May 11

Recall: $V(X) = \mathbb{E}([X - \mathbb{E}(X)]^2) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \geq 0$

Properties:

① $V(X+a) = V(X)$ where a is a fixed constant

② $V(\alpha X) = \alpha^2 V(X)$, $\alpha \in \mathbb{R}$ is a constant

Usual Discrete Distributions

① Bernoulli Distribution:

Setup: A random experiment leads to 2 outcome {"success", "failure"}

set $P(\text{"success"}) = p$, $0 < p < 1$.

Define a random variable X as $X(\text{"success"}) = 1$ and

$X(\text{"failure"}) = 0$.

p.f. of X:

x	0	1
$p_X(x)$	$1-p$	p

→ we say X has a Bernoulli distribution with parameter p.

Notation: $X \sim Ber(p)$

↳ " is distributed as"

Remark: If $X \sim Ber(p)$ then $Y = 1 - X \sim Ber(1-p)$

func. of X (random var.)

Prop: If $X \sim Ber(p)$, then $\mathbb{E}(X) = p$ and $V(X) = p(1-p)$

Proof: $\mathbb{E}(X) = 0\, p_X(0) + 1 \cdot p_X(1) = 0(1-p) + 1 \cdot p = p$

$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - p^2$

$\mathbb{E}(X^2) = \sum_x x^2 p_X(x) = 0^2(1-p) + 1^2 p = p$

$V(X) = p - p^2 = p(1-p)$

② Binomial Distributions (R.V)

- Set up: A random experiment leads to 2 outcomes S = success, F = failure and $P(S) = p$, $0 < p < 1$. Repeat the above exp. n times.
  - Define X = # of success.
  - we shall assume that the triangles are independent

- Ex: Toss a fair coin 10 times and X = # of tails.

Note that $X \in \{0, 1, 2, \cdots, n\}$

$P_X(0) = Pr(X=0) = P(\text{"}\underbrace{FF\cdots F}_{n \text{ times}}\text{"}) = (1-p)^n$

$P_X(1) = Pr(X=1) = P(\text{"}\underbrace{FF\cdots F}_{n-1}S\text{"} \text{ or } \text{"}\underbrace{F\cdots F}_{n-2}SF\text{"} \text{ or } \cdots \text{"}S\underbrace{F\cdots F}_{n-1}\text{"})$

$\rightarrow P(X=1) = C_1^n \, p(1-p)^{n-1}$

$P_X(2) = Pr(X=2) = P(\text{"}SS\underbrace{F\cdots F}_{n-2}\text{"}$ .... there are C_2^n different positions for the pair of successes.

$\rightarrow P_X(2) = C_2^n \, p^2(1-p)^{n-2}$

More generally, for any $k \in \{0, 1, 2, \cdots n\}$

$\rightarrow P_X(k) = P(X=k) = C_k^n \, p^k (1-p)^{n-k}$

Remark: $\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} C_k^n \, p^k (1-p)^{n-k} = [p + (1-p)]^n = 1^n = 1$

A r.v. X which has a p.f. such as above X is said to have a Binomial distribution w/ parameters n and p.

Notation: $X \sim \text{Binomial}(n, p)$

Ex: A coin s.t. $P(H) = \frac{1}{3}$. Flip the coin 5 times. Prob. of obtaining exactly 3 tails?

$X = \#$ of H. $X \sim \text{Binomial}(n=5, p=\frac{1}{3})$

$P(3 \text{ tails}) = P(X=2) = C^5_2 (\frac{1}{3})^2 (\frac{2}{3})^3 = 10 \cdot \frac{8}{3^5} = \frac{80}{243}$

$\hookrightarrow Y = \#$ of T. $P(Y=3)$.

Remark: If $X \sim \text{Bin}(n, p)$ then $Y = n - X \sim \text{Bin}(n, 1-p)$

Prop: If $X \sim \text{Bin}(n, p)$ then

① $\mathbb{E}(X) = np$

② $V(X) = np(1-p)$

Proof: $\boxed{k\, C^n_k} = k \frac{n!}{(n-k)!\,k!} = \boxed{\frac{n!}{(n-k)!\,(k-1)!}}$ (if $k \neq 0$) $= \frac{n\,(n-1)!}{(n-k)!\,(k-1)!}$

$= n\, C^{n-1}_{k-1}$ if $k \neq 0$.

Exercise: $\boxed{k(k-1)\, C^n_k = n(n-1)\, C^{n-2}_{k-2}}$

$k(k-1)(k-2)\, C^n_k = \ldots$ (back to the proof of prop.)

$\rightarrow$ ① $\mathbb{E}(X) = \sum_{k=0}^{n} k\, P_X(k) = \sum_{k=0}^{n} k\, C^n_k\, p^k (1-p)^{n-k}$

$= 0 + \sum_{k=1}^{n} k\, C^n_k\, p^k (1-p)^{n-k} = \sum_{k=1}^{n} n\, C^{n-1}_{k-1}\, p^k (1-p)^{n-k}$

$= n \sum_{k=1}^{n} \underbrace{C^{n-1}_{k-1}}_{=\ell}\, p^k (1-p)^{n-k} \overset{(\ell = k-1)}{=} n \sum_{\ell=0}^{n-1} p^{\ell+1} (1-p)^{(n-1)-\ell}$

$= np \sum_{\ell=0}^{n-1} C^{n-1}_{\ell}\, p^{\ell} (1-p)^{(n-1)-\ell} = np[p + (1-p)]^{n-1} = np$

② $V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \mathbb{E}(X^2) - n^2p^2$

Note that $X^2 = X(X-1) + X$

$$\mathbb{E}(X^2) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) = \mathbb{E}(X(X-1)) + np$$

$$V(X) = \mathbb{E}(X(X-1)) + np - n^2p^2$$

$$\rightarrow \mathbb{E}(X^2) = \sum_{K=0}^{n} \underline{K^2 C_K^n}\, p^K (1-p)^{n-K}$$

$$\hookrightarrow K^2 C_K^n = K \cdot K C_K^n = K C_{K-1}^{n-1} = K \frac{(n-1)!}{(n-K)!(K-1)!}$$

$$\mathbb{E}(X(X-1)) = \sum_{K=0}^{n} \underline{K(K-1) C_K^n}\, p^K (1-p)^{n-K} = 0 + 0 + \sum_{K=2}^{n} \underline{n(n-1) C_{K-2}^{n-2}}\, p^K (1-p)^{n-K}$$

$$= n(n-1) \sum_{K=2}^{n} C_{K-2}^{n-2}\, p^K (1-p)^{n-K} = n(n-1) \sum_{K=2}^{n} C_{K-2}^{n-2}\, p^K (1-p)^{n-K}$$

$$(l = K-2)$$

$$= n(n-1) \sum_{l=0}^{n-2} C_l^{n-2}\, p^{l+2} (1-p)^{(n-2)-l} = n(n-1)p^2 \sum_{l=0}^{n-2} C_l^{n-2}\, p^l (1-p)^{(n-2)-l}$$

$$= n(n-1)p^2$$

$$V(X) = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

Exercise: $X \sim Bin(n,p)$. Find $\mathbb{E}(X^3)$

Hint: $X^3 = X(X-1)(X-2) + \underline{3}X^2 - \underline{2}X^2$

$\rightarrow K(K-1)(K-2) C_K^n = n(n-1)(n-2) C_{K-3}^{n-3}$

$\rightarrow \mathbb{E}(X(X-1)(X-2)) = n(n-1)(n-2)p^3$

Ex: You have to pay \$2 to play the following game. Flip a coin 5 times. Each time the coin turns up H you win \$3, each time it turns up T, you lose \$1. Given that $P(H) = 0.4$, find the <u>mean</u> & variance of you pay off.

↳ = expected value

$X = \#$ of H, $X \sim Bin(n=5, p)$

$Y =$ payoff. $y = -2 + 3X - (5-X) = 4X - 7$

$\mathbb{E}(X) = 4\mathbb{E}(X) - 7 = 20p - 7$

$V(Y) = V(4X) = 16V(X) = 16 \times 5p(1-p) = 80p(1-p)$

② Geometric distribution

Recall: $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ whenever $|x|<1$

→ set up: An experiment leads to either success = S or failure = f with $P(S) = p$. Repeat the exp. until the first S occurs.

Let X = # of trials needed for first S.

Note: $X \in \{1, 2, 3, \dots, \dots\} = \mathbb{N}^*$ ← set of positive ints. and

$$P_X(k) = P(X=k) = P(\text{"}\underbrace{F\dots F}_{k-1}S\text{"}) = (1-p)^{k-1}p$$

→ $X \in \{1, 2, \dots\} = \mathbb{N}^*$. $\forall k \in \mathbb{N}^*$.

$P_X(k) = P(X=k) = p(1-p)^{k-1}$

- A r.v. X with the p.f. described above is said to have a geometric distribution with parameter p.
- Notation: $X \sim \text{Geometric}(p)$

Remark: $\sum_{k=1}^{+\infty} P_X(k) = \sum_{k=1}^{+\infty} p(1-p)^{k-1} = p\sum_{k=1}^{+\infty}(1-p)^{k-1} \overset{(l=k-1)}{=} p\sum_{l=0}^{+\infty}(1-p)^l$

$= p\frac{1}{1-(1-p)} = \frac{p}{p} = 1$

prop: $X \sim \text{Geometric}(p)$ then:

① $\mathbb{E}(X) = \frac{1}{p}$ ② $V(X) = \frac{1}{p}\left(\frac{1}{p} - 1\right)$

Proof:

① $\mathbb{E}(X) = \sum_{k=1}^{+\infty} k\,P_X(k) = \sum_{k=1}^{+\infty} k\,p(1-p)^{k-1}$ (converges use ratio test)

→ Recall: $\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x}$ whenever $|x|<1$

$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{+\infty}\frac{d}{dx}(x^k) = \sum_{k=1}^{+\infty} k\,x^{k-1}$ if $|x|<1$

$\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \sum_{k=0}^{+\infty}\frac{d^2}{dx^2}(x^k) = \sum_{k=2}^{+\infty} k(k-1)x^{k-2}$ if $|x|<1$.

$\mathbb{E}(X) = p \sum_{k=1}^{+\infty} k(1-p)^{k-1} = p\frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$

② $V(X) = \mathbb{E}(X(X-1)) + \mathbb{E}(X) - (\mathbb{E}(X))^2 = \mathbb{E}(X(X-1)) + \frac{1}{p} - \frac{1}{p^2}$

$\mathbb{E}(X(X-1)) = \sum_{k=1}^{+\infty} k(k-1)\,p(1-p)^{k-1} = 0 + p\sum_{k=2}^{+\infty} k(k-1)(1-p)^{k-1} =$

$p(1-p)\sum_{k=2}^{+\infty} k(k-1)(1-p)^{k-2} = p(1-p)\cdot\frac{2}{p^3} = \frac{2(1-p)}{p^2} = \frac{2}{p^2} - \frac{2}{p} \cdot V(X)$

$= \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p}\left(\frac{1}{p} - 1\right)$

Ex (3.70 p.119)

a) S = find a productive well. $P(S) = 0.2 = p$.

$X \sim$ Geometric (p)

$P(X=3) = (0.8)^2(0.2) = \ldots$

b) $P(X \geq 11) = 1 - P(\underline{X \leq 10})$

$P(\underbrace{\text{"}FF\cdots F\text{"}}_{10 \text{ times}}) = (1-p)^{10}$

$P(X \geq 11) = \sum_{k=11}^{+\infty} (1-p)^{k-1} p$. Let $\ell = k - 11$

$\rightarrow \sum_{\ell=0}^{+\infty} (1-p)^{\ell+10} p = p(1-p)^{10} \sum_{\ell=0}^{+\infty} (1-p)^{\ell} = p(1-p)^{10}\frac{1}{1-(1-p)} = \frac{p}{p}(1-p)^{10}$

$= (1-p)^{10}$

$\mathbb{E}(X(X-1)) = \sum_{k=1}^{+\infty} k(k-1)\,P_X(k) = 0 + \sum_{k=2}^{+\infty} k(k-1)\,P_X(k)$

$= p(1-p)\underline{\sum_{k=2}^{+\infty} k(k-1)\cdot(1-p)^{k-2}} = \frac{p(1-p)\,x^2}{p^3}$

Recall: $\frac{2}{(1-x)^3} = \sum_{k=2}^{+\infty} k(k-1)\,x^{k-2}$

2.137 : $\boxed{\begin{matrix} i \text{ white} \\ 5-i \text{ black} \end{matrix}}$ urn ⓘ B_i: urn i is selected. $P(B_i) = \frac{1}{5}$.

E: 2 white balls selected. $P(E) = \sum_{k=1}^{5} P(E \mid B_k)\, P(B_k)$

$P(E \mid B_1) = 0$. $P(E \mid B_k) = \dfrac{C^k_2}{C^5_2} = \dfrac{\frac{k(k-1)}{2}}{10} = \dfrac{k(k-1)}{20}$

$P(E) = \frac{1}{100} \sum_{k=2}^{5} k(k-1) = \frac{1}{100}[2+6+12+20]$

3.83 : $P(S) = \frac{1}{n}$. $P(\text{"FFFFFS"}) = (1-\frac{1}{n})^5 \frac{1}{n}$

Ω = set of outcomes

$P(\Omega)$ = set of all subsets of Ω = set of all events

Exercise : If card$(\Omega) = n$, then card$(P(\Omega)) = 2^n$

Ex: 4 Balls (1G, 1Y, 1R, 1B) into 4 Hats (1G, 1Y, 1R, 1B)

X = numbers of balls which are placed into hats of same color

p.f. of X.

$P(X=4) = \frac{1}{24}$