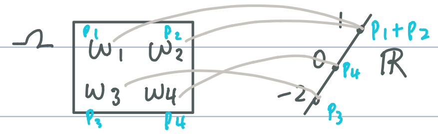
May 7

(Ω, P) a probability space



Def (Random variable): let (Ω, P) be a probability space.

A random var. X is a function $X : \Omega \to \mathbb{R}$ which assigns to every outcome $w \in \Omega$, a real number $X(w) \in \mathbb{R}$

Ex.

① Toss a fair coin. $\Omega = \{H, T\}$. Define X as follows:

$X(H) = 2$, $X(T) = -4$.

Interpretation: If H you win 2, If T you loose 4.

② Roll a die. $\Omega = \{w_1 \sim w_6\}$. Set $X(w_1) = X(w_3) = W(X_5) = -2$.

$X(W_2) = X(W_4) = X(W_6) = 1$

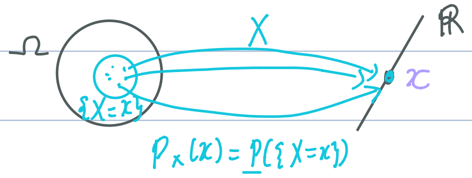
③ Toss a coin until H appears.

$\Omega = \{w_1, w_2, \dots, w_n, \dots\}$

$w_n = $ "$\underbrace{T \dots T}_{n-1}H$". Let X be the number of tosses needed to obtain H. $X(w_n) = n$.

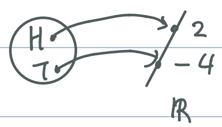
Def (Probability Func.): Given a random variable X defined on a probability space (Ω, P). The probability func. of X denoted p_X, is the function $p_X : \mathbb{R} \to (0,1)$ defined $p_X(x) = \underline{P}(\{X = x\})$.

The event $\{X = x\} \subset \Omega$ is the subset of all outcomes ω (in Ω) such that $X(\omega) = x$



Ex:

① Toss a fair coin $\Omega = \{H, T\}$. $X(H) = 2$ and $X(T) = -4$. What is the prob. func. (p.f.) of X?



If $x = 0$, what is $p_X(0) = ?$

$\to \{X = 0\} = \emptyset$, $p_X(0) = \underline{P}(\emptyset) = 0$

$p_X(1) = 0$

$p_X(2) = \underline{P}(\{X = 2\}) = \underline{P}(\{H\}^X) = \frac{1}{2}$; $p_X(-4) = \frac{1}{2}$

The probability function p.f. of X can be summarized as

$$p_X(x) = \begin{cases} \frac{1}{2} & \text{if } x = -4 \text{ or } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

x	-4	2
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{2}$

Remark: It is worth noting that $p_X(x) = 0$ whenever $x \notin \text{Range}(X)$

Range $= \{-4, 2\}$

② Roll a die $\Omega = \{w_1, w_2, w_3, w_4, w_5, w_6\}$.

$P(\{w_1\}) = P(\{w_4\}) = P(\{w_6\}) = \frac{1}{6}$

$P(\{w_3\}) = \frac{1}{4}$ $\quad P(\{w_5\}) = \frac{1}{6}$ $\quad P(\{w_2\}) = \frac{1}{12}$

$X(w_1) = X(w_3) = X(w_5) = -2$

$X(w_2) = X(w_4) = X(w_6) = 1$ $\qquad$ $\text{Range}(X) = \{1, -2\}$

The p.f. of X can be summarized as

x	-2	1
$P_X(x)$	$\frac{7}{12}$	$\frac{5}{12}$

$P_X(-2) = P(\{X = -2\})$

$= P(\{w_1, w_3, w_5\}) = \frac{7}{12}$

③ A coin is such that $P(H) = p$ $\quad p \in (0,1)$. Toss the coin until H appears. $\Omega = \{w_1, w_2, \cdots, w_m, \cdots\}$

$X(w_n) = n$. p.f. of X. $\text{Range}(X) = \{1, 2, 3, \ldots\} = \mathbb{N}^*$ set of positive integers

$$P_X(x) = \begin{cases} 0 \text{ whenever } x \notin \mathbb{N}^* \\ (1-p)^{n-1} p \text{ if } x = n \text{ (n is a positive integer)} \end{cases}$$

Remark: If Ω is a discrete samplespace and X is a random var. then $P_X(x) = 0$ whenever $x \notin \text{Range}(X)$. Range (X) is called the support of P_X. Since Ω is discrete, Range (X) is also discrete and we have $\sum_{x \in \text{Range}(X)} P_X(x) = 1$.

Proposition: let p be a func. defined on $\mathbb{R}$ s.t. the set

① $\{x \mid p(x) \neq 0\}$ is discrete

② $p(x) \geq 0 \quad \forall x$

③ $\sum_x p(x) = 1$.

There exists a random var. X s.t. its p.f is p.

Definition (Expected Value): let X be a discrete random var. and p_X its p.f. The expected value of X is denoted $E(X)$ and defined as $\mathbb{E}(X) = \sum_{x} x\, p_X(x)$ → whenever RHS is well defined

this sum either a finite sum or a series

Ex:

① X is a r.v. with p.f

x	-1	0	1
$p_X(x)$	$1/2$	$1/3$	$1/6$

$$\mathbb{E}(X) = \sum_{x} x\, p_X(x) = (-1)\tfrac{1}{2} + 0\left(\tfrac{1}{3}\right) + 1\left(\tfrac{1}{6}\right) = -\tfrac{1}{2} + \tfrac{1}{6} = \boxed{-\tfrac{1}{3}}$$

② consider a discrete r.v X with p.f defined as:

$$p_X(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{N}^* \\ \frac{1}{n(n+1)} & \text{if } x = n \text{ (pos. int.)} \end{cases}$$

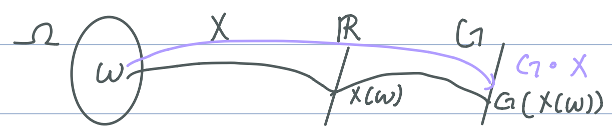
$1 \quad 2 \quad 3 \quad n$, $\frac{1}{n(n+1)}$

(Note: $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1$)

What about $\mathbb{E}(X)$?

$\mathbb{E}(X)$ (if defined) would be: $\sum_{n=1}^{+\infty} n\, p_X(n) = \sum_{n=1}^{+\infty} \frac{1}{n+1}$ ← divergent series so $\mathbb{E}(X)$ not defined.

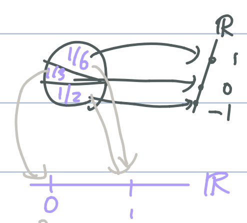
Prop: Let X be a discrete r.v. and p_X be its p.f. Let $G: \mathbb{R} \to \mathbb{R}$ be a function. Then $\mathbb{E}(G(X)) = \sum_{x} G(x)\, p_X(x)$



Ex: X r.v. with p.f.

x	-1	0	1
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$Y = X^2$



y	0	1
$p_Y(y)$	$\frac{1}{3}$	$\frac{2}{3}$

$\mathbb{E}(Y) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$

Using proposition instead: $\mathbb{E}(Y) = \mathbb{E}(X^2) = \sum_{x} x^2 p_X(x) =$

$= (-1)^2 \frac{1}{2} + 0^2 \frac{1}{3} + 1^2 \frac{1}{6} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

Q: $\mathbb{E}(X^2) = (\mathbb{E}(X))^2$? $\rightarrow$ FALSE ! never do $\mathbb{E}(F(X)) = F(\mathbb{E}(X))$

Properties:

① If c is a constant, $\mathbb{E}(c) = c$.

② If α is a constant and F is a function, $\mathbb{E}(\alpha F(X)) = \alpha \mathbb{E}(F(X))$.

③ If F and G are two functions then,

$$\mathbb{E}(F(X) + G(X)) = \mathbb{E}(F(X)) + \mathbb{E}(G(X))$$

Def (variance of a r.v.): let X be a discrete r.v. The variance of X is defined as $V(X) = \mathbb{E}([X - \mu_X]^2)$ where $\underbrace{\mu_X}_{\text{first var.}} = \mathbb{E}(X)$.

Note that $V(X) \geq 0$.

Prop: $V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Proof: Set $\mu_X = \mathbb{E}(X)$. $V(X) = \mathbb{E}([X - \mu_X]^2) = \mathbb{E}(X^2 - 2\mu_X X + \mu_X^2) =$

$\mathbb{E}(X^2) + \mathbb{E}(-2\mu_X X) + \mathbb{E}(\mu_X^2) = \mathbb{E}(X^2) \underbrace{- 2\mu_X}_{\text{constant}} \mathbb{E}(X) + \mu_X^2$

$= \mathbb{E}(X^2) - 2\mu_X^2 + \mu_X^2 = \mathbb{E}(X^2) - \mu_X^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$

Ex:

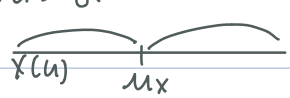
X	-1	0	1
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$\mathbb{E}(X) = -1/3$

$\mathbb{E}(X^2) = \frac{2}{3}$

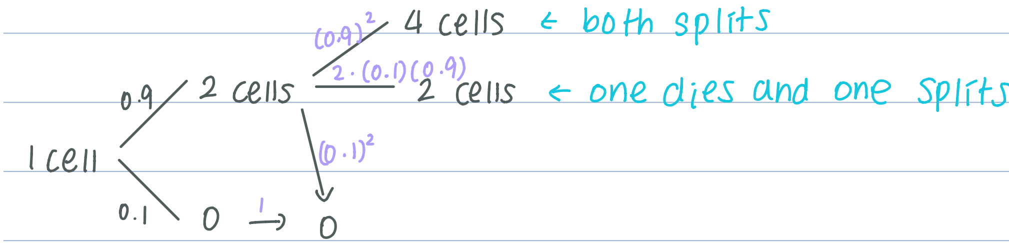
$V(X) = \frac{2}{3} - \left(-\frac{1}{3}\right)^2 = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}$

$V(X)$ gives an indication about the "spread" of X.



Standard deviation $= \sqrt{V(X)}$

Ex (3.8, p.90)



$X \in \{0, 2, 4\}$

$P_X(0) = 0.1 + (0.1)^2 \cdot 0.9 = 0.1 + 0.009 = 0.109$

$P_X(2) = 0.9 \cdot 2 \cdot (0.1)(0.9) = \quad \ldots$

$P_X(4) = (0.9)^3 \;\ldots$