May 4

Review of Set Theory

A set : a collection of objects called <u>elements</u> of the set.

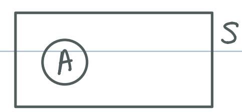
- Ex: A = set of positive integers less than 6

  $A = \{1, 2, 3, 4, 5\}$

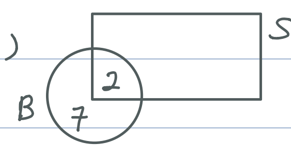
  $1 \in A$ but $7 \notin A$

Subset of a set

- Given a set S; we say that a set A is a subset of S if whenever $x \in A$, $x \in S$



- Ex: $S = \{1, 2, 3, 4\}$, $A = \{1, 2\}$ → A is a subset of S $(A \subseteq S)$
  - $B = \{2, 7\}$ → B is not a subset of S $(B \not\subseteq S)$



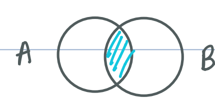
Remark

① $S \subseteq S$

② The empty set denoted $\emptyset$ is a subset of every set

Operations on Sets

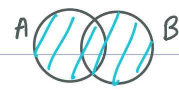
① Intersection : given two sets A, B, the intersection of A and B is the set denoted $A \cap B$ and defined as :



$A \cap B = \{x \mid x \in A$ and $x \in B\}$

- Ex: $A = \{1, 2, 3, 4\}$, $B = \{4, 7, 3\}$ → $A \cap B = \{4, 3\}$

② Union: given two sets A, B, the union of A and B is the set denoted $A \cup B$ and defined as:



$A \cup B = \{x \mid x \in A$ or $x \in B\}$ "inclusive"

• Ex: $A = \{1, 2, 3\}$, $B = \{3, 4\}$ → $A \cup B = \{1, 2, 3, 4\}$

③ Complement of a set: start with a reference set S. Let A be a subset of S. The complement of A w.r.t S is the set denoted:

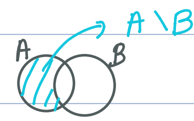
$A^c = \{x \in S \mid x \notin A\}$

• Ex: $S = \{1, 2, 3, 4, 5\}$, $A = \{2, 5\}$ → $A^c = \{1, 3, 4\}$

Remark:

① "Difference" between two sets

$A \setminus B = \{x \mid x \in A$ and $x \notin B\}$



$A \setminus B = A \setminus (A \cap B)$

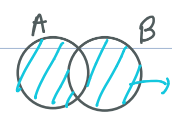
If A, B are subsets of S, then $A \setminus B = A \setminus (A \cap B) = A \cap B^c$

$A^c = S \setminus A$ if $A \subseteq S$

② Symmetric Difference

$A \triangle B = (A \setminus B) \cup (B \setminus A)$

$A \triangle B = B \triangle A$



Properties

① $A \cap \emptyset = \emptyset$

② $A \cap B = B \cap A$; $(A \cap B) \cap C = A \cap (B \cap C)$ (symmetry)

③ $A \cup \emptyset = A$

④ $A \cup B = B \cup A$; $(A \cup B) \cup C = A \cup (B \cup C)$

⑤ $\emptyset^c = S$

⑥ $(A^c)^c = A$

⑦ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

⑧ $(A \cap B)^c = A^c \cup B^c$

⑨ $(A \cup B)^c = A^c \cap B^c$

Random Experiment

Random: experiment where the outcome is not known for sure, but we know that it belongs to the set of all possible outcomes S.

- Random ≠ Deterministic
- Ex: A bug starting at $t=0$ from $x=0$ is moving on the x-axis according to the equation $x(t) = 2t$ → Deterministic Exp.
- Ex: Flip a coin (H, T). we cannot know for sure what the outcome will be, but we know it belongs to the set $\underbrace{S = \{H, T\}}_{\text{sample space or set of all outcomes}}$
- Ex:

① Roll a dice : $S = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ where w_i : the result is i $(i = 1 \sim 6)$

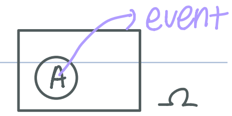
② Toss a coin until H appears : $S = \{\overset{w_1}{H}, \overset{w_2}{TH}, \overset{w_3}{TTH}, \overset{w_4}{TTTH} \cdots, \overset{w_n}{\ }\}$

typical outcome

$w_n = \underbrace{T \cdots T}_{n-1 \text{ time}} H$ (countably ∞)

Remark: The sample space S is also denoted Ω

Definition (Events ..): Given a random experiment and associated sample space Ω; we call event any subset of Ω



Terminology

① $A = \emptyset$ is called the null event

② $A = \Omega$ is the certain event

③ Any event A which is a singleton (i.e. has exactly one outcome) is called elementary event

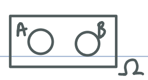
④ For any event A, A^c is called the complementary event of A and denoted also $\bar{A}$

Ex: Roll a die. $\Omega = \{\omega_1, \omega_2, \cdots, \omega_6\}$, $A = \{\omega_1\}$ is an elementary event

$B = \{\omega_1, \omega_2\}$ is an event, but not an elementary event

$B^c = \{\omega_3, \omega_4, \omega_5, \omega_6\}$

Definition (Disjoint Events): A and B are disjoint if $A \cap B = \emptyset$



Definition (Probability): Let Ω be a sample space. A probability is a function $\underline{P} : P(\Omega) \to [0, 1]$

($P(\Omega)$ is the power set of Ω ie. the set of all subsets of Ω)

such that: ① $P(\Omega) = 1$

② If $A_1, A_2, \cdots, A_n \cdots$ is a sequence of pairwise disjoint events (ie. $A_i \cap A_j = \emptyset$), then

Ω  , $P(\bigsqcup_n A_n) = \sum_n P(A_n)$

finite or ∞

Remark: If $A_1, A_2, \cdots, A_N$ is a finite sequence of pairwise disjoint set, then $P(\bigsqcup_{n=1}^{N} A_n) = \sum_{i=1}^{n} P(A_n)$

Ex: ① Toss a coin : $\Omega = \{H, T\}$. set $P(\{H\}) = p \in (0,1)$ and

$P(\{T\}) = 1 - p$.

$\mathcal{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \Omega\}$ → power set of Ω

$P(\emptyset) = 0$, $P(\Omega) = 1$

② Roll a die : $\Omega = \{w_1, w_2, w_3, w_4, w_5, w_6\}$

let $p_1, p_2, p_3, p_4, p_5, p_6$ be number in $(0,1)$ s.t. $\sum_{i=1}^{6} p_i = 1$.

Set $P(\{w_i\}) = p_i$.

In so doing we define a probability as $\mathcal{P}(\Omega)$

③ Take Ω to be infinite but countable. (Roll a coin until it appears)

$\Omega = \{w_1, w_2, w_3, \cdots, w_n, \cdots\}$

Set $P(\{w_n\}) = \frac{c}{n(n+1)}$. $c \longrightarrow$ must have $c > 0$

$1 = P(\Omega) = P(\bigcup_n \{w_n\}) = \sum_{n=1}^{+\infty} \frac{c}{n(n+1)} = c \sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$.

Recall $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1$ (Telescoping series) $\Rightarrow \boxed{c = 1}$

i.e. $P(\{w_n\}) = \frac{1}{n(n+1)}$

Properties

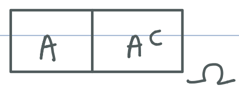
① $P(\emptyset) = 0$

② $P(\bar{A}) = 1 - P(A)$ → A^c

Proof

① $\Omega = \Omega \cup \emptyset$ (Ω and $\emptyset$ are disjoint)

$1 = P(\Omega) = P(\Omega \cup \emptyset) = P(\Omega) + P(\emptyset) \Rightarrow P(\emptyset) = 0$

② 

$\Omega = A \cup A^c$

$A \cap A^c = \emptyset$

$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

$P(A^c) = 1 - P(A)$