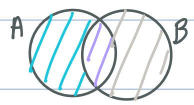
May 5

Proposition: Let Ω be a sample space and P be a probability in $\mathcal{P}(\Omega)$. If $A, B \in \mathcal{P}(\Omega)$, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:



$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$

The 3 events in the RHS are pairwise disjoints

$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$

$A = (A \setminus B) \cup (A \cap B)$, $P(A) = P(A \setminus B) + P(A \cap B)$

↳ $P(A \setminus B) = P(A) - P(A \cap B)$

Similarly, $P(B \setminus A) = P(B) - P(A \cap B)$.

Thus, $P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A) = P(A) + P(B) - P(A \cap B)$

Exercise: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Discrete Sample Space: A sample space Ω is said to be discrete if Ω is finite or infinite but countable.

Fact: Given a discrete sample space $\Omega = \{\overset{p_1}{w_1}, \overset{p_2}{w_2}, \ldots, w_n, \ldots\}$.

A probability P on $\mathcal{P}(\Omega)$ is completely, defined by a sequence $p_1, p_2, p_3, \ldots, p_n, \ldots$ of numbers in $(0,1)$ s.t. $\sum_n p_n = 1$ and by setting $P(\{w_n\}) = p_n$.

For any event $A \in \mathcal{P}(\Omega)$, we can write $A = \bigcup_{w_i \in A} \{w_i\}$

breaking A as union of elementary events.

Thus $P(A) = \sum_{w_i \in A} p_i$ (Recall $p_i = P(\{w_i\})$)

Ex: ① A die is s.t. $p_1 = p_2 = p_3 = \frac{1}{6}$ and $p_4 = \frac{1}{4}$, $p_5 = \frac{1}{6}$ and $p_6 = \frac{1}{12}$. $\Omega = \{w_1, w_2, w_3, w_4, w_5, w_6\}$. $p_i = P(\{w_i\})$

A: "The outcome is even". $P(A) = ?$

$A = \{w_2, w_4, w_6\}$. $P(A) = p_2 + p_4 + p_6 = \frac{1}{6} + \frac{1}{4} + \frac{1}{12} = \frac{1}{2}$

② Toss a coin until H appears. We have said that $\Omega = \{w_1, w_2, \dots, w_n \dots\}$. $w_n = "\underbrace{T \dots T}_{n-1} H"$. Set $p_n = P(\{w_n\}) = c\left(\frac{1}{3}\right)^n$.

Find $c \Rightarrow \sum_n p_n = 1 \Longrightarrow c \sum_{n=1}^{+\infty} \left(\frac{1}{3}\right)^n = 1$.

$$c = \frac{1}{\sum_{n=1}^{+\infty} \left(\frac{1}{3}\right)^n} = \frac{1}{\frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right)} = \frac{1}{\frac{1}{2}} = 2$$

$p_n = P(\{w_n\}) = 2\left(\frac{1}{3}\right)^n \quad n = 1, 2, \dots$

A: "H appears for the first time at an even number of tosses"

$A = \{w_2, w_4, w_6 \dots, w_{2n}, \dots\}$

$$P(A) = \sum_{n=1}^{+\infty} P(\{w_{2n}\}) = \sum_{n=1}^{+\infty} 2\left(\frac{1}{3}\right)^{2n} = 2\sum_{n=1}^{+\infty} \left(\frac{1}{9}\right)^n = \frac{2}{9}\left(\frac{1}{1-1/9}\right)$$
$$= \frac{2}{8} = \frac{1}{4}$$

$\boxed{\text{Equi}}$probability

- Let Ω be finite sample space.
- $\Omega = \{w_1, w_2, w_3, \dots w_N\}$ where $N = \mathrm{Card}(\Omega)$

\# of outcomes in Ω

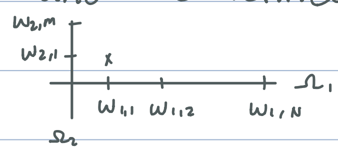
Equiprobability means: The w_i have the same probability

- ie. $p_1 = p_2 = \cdots = p_N = \frac{1}{N} = \frac{1}{card(\Omega)}$
- For any event A, we have $P(A) = \sum_{w_i \in A} P(\{w_i\}) = \sum_{w_i \in A} \frac{1}{N}$

$= card(A) \cdot \frac{1}{N} = \frac{card(A)}{card(\Omega)} = P(A)$

Counting Tools

① Cartesian Product : Given two sample spaces Ω_1 and Ω_2, the cartesian product of Ω_1 and Ω_2 defined as $\Omega_1 \times \Omega_2 = \{(w_1, w_2) \mid w_1 \in \Omega_1, w_2 \in \Omega_2\}$



$card(\Omega_1 \times \Omega_2) = card(\Omega_1) \cdot card(\Omega_2)$

Ex. Roll 2 fair die. what is the probability that a sum strictly less than 5 turns up?

First die $\Omega_1 = \{w_1, w_2, w_3, w_4, w_5, w_6\}$

$\Omega_2 =$ 〃 $= \Omega_1$

$card(\Omega_1 \times \Omega_2) = 6 \times 6 = 36$

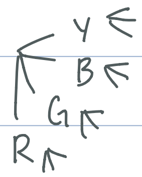
$card(A) = 6$. $P(A) = \frac{6}{36} = \frac{1}{6}$

② Permutation: A permutation of r elements chosen from n elements is equivalent to drawing successively w/out replacement (w.o.R) r objects from n distinct objects

- How many such permutations are there?

Ex: A urn contains 4 balls (1Y, 1G, 1B, 1R). Draw successively w.o.R 2 balls from the urn. (Y G ≠ G Y)

How many permutations? $4 \times 3 = 12$



- In the general setting, the total # of permutations of r objects chosen from n objects is denoted P_r^n and

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Recall: $n! = 1 \times 2 \times \cdots \times n$, $0! = 1$

Ex.

① A urn contains 4 balls (1Y, 1G, 1B, 1R). Draw successively 3 balls from the urn w.o.R. What is the probability that the last ball drawn is green?

→ The sample space is the set of all permutations of 3 balls selected from the urn.

$card(\Omega) = P_3^4 = 4 \times 3 \times 2 = 24$

$A =$ " _ _ G " ← $card(A) = P_2^3 = 3 \times 2 = 6$

$P(A) = \frac{1}{4}$

$B =$ "one of the selected ball is Green" $\overbrace{G\ \_\ \_}^{A_1}$ or $\overbrace{\_\ G\ \_}^{A_2}$ or $\overbrace{\_\ \_\ G}^{A_3}$

$B = A_1 \cup A_2 \cup A_3$ (disjoint)

$P(B) = P(A_1) + P(A_2) + P(A_3) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

② By permuting randomly the letters of the word 'BELLA'. What is the prob. that the 2 L will be together?

$B E L_1 L_2 A = B E L_2 L_1 A$

- If the L are distinct, then we would have $5! = 120$ permutations
- Since 2 Ls not distinct, permutations : $\frac{120}{2} = 60$
- $\underbrace{LL\_\ \_\ \_}_{3! = 6}$ or $\underbrace{\_\ LL\_\ \_}_{6}$ or $\underbrace{\_\ \_\ LL\_}_{6}$ or $\underbrace{\_\ \_\ \_\ LL}_{6}$
- $card(A) = 4 \times 6$. $P(A) = \frac{4 \times 6}{60} = \frac{2}{5}$

③ **Combination**: A comb. of r elements chosen from n elements is equiv. to drawing simultaneously r objects from a set of n objects. Equivalently it is a subset of size r chosen from the set of n elements.

The total # of such combinations is C_r^n or $\binom{n}{r}$

Ex: $S = \{a, b, c, d\}$. select $r = 2$ letters from S (at same time)

From permutation to combination

$P_2^4 = 4 \times 3 = 12$ perm. of 2 chosen from 4.

$\left.\begin{matrix} a\ b \\ b\ a \end{matrix}\right\} \{a, b\}$ $\qquad \frac{\#\text{perm}}{2} = \#\text{comb}$

perm $\qquad$ # comb

In the general setting $C_r^n = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$

Ex: $C_3^5 = \frac{5!}{(5-3)!\,3!} = \frac{5 \times 4}{2!} = 10$

Ex: A deck of 52 cards. A hand is a comb. of 5 cards chosen from 52 cards.

If a hand is selected at random, what is the probability that it will contain at least one king

A: "at least one king" . $\bar{A}$ = "0 king".

$P(A) = 1 - P(\bar{A})$

Ω is the set of all "hands". $\text{card}(\Omega) = C_5^{52}$.

$\bar{A}$: 0 king $\Longleftrightarrow$ selects 5 from $52 - 4 = 48$

$\text{card}(\bar{A}) = C_5^{48}$

$$P(\bar{A}) = \frac{C_5^{48}}{C_5^{52}} = \frac{\cancel{48} \times 47 \times 46 \times 45 \times 44}{52 \times 51 \times 50 \times 49 \times \cancel{48}}$$

Properties

① $C_0^n = 1$; $C_n^n = 1$

② $C_r^n = C_{n-r}^n$

A	A^c
r	$n-r$

③ $C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$

Binomial Expansion

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$

$(a+b)^n = a^n + C_1^n ab^{n-1} + C_2^n a^2 b^{n-2} + \dots + C_k^n a^k b^{n-k} + \dots + C_n^n b^n$

$(a+b)^n = \sum_{k=0}^{n} C_k^n a^k b^{n-k}$

Ex: $(a+b)^4 = C_0^4 a^0 b^4 + C_1^4 a^1 b^3 + C_2^4 a^2 b^2 + C_3^4 a^3 b + C_4^4 a^4 b^0 =$

$b^4 + 4ab^3 + 6a^2b^2 + 4a^3b + b^4$