May 6

Recall: $C_0^n = C_n^n = 1$ and $C_r^n = C_{n-r}^n$

$C_r^n = C_{r-1}^{n-1} + C_r^{n-1}$ $\qquad C_r^n = \dfrac{n!}{(n-r)!\,r!}$

↳ pascal Triangle

"$r \le n$"

| n \ r | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | X | X | X | X |
| 1 | 1 + | 1 | X | X | X |
| 2 | 1 + | 2 + | 1 | X | X |
| 3 | 1 + | 3 + | 3 + | 1 | X |
| 4 | 1 | 4 | 6 | 4 | 1 |

$C_0^n = C_3^n = 1$

$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$

Ex: what is the coeff. of $x^3$ in expansion of $(\overset{a}{x} + \overset{b}{2\sqrt{x}})^4$

- Recall: $(a+b)^4 = a^4 + 4ab^3 + \underbrace{6a^2b^2}_{x^3} + 4a^3b + b^4$
- Coeff $= 6 \times 2^2 = 24$

Conditional Probability : Let $(\Omega, P)$ be a probability space.

Let $A$ be an event s.t $P(A) > 0$ ($A$ is not the null event).

For any event $B$, the conditional probability of $B$ given $A$ is defined as $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$

Remark: $P(A|A) = 1$ . $P(\Omega|A) = 1$

Ex: Roll a fair die :

A: "result even" B: "result 1 or 2 or 5" C: "result 1 or 2"

↳ subset of B

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{3}$

$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$ → outcome is "2"

$P(A|B) = \frac{P(B \cap A)}{P(B)} = \frac{1/6}{1/2} = \frac{1}{3}$

$P(B|C) = \frac{\underbrace{P(B \cap C)}_{= P(C)}}{P(C)} = \frac{1/3}{1/3} = 1 \neq P(B)$

$P(C|B) = \frac{P(B \cap C)}{P(B)} = \frac{1/3}{1/2} = \frac{2}{3}$

$P(A|C) = \frac{1}{2}$, $P(C|A) = \frac{1}{3}$

Remark: In general $P(A|B) \neq P(B|A)$

$P(A \cap B) = P(A|B)\,P(B)$

Independence: Given a probability space $(\Omega, P)$, two events A and B are independent if ① $P(A|B) = P(A)$ or ② $P(B|A) = P(B)$ or ③ $P(A \cap B) = P(A)\,P(B)$

Remark:

① $P(A|B) = P(A) \iff \frac{P(A \cap B)}{P(B)} = P(A) \iff$ ③ $P(A \cap B) = P(A)\,P(B)$

② $\iff P(B|A) = \frac{P(A \cap B)}{P(A)} = P(B)$

Ex: (use the prev. ex)

$P(A \cap B) = \frac{1}{6} \neq \frac{1}{2} \times \frac{1}{2} = P(A)\,P(B)$ → A & B not independent

$P(B|C) = 1 \neq P(B)$ thus B and C are not independent.

$P(A|C) = \frac{1}{2} = P(A)$ thus A and C are independent

independent ≠ disjoint

$P(A \cap B) = 0 \iff P(A)P(B) \iff A$ or $B$ is the null event

Remark: Repeated Trials (Roll a die twice or n times).

$A_1$ is based on the outcomes on the 1st trial

$A_2$ " 2nd trial

→ then $A_1$ and $A_2$ independent.

Ex: A coin is s.t. $P(H) = p$ $(0 < p < 1)$. Roll the coin until H appears. $W_n = "\underbrace{T \ldots T}_{n-1}H"$ using independence

$P(\{W_n\}) = (1-p)^{n-1} p$ $(p = \frac{2}{3}, \ 1-p = \frac{1}{3})$

Property: If A & B are independent, then $A^c$ & B independent.

$\Rightarrow A^c$ & $B^c$ independent

Proof: $P(A^c \cap B) = P(B \setminus A) = P(B) - P(A \cap B) = P(B) - \underbrace{P(A)P(B)}_{\text{b/c indep.}}$

$= P(B) - (1 - P(A)) = P(B)P(A^c)$

Thus B & $A^c$ independent, so are $B^c$ & A ; $A^c$ & $B^c$.

Ex: 20% of people in a given population have disease A, 30% have disease B. One can contract disease A independently of contracting disease B. What proportion of the population has neither disease?

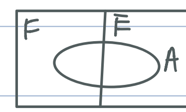
→ $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$ since A and B are independent.

$0.7 \times 0.8 = 0.56$

Exercise: suppose that the prob. that a selected individual from a population has the flu is $p$ $(0<p<1)$. A person who has the flu is thrice (x3) as likely to have pneumonia as a person who doesn't have the flu. Prob. that a selected individual has the flu, given that he has pneumonia.

$F$: person w/ flu $\qquad$ $A$: person w/ pneumonia

$P(F) = p$ $\qquad$ $P(A \mid F) = 3P(A \mid \bar{F})$



N.T.S: $P(F \mid A)$?

$\frac{P(A\cap F)}{P(F)} = 3\frac{P(A\cap \bar{F})}{P(\bar{F})}$

$A = (A\cap F) \cup (A \cap \bar{F})$

$\frac{3(A\cap F)}{p} = \frac{3P(A\cap \bar{F})}{1-p}$

$P(A) = P(A\cap F) + P(A\cap \bar{F})$

$P(A\cap \bar{F}) = \frac{1-p}{3p}P(A\cap F)$

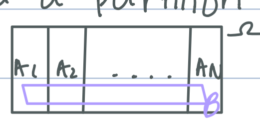
$\Rightarrow P(A) = P(A\cap F) + P(A\cap \bar{F})$

$P(A) = P(A\cap F) + \frac{1-p}{3p}P(A\cap F) = \frac{(2p+1)}{3p}P(A\cap F)$

$\Rightarrow P(F \mid A) = \frac{3p}{2p+1}$

Baye's Rule:

Partition: given a sample space $\Omega$, a finite sequence $A_1, A_2, \dots, A_N$ of events is called a partition if the $A_i$'s are pairwise and $\Omega = \bigcup_{i=1}^{N} A_i$




Given a partition $A_1, A_2, \dots, A_N$ of $\Omega$ and $B$ an event.

$B = B\cap\Omega = B\cap(A_1 \cup A_2 \dots \cup \dots \cup A_N)$

$= (B\cap A_1) \cup (B\cap A_2) \dots \cup (B\cap A_N)$

$= \bigsqcup_{i=1}^{N} (B\cap A_i)$. (Note the event $B\cap A_i$ are pairwise & disjoint)

It follows that $P(B) = \sum_{i=1}^{N} P(B \cap A_i) = \sum_{i=1}^{N} P(B|A_i)P(A_i)$

$$\star\ P(B) = \sum_{i=1}^{N} P(B|A_i)\,P(A_i)$$

- For $\underline{k \text{ fixed}}$, $P(A_k|B)$ ?

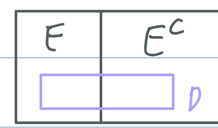
$$P(A_k|B) = \frac{P(A_k \cap B)}{P(B)} = \frac{P(B|A_k)P(A_k)}{P(B)} = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{N} P(B|A_i)P(A_i)}$$

Ex:

① Urn A contains 3 red balls & 2 green.
Urn B " 1R & 4G. A ball is selected from Urn A and placed into Urn B. Then A ball is selected from Urn B. What is the prob. that the ball selected from Urn A is Green given that the ball selected from Urn B is red.

Let $D$: ball from B is red
$E$: ball from A is green

$= P(E|D) = \frac{1}{4}$



(Null event)

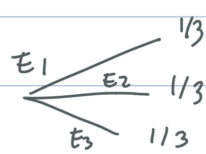
$P(E) = \frac{2}{5}$, $P(E^c) = \frac{3}{5}$

$P(D|E) = \frac{1}{6}$, $P(D|E^c) = \frac{2}{6}$

$P(D) = P(D|E)P(E) + P(D|E^c)P(E^c) = \frac{1}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{3}{5}$

$P(E \cap D) = \frac{1}{6} \times \frac{2}{5}$

② Refer to 2.132 (p. 74)




$E_i$ = plane went down in region $i$

$P(E_i) = \frac{1}{3}$

D: the search in region 1 is unsuccessful

$P(D|E_1) = 1 - \alpha_1 \qquad P(D|E_2) = 1 \qquad P(D|E_3) = 1$

$P(D) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}\alpha_1 = \frac{2}{3} + \frac{\alpha_1}{3}$

$P(E_1|D) = \dfrac{\frac{1}{3}\alpha_1}{\frac{2}{3} + \frac{\alpha_1}{3}} = \dfrac{\alpha_1}{2+\alpha_1}$

$P(E_2|D) = \dfrac{\frac{1}{3}}{\frac{2}{3} + \frac{\alpha_1}{3}} = \dfrac{1}{2+\alpha_1} = P(E_3|D)$