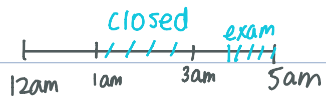
May 19

Midterm - Ch. 2 & 3 (4 Questions)

Ex (4.50):



X: time when the call is made

$X \sim \text{Uniform}(0, 5)$ in hours

$P(0<X<1 \text{ or } 3<X<4) = P(0<X<1) + P(3<X<4) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5} = 0.4$

② Normal Distribution

- Fact: $\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$

(Hint: use a change of variable)

↳ Exercise: using the fact above, prove $\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}_{(*)} dx = \sigma\sqrt{2\pi}$

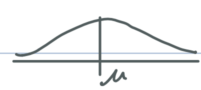
→ σ and μ are constant $t = \frac{x-\mu}{\sigma}$

Therefore, given $\sigma > 0$ and μ fixed, the function

to make 1

$f(x) = \underbrace{\frac{1}{\sigma\sqrt{2\pi}}}_{\text{normalizing constant}} \underbrace{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}_{\text{kernel}}$, $\underbrace{x \in \mathbb{R}}_{\text{support}}$ is a probability density function of some continuous r.v. X.

X is said to have a normal distribution with parameters μ and σ^2 (we will see that μ is the mean of X and σ^2 is the variance of X)



Notation: $X \sim N(\mu, \sigma^2)$

Prop: If $X \sim N(\mu, \sigma^2)$, then $X = \sigma Z + \mu$ where $Z \sim N(0,1)$.

Z is called a standard normal r.v.

Proof: Assume $X \sim N(\mu, \sigma^2)$.

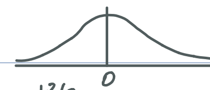
set $Z = \frac{1}{\sigma}(X-\mu) = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$.

Therefore the pdf of Z : $f_Z(z) = \frac{1}{|\sigma|} f_X(\sigma z + \mu) = \sigma f_X(\sigma z + \mu)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $z \in \mathbb{R}$ thus $Z \sim N(0,1)$

Properties of $Z \sim N(0,1)$

① pdf of Z is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $z \in \mathbb{R}$



② cdf of Z $F_Z(z) = P(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$ (integral of pdf)

(*) $F_Z(0) = P(Z \le 0) = \frac{1}{2}$ (**) $F_Z(-z) + F_Z(z) = 1$ (b/c its summ.)

Ex (4.66) :

σ = standard dev.

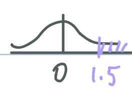
$X \sim N(\mu = 3.0005, \sigma^2 = (0.001)^2)$

$P(\text{scrap}) = P(|X - 3.000| > 0.002) = 1 - P(-0.002 < X - 3 < 0.002)$

$= 1 - P(-0.0025 < X - 3.0005 < 0.0015) =$

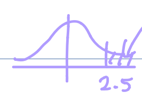
$1 - P(-2.5 < Z = \frac{X - 3.0005}{0.001} < 1.5) = 1 - P(-2.5 < Z < 1.5) =$

$1 - (F_Z(1.5) - F_Z(-2.5))$ (use a table or a software) p.848

$\rightarrow F_Z(1.5) \rightarrow$ 

← area from the table = 0.0668

$F_Z(1.5) = 1 - 0.0668 = 0.9332$

※ Try $F_Z(-2.5) = 1 - F_Z(2.5)$ 

Prop: If $X \sim N(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$ and $V(X) = \sigma^2$

Proof: $X = \sigma Z + \mu$ where $Z \sim N(0,1)$.

$\mathbb{E}(X) = \sigma \mathbb{E}(Z) + \mu$ and $V(X) = \sigma^2 V(Z)$.

Thus, it is enough to prove that $\mathbb{E}(Z) = 0$ and $V(Z) = 1$.

① $\mathbb{E}(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z e^{-\frac{z^2}{2}}}_{\text{odd func.}} dz = 0$

② $V(Z) = \mathbb{E}(Z^2) - \underbrace{(\mathbb{E}(Z))^2}_{0} = \mathbb{E}(Z^2)$

$\mathbb{E}(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z^2 e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{+\infty} z \, d\left(\frac{-e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\right) = z \frac{e^{-\frac{z^2}{2}}}{\sqrt{\pi}} \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$

↳ Note: $\lim_{z\to+\infty} z e^{-z^2/2} = 0 = \lim_{z\to-\infty} z e^{-z^2/2}$

$\mathbb{E}(z^2) = \int_{-\infty}^{+\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \, dz = 1$

$\mathbb{E}(x^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$

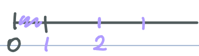
Exercise: let $Z \sim N(0,1)$; Find a formula for $\mathbb{E}(Z^n)$

→ Hint: $\mathbb{E}(Z^{2n+1}) = 0$

?

③ Gamma distribution → Gamma function

Set $\gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$ converges if $\alpha > 0$

Property: $\gamma(\alpha+1) = \alpha\,\gamma(\alpha)$  $\gamma(\frac{3}{2}) = \frac{1}{2}\gamma(\frac{1}{2})$

Proof: $\gamma(\alpha+1) = \int_0^{+\infty} x^\alpha e^{-x} dx = \int_0^{+\infty} x^\alpha \, d(-e^{-x}) = x^\alpha e^{-x}\Big|_0^{+\infty} + \alpha \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$

$= \alpha\,\gamma(\alpha)$

Applications:

① $\gamma(1) = \int_0^{+\infty} e^{-x} dx = 1$ $\quad \gamma(2) = \gamma(1+1) = 1 \cdot \gamma(1) = 1$

$\gamma(3) = \gamma(2+1) = 2\gamma(2) = 2$ $\quad \gamma(4) = 3\gamma(3) = 6$

... $\gamma(n) = (n-1)!$ (for n integer)

② Given $\gamma(\frac{1}{2}) = \sqrt{\pi}$, $\gamma(\frac{7}{2})$?

$\gamma(\frac{7}{2}) = \frac{5}{2}\gamma(\frac{5}{2}) = \frac{5}{2}\cdot\frac{3}{2}\gamma(\frac{3}{2}) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\gamma(\frac{1}{2}) = \frac{15}{8}\sqrt{\pi}$

→ Exercise: Find $\gamma\left(\frac{2n+1}{2}\right)$

ζ

x not integer

$\Gamma(x) = (x-1)(x-2)\cdots(x-[x])\,\Gamma(\underbrace{x-[x]}_{\in(0,1)})$ where $[x]$ is the largest integer $\leq x$.

*Remark: for $a>0$ and $b>0$. $\int_0^{+\infty} e^{-bx} x^a\,dx \overset{(t=bx)}{=} \int_0^{+\infty} e^{-t}\left(\frac{t}{b}\right)^a \frac{dt}{b} =$
$\frac{1}{b^{a+1}}\int_0^{+\infty} t^a e^{-t}\,dt = \frac{1}{b^{a+1}}\Gamma(a+1)$

** $\int_0^{+\infty} e^{-bx} x^a\,dx = \frac{\Gamma(a+1)}{b^{a+1}}$

$\to \int_0^{+\infty} x^{10} e^{-2x}\,dx = \frac{\Gamma(11)}{2^{11}} = \frac{10!}{2^{11}}$

Back to Gamma Distribution

The following function $f(x) = \begin{cases} c\,x^{\alpha-1} e^{-x/\beta} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$ is a pdf if

$c = \frac{1}{\int_0^{+\infty} x^{\alpha-1} e^{-x/\beta}} = \frac{1}{\Gamma(\alpha)\beta^\alpha}$

The function $f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} & \text{if } x>0 \\ 0 & \text{otherwise} \end{cases}$ is a pdf for $\alpha>0$

and $\beta>0$ fixed.

It describes a continuous r.v. X which is said to have a Gamma distribution with parameters α, β.

Notation: $X \sim \text{Gamma}(\alpha, \beta)$ ← scale parameter (β)

↑ shape parameter (α)

Remark: If $X \sim \text{Gamma}(\alpha,\beta)$, then $X = \beta Y$ where $Y \sim \text{Gamma}(\alpha, 1)$

$\alpha = \frac{1}{2}$, $\beta = 1$ $\to$ cdf $= F_X(x) = \frac{1}{\sqrt{\pi}}\int_0^x t^{-1/2} e^{-t}\,dt$ ($x>0$)

Prop: If $X \sim$ Gamma, then $\mathbb{E}(X) = \beta\alpha$ and $V(X) = \beta^2\alpha$

Proof:

① $$\mathbb{E}(X) = \frac{1}{\gamma(\alpha)\beta^\alpha}\int_0^{+\infty} x\, x^{\alpha-1} e^{-x/\beta}\,dx = \frac{1}{\gamma(\alpha)\beta^\alpha}\int_0^{+\infty} x^{\alpha} e^{-x/\beta}\,dx = \frac{1}{\gamma(\alpha)\beta^\alpha} = \beta^{\alpha+1}\gamma(\alpha+1)$$

$$= \beta\frac{\gamma(\alpha+1)}{\gamma(\alpha)} = \alpha\beta$$

② $$\mathbb{E}(X^2) = \frac{1}{\gamma(\alpha)\beta^\alpha}\int_0^{+\infty} x^{\alpha+1} e^{-x/\beta}\,dx = \frac{\gamma(\alpha+2)\beta^{\alpha+2}}{\gamma(\alpha)\beta^\alpha} = \beta^2\frac{\gamma(\alpha+2)}{\gamma(\alpha)} = \beta\alpha(\alpha+1)$$

$$V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \beta^2(\alpha^2+\alpha) - \alpha^2\beta^2 = \alpha\beta^2$$

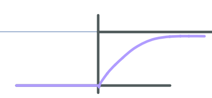
Exercise: $X \sim \text{Gamma}(\alpha,\beta)$. $\mathbb{E}(X^n) = \beta^n\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$

Exponential Distribution: These are Gamma distributions with $\alpha = 1$.

Notation $X \sim$ Exponential (mean $= \beta$) iff the pdf of X is

$$f_X(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

cdf of X: $$F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-x/\beta} & \text{if } x \ge 0 \end{cases}$$



$\mathbb{E}(X) = \beta$ and $V(X) = \beta^2$

Memoryless Property: A continuous r.v. X has an exponential distribution iff: $P(X > a+b \mid X > a) = P(X > b)$ for $a > 0$, $b > 0$ $= \frac{S(a+b)}{S(a)} = S(b)$

→ Indeed if $X \sim$ Exponential (mean $= \beta$),

$$P(X > a+b \mid X > a) = \frac{P(X > a+b \cap X > a)}{P(X > a)} = \frac{P(X > a+b)}{P(X > a)} = \frac{1 - F_X(a+b)}{1 - F_X(a)}$$

$$= \frac{e^{-(a+b)/\beta}}{e^{-a/\beta}} = e^{-b/\beta} = P(X > b)$$