May 14

Q: Flip a coin $X \sim Bin(p = \frac{1}{3}, n = 10)$. Start with a future $= P$.
If H then future tripled, if T then future is halved.

$X = \#$ of H. $\quad \underbrace{H \cdots H}_{X} \underbrace{T \cdots T}_{n-X}$.

$P(3)^X (\frac{1}{2})^{n-X} = \frac{P}{2^n}$.

$\frac{3}{2} - x = \frac{P}{2^n} 6^X = Y$

$E(Y) = \frac{P}{2^n} \mathbb{E}(6^X) = \cdots$ (Hint: use m.g.f)

$k \in \mathbb{N}$.

$$P(X = X_1 + X_2 = k) = P(X_1 = 0 \text{ and } X_2 = k) + P(X_1 = 1 \text{ and } X_2 = k-1) + \cdots + P(X_1 = k \text{ and } X_2 = 0)$$

$$= \sum_{i=0}^{k} P(X_1 = i \overset{\cap}{\text{and}} X_2 = k-i) = \sum_{i=0}^{k} \frac{e^{-\lambda_1}}{i!} \lambda_1^i e^{-\lambda_2} \frac{\lambda_2^{k-i}}{(k-i)!}$$

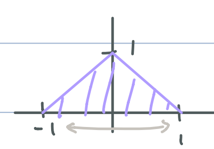
$$= \frac{e^{-(\lambda_1 + \lambda_2)}}{k!} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} = \cdots$$

Recall: A continuous r.v. X is completely defined by its p.d.f f_X which is a function such that:

① $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$

② $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Ex: $f_X(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$



Note that the support of X is $(-1, 1)$ $(-1, 1]$

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ \int_{-1}^{x} (1+t)\, dt = x + 1 + \frac{1}{2}(x^2 - 1) & \text{if } -1 \leq x \leq 0 \\ \frac{1}{2} + \int_0^x (1-t)\, dt = \frac{1}{2} + x - \frac{1}{2}x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt \leftarrow$ $P(\frac{1}{2} \leq X \leq 2) = F(2) - F_X(\frac{1}{2})$

$= 1 - F_X(\frac{1}{2}) = \cdots$

Expected Value

- Def: Let X be a continuous r.v. and f_X its pdf.

  ① $\mathbb{E}(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx$ whenever the RHS is absolutely convergent

  ② If $g: \mathbb{R} \to \mathbb{R}$, then $\mathbb{E}(g(X)) = \int_{-\infty}^{+\infty} g(x) f_X(x)\,dx$ whenever RHS is absolutely convergent

Ex: ① Cauchy r.v. $\to \mathbb{E}(|X|^p)$ is defined iff. $p<1$

- Recall: $f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ $\quad x \in \mathbb{R}$.

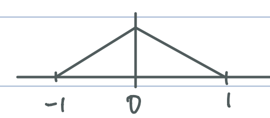
$E(X)$? $\to \underline{\int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{|x|}{1+x^2}\,dx}$ (does this converge?) $\to$ break into two

$\to \underbrace{\int_0^{+\infty} \frac{|x|}{1+x^2}\,dx}_{\text{diverges}} = \underbrace{\int_0^{+\infty} \frac{x}{1+x^2}\,dx}_{\text{diverges}}$ $\quad \frac{x}{1+x^2} \sim \frac{1}{x}$ as $x \to +\infty$

, so diverges

- X has a cauchy distribution, then $\mathbb{E}(X) = \text{DNE}$

② $f_X(x) = \begin{cases} 1-|x| & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$. $\mathbb{E}(X) = ?$

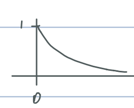


$\int_{-\infty}^{+\infty} |x| f_X(x)\,dx = \int_{-1}^{1} |x|(1-|x|)\,dx$ ← definite integral is continuous, thus integral exists.

$\mathbb{E}(X) = \int_{-1}^{1} \underline{x(1-|x|)}\,dx = 0$

↳ odd function on a symmetric domain.

③ $f_X(x) = \begin{cases} e^{-x} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$



$\int_{-\infty}^{+\infty} |x| f_X(x)\,dx = \int_{-\infty}^{+\infty} x e^{-x}\,dx$ $\quad \lim_{x\to\infty} x^4 e^{-x} = 0$ if x is large: $x^4 e^{-x} < 1$

$E(X) = \int_0^{+\infty} \underset{u}{x}\, \underset{dv}{e^{-x}dx} = uv - \int v\,du$ $\quad$ comparison test ← $\underline{x e^{-x}} < \frac{7}{x^3}$

$= \underbrace{\cancel{-xe^{-x}\Big|_0^{+\infty}}}_{0} - \int_0^{+\infty} -e^{-x}\,dx = \int_0^{+\infty} e^{-x}\,dx = 1$

$\therefore \mathbb{E}(X) = 1$

Exercise: Find $\mathbb{E}(X^n) = I_n$

$\to$ Hint: Find a recurrence relation for I_n

Properties:

① If C is a constant $\mathbb{E}(C) = C$

② If F and G are 2 functions and α, β are 2 scalars, then

$\mathbb{E}(\alpha F(X) + \beta G(X)) = \alpha\mathbb{E}(F(X)) + \beta\mathbb{E}(G(X))$.

Def (Variance): If X is a continuous r.v. the variance of X is defined as $V(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$

Remark: $V(X) = \underline{\mathbb{E}(X^2) - (\mathbb{E}(X))^2} \geq 0$

Ex: $f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$. $V(X) = ?$

$\mathbb{E}(X) = 1$

$\mathbb{E}(X^2) = \int_0^{+\infty} \underbrace{x^2}_{u} \underbrace{e^{-x} dx}_{dv} = -x^2 e^{-x} \Big|_0^{+\infty} - \int_0^{+\infty} -e^{-x^2} d(x^2) = \int_0^{+\infty} 2x e^{-x} dx = 2\int_0^{+\infty} x e^{-x} dx = 2$

$V(X) = 2 - 1^2 = 1$

Prop: Let X be continuous variable and f_X is its pdf. Let $\alpha, \beta \in \mathbb{R}$ $(\alpha \neq 0)$. Set $Y = \alpha X + \beta$. The pdf of Y is given by $\underline{f_Y(y) = \frac{1}{|\alpha|} f_X\left(\frac{y-\beta}{\alpha}\right)}$

↳ how 2 pdfs are connected

Proof: Case $\alpha > 0$.

cdf of Y?

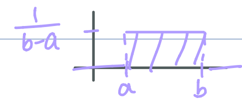
since $\alpha > 0$

$\rightarrow F_Y(y) = P(Y \leq y) = P(\alpha X + \beta \leq y) = P(\alpha X \leq y - \beta) = P\left(X \leq \frac{y-\beta}{\alpha}\right)$

$f_Y(y) = \frac{d}{dy}(F_Y(y))$

$f_Y(y) = \frac{1}{\alpha} \frac{d}{dx}(F)\left(\frac{y-\beta}{\alpha}\right) = \frac{1}{\alpha} f_X\left(\frac{y-\beta}{\alpha}\right)$

Usual families of continuous r.v.

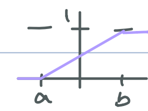


① Uniform distributions

- Let $I = (a, b)$ be a finite interval. Set $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in (a,b) \\ 0 & \text{otherwise} \end{cases}$

→ X is said to have a uniform distribution over (a,b).

- Notation: $X \sim \text{Uniform}(a,b)$
- If $X \sim \text{Uniform}(a,b)$ then $F_X(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b \end{cases}$

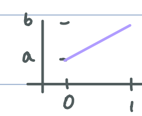


- If $X \sim \text{Uniform}(a,b)$ and I is an interval, then

$$P(X \in I) = \frac{\text{length}(I \cap (a,b))}{b-a}$$

Ex: $X \sim \text{Uniform}(-3, 7)$

$$P(X \in (2,14)) = \frac{\text{length}((2,14) \cap (-3,7))}{10} = \frac{5}{10} = \frac{1}{2}$$



Proposition: let $X \sim \text{Uniform}(a,b)$, then $Y = \cdot\, X + \cdot$ where $X \sim \text{Uniform}(0,1)$

Proof: Start with $Y \sim \text{Uniform}(a,b)$

Set $X = \frac{1}{b-a}(Y-a) = \underbrace{\frac{1}{b-a}}_{\alpha} Y - \underbrace{\frac{a}{b-a}}_{\beta}$. → what is the pdf of X?

$f_X(x) = \frac{1}{|\alpha|} f_Y\left(\frac{x-\beta}{\alpha}\right)$

$\frac{1}{|\alpha|} = b-a$. $\frac{x-\beta}{\alpha} = (b-a)\left[x + \frac{a}{b-a}\right] = (b-a)x + a$

$f_X(x) = (b-a)\, f_Y(\underline{(b-a)x} + a) = \begin{cases} (b-a) \times \frac{1}{(b-a)} & \text{if } a < (b-a)x + a < b \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$ (ie. $X \sim \text{Uniform}(0,1)$)

Prop: let $X \sim \text{Uniform}(a,b)$

① $E(X) = \frac{b+a}{2}$

② $V(X) = \frac{(b-a)^2}{12}$

Proof: Recall $X = (b-a)Y + a$ where $Y \sim \text{Uniform}(0,1)$.

Thus $\mathbb{E}(X) = (b-a)\,\mathbb{E}(Y) + a$ and $V(X) = (b-a)^2 V(Y)$

$\mathbb{E}(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y\,dy = \frac{1}{2}$.

$\mathbb{E}(Y^2) = \int_0^1 y^2 dy = \frac{1}{3} \rightarrow V(Y) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

$V(X) = (b-a)^2 \quad V(Y) = \frac{(b-a)^2}{12}$

$\mathbb{E}(X) = (b-a)\,\mathbb{E}(Y) + a = \frac{b-a}{2} + a = \frac{a+b}{2}$

$X \sim \text{Prisson}(\lambda = 4)$

$P(X=K) = e^{-4}\frac{4^K}{K!}$

$\lambda(K) = \frac{P(X=K-1)}{P(X=K)} \quad K = 1, \ldots$

$\lambda(K) < 1 \Rightarrow P(K-1) < P(K)$

$\lambda(K) > 1 \Rightarrow P(K-1) > P(K)$

$\uparrow$

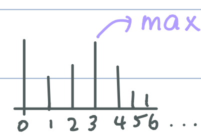
$\lambda(K) = \frac{4^{K-1}}{(K-1)!} \cdot \frac{K!}{4^K} = \frac{K}{4} \rightarrow$ If $K < 4$, $\lambda(K) < 1$.

ie. $P(K-1) < P(K)$. $\quad P(0) < P(1) < P(2)$.

$\hookrightarrow$ If $K > 4$, $\lambda(K) > 1$.

ie. $P(K-1) > P(K)$

$P(4) > P(5) > P(6) \ldots$



Q: (162)

$m_X(t) = e^{\lambda(e^t - 1)}$

$r(t) = \ln(m_X(t)) = \lambda(e^t - 1)$

$r'(t) = \lambda e^t \rightarrow r'(0) = \lambda$

$r(t) = \ln(m_X(t))$

$r'(t) = \frac{m'_X(t)}{m_X(t)}$

$r'(0) = \frac{m'_X(0)}{m_X(0)} \underset{\leftrightarrow}{=} \frac{m'_X(0)}{1} = m'_X(0) = \mathbb{E}(X)$

$r'(t) = \frac{m'_X(t)}{m_X(t)} \quad r''(t) = \frac{m''_X(t)\,m_X(t) - (m_X(t))^2}{(m_X(t))^2}$. $\quad r''(0) = m''_X(0) \ldots = V(X)$.

Ex (3.157)

b) $X = Y - 2$

$m_X(t) = \mathbb{E}(e^{tX}) = \mathbb{E}(e^{t(Y-2)}) = \mathbb{E}(e^{-2t} e^{tY}) = e^{-2t} m_Y(t)$