1. Consider two discrete random variables X, Y with joint probabilities given by the contingency table:

$P(X,Y)$	$Y=-1$	$Y=0$	$Y=-1$
$X=-1$	.05	.10	.15
$X=0$	.15	.15	.10
$X=+1$	.15	.00	.15

(a) [2 marks] Find the *Minimum Mean Square Error* (MMSE) predictor of Y given X, i.e. the conditional expectation $g(X) = \mathbb{E}[Y|X]$, and the MSE it achieves, i.e. $\mathbb{E}[(Y - g(X))^2]$.

$$f_{y|X=-1} = \begin{cases} 1/6, & Y=-1 \\ 1/3, & Y=0 \\ 1/2, & Y=+1 \end{cases}$$

$$\mathbb{E}[Y|X=-1] = \sum_{y=-1}^{1} y f(y|X=-1)$$

$$= -1 \cdot 1/6 + 0 \cdot 1/3 + 1 \cdot 1/2$$

$$= -1/6 + 1/2 = 1/3$$

$$f_{y|X=0} = \begin{cases} 15/40 = 3/8, & Y=-1 \\ 15/40 = 3/8, & Y=0 \\ 1/4 = 2/8, & Y=1 \end{cases}$$

$$\mathbb{E}[Y|X=0] = -3/8 + 2/8 = -1/8$$

$$f_{y|X=1} = \begin{cases} 1/2, & Y=-1 \\ 0, & Y=0 \\ 1/2, & Y=1 \end{cases}$$

$$\mathbb{E}[Y|X=1] = 0$$

$$\frac{1}{9} - \frac{1}{24}$$

$$g(x)\begin{cases} g(X=-1) = 1/3 \\ g(X=0) = -1/8 \\ g(X=1) = 0. \end{cases}$$

$$\mathbb{E}[(Y-g(X))^2] = \mathbb{E}[Y^2 - 2Yg(X) + (g(X))^2]$$

$$\mathbb{E}[Y] = \sum_{x,y}\sum y\, P(X,Y)$$

$$= (-1)(0.05) + (-1)(0.15) + (-1)(0.15)$$
$$+ 0.15 + 0.1 + 0.15 = 0.05$$

$$\mathbb{E}[Y^2] = 0.05 + 0.15 + 0.15 + 0.15 + 0.1 + 0.15$$
$$= 0.65.$$

$$\mathbb{E}[Y^2] - 2\,\mathbb{E}[Y g(X)] + \mathbb{E}[(g(X))^2]$$

$X = -1$,

$$= 0.65 - 2 \cdot \frac{1}{3} \cdot 0.05 + \frac{1}{9}$$

$$= \frac{131}{180} \approx 0.73$$

$X = 0$,

$$= 0.65 - 2 \cdot \left(-\frac{1}{8}\right) \cdot 0.05 + \frac{1}{64} = \frac{217}{320} \approx 0.68$$

$X=1,$

$= 0.65$

$$\text{MSE} = \begin{cases} \frac{131}{180}, & X=-1 \\ \frac{217}{320}, & X=0 \\ \frac{13}{20}, & X=1 \end{cases}$$

(b) [2 marks] Find the *Best Linear Predictor* (BLP) of Y given X, i.e. $\hat{Y} = a + bX$, for the BLP coefficients a, b, and the MSE it achieves.
(Note: This is an example where the MMSE predictor and the BLP are different.)

The coefficients of the BLP are found by solving $\mathbb{E}[(Y - g(X))\,Y_n] = 0, \ \forall_{n=1,2,3}$.

$Y_1 = 1,\ Y_2 = 0,\ Y_3 = 0.$

$\Rightarrow \mathbb{E}[(Y - (a+bX))\,Y_n],$

$\mathbb{E}[Y \cdot Y_n] - a\,\mathbb{E}[Y_n] - b\,\mathbb{E}[X \cdot Y_n] = 0.$

$Y_n\,[\mathbb{E}[Y] - a - b\,\mathbb{E}[X]] = 0.$

$\mathbb{E}[Y] = a.$

$a = 0.05.$

MSE

$$\mathbb{E}[(Y - 0.05)^2]$$

$$= \mathbb{E}[Y^2] - 0.1\,E[Y] + 0.025$$

$$= 0.65 - 0.005 + 0.025$$

$$= 0.6475.$$

2. Consider the AR(1) model $X_t = \phi X_{t-1} + W_t,\;\; W_t \sim \mathrm{WN}(0, \sigma_w^2)$.

(a) [3 marks] Find the covariance between the 1- & 2-step-ahead BLP errors, i.e. find

$$\mathrm{Cov}\left[(X_{n+1} - X_{n+1}^n)(X_{n+2} - X_{n+2}^n)\right]$$

as a function of (ϕ, σ_w^2).
(Note: this should be *non-zero*; generally the different-step-ahead forecasts will be correlated.)

$$X_{n+1} = \alpha_1 X_n + W_n$$

$$\mathbb{E}[X_{n+1} X_n] = \alpha_1 E[X_n^2]$$

$$\gamma(1) = \alpha_1 \gamma(0)$$

$$\alpha_1 = \frac{\gamma(1)}{\gamma(0)} = \phi.$$

$$\hat{X}_{n+1} = \phi X_n$$

$$\hat{X}_{n+2}^n = \phi^2 X_n.$$

$$\mathrm{Cov}\left[(X_{n+1} - \phi X_n)(X_{n+2} - \phi^2 X_n\right]$$

$$= \mathrm{Cov}(X_{n+1}, X_{n+2}) - \phi^2 \mathrm{cov}(X_{n+1}, X_n)$$

$$- \phi \,\mathrm{cov}(X_n, X_{n+2}) + \phi^3 \mathrm{cov}(X_n, X_n)$$

$$= \gamma(1) - \phi^2 \gamma(1) - \phi \gamma(2) + \phi^3 \gamma(0)$$

$$= \phi \cdot \frac{\sigma_w^2}{1-\phi^2} - \phi^2 \cdot \frac{\phi \sigma_w^2}{1-\phi^2} - \cancel{\phi \cdot \frac{\phi^2 \cdot \sigma_w^2}{1-\phi^2}} + \cancel{\phi^3 \cdot \frac{\sigma_w^2}{1-\phi^2}}$$

$$= \frac{\phi \sigma_w^2}{\cancel{1-\phi^2}} (\cancel{1-\phi^2}) = \phi \cdot \sigma_w^2$$

(b) [3 marks] Find the covariance between the subsequent 1-step-ahead BLP errors, i.e. find $\text{Cov}\left[(X_n - X_n^{n-1})(X_{n+1} - X_{n+1}^{n})\right]$ as a function of (ϕ, σ_w^2).

(Note: These are similar to the model residuals *given perfect knowledge of the parameters.*)