.95)



cdf

$F_Y(y) = \begin{cases} 0 & \text{if } y \le 0 \\ 1 - e^{-y/\beta} & \text{if } y \ge 0 \end{cases}$

$Y \sim \text{Exponential}(\text{mean} = \beta)$

for $k = 1, 2, \cdots$, $P(X = k) = P(k-1 \le Y < k) = F_Y(k) - F_Y(k-1) = e^{-\frac{1}{\beta}(k-1)} - e^{-\frac{1}{\beta}k}$

$= (e^{-1/\beta})^{(k-1)} \underbrace{(1 - e^{-1/\beta})}_{p}$

$X \sim \text{Geometric}(p = 1 - e^{-1/\beta}) = p(1-p)^{k-1}$

Recall: $X \sim \text{Gamma}(\alpha, \beta)$ iff the pdf of X is $f_X(x) = \begin{cases} \Gamma(\alpha)\beta^{\alpha} x^{\alpha-1} e^{-x/\beta} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

Exercise: $Z \sim N(0, 1)$. Set $X = Z^2$. What is the distribution of X? $(X \ge 0)$

cdf of X. $F_X(x) = P(X = Z^2 \le x)$. $F_X(x) = 0$ whenever $x \le 0$.

Let $x \ge 0$. $F_X(x) = P(Z^2 \le x) = P(-\sqrt{x} \le Z \le \sqrt{x}) = F_Z(\sqrt{x}) - F_Z(-\sqrt{x})$

$= 2F_Z(\sqrt{x}) - 1$

cdf: $F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 2F_Z(\sqrt{x}) - 1 & \text{if } x \ge 0 \end{cases}$

pdf of X: $f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{\sqrt{x}} F_Z'(\sqrt{x}) = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} e^{-x/2} & \text{if } x > 0 \end{cases}$ $(c \cdot x^{-1/2} e^{-x/2})$

$X \sim \text{Gamma}(\alpha = 1/2, \beta = 2)$ also $\frac{1}{2^{1/2}\Gamma(1/2)} = \frac{1}{\sqrt{2\pi}} \Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$

Def: $X = Z^2$ has a chi-square distribution with one degree of freedom.

- Notation: $X = Z^2 \sim \chi^2_{(1)}$

$X \sim \chi^2(p) \iff X \sim \text{Gamma}(\alpha = \frac{p}{2}, \beta = 2)$

degree of freedom (p = positive integer)

Note: if $X \sim \chi^2(p)$, then the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{\gamma(p/2)\, 2^{p/2}} x^{p/2-1} e^{-x/2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$E(X) = p$ and $V(X) = 2p$

Ex (refer 4.112 (a)): $X \sim \chi^2(p)$. What is $E(X^n)$?

$$E(X^n) = \frac{1}{\gamma(p/2)\, 2^{p/2}} \int_0^{+\infty} x^{p/2-1+n} e^{-x/2} dx = \frac{1}{\gamma(p/2)\, 2^{p/2}} \gamma(n+\tfrac{p}{2})\, 2^{n+p/2}$$

$$= 2^n \tfrac{p}{2} \cdot (\tfrac{p}{2}+1) \cdot (\tfrac{p}{2}+2) \cdots (\tfrac{p}{2}+(n-1))$$

③ Beta Distribution

The Beta function: $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$ defined for $\alpha > 0, \beta > 0$.

Ex: $B(\alpha=3, \beta=1) = \int_0^1 x^2 dx = 1/3$

Fact: If $\alpha > 0$ and $\beta > 0$, then $B(\alpha, \beta) = \frac{\gamma(\alpha)\gamma(\beta)}{\gamma(\alpha+\beta)}$

Examples:

① $\int_0^1 x^4 (1-x)^7 dx = B(\alpha=5, \beta=7) = \frac{\gamma(5)\gamma(7)}{\gamma(12)} = \frac{4!\,6!}{11!} = \frac{4!}{11 \times 10 \times 9 \times 8 \times 7}$

② $\int_0^1 \sqrt{\frac{(1-x)^3}{x}} dx = B(\alpha = \tfrac{1}{2}, \beta = \tfrac{5}{2}) = \frac{\gamma(1/2)\gamma(5/2)}{\gamma(3)} = \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2} = \frac{3\pi}{8}$

The following function $f_X(x) = \begin{cases} C \cdot x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$ is the pdf of the called Beta Distribution with parameters α, β.

Notation: $X \sim \text{Beta}(\alpha, \beta)$

Remark: $X \sim \text{Uniform}(0,1) \iff X \sim \text{Beta}(\alpha=1, \beta=1)$

Prop: If $X \sim Beta(\alpha, \beta)$ then ① $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$ ② $V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Proof:

① $\mathbb{E}(X) = \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1}dx = \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \cdot \frac{\gamma(\alpha+1)\gamma(\beta)}{\gamma(\alpha+\beta+1)} = \frac{\gamma(\alpha+1)}{\gamma(\alpha)} \cdot \frac{\gamma(\alpha+\beta)}{\gamma(\alpha+\beta+1)}$

$= \frac{\alpha}{\alpha+\beta}$

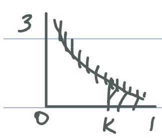
② $\mathbb{E}(X^2) = \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \int_0^1 x^{\alpha+1}(1-x)^{\beta-1}dx = \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \frac{\gamma(\alpha+2)\alpha(\beta)}{\gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} \cdots$

$V(X) = \mathbb{E}(X^2) - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \cdots = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Exercise: $\mathbb{E}(X^n) = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+(n-1))}$

Ex (refer 4.128)

$f_Y(y) = \begin{cases} 3(1-y)^2 & \text{if } 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$ $\qquad Y \sim Beta(\alpha=1, \beta=3)$



Find k such that $P(Y>k) = 10\% = 0.1$

$\int_k^1 3(1-y)^2 dy = 0.1 = (1-k)^3$

$1-k = \sqrt[3]{0.1} \rightarrow k = 1-\sqrt[3]{0.1}$

Moment generating function (mgf)

Def: Let X be a continuous r.v. and f_X its pdf.

The mgf of X defined as: $m_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$

Remark:

① m_X is always defined for $t=0$; $m_X(0)=1$. Also the domain of m_X is an interval.

② m_X is a unique, identifier of f_X $\qquad m_X \leftrightarrow f_X$

③ whenever the domain of m_X contains an interval of the fun $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then $\frac{d^n}{dt^n}(m_X(t))\Big|_{t=0} = \mathbb{E}(X^n)$

Recall: if $X = aY + b$ then $m_X(t) = e^{bt} m_Y(at)$

Ex: ① mgf of uniform distributions: If $X \sim \text{Uniform}(a,b)$ then $X = (b-a)Y + a$ where $Y \sim \text{Uniform}(0,1)$ so that $m_X(t) = e^{at} m_Y((b-a)t)$

$$m_Y(p) = \mathbb{E}(e^{Y}) = \int_0^1 e^{sy}\,dy = \begin{cases} \frac{1}{p}(e^{s}-1) & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$$

$$m_X(t) = \begin{cases} \frac{1}{(b-a)t}[e^{bt} - e^{at}] & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

If $m_X(t) = \frac{1}{3t}(e^{5t} - e^{2t})$ for $t \neq 0$ then $X \sim \text{Uniform}(2,5)$

② Normal distribution: $X \sim N(\mu, \sigma^2) \iff X = \sigma Y + \mu$ where $Y \sim N(0,1)$

$m_X(t) = e^{\mu t} m_Y(\sigma t)$

$m_Y(s) = \mathbb{E}(e^{sY}) = \int_{-\infty}^{+\infty} e^{sy} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy$ (defined for every s)

$$m_Y(n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y^2 - 2sy)}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y-s)^2 + \frac{1}{2}s^2}\,dy$$

$$= \frac{e^{s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y-s)^2}\,dy \overset{(u=y-s)}{=} e^{s^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2}\,du = e^{s^2/2}$$

$m_Y(s) = e^{s^2/2}$ $\forall s \in \mathbb{R}$. Therefore $m_X(t) = e^{(\frac{1}{2}\sigma^2 t^2 + \mu t)}$ $\forall t \in \mathbb{R}$.

For example, if $m_X(t) = e^{t(3+4t)}$ $t \in \mathbb{R}$, $X \sim N(\mu = 3, \sigma^2 = 8)$

③ mgf of Gamma distributions

$X \sim \text{Gamma}(\alpha, \beta)$

$$m_X(t) = \mathbb{E}(e^{tX}) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{+\infty} e^{tx} x^{\alpha-1} e^{-x/\beta}\,dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{+\infty} x^{\alpha-1} e^{-x[\frac{1}{\beta} - t]}\,dx$$

converges iff $\frac{1}{\beta} - t > 0$ ie. $\boxed{t < \frac{1}{\beta}}$

whenever $t \in (-\infty, \frac{1}{\beta})$ then $m_X(t) = \frac{1}{\gamma(\alpha)\beta^{\alpha}} \; \Gamma(\alpha) \cdot \frac{\beta^{\alpha}}{(1-\beta t)^{\alpha}}$

$\boxed{m_X(t) = \frac{1}{(1-\beta t)^{\alpha}} \quad t \in (-\infty, \frac{1}{\beta})}$. $\int_0^{+\infty} x^a e^{-bx} dx = \frac{\gamma(a+1)}{b^{a+1}}$ $b > 0$ and $a > -1$