Enriched Immersed Finite Element Method for Hele-Shaw Flows

Choi, Yoonjeong
Department of Mathematical Science

2019. 06. 17

Contents

1. Hele-Shaw Equation

Hele-Shaw equation is a special case of Stokes equation, which describes flows between two parallel plates with small gap.

We assume the situation when one fluid is injected to the other fluid so the injected fluid pushes away the other fluid where two fluids are not mixed.

The problem consists of two part : One is for pressure and velocity. The other is for interface.

1. Hele-Shaw Equation

Note that the Hele-Shaw equation is a moving interface elliptic problem.

One can apply the standard IFEMs to find the pressure when the time is fixed. However, since the interface moves along the velocity field, an accurate approximation of the velocity is important.

Thus we propose a new finite element method to approximate the pressure and velocity of the Hele-Shaw flows.

1. Hele-Shaw Equation

1.1. Governing equation for pressure and velocity

Let Ω be the domain which is separated into $\Omega^+(t)$ and $\Omega^-(t)$ by an interface $\Gamma(t)$. Then, for fixed time t, the pressure p and the velocity field $\mathbf{u}$ are determined by

$$\begin{aligned}
\mathrm{div}\mathbf{u}^s &= f^s, \quad \text{in } \Omega^s(t), \quad s = -, + \\
\mathbf{u}^s &= -\beta^s \nabla p^s, \quad \text{in } \Omega^s(t), \quad s = -, + \\
[p]_{\Gamma(t)} &= \tau\kappa, \\
[\beta \nabla p \cdot \mathbf{n}]_{\Gamma(t)} &= 0, \\
p &= g, \quad \text{in } \partial\Omega,
\end{aligned}$$

where the coefficient β^s is a positive constant determined by the viscosity of the fluid in Ω^s, τ is the surface tension and κ is the curvature of the interface $\Gamma(t)$.

1.2. Governing equation for interface

To describe the moving interface, we use the level set method. Let $\Phi(t, X) = \Phi(t, x, y),\ X = (x, y)$ be a function whose zero level set describes the interface $\Gamma(t)$ at time t i.e

$$\begin{aligned} \Phi(t, X) > 0 \quad &\text{for} \quad \mathrm{X} \in \Omega^{-}(\mathrm{t}) \\ \Phi(t, X) = 0 \quad &\text{for} \quad \mathrm{X} \in \Gamma(\mathrm{t}) \\ \Phi(t, X) < 0 \quad &\text{for} \quad \mathrm{X} \in \Omega^{+}(\mathrm{t}) \end{aligned}$$

By differentiating $\Phi(t, X(t)) = c$ with respect to time t, we obtain the equation of the level set function

$$\Gamma(t) = \{\Phi(t, X) = 0\}$$
$$\frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \nabla \Phi = 0$$

1.3. Overall Procedures

We propose the overall procedures. We solve the pressure equations and the level set equation sequentially. Suppose that we obtain $p^n, \mathbf{u}^n$ and Φ^n at the time t_n.

1. Solve for p^{n+1} and $\mathbf{u}^{n+1}$:

$$\begin{aligned}
\mathrm{div}\mathbf{u}^{n+1} &= f, \quad \text{in } \Omega^{\pm}(t_n), \\
\mathbf{u}^{n+1} &= -\beta \nabla p^{n+1}, \quad \text{in } \Omega^{\pm}(t), \\
[p^{n+1}]_{\Gamma(t_n)} &= \tau\kappa, \\
[\beta \nabla p^{n+1} \cdot \mathbf{n}]_{\Gamma(t_n)} &= 0,
\end{aligned}$$

2. Solve for Φ^{n+1} :

$$\frac{\Phi^{n+1} - \Phi^n}{\partial t} + \mathbf{u}^{n+1} \cdot \nabla \Phi^n = 0.$$

3. Update interface location $\Gamma(t^{n+1}) = \{X | \Phi^{n+1}(X) = 0\}$.

2. Numerical Methods for pressure and velocity

In this chapter, we propose new numerical methods to solve for pressure and velocity equations when time $t = t_n$ is fixed.

For simplicity, we write $p = p^n$, $\mathbf{u} = \mathbf{u}^n$, $\Gamma = \Gamma(t_n)$ and $\Omega^{\pm} = \Omega^{\pm}(t_n)$. Moreover, in this chapter, we consider the following elliptic interface problem which is a general case for the governing equations for pressure and velocity

$$\begin{aligned} -\nabla \cdot \beta \nabla p &= f, \quad \text{in } \Omega^s, \quad s = -, + \\ [p]_\Gamma &= J_1, \\ [\beta \nabla p \cdot \mathbf{n}]_\Gamma &= J_2, \\ p &= g, \quad \text{in } \partial\Omega, \end{aligned}$$

We introduce some notations for function spaces and their norms. For any bounded subdomain D, we denote $D^+ = D \cap \Omega^+$, $D^- = D \cap \Omega^-$ and define $H^m(D)$, $H_0^1(D)$, $H^m(\partial D)$ to be the ordinary Sobolev spaces of order m with the norm $||\cdot||_{m,D}$ and the semi-norm $|\cdot|_{m,D}$. For $m = 1, 2$, the space $\tilde{H}^m(D)$ is defined as

$$\tilde{H}^m(D) := H^m(D^+) \cap H^m(D^-),$$

with norms (semi-norms)

$$||u||^2_{\tilde{H}^m(D)} := ||u||^2_{H^m(D^+)} + ||u||^2_{H^m(D^-)}.$$

The subspace $\tilde{H}_0^1(D)$ is defined as

$$\tilde{H}_0^1(D) := \{u \in \tilde{H}^1(D) \,|\, u = 0 \text{ on } \partial D\}.$$

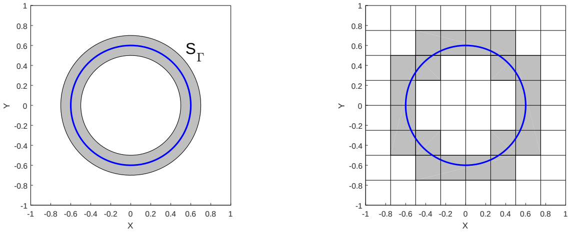
We define spaces $\mathcal{U}(\Omega)$ and $\mathcal{U}_0(\Omega)$ for jump conditions as follows:

$$\begin{aligned} \mathcal{U}(\Omega) \quad &:= \{p \in \tilde{H}^2(\Omega) \,|\, [p]_\Gamma = J_1,\ [\beta \nabla p \cdot \mathbf{n}]_\Gamma = J_2\} \\ \mathcal{U}_0(\Omega) \quad &:= \{p \in \mathcal{U}(\Omega) \,|\, p = 0 \text{ on } \partial D\ \} \end{aligned}$$

2.1. Discontinuous Bubble(DB) for nonhomogeneous jumps

Since $\mathcal{U}(\Omega)$ is not a vector space, we need to subtract the nonhomogeneous jump conditions of the pressure.

To do this, we introduce a *bubble function* p^* so that we can divide the pressure into $p = p^0 + p^*$ where $p^0 \in H_0^1(\Omega)$ and p^* satisfies the jump conditions. Here, p^* is supported on S_Γ which is a thin tube containing the interface Γ.



2.2. DB-IFEM

In this section, we describe how to discretize p^0 and p^*.

Let $\mathcal{T}_h$ be a triangulation of Ω consisting of rectangles. Let $T \in \mathcal{T}_h$ be an *interface element* if T is cut by the interface Γ and be a *non-interface element* otherwise. We introduce some notations:

$\mathcal{T}_h^* =$ the set of all interface elements in $\mathcal{T}_h$

$\mathcal{T}_h^N =$ the set of all non-interface elements in $\mathcal{T}_h$

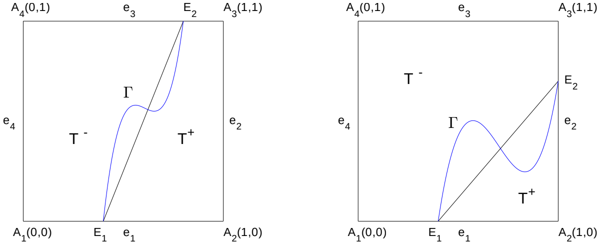
$\mathcal{S}_h = \bigcup_{T \in \mathcal{T}_h^*} T =$ the union of all interface elements

For discretizing p^0, we let $S_h(T)$ be a space of standard bilinear functions on T with nodal d.o.f for each element T.

2.2. DB-IFEM

For every non-interface element $T \in \mathcal{T}_h^N$, we just consider basis functions in $S_h(T)$ as same as the standard $Q1$-conforming space.

For an interface element $T \in \mathcal{T}_h^*$, we modify basis functions.



We define the piecewise bilinear function with nodal values A_i, $i = 1, 2, 3, 4$ as

$$\hat{\phi}(x,y) := \begin{cases} \hat{\phi}^+(x,y) = a^+ + b^+x + c^+y + d^+xy, & (x,y) \in T^+, \\ \hat{\phi}^-(x,y) = a^- + b^-x + c^-y + d^-xy, & (x,y) \in T^- \end{cases} \quad (1)$$

2.2. DB-IFEM

The coefficients of $\hat{\phi}$ are determined by nodal and jump conditions:

$$\hat{\phi}(A_i) = V_i, \quad i = 1,2,3,4 \tag{2a}$$

$$\hat{\phi}^+(E_i) = \hat{\phi}^-(E_i), \quad i = 1,2 \tag{2b}$$

$$d^+ = d^- \tag{2c}$$

$$\int_{\overline{E_1E_2}} \beta^+ \nabla \hat{\phi}^+ \cdot \mathbf{n}_{\overline{E_1E_2}} = \int_{\overline{E_1E_2}} \beta^- \nabla \hat{\phi}^- \cdot \mathbf{n}_{\overline{E_1E_2}} \tag{2d}$$

where $V_i, \ i = 1,2,3,4$ are nodal values.

Hence we define $Q1$-conforming base IFEM space $\hat{S}_h(\Omega)$ as follow:

$$\begin{cases} \phi|_T \in S_h(T) & \text{if } T \text{ is a noninterface element,} \\ \phi|_T \in \widehat{S}_h(T) & \text{if } T \text{ is an interface element,} \\ \phi|_{T_1}(X) = \phi|_{T_2}(X) & \text{if } T_1 \text{ and } T_2 \text{ are adjacent elements} \\ & \text{and } X \text{ is a common node of } T_1 \text{ and } T_2, \\ \phi(X) = 0 & \text{if } X \text{ is a node on the boundary edges.} \end{cases}$$

2.2. DB-IFEM

For discretizing p^*, we let $p_h^*(T) = 0$ for a non-interface element T.

For an interface element T, we define $p_h^*|_T$ as a piecewise bilinear function of a form given in (1) satisfying

$$p_h^*(A_i) = 0, \quad i = 1, 2, 3, 4 \tag{3a}$$

$$[p_h^*(E_i)]_{\overline{E_1 E_2}} = J(E_i), \quad i = 1, 2, \tag{3b}$$

$$d^+ = d^- \tag{3c}$$

$$\int_{\overline{E_1 E_2}} \beta^+ \nabla p_h^*|_{T^+} \cdot \mathbf{n}_{\overline{E_1 E_2}} = \int_{\overline{E_1 E_2}} \beta^- \nabla p_h^*|_{T^-} \cdot \mathbf{n}_{\overline{E_1 E_2}} \tag{3d}$$

2.2. DB-IFEM

Once we determine p_h^*, we discretize the weak form : Find $p_h^0 \in \hat{S}_h(\Omega)$ such that

$$a_h(p_h^0, q_h) = l(q_h) - a_h(p_h^*) \quad \text{for every } q_h \in \hat{S}_h(\Omega)$$

where

$$a_h(p_h, q_h) = \sum_{T \in \mathcal{T}_h} \left(\int_{T^+} \beta \nabla p_h \cdot \nabla q_h \mathrm{d}\mathbf{x} + \int_{T^-} \beta \nabla p_h \cdot \nabla q_h \right)$$

$$l(q_h) = \sum_{T \in \mathcal{T}_h} \int_T f q_h + \sum_{T \in \mathcal{T}_h^*} \int_{\Gamma^*} J_2 q_h$$

2.3. Enriched DB-IFEM

Although the discontinuous bubble IFEM scheme can solve the problem for pressure, this scheme has restriction to compute the Darcy velocity accurately.

We use the concept of enriched Galerkin methods for computation of Darcy velocity and local conservation of the velocity field.

The trial space for pressure is enriched by bubble functions.

We define the enriched IFEM space as

$$\hat{E}_h(\Omega) := \hat{S}_h(\Omega) \oplus P_h^0(\Omega)$$

where $P_h^0(\Omega)$ is the space of piecewise constant functions on T for $T \in \mathcal{T}_h$.

2.3. Enriched DB-IFEM

Note that functions in $\hat{E}_h(\Omega)$ are discontinuous across the edges. Thus we need to modify the weak form in DB-IFEM.

Before we propose the Enriched DB-IFEM scheme, we introduce some notations.
$\mathcal{E}_h^0$ = set of interior edges
$\mathcal{E}_h^\partial$ = set of boundary edges
$\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$ = set of all edges

Let $e = \partial T_1 \cap \partial T_2$ be the common edge of two adjacent elements T_1 and T_2. We define the jump and the average along the edge as

$$[\phi]_e = \begin{cases} \phi_1 - \phi_2 & \text{if } e \in \mathcal{E}_h^0 \\ \phi|_e & \text{if } e \in \mathcal{E}_h^\partial \end{cases} \quad , \quad \{\phi\}_e = \begin{cases} \frac{\phi_1+\phi_2}{2} & \text{if } e \in \mathcal{E}_h^0 \\ \phi|_e & \text{if } e \in \mathcal{E}_h^\partial \end{cases}$$

2.3. Enriched DB-IFEM

We propose our Enriched DB-IFEM scheme : Find $p_h^0 \in \hat{E}_h(\Omega)$ such that

$$a_h(p_h^0, q_h) = l(q_h) - a_h(p_h^*) \quad \text{for every } q_h \in \hat{E}_h(\Omega) \tag{4}$$

where

$$\begin{aligned} a_h(p_h, q_h) = & \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla p_h \cdot \nabla q_h - \sum_{e \in \mathcal{E}_h} \int_e \{\beta \nabla p_h \cdot \nabla n_e\}_e [q_h]_e \\ & + \sum_{e \in \mathcal{E}_h} \frac{\sigma}{|e|} \int_e [p_h][q_h] - \sum_{e \in \mathcal{E}_h} \int_e [p_h]\{\beta \nabla q_h \cdot \mathbf{n_e}\} \end{aligned}$$

$$l(q_h) = \sum_{T \in \mathcal{T}_h} \int_T f q_h + \int_\Gamma J_2 q_h + \sum_{e \in \mathcal{E}_h^\partial} \frac{\sigma}{|e|} \int_e g q_h - \sum_{e \in \mathcal{E}_h^\partial} \int_e g\,(\beta \nabla q_h \cdot \mathbf{n_e})$$

Since the bilinear form a_h in (4) is continuous and coercive, there exists a unique solution p_h^0 for the problem (4) by Lax-Milgram theorem.

2.3. Enriched DB-IFEM

To find the velocity $\mathbf{u}_h$, we let $V_h(\Omega)$ be the lowest order of Raviart-Thomas element. We construct $\mathbf{u}_h \in V_h(\Omega)$ locally as

$$(u_n, n)_e = \left(-\{\beta \nabla p_h^0 \cdot \mathbf{n}\}_e + \frac{\sigma}{|e|}[p_h^0] \right) + \left(-\{\beta \nabla p_h^* \cdot \mathbf{n}\}_e + \frac{\sigma}{|e|}[p_h^*] \right) \tag{5}$$

for each edge e of the elements in $\mathcal{T}_h$.

Then the velocity $\mathbf{u}_h$ is locally conservative by following Proposition.

2.3. Enriched DB-IFEM

Proposition(Local mass conservation)

For every $T \in \mathcal{T}_h$,

$$\int_{\partial T} \mathbf{u}_h \cdot \mathbf{n} = \int_T f$$

Proof Take a test function $\phi_h = \mathcal{X}_T$ in (4) and use the definition of $\mathbf{u}_h$ in (5).

3. Numerical Methods for interface

We assume that the domain Ω is a rectangle and that the domain is discretized by structured grid. For simplicity, we assume that the grid is uniform i.e $\Delta x = \Delta x_i$ and $\Delta y = \Delta y_j$ for all i, j.

We approximate $\Phi_h(t, x, y)$ at center points of each element $T \in \mathcal{T}_h$ at discretized time $t^n = n\Delta t$.
We let $\Phi^n_{i,j}$ be a numerical approximation to the solution $\Phi(t^n, x_i, y_j) = \Phi(n\Delta t, (i + \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta y)$.

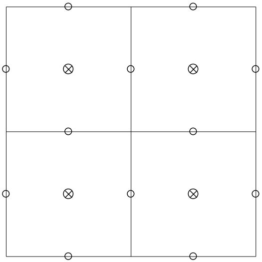


Figure: DOFs of Φ_h located at cell center $\otimes$ and DOFs of $\mathbf{u}_h$ located at edges $\circ$

3.1. Interface Reconstruction
Since only values of level set functions evaluated at centers of elements are given, we need to find points on the interface with these values.

We propose how to reconstruct the interface from the values $\Phi^n_{i,j}$ when the time $t = t_n$ is fixed.

We say that a center point $X = (x_i, y_j)$ of an element is a *control point* if $\Phi^n(X) < 0$ and the number of adjacent elements whose centers have positive values of Φ^n is 1,2, or 3.

We find interface points by projection of control points.

3.1. Interface Reconstruction

Let $X = (x_i, y_j)$ be a control point. We let $\mathbf{n}$ be the unit gradient vector of Φ:

$$\mathbf{n} = \frac{\nabla\Phi}{||\nabla\Phi||}$$

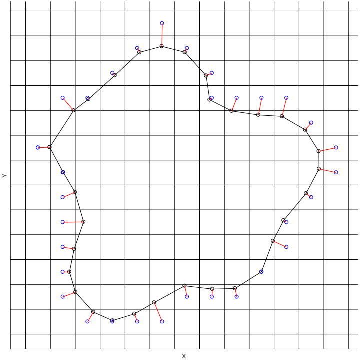
Then we can find the interface point X^*, which is the projection of X along $\mathbf{n}$:

$$X^* = X + t\mathbf{n}$$

Here, t is the positive solution of the quadratic equation:

$$\frac{1}{2}(\mathbf{n}^T He(\Phi)\mathbf{n})t^2 + ||\nabla\Phi(X)||t + \Phi(X) = 0$$

where $He(\Phi)$ is the Hessian matrix evaluated at the point X.



Here, blue circles are control points and black circles are result of the projection(red lines)

3.2. Evolution of Interface with WENO

Once we obtain the velocity $\mathbf{u}$ from the Hele Shaw equations at time t, we can update the interface $\Gamma(t)$

This equation is a Hamilton-Jacobi equation, so we can rewrite above equation as

$$\Phi_t + H(\Phi_x, \Phi_y) = 0 \tag{6}$$

where $H(\phi, \psi) = u\phi + v\psi$ and $\mathbf{u} = (u, v)$

To solve the equation, we apply the fifth-order WENO scheme with the local Lax-Friedrichs(LLF) flux

3.2. Evolution of Interface with WENO

Now suppose that $\Phi^n_{i,j}$ is a numerical approximation of (6) at $t = t^n$. Then the WENO scheme is

$$\Phi^{n+1}_{i,j} = \Phi^n_{i,j} - \Delta t \hat{H}(\Phi^+_{x,i,j}, \Phi^-_{x,i,j}, \Phi^+_{y,i,j}, \Phi^-_{y,i,j}) \tag{7}$$

where $\hat{H}$ is the LLF flux and $\Phi^{\pm}_{x,i,j}$ and $\Phi^{\pm}_{y,i,j}$ are fifth-order WENO approximations of the partial derivatives Φ_x and Φ_y at (x_i, y_j) respectively.

Here, the LLF flux is

$$\hat{H}(\phi^+, \phi^-, \psi^+, \psi^-) = (u - |u|)\frac{\phi^+}{2} + (u + |u|)\frac{\phi^-}{2} + (v - |v|)\frac{\psi^+}{2} + (v + |v|)\frac{\psi^-}{2}$$

3.2. Evolution of Interface with WENO

For the fifth-order WENO approximations $\Phi^{\pm}_{x,i,j}$ and $\Phi^{\pm}_{y,i,j}$, we only describe WENO approximations $\Phi^{\pm}_{x,i,j}$ of x-direction partial derivative. One can find $\Phi^{\pm}_{y,i,j}$ in similar way.

For simplicity, we denote $\phi(x) = \Phi_x(t^n, x, y)$ when $y = y_j$ and $t = t_n$ are fixed.

We define the cell averages of ϕ as

$$\overline{\phi}_i = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \phi(x)\mathrm{d}x \tag{8}$$

3.2. Evolution of Interface with WENO

We denote 3 candidate stencils of the point x_i by $S_k(i)$, $k = 0, 1, 2$ where

$$S_k(i) = \{x_{i-\frac{1}{2}-k}, x_{i-\frac{1}{2}-k+1}, x_{i-\frac{1}{2}-k+2}\}$$

i.e $S_0(i) = \{x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}+1}, x_{i-\frac{1}{2}+2}\}, S_1(i) = \{x_{i-\frac{1}{2}-1}, x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}+1}\}, S_2(i) = \{x_{i-\frac{1}{2}-2}, x_{i-\frac{1}{2}-1}, x_{i-\frac{1}{2}}\}$.

Then there exist a interpolation polynomial $p_k(x)$ of degree 2 on each stencil $S_k(i)$ such that

$$\overline{\phi}_j = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_k(x)\mathrm{d}x, \qquad j = i-\frac{1}{2}-k, i-\frac{1}{2}-k+1, i-\frac{1}{2}-k+2$$

Let $\phi_i^{(k)-} = p_k(x_i)$ and $\phi_{i-1}^{(k)+} = p_k(x_{i-1})$.

Then there exist coefficients $c_{k,j}$ so that

$$\phi_i^{(k)-} = \sum_{l=0}^{2} c_{k,l}\overline{\phi}_{i-k+j}, \quad \phi_{i-1}^{(k)+} = \sum_{l=0}^{2} c_{k-1,l}\overline{\phi}_{i-k+j}$$

3.2. Evolution of Interface with WENO

The desired WENO scheme is as follow:

$$\phi_i^- = \sum_{k=0}^{2} \omega_k \phi_i^{(k)-}, \phi_{i-1}^+ = \sum_{k=0}^{2} \tilde{\omega}_k \phi_{i-1}^{(k)+} \quad and \quad \sum_{k=0}^{2} \omega_k = \sum_{k=0}^{2} \tilde{\omega}_k = 1$$

where ϕ_i^- and ϕ_{i-1}^+ are our WENO approximations of $\Phi_{x,i,j}^-$ and $\Phi_{x,i-1,j}^+$

Here, the weights ω_k and $\tilde{\omega}_k$ are constants which are assigned zero when the candidate stencil has a discontinuity to avoid discontinuous cells.

The weights are given as follow:

$$\omega_k = \frac{\alpha_k}{\sum_{s=0}^{2} \alpha_s}, \quad \tilde{\omega}_k = \frac{\tilde{\alpha}_k}{\sum_{s=0}^{2} \tilde{\alpha}_s}, \quad k = 0, 1, 2$$

where

$$\alpha_k = \frac{d_k}{(\epsilon + \beta_k)^2}, \quad \tilde{\alpha}_k = \frac{\tilde{d}_k}{(\epsilon + \beta_k)^2}$$

3.2. Evolution of Interface with WENO

The parameters d_k and $\tilde{d}_k$ are coefficients satisfying

$$\phi_i^- = \sum_{k=0}^{2} d_k \phi_i^{(k)-}, \quad \phi_{i-1}^+ = \sum_{k=0}^{2} \tilde{d}_k \phi_{i-1}^{(k)+}$$

when the function ϕ is smooth in all stencils.

The parameter β_k is a sum of the square of L^2 norms of all derivatives of p_k over $[x_i, x_{i+1}]$ so that the total variation of the interpolation polynomial p_k in the stencil $S_k(i)$ is minimized.
i.e β_k measures smoothness of ϕ in the stencil $S_k(i)$.

Finally, if ϕ is continuous then (8) can be rewritten as

$$\overline{\phi}_i = \frac{1}{\Delta x}(\Phi_{i+1,j} - \Phi_{i,j}) \tag{9}$$

Thus if we approximate $\overline{\phi}_i$ as above, we obtain the WENO approximations $\Phi^{\pm}_{x,i,j}$.

4. Numerical Results

We provide some numerical experiments of Hele-Shaw problems with various interface Γ and coefficients β and $d_0 = \frac{2\tau\pi\beta^+}{\int_\Omega f}$.

In our experiments, we assume that the domain is $\Omega = [-2, 2]^2$ and we consider a uniform triangulation $\mathcal{T}_h$ by rectangles whose size is h.

The source term and the boundary condition are given as

$$f = \begin{cases} \frac{6V_0}{\alpha^2}(\alpha - r), & \text{if } r \le \alpha \\ 0, & \text{otherwise}, \end{cases}$$

$$p = -\frac{V_0\alpha}{\beta^+}\log(r), \quad \text{on } \partial\Omega,$$

where V_0 is a controlling parameter and α is the radius for the source.

4. Numerical Results

4.1. Example 1

In this example, we apply our method to a benchmark problem given as

$$p(r) = \begin{cases} \frac{V_0}{\beta^-}\left(\frac{2r^3}{3\alpha^2} - \frac{3r^2}{2\alpha}\right) + C_1, & \text{if } 0 \le r \le \alpha \\ -\frac{V_0\alpha}{\beta^-}\log(r) + C_0 & \text{if } \alpha < r \le r_\Gamma \\ -\frac{V_0\alpha}{\beta^+}\log(r), & \text{otherwise,} \end{cases}$$

where $r_\Gamma = \sqrt{2\alpha V_0 t + r_0^2}$ and r_0 is the radius of the initial interface.

C_0 and C_1 are given as

$$C_0 = \frac{\tau}{r_\Gamma} + V_0\alpha\log(r_\Gamma)\left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right)$$

$$C_1 = C_0 - \frac{V_0\alpha\log(\alpha)}{\beta^-} + \frac{5V_0\alpha}{6\beta^-}$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.1$, $d_0 = 2.5 \cdot 10^{-3}$.
Target time is $T = 10$
We set the initial interface as circle $x^2 + y^2 = 0.4^2$.

4. Numerical Results

4.1. Example 1

Table: Errors for pressure and velocity

Element	$\|\|p - p_h\|\|_{L^2(\Omega)}$	order	$\|\|p - p_h\|\|_{1,h}$	order	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega)}$	order
16^2	$1.844E-2$		$2.731E-2$		$7.984E-2$	
32^2	$2.632E-3$	2.808	$4.215E-3$	2.696	$3.236E-2$	1.303
64^2	$1.093E-3$	1.268	$1.820E-3$	1.211	$1.183E-2$	1.451
128^2	$9.564E-5$	3.514	$6.675E-4$	1.447	$5.318E-3$	1.154

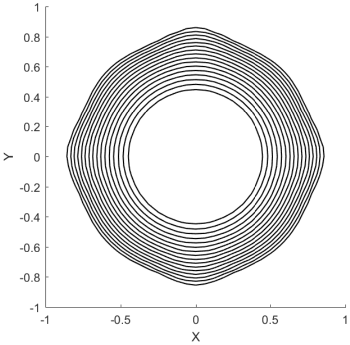


Figure: Interfaces

4. Numerical Results

4.1. Example 1

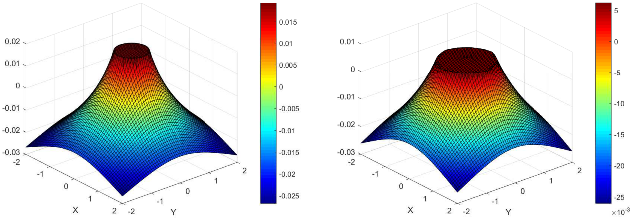


Figure: Pressures at time t=0,10

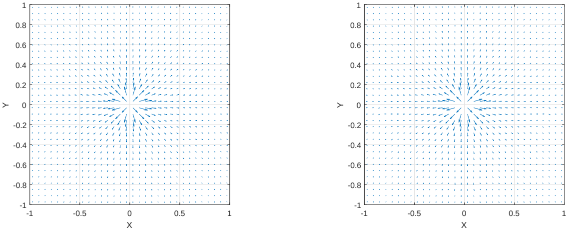


Figure: Velocity at time t=0,10

4.2. Example 2

In Example 2 and Example 3, we consider more complex interface conditions.

In Example 2, two initial interfaces are given by

$$r1 = r_0 + 0.1\sin(3\theta)$$
$$r2 = r_0 + 0.1\sin(5\theta)$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.3$, $d_0 = 2.3 \cdot 10^{-3}$ and $r_0 = 0.7$. Target time is $T = 8$

The boundary condition is given as

$$p = -\frac{V_0\alpha}{\beta^+}\log(r) + 0.1, \quad \text{on } \partial\Omega,$$

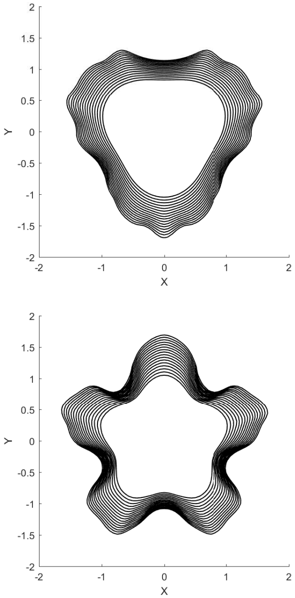


Figure: Interfaces

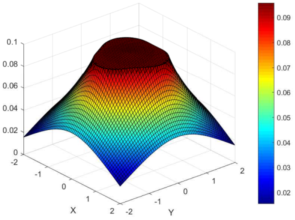


Figure: Pressures for $r1$ at time t=0,8

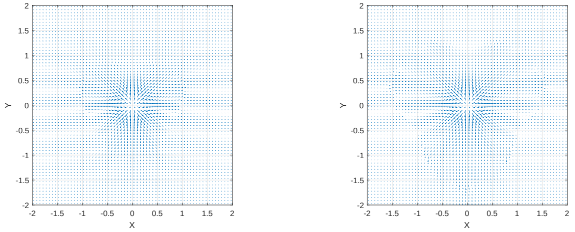


Figure: Velocity for $r1$ at time t=0,8

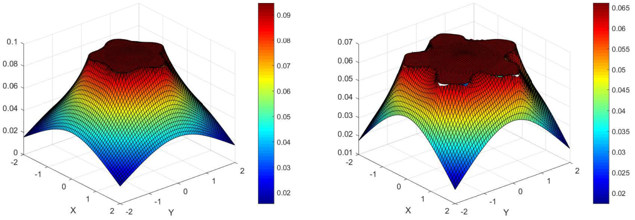


Figure: Pressures for $r2$ at time t=0,8

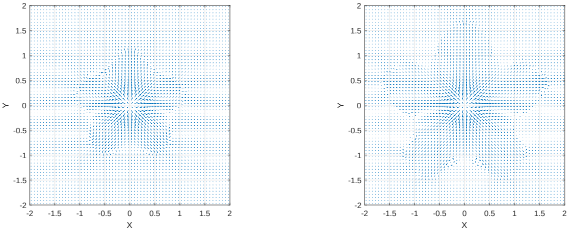


Figure: Velocity for $r2$ at time t=0,8

4.3. Example 3

In Example 3, two initial interfaces are given by

$$r1 = r_0 + 0.05(\sin(2\theta) + \cos(3\theta))$$
$$r2 = r_0 + 0.05(\sin(5\theta) + \cos(3\theta))$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.3$, $d_0 = 2.1 \cdot 10^{-3}$ and $r_0 = 0.55$. Target time is $T = 8$

The boundary condition is given as

$$p = -\frac{V_0 \alpha}{\beta^+} \log(r) + 0.1, \quad \text{on } \partial\Omega,$$

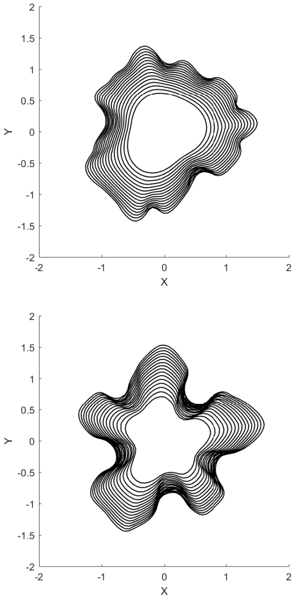


Figure: Interfaces

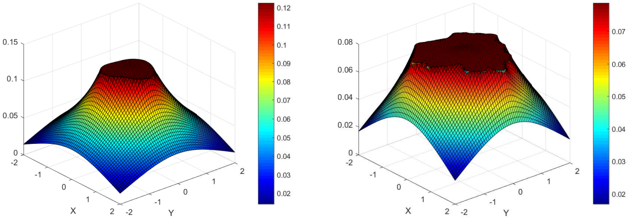


Figure: Pressures for $r1$ at time t=0,8

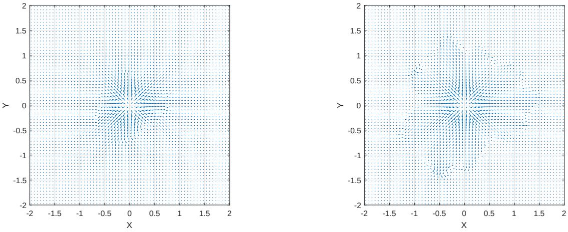


Figure: Velocity for $r1$ at time t=0,8

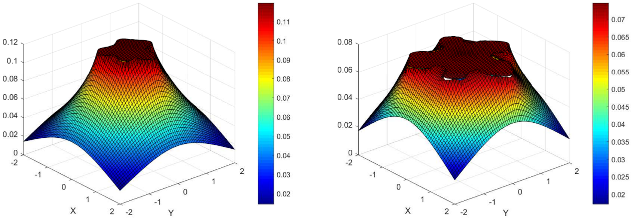


Figure: Pressures for $r2$ at time t=0,8

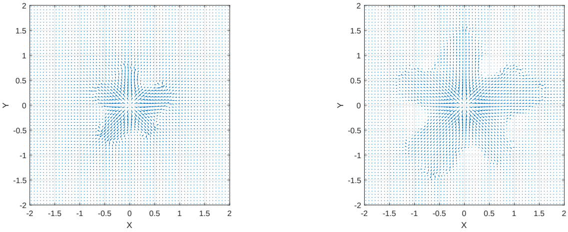


Figure: Velocity for $r2$ at time t=0,8

THANK YOU