Research Proposal

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Research Title: Optimal quantization for European option pricing under non-log-normal asset dynamics

1 Introduction

In this research, our goals are:

1. To investigate optimal quantization methods and their application in option pricing, with a focus on European options
2. To numerically solve the Black-Scholes Partial Differential Equation (BS-PDE) and Stochastic Differential Equation (SDE) using the quantization method.
3. To examine how the closed-form solution of the Black-Scholes model changes (or if the closed-form can still be obtained) when the underlying asset does not follow a classical log-normal distribution, and to approximate option prices using quantization under these alternative models
4. To investigate how to determine error for a given quantization.
5. To evaluate the efficiency and accuracy of the quantization method compared to other numerical methods (e.g., finite difference, Monte Carlo).

2 Background & Context

2.1 Black Scholes-PDE

The Black-Scholes PDE is given by:

$$rf = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S}$$

We solve this equation for the option price function $f(t, S)$, the classical to solve this equation involves with the assumption of log form of asset price by substituting

$$x = T - t,$$
$$u = \ln\left(\frac{S}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t),$$

into

$$f(S, t) = e^{-rx} y(u, x)$$

When the boundary condition is:

$$f(S_T, T) = \begin{cases} S_T - X & \text{if } S_T \geq X, \\ 0 & \text{if } S_T < X, \end{cases}$$

, where X is the strike price of the option.

Then we investigate the movement of stock price and the payoff of a European call option. However, here we want to investigate the case where the stock price no longer follow lognormal distribution. Therefore, we introduce quantization method.

2.2 Quantization Method

We want to apply quantization method to solve this problem. We apply the quantization method to approximate a continuous random variable X by a discrete random variable $\widehat{X}^{\Gamma}$, which takes values on a finite grid (quantizer) Γ with cardinality n. That is,

$$\Gamma := \{\gamma_1, \gamma_2, \ldots, \gamma_n\},$$

where each $\gamma_i \in \mathbb{R}^d$ is called a codeword.

Let $\widehat{X}^{\Gamma} = \mathrm{Proj}_{\Gamma}(X)$ denote the projection of X onto the grid Γ, according to a chosen projection rule (e.g., nearest neighbor).

The nearest neighbour projection operator $\pi_{\Gamma} : \mathbb{R} \to \Gamma$ is defined as

$$\pi_{\Gamma}(X) = \gamma^i \in \Gamma \quad \text{such that} \quad \|X - \gamma_i\| \leq \|X - \gamma_j\| \quad \text{for all } j = 1, \ldots, N,$$

where equality holds only for $i < j$.

Then, the L^p-quantization error is defined by:

$$\|X - \widehat{X}^{\Gamma}\|_p^p = \mathbb{E}\left[\min_{1 \leq i \leq n} |X - \gamma_i|^p\right].$$

Then we define Voronoi region as

$$R_i(\Gamma) = \left\{x \in \mathbb{R} | \pi_{\Gamma}(x) = \gamma^i\right\}$$

Then we wish to find the quantizer Γ such that X best approximates $\tilde{X}$ and it is expressed by the distortion function as

$$D(\Gamma) = \mathbb{E}\left[\||X - \tilde{X}\||^2\right]$$

We require the Γ that minimizes $D(\Gamma)$.

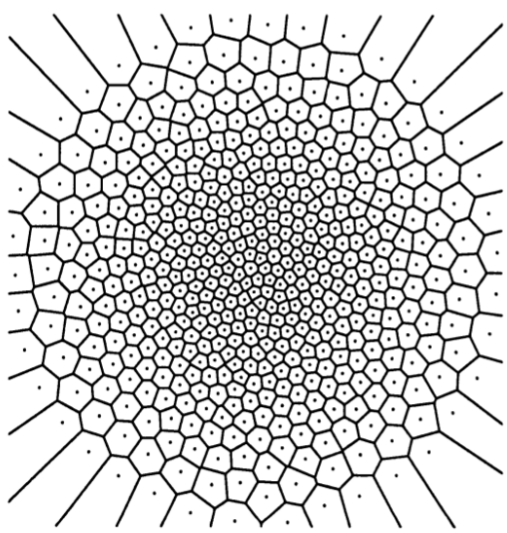


Figure 1: Rachev, S.T. (2011), An example of Voronoi region with 500 quantizer in L^2 space

2.3 Stochastic Differential Equation(SDE)

Stochastic differential equation describes the movement of the stock, followed by

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

where W_t follows a Wiener process. We can apply quantization to simulate paths using the Euler–Maruyama scheme. This approach is discussed in the paper by Pagès, G. and Sagna, A. (2015). Using quantization directly to solve SDEs is known as *recursive marginal quantization (RMQ).*

3 Literature review

For log-normal distributions (as in Black-Scholes), precomputed optimal grids exist for various sizes (e.g., 10, 50, 100 quantizers). However, since our research focuses on non-classical distribution, we must choose the number of quantizers that will be used in the computation grids depending on the dimension of the problem and desired accuracy (e.g.,L^2 error). This will also depend on the desired computation power.

Hulley, H, & Platen, E (2012) present a Minimal Market Model based on squared Bessel processes. This model provides closed-form pricing formulas for European contingent claims using the non-central chi-squared distribution. The stochastic process follows:.

Let X_t be a squared Bessel process:

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dW_t,$$

and

$$\varphi(t) := \frac{4\eta}{\alpha}\left(e^{\eta t} - 1\right) \tag{1}$$

The discounted stock price is

$$\bar{S}_t := \frac{S_t}{B_t}. \tag{2}$$

It can be shown that $\bar{S}_t$ is equal in distribution to a time-transformed squared Bessel process of dimension 4:

$$\bar{S}_t \stackrel{d}{=} X_{\varphi(t)}, \tag{3}$$

They also mention that efficient algorithms exist for computing the non-central chi-squared distribution func tion, making the pricing and hedging formulas computationally feasible.

Graf, S. & Luschgy, H. (2000) present optimal quantizers for various univariate distributions, including the uniform distribution, spherical distribution, and the d-dimensional standard normal distribution. Their work lays the theoretical foundation for the well-known Lloyd algorithm, which is widely used to construct optimal quantizers. To approximate an n-stationary set of centers for a probability distribution P of order r, the following iterative method was used:

Step 1: Initialization:

Choose an initial set $\alpha^{(0)} = \{a_1^{(0)}, \ldots, a_n^{(0)}\} \subset \mathbb{R}^d$, for example by sampling from P. Compute the initial distortion:

$$e_0 = \mathbb{E}_P\left[\min_{a\in\alpha^{(0)}} \|X - a\|^r\right].$$

Step 2: Voronoi Partition:

For each $x \in \mathbb{R}^d$, assign it to the closest center in $\alpha^{(i)}$. This defines the Voronoi cells:

$$V_a = \left\{x \in \mathbb{R}^d : \|x - a\| \leq \|x - a'\| \quad \forall a' \in \alpha^{(i)}\right\},$$

which partition the space as $\mathcal{A}^{(i)} = \{V_a\}_{a\in\alpha^{(i)}}$.

Step 3: Update Centers:

For each cell $V \in \mathcal{A}^{(i)}$ with $P(V) > 0$, compute the new center:

$$a_V^{(i+1)} = \arg\min_{a\in\mathbb{R}^d} \int_V \|x - a\|^r\, dP(x).$$

This is the conditional r-moment minimizing center:

- For $r = 2$: the conditional mean on V
- For $r = 1$: the conditional median

Set $\alpha^{(i+1)} = \{a_V^{(i+1)}\}_{V \in \mathcal{A}^{(i)}}$.

Step 4: Check for Convergence:
Compute the new distortion:

$$e_{i+1} = \mathbb{E}_P \left[\min_{a \in \alpha^{(i+1)}} \|X - a\|^r \right].$$

If the relative change in distortion is below a threshold $\varepsilon > 0$, i.e.,

$$\frac{e_i - e_{i+1}}{e_i} < \varepsilon,$$

then stop. Otherwise, set $i \leftarrow i + 1$ and repeat from Step 2.

Gilles Pag'es and Abass Sagna (2015) show the marginal quantization method for the pricing of an European Put option in a local volatility model (as well as in the Black-Scholes model) and compare the results with those obtained from the Monte Carlo method.

A comparison with Monte Carlo simulations shows that the proposed method may sometimes be more efficient. In this paper, they discretize the process using Euler scheme with n= 120 discretization steps and the gird size was allocated as $N_k = 300$ and $N_k = 400$.

4 Methodology

4.1 Calculating the Weights and Index Jumps for Quantized Processes

One of the most important aspects of this research is determining how to obtain optimal or at least effective grids Γ_k for every $k = 0, \ldots, n$, along with their associated weights and transition probabilities.

According to Pagès and Sagna (2015), in a general framework, as long as the stochastic process $(\bar{X}_{t_k})_k$ (or the underlying diffusion process $(X_t)_{t \geq 0}$) can be simulated, one may employ a zero-search stochastic gradient algorithm known as Competitive Learning Vector Quantization (CLVQ).

In the one-dimensional case, one may instead rely on deterministic procedures such as Lloyd's algorithm. For certain scalar distributions (e.g., the normal, exponential, or Weibull distributions), the Newton–Raphson algorithm can also be applied to compute the quantizers efficiently.

4.2 Implementation Schemes

We implement a distortion-minimizing algorithm using numerical tools in R, Python, or MATLAB. The main components of the implementation are:

- Newton iteration: A widely used method for optimizing the placement of quantizers. Previous studies have successfully applied this technique to the Gaussian and non-central chi-squared distributions with one degree of freedom. It would be also useful for static grid placement when the underlying distribution admits a closed-form expression.

- Euler–Maruyama scheme: Used for discretizing stochastic differential equations that define the underlying diffusion process.

- Numerical computation of quadratic optimal quantizers for the marginal random variable $\bar{X}_{t_{k+1}}$, given the probability distribution of $\bar{X}_{t_k}$. This involves using algorithms such as:

  - Competitive Learning Vector Quantization (CLVQ)
  - Randomized Lloyd's algorithm

  Both of these algorithms require computation of the gradient of the distortion function.

- Lloyd's algorithm: Efficient for scalar cases or low-dimensional problems when the probability distribution is known.

4.3 Numerical Evidence

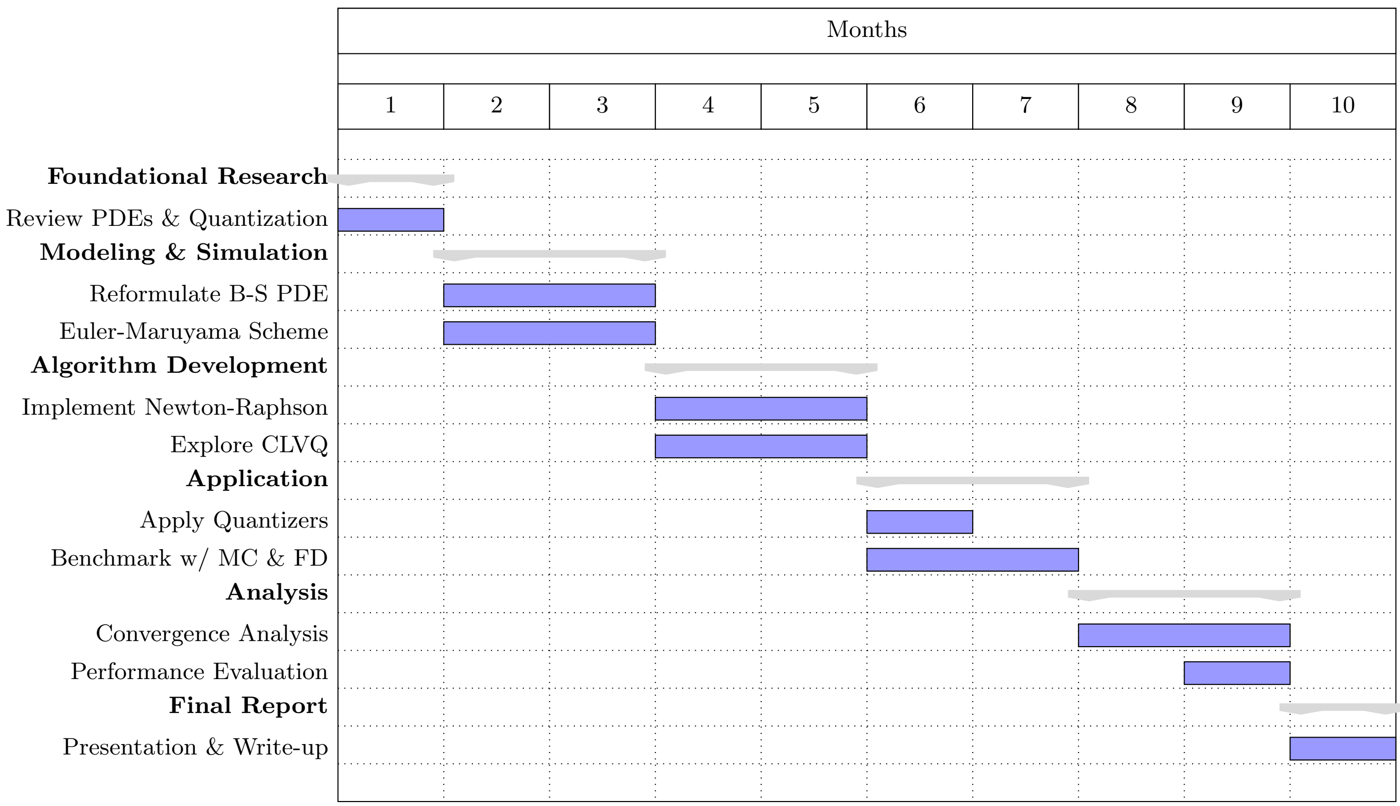
We show numerical evidence of improved weak order convergence of the proposed quantization-based optimization method.

- A comparison with Monte Carlo Simulation and Finite Diffrerence method

Monte Carlo Simulation and Finite difference method will be used as a benchmark to show optimization technique is more efficient(for a given computation time), accurate (lower error), and better adapted to the irregular features of non-log distributions.

Timeline

The research will be conducted for 9 months (37 weeks) over the course of the semesters including a summer break.



5 References

1. Hulley, H. & Platen, E. (2012), 'Hedging for the long run', *Mathematics and Financial Economics*, vol. 6, no. 2, pp. 105–124.
2. Pagès, G. & Sagna, A. (2015). Recursive marginal quantization of the Euler scheme of a diffusion process. *Applied Mathematical Finance*, 22(5), 463–498. https://doi.org/10.1080/1350486x.2015.1091741
3. Graf, S. & Luschgy, H. (2000). *Foundations of Quantization for Probability Distributions.* Springer, Berlin; Heidelberg; New York.
4. Rachev, S.T. (2011). Chapter 7: Optimal Quantization Methods and Applications to Numerical Problems in Finance. In *Handbook of Computational and Numerical Methods in Finance.* Springer Science & Business Media.