May 25

Final: $\frac{1}{3}$ quiz, $\frac{2}{3}$ take home 48 hr

Independent Random Variables

① Discrete case: two discrete r.v. X, Y are said to be independent if

$P(X=x, Y=y) = P(X=x)\,P(Y=y)$ for every x, y.

i.e. $P(x,y) = P_X(x)\,P_Y(y) \quad \forall x, y$

Ex:

$y \backslash x$	-1	0	1	$P_Y(y)$
0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
$P_X(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	

$Y \sim Ber(p=\frac{1}{2})$

$P(0,1)=0$ BUT $P_X(0)\,P_Y(1) = \frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12} \neq 0$

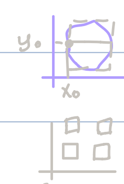
↳ Thus X, Y are not independent

$y \backslash x$	-1	0	1	$P_Y(y)$
0	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{6}\cdot\frac{1}{2}$	$\frac{1}{3}\cdot\frac{1}{2}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{2}$
$P_X(x)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	

$\Rightarrow$ X, Y independent

② X and Y are continuous: Two continuous r.v. X, Y are said to be independent if $\underbrace{f(x,y)}_{\text{jpdf}} = f_X(x)\,f_Y(y)$ $(*)$

Remark: $(*)$ implies that the support of jpdf is a cartesian product of the support of X and the support of Y.



$f(x_0, y_0) = 1$

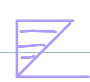
But $f_X(x_0) \neq 0$ and $f_Y(y_0) \neq 0$

independence only called when rectangle or union of rectangles

Ex: ① $f(x,y) = \begin{cases} 8x & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f(\frac{3}{4}, \frac{1}{4}) = 0$ but $f_X(\frac{3}{4}) \neq 0$ and $f_Y(\frac{1}{4}) \neq 0$.

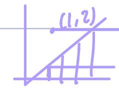
X and Y are not independent.



$f_Y(\frac{1}{4}) \neq 0$

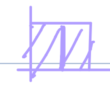
② $f(x,y) = \begin{cases} \frac{1}{x} e^{-x} & \text{if } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$ $\boxed{1}$ → what is the marginal density of X & Y?

$f(1,2) = 0$ but $f_X(1) \neq 0$ $f_Y(2) \neq 0$



Thus, X, Y are not independent

③ $f(x,y) = \begin{cases} \overset{\to\, = 1}{C}(x+y) & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

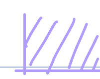


$f_X(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ $f_Y(y) = \begin{cases} y + \frac{1}{2} & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

clearly $f(x,y) \neq f_X(x) f_Y(y)$

Remark: If the support of f (the jpdf) is a rectangle AND the kernel of f is a product of a function of x and a function of y, then X and Y are independent.

Ex: $f(x,y) = \begin{cases} \overset{c=2}{C}\, \underline{x y^2 e^{-(2x+y)}} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$



The support of f is a rectangle and $x y^2 e^{-(2x+y)} = (x e^{-2x})(y^2 e^{-y})$.

Thus X and Y are independent. $X \sim \text{Gamma}(\alpha = 2, \beta = \frac{1}{2})$

$C = 4 \cdot \frac{1}{2} = 2$ $Y \sim \text{Gamma}(\alpha = 3, \beta = 1)$

Expected Value: let X, Y be 2 r.v. and $F: \mathbb{R}^2 \to \mathbb{R}^2$

① Discrete case → $\mathbb{E}(F(X,Y)) = \sum_x \sum_y F(x,y)\, p(x,y)$ where $p(x,y)$ is the pf of (X,Y)

② Continuous case → $\mathbb{E}(F(X,Y)) = \iint_{\mathbb{R}^2} F(x,y)\, f(x,y)\, dA$ where f is the jpdf of (X,Y)

Remark (continuous case): Using the definition above,

$$\mathbb{E}(X) = \iint_{\mathbb{R}} x\, f(x,y)\, dA = \int_{-\infty}^{-\infty} \Big(\underbrace{\int_{-\infty}^{+\infty} x f(x,y)\, dy}\Big)\, dx = \int_{-\infty}^{+\infty} x \Big(\underbrace{\int_{-\infty}^{+\infty} f(x,y)\, dy}\Big)\, dx$$

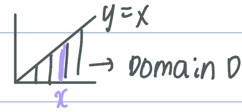
Ex: ①

$y \backslash x$	-1	0	1
0	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
1	$\frac{1}{4}$	0	$\frac{1}{4}$

$\longrightarrow$ ones for computation

$\mathbb{E}(XY) = (1)(-1)\frac{1}{4} + (1)(1)\frac{1}{4} = 0$

② $f(x,y) = \begin{cases} \frac{1}{x} e^{-x} & \text{if } 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$



$$\mathbb{E}(XY^2) = \iint_D x y^2 f(x,y)\, dA = \int_0^{+\infty} \Big(\int_0^x x y^2 \frac{1}{x} e^{-x}\, dy \Big)\, dx = \int_0^{\infty} \Big(\frac{x}{x} e^{-x} \int_0^x y^2\, dy \Big)\, dx$$

$$= \int_0^{+\infty} \frac{x^3}{3} e^{-x}\, dx = \frac{1}{3}$$

$E(\mathbb{E}(X), \mathbb{E}(Y))\, r(4) = 2$ $\qquad \cancel{f(\mathbb{E}(X), \mathbb{E}(Y))}$

Properties

① $\mathbb{E}(F(X,Y) + G(X,Y)) = \mathbb{E}(F(X,Y)) + \mathbb{E}(G(X,Y))$

② If $\alpha \in \mathbb{R}$, $\mathbb{E}(\alpha F(X,Y)) = \alpha \mathbb{E}(F(X,Y))$

Proposition: If X and Y are independent, then $\mathbb{E}(F(X)G(Y)) =$

$\mathbb{E}(F(X))\, \mathbb{E}(G(Y))$

Proof (continuous): $\mathbb{E}(F(X)G(Y)) = \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} F(x) G(y) f(x,y)\, dy \Big)\, dx$

$$= \int_{-\infty}^{+\infty} \Big(\int_{-\infty}^{+\infty} F(x) G(y) f_X(x) f_Y(y)\, dy \Big)\, dx = \int_{-\infty}^{+\infty} F(x) f_X(x) \Big(\underbrace{\int_{-\infty}^{+\infty} G(y) f_Y(y)\, dy}_{\mathbb{E}(G(Y))} \Big)\, dx$$

$$= \mathbb{E}(G(Y)) \int_{-\infty}^{+\infty} F(x) f_X(x)\, dx = \mathbb{E}(G(Y))\, \mathbb{E}(F(X))$$

Ex: $f(x,y) = \begin{cases} 2xy^2 e^{-(2x+y)} & \text{if } x > 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{E}(X^3 Y^4)$? $\quad$ X and Y are independent (see before)

$\rightarrow X \sim \text{Gamma}(\alpha=2, \beta=\frac{1}{2})$ $\qquad Y \sim \text{Gamma}(\alpha=3, \beta=1)$

$\mathbb{E}(X^3 Y^4) = \mathbb{E}(X^3)\,\mathbb{E}(Y^4) = (\frac{1}{2})^3 \gamma(5)\, \gamma(3+4) = \frac{4!\,6!}{8}$

Corollary (Mgf of the sum of independent r.v.): assume X, Y re independent. The mgf of $X+Y$ is $m_{X+Y}(t) = \mathbb{E}(e^{t(X+Y)}) = \mathbb{E}(e^{tX}e^{tY}) = \mathbb{E}(e^{tX})\,\mathbb{E}(e^{tY}) = m_X(t)\, m_Y(t)$

$m_{X+Y}(t) = m_X(t) + m_Y(t)$

Applications

① sum of independent Poisson r.v.: $X_1 \sim \text{Poisson}(\lambda_1)$ $\quad X_2 \sim \text{Poisson}(\lambda_2)$

X_1 and X_2 are independent. $X = X_1 + X_2$.

$m_X(t) = m_{X_1}(t)\, m_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$

$m_X(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}$ therefore $X_1 + X_2 = X \sim \text{Poisson}(\lambda_1+\lambda_2)$

② Sum of independent r.v.: $X_1 \sim N(\mu_1, \sigma_1^2)$ $\quad X_2 \sim N(\mu_2, \sigma_2^2)$.

X_1, X_2 are independent. $X = X_1 + X_2$: $m_X(t) = m_{X_1}(t)\, m_{X_2}(t)$

$= e^{\sigma_1^2 \frac{t^2}{2} + \mu_1 t^2}\, e^{\sigma_2^2 \frac{t^2}{2} + \mu_2 t} = e^{(\sigma_1^2+\sigma_2^2)\frac{t^2}{2} + (\mu_1+\mu_2)t}$.

therefore, $X = X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2)$.

③ Sum of 2 independent Gamma with same β:

$X_1 \sim \text{Gamma}(\alpha_1, \beta)$ and $X_2 \sim \text{Gamma}(\alpha_2, \beta)$. X_1 & X_2 independent.

$X = X_1 + X_2$. $\quad m_X(t) = m_{X_1}(t)\, m_{X_2}(t) = (1-\beta_1 t)^{-\alpha_1} (1-\beta_2 t)^{-\alpha_2}$, $\quad t < \frac{1}{\beta}$

$m_X(t) = (1-\beta t)^{-(\alpha_1+\alpha_2)}$ $\quad t < \frac{1}{\beta}$.

ie. $X = X_1 + X_2 \sim \text{Gamma}(\alpha_1+\alpha_2, \beta)$

$X_1 \sim$ Exponential, $\quad X_2 \sim \chi^2_{(p)}$