FMAT 3888 Monte Carlo and Finite Difference Methods in option pricing

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1 Introduction

In many option pricing scenarios, the rapid and precise computation of numerous prices within constrained time frames is often a critical requirement. The two most commonly used methods for option pricing are Monte Carlos, Finite Difference. In order to compute option price using these methods, it is very important to consider these two questions- How accurate are these methods, and what level of stability do they exhibit when employed for option pricing? These questions are essential, since any error in these pricing methods can have significant financial implications, potentially leading to substantial losses.

In this report, we will discuss the theoretical foundations of both methods and extend it to evaluate their accuracy and stability. Additionally, we will consider the advantages and disadvantages of each approach, leading to a comprehensive comparison that will aid in determining their applicability in different financial contexts.

In section 5, we will take a practical approach by demonstrating the performance of both Monte Carlo and Finite Difference methods using a specific numerical scenario and examine how stability and accuracy are emphasised for both methods to be applied in the option pricing scenario.

2 Monte Carlo (MC) Method

2.1 Monte Carlo Concept

The Monte Carlo method is a random sampling-based technique that works based on the law of large numbers. The Law of Large Numbers states that as the number of independent, identically distributed random variables $X_1, X_2, \ldots, X_i$ increases, the sample mean $\bar{X}_i$ approaches the expected value μ:

Then, to compute $Ef(X)$, we can generate N independent identical distribution copies of X, ($X_1, \ \ldots, \ X_N$)

$$Ef(X) \approx \frac{1}{N}\Sigma_{i=1}^{N}(f(X_i))$$

As a starting point, for example, we can use Monte Carlo to price a vanilla European call option. The formula for a European call option price in the Black-Scholes model is given by:

$$S_T = S_0 e^{(\frac{-1}{2}\sigma^2 t + \sigma\sqrt{t}X)} \tag{1}$$

where $X \sim \mathcal{N}(0,1)$, S_0 = Initial stock price, r=interest rate, σ = volatility.

and we must generate the option payoffs according to the expressions:

$$max[0, S_T - K]$$

In option pricing, we need to estimate the expected value of the discounted payoff of the option:

$$S = e^{-rT}\hat{E}[S_T]$$

Therefore, we can compute the expected payoff of a European call for Monte Carlo with this following mathematics expression:

$$e^{-rT}E[S_0 e^{((r\frac{-1}{2}\sigma^2)t+\sigma\sqrt{t}X} - K)^+], \tag{2}$$

Equation (1) yields theoretical value of the option based on the BS model and we can compare this to the result obtained by MC method (2). In this case, we only need to consider the path of just two points: the initial price and the price at expiration. Although pricing a European call by MC(2) may not seem as efficient as using the Black Scholes equation (1), Comparing the results from MC with the theoretical formula has statistical meanings such as calculating error derived by MC and testing efficiency, (which is also used later in section 5)

2.2 Efficiency and accuracy

Testing the efficiency of Monte Carlo methods in option pricing involves evaluating their accuracy and computational performance compared to alternative methods. **Accuracy** of MC method is of order $\frac{1}{\sqrt{N}}$. It means in order to reduce the error to $\frac{1}{10}$ of its original value, the number of samples need to be increased by a factor of 100 to keep the same level of accuracy. Therefore, It seems not efficient. This problem can be solved by variance reduction technique -Antithetic variates

Antithetic variates: The idea behind antithetic variates is to generate negatively correlated, symmetrically distributed f(X) and f(-X), so that $Cov[f(x), f(-x)] \leq 0$ and (given they are i.i.d..) combine them, we get $\frac{f(x)+f(-x)}{2}$. Then, Ef(X) = Ef(-x) = E[$\frac{f(x)+f(-x)}{2}$]

For example, to reduce the variance using Antithetic with N simulations,

$$\frac{1}{N}Var[\frac{f(x)+f(-x)}{2}] = \frac{Varf(x)+Cov[f(x),f(-x)]}{2N} \tag{3}$$

When using the antithetic variance reduction method in MC simulations, the same level of accuracy in estimates can be achieved using at most half the number of simulations required by the basic MC method (2N), assuming all the assumptions are met, compared to what is needed in the basic MC method(2N).

For basic MC with 2N simulations,

$$Var(\frac{1}{2N}\sum f(x_i)) = \frac{1}{2N}Var[f(x)] \tag{4}$$

since $Cov[f(x), f(-x)] \leq 0$, therefore (3) $\leq$ (4), when $f(x)$ is monotonic and comes from a symmetric distribution, negatively correlated with $f(-x)$. Antithetic with N has less variance than MC with 2N, thus, reduced error. However, this method may not yield a variance reduction when monotonicity condition is not satisfied. It is important to check negative correlation in both random numbers and the outer samples. Furthermore, in option pricing, this can be used, for example, in [1] the BS model by generating the original asset price paths S_i, and its antithetic counterpart $\tilde{S}_i$ and the independently correlated payoffs can be calculated

2.3 Generating Random Paths

Unlike the European call options, many other financial models require trajectories of underlying stock prices $[S_t, o \leq t \leq T]$. Path-Dependent Derivatives: Many financial derivatives, like Asian options, barrier options, and American options, have payoffs that depend on the entire path of the underlying asset's prices. In these cases, simulating a single final price, as in European options, is not sufficient. We need to simulate a series of prices over time to determine the derivative's value accurately. One common process used for this purpose is Euler Method with the following equation:

The Black-Scholes stochastic differential equation (SDE) is given as:

$$dS(t) = rS(t)dt + \sigma S(t)B_t \tag{5}$$

where $(B_t)_t \geq 0$

To discretize this equation using the Euler method with a time step Δt, we get:

$$S(t+\Delta t) = S(t) + rS(t)\Delta t + \sigma S(t)\sqrt{\Delta t} \cdot Z, \tag{6}$$

where $Z \sim \mathcal{N}(0,1)$

In order to generate a sample path of $S(t)_{0\leq t\leq T}$, we choose interger M (the number of time steps), N(the number of paths) and calculate (time increment) $\Delta t = \frac{T}{M}$ and generate M iid copies of N(0,1), denote $Z_i, i = 1, 2, ..M$. Then, from the initial stock price, set as $S_0 = S(1)$, we can generate a sample path by keeping M updated from 1 to the chosen interger M with loop and store the N simulated stock price paths in S

$$\text{for } k = 1 : M \text{ (loop)}$$
$$S(k+1) = S(k) + S(k) \cdot r \cdot \Delta t + S(k) \cdot \sigma \cdot \sqrt{\Delta t} \cdot \mathbb{Z}(k)$$

2.4 Advantage and Disadvantage

This tool offers several notable advantages, including its user-friendly nature, and adaptability to various scenarios. For instance, it can effectively handle complex factors like stochastic volatility and many complicated exotic options. Furthermore, it is efficient for tackling high-dimensional problems, where traditional partial differential equation (PDE) frameworks may be inefficient. However, there are certain challenges associated with the tool, particularly when applied to American options that involve early exercise decisions. Unlike simulations, which progress forward in time, American options require a backward-in-time approach. Additionally, another disadvantage of this tool is the computational burden it imposes. As one increases the number of simulations to refine confidence interval estimates, the computational demands grow significantly. While variance reduction techniques can help improve this issue to some extent, they also reqire additional effort and knowledge.

3 Finite Difference Methods

Finite Difference (FD) Methods are numerical techniques used to solve and approximate partial differential equations (PDEs) that describe the evolution of option prices over time. In this report, **1)Explicit method, 2)Implicit method, 3) Crank-Nicolson (CN) method** will be discussed. In this report, the Black-Scholes partial differential equation will be used to show how each of FD methods works

$$f(t, S) = \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0, \tag{7}$$

To solve a PDE by finite difference method, we must set up a discrete grid (t,S) with respect to time and asset price. $f_{i,j} = f(i\delta t, j\delta S)$

$$t = (0, \delta t, 2\delta t, ..N\delta t) = T, S = (0, \delta S, 2\delta S, ..M\delta S) = S_{max}$$

We then can apply the approximation schemes (ex, forward, backward..), and approximate the partial derivatives in equation 7

$$\text{Forward Approximation:} \frac{\partial f}{\partial t} \approx \frac{f_{i+1} - f_{i,j}}{\delta t}, \text{derived from} f'(x) = \frac{1}{\Delta x}(f(x+\Delta x) - f(x)),$$

$$\text{Backward Approximation:} \frac{\partial f}{\partial t} \approx \frac{f_{i,j} - f_{i-1,j}}{\delta t}, \text{derived from} f'(x) = \frac{1}{\Delta x}(f(x) - f(x-\Delta x)),$$

$$\text{Central Approximation:} \frac{\partial f}{\partial t} \approx \frac{f_{i+1,j} - f_{i-1,j}}{2\delta t}, \text{derived from} f'(x) = \frac{1}{\Delta 2x}(f(x+\Delta x) - f(x-\Delta x)),$$

$$\text{Thus, for equation 7, we need 2nd derivative approximation:} \frac{\partial^2 f}{\partial S^2} \approx \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{(\Delta S)^2},$$

The same thing can be done using each approximation with respect to first derivatives of asset price $\frac{\partial f}{\delta S}$. Depending on the combination of schemes for discretizing the equation, we end up with different methods, explicit, implicit, or Crank-Nicolson which we will explore in the following sections. The choice of this method affects the stability and accuracy of the solution.

3.1 Explicit Method

As a first attempt to solve equation 7, we can consider European call option and demonstrate what explicit method is. We use a backward approximation for $\frac{\partial f}{\delta t}$, Central approximation for $\frac{\partial f}{\delta S}$ and 2nd derivative approximation for $\frac{\partial^2 f}{\delta S^2}$. After substituting these into equation 7 and rearranging the equation, the explicit finite difference equation becomes:

$$f_{i-1,j} = a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} \quad i = 0, 1 \ldots N-1, \; j = 1, 2, \ldots M-1 \tag{8}$$

Where the coefficients a, b, and c are defined as:

$$\begin{aligned} a_j &= -\frac{1}{2}\delta(t)(\sigma^2 i^2 - rj) \\ b_j &= 1 - \delta(t)(\sigma^2 i^2 + r) \\ c_j &= \frac{1}{2}\delta(t)(\sigma^2 i^2 + rj) \end{aligned}$$

Therefore, In the explicit scheme, we can calculate $f_{N-1,j}$ from a explicitly known combination of three terms $f_{N,j+1}/f_{N,j}/f_{N,j-1}$

3.1.1 Efficiency and stability

Based on the truncation error introduced when approximating the partial derivative, explicit scheme is known to be first-order accurate $O(\delta t)$, $O(\delta S^2)$. Thus, the number of computations required to achieve same accuracy is $O(\delta S^2)$ is $O(N^3)$. More importantly, there is a strict stability condition for the grid size(This will be discussed in more detail in section 5-2), otherwise the method gives unreliable result.Hence, using explicit scheme seems not as efficient in option pricing dynamics. In order to compensate these, Implicit and CN methods will be introduced in the following section.

3.2 Implicit, Crank-Nicolson Scheme

Follwoing the similar steps discussed above, implicit method yields

$$d_j f_{i,j-1} + e_j f_{i,j} + g_j f_{i,j+1} = f_{i+1,j} \tag{9}$$

where d,e,g are coefficient in a similar form.

Unlike the explicit method, implicit method $f_{N,j}$ has to be solved with three unknown terms, $f_{N-1,j+1}/f_{N-1,j}/f_{N-1,j-1}$ hence requires an additional complication of solving a systems of linear M-1 equations for time layer i with tridiagonal matrix that includes the coefficients.

$$\begin{bmatrix} e_1 & g_1 & 0 & 0 & \cdots & 0 & \\ d_2 & e_2 & g_2 & 0 & \cdots & 0 & \\ 0 & d_3 & e_3 & g_3 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & g_{M-2} & \\ 0 & 0 & 0 & 0 & \cdots & d_{M-1} & e_{M-1} \end{bmatrix} \begin{bmatrix} f_{i1} \\ f_{i2} \\ f_{i3} \\ \vdots \\ f_{i,M-2} \\ f_{i,M-1} \end{bmatrix} = \begin{bmatrix} f_{i+1,1} \\ f_{i+1,2} \\ \vdots \\ \vdots \\ \vdots \\ f_{i+1,M-1} \end{bmatrix} - \begin{bmatrix} d_1 f_{i+1,0} \\ 0 \\ 0 \\ \vdots \\ 0 \\ g_{M-1} f_{i+1,M} \end{bmatrix}$$

Similarly, The Crank-Nicolson method also involves solving tridiagonal inverse matrix. It is basically a combination of both explict and implicit method, hence this method considers the three terms on the both sides of f_{N-1} and f_N, covering all six points. Due to this characteristics, the Crank-Nicolson method results in second-order accuracy $O(\delta t^2)$, $O(\delta S^2)$. Both methods are unconditionally stable without interruption of grid size. Thus, the number of computations required to achieve same accuracy is $O(\delta S^2)$ is $O(N^2)$, which is less than explicit method.

3.3 Advantage, Disadvantage

Explicit method is easier to compute therefore is efficient in simple financial models that do not require solving linear equations system, but is dependent on stability constrains (section 5.2). These constraints can be very restrictive, leading to small time steps. Whereas, implicit and CN methods are unconditionally stable, handling large time step sizes and can be more accurate for complex options or options with early exercise features. However, due to the characteristics of solving a system of linear equations, the computation can be complex and challenging to interpret in numerical methods. The CN scheme is especially advantageous for accuracy compared to the other 2 schemes, hence it may seem like the most efficient method when explicit scheme is not available.

3.4 Applicability

FD methods are well-suited for pricing American options, which can be exercised at any time before expiration. The charqcteristics of FD methods such as being able to compute going backward in time allows for the estimation of optimal exercise, making them a valuable tool for pricing and analyzing American-style derivatives.

For the Path-Independent Options in low dimension, FD methods can also be suitable for pricing, as it gives deterministic solution, not relying on the probabilisic random sampling like MC method. This will be extended more in the following section.

4 Comparison of MC and FD

MC relies on generating random paths for the underlying asset's price, S_t, and use these paths to estimate option price, therefore, If the payoff of a financial derivative depends on trajectory of stock price(eg. lookback option or other path-dependent options) MC would be preferred over FD. If the payoff a financial derivative depends solely on S_T, as in the case where $f(S_1^T, S_2^T, ..., S_d^T)$, where d is the dimensionality of the problem, the choice between FD and MC methods becomes an important consideration.

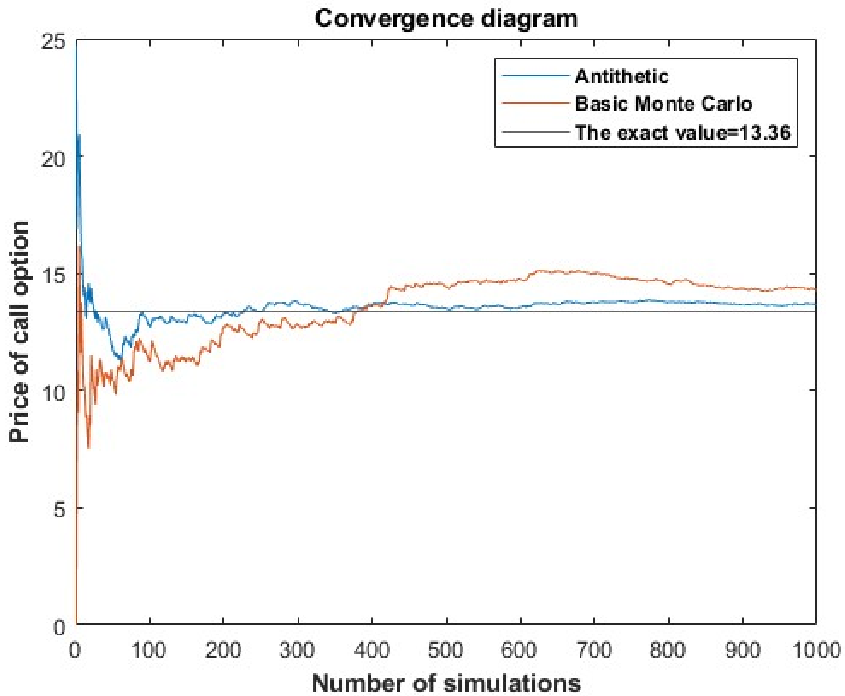
In general, MC works bettwe in high-dimension case, where $d > 3$, however, it provides option price for specific initial conditions $V((0, S_0^*)$ rather than a comprehensive grid of prices across time and asset prices $(V((t, S))_{t,s}$ which FD provides. Therefore, in low-dimensional problem ($d <= 3$) and to provide accurate and precise solutions at various points in V(t,S), FD method would be preferred.

5 Numerical Examples

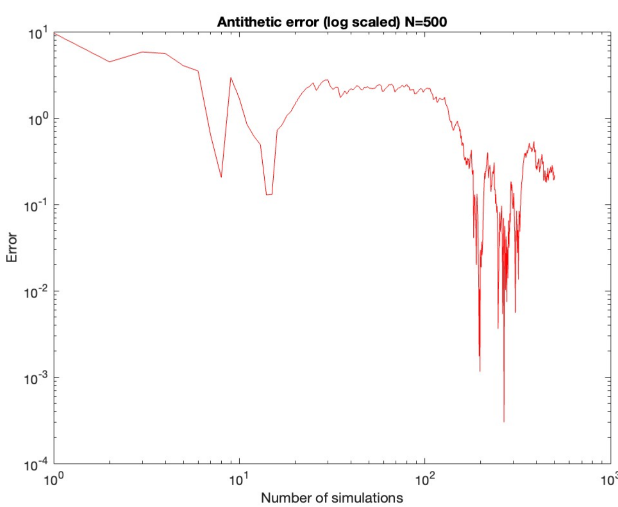
5.1 MC and its variations

The performance of MC and its variation methods were considered against the BS model for the European call option (based on equation (2)) with the following parameters:

$$K = 100, T = 1, r = 0.07, \sigma = 0.25, S_0 = 100, N = 1000$$



The convergence graph shows that using the antithetic method with N=1000 simulations, the price of European call option converges faster and closer to the true value (13.36) compared to the basic MC method. Thus, there is more consistency in pattern (smaller variability) using antithetic method. This shows that the antithetic method effectively reduces the variance of the estimates, providing more accurate results in option pricing. To be more specific, we can analyse accuracy of each method numerically by comparing error values.



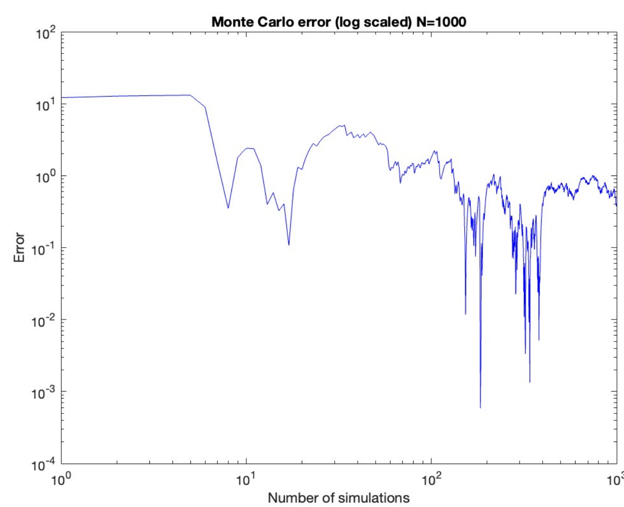


Table 1 displays the mean error values for two different simulation methods (antithetic and basic MC) at two different sample sizes (N=1000 and N=500)

When comparing the antithetic and basic MC, we can observe a consistent trend where antithetic method consistently yields lower mean errors for both cases where N=1000 and N= 500 compared to basic MC. Thus, importantly, the table shows antithetic with N=500

Table 1: Error Comparison: Antithetic vs. Basic MC

Simulation Type	**Number of Simulations (N)**	**Mean Error**
Antithetic	1000	0.4628
Basic MC	1000	0.8257
Antithetic	500	0.7532
Basic MC	500	0.9505

yields mean error value of 0.75 which is still lower than Basic MC with N=1000 that resulted in 0.83. This is consistent with the analysis in (3),(4), as we discussed antithetic method can still achieve the same level of accuracy with at most half number(N) of simulations that are required in MC (2N). Antithetic method, indeed, demonstrates its efficiency and accuracy over basic MC in convergence and reducing estimation errors.

Thus, the impact of sample size on mean error values is also highlighted. As we decrease the sample size from N=1000 to N=500, we observe that the mean errors increase for both simulation methods. This phenomenon is consistent with the concept of the law of large numbers, indicating that larger sample sizes lead to more accurate estimates.

5.2 FD methods

The performance of FD methods were considered by the Black-Scholes PDE (7) with the same parameters given in 5.1 but we approximate the equation (7) using the heat equation (10) and introducing the following transformations with the change of variables new variables:

$$x = \ln\left(\frac{S}{K}\right), \qquad t = \frac{\sigma^2}{2}(T - \tilde{t}), \qquad q = \frac{2r}{\sigma^2}, \qquad u(t,x) = V\left(T - \frac{2t}{\sigma^2}, Ke^x\right).$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{10}$$

$$\Delta t = \frac{\sigma^2 T}{2M}, \Delta x = \frac{6\sigma\sqrt{T}}{N}, \lambda = \frac{\Delta t}{\Delta x^2}, -3\sigma\sqrt{t} \le x \le 3\sigma\sqrt{t}$$

It now describes how option price $u(\Delta t, x)$ evolves with time t and the transformed stock price x

Table 2: Option Value Comparison Table $\lambda = 0.1389, M = 1000, N = 100$

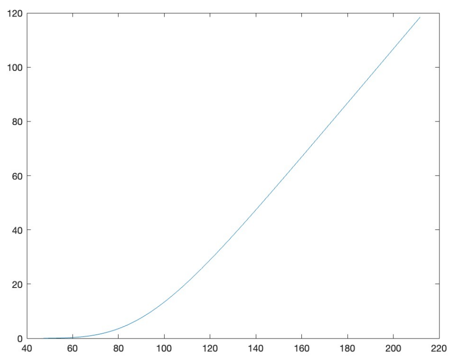
S_0	Explicit Method	Implicit Method	Crank-Nicolson Method
60	0.2904	0.2922	0.2913
100	13.3610	13.3597	13.3604
150	57.0840	57.0854	57.0847

This table shows the call option prices for stock prices of 60, 100, and 150 using various FD methods while adjusting the number of time steps N. when we set N to 100, every

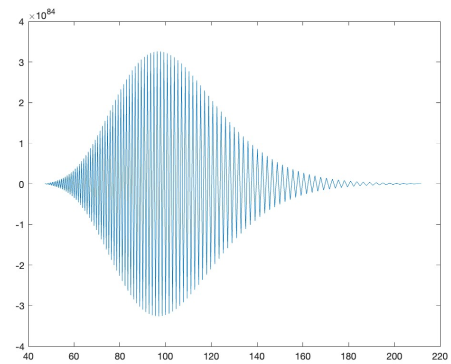
Table 3: Option Value Comparison Table $\lambda = 0.5556, M = 1000, N = 200$

S_0	Explicit Method	Implicit Method	Crank-Nicolson Method
60	-4.1154e+77	0.2917	0.2908
100	1.0612e+83	13.3624	13.3630
150	4.4075e+83	57.0849	57.0842

method showed efficient convergence to the true value. However, as it increased to 200, we encountered a challenge in the explicit method, rendering the option price estimates unreliable. To visualise this, we can refer to the following graphs



(a) $\lambda = 0.1389$



(b) $\lambda = 0.5556$

The option value increases with time accordingly as shown in (a) for implicit and CN schemes without interruption of the size of N. However the graph for explicit scheme turns into (b) when N=200. This is because N affects the Δx by decreasing the grid size, which then leads to an increase of λ. The graph clearly shows that once λ exceeds a certain critical value (0.5), the explicit method becomes unstable, As the errors grow uncontrollably, leading to unreliable results, whereas other methods are not controlled by the stability limit λ.

5.3 Conclusion

In conclusion, this report has explored two widely used methods for option pricing: Monte Carlo and Finite Difference. Through a thorough examination of their theoretical foundations, accuracy, and stability, we have gained valuable insights into the strengths and weaknesses of each approach.

Our analysis has revealed that while Monte Carlo methods excel in path-dependent options and high dimensionality, Finite Difference methods offer precise results and can be computationally efficient for certain scenarios such as American-styled option.

we have successfully assessed the advantages and efficiency of employing antithetic variables in Monte Carlo simulations for option pricing, thus reaffirmed the importance of sample size on accuracy in MC. We have also highlighted the stability conditions in FD method, especially the importance of choosing the right grid size in explicit method.

Improvement on this study can be done by extending the application of Monte Carlo and Finite Difference methods beyond European-style options. Exploring these methods in the context of a broader range of financial derivatives, including American options, exotic options can offer valuable insights into their effectiveness and superiority in different financial scenarios.

References

[1] Miller, J, Edelman, D Appleby, J 2007, Numerical Methods for Finance, CRC Press.

[2] Option Pricing - Finite Difference Methods n.d., www.goddardconsulting.ca.

[3] Igor Vidić (2013). Numerical methods for option pricing.

[4] Fadugba, Sunday Nwozo, Chukwuma Babalola, Bayowa. (2012). The Comparative Study of Finite Difference Method and Monte Carlo Method for Pricing European Option. 2.