1.1 A typical 2nd order elliptic pde

Let Ω be the unit square $[0,1]^2$. We partition the domain by $n \times n$ uniform grid and let $h = 1/n$. We let $\Omega_h = \{(x_i, y_j) = (hi, hj) \in \Omega\}$ be the set of all grid points. The matrix equation appearing in solving

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

by finite difference method with uniform grid is $A\mathbf{x} = \mathbf{b}$, where

$$A = \frac{1}{h^2}\begin{bmatrix} B & -I & 0 & \cdots & \\ -I & B & -I & 0 & \\ & -I & \ddots & \ddots & \\ & & \ddots & B & -I \\ & \cdots & 0 & -I & B \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 & 0 & \cdots & & \\ -1 & 4 & -1 & 0 & & \\ 0 & -1 & \ddots & \ddots & 0 \\ & & \ddots & 4 & -1 \\ & \cdots & 0 & -1 & 4 \end{bmatrix} \tag{1.2}$$

where A is $(n-1) \times (n-1)$ block matrix, B is $(n-1) \times (n-1)$ matrix, $\mathbf{x}$ is the solution vector $[u_{1,1}, u_{1,1}, \cdots, u_{n-1,n-1}]$ and $\mathbf{b}[i] = f(x_i)+$ BC's for $i = 1, 2, \cdots, (n-1) \times (n-1)$.

Theorem 1.1.1. *The eigenvectors of A are $\mathbf{v}^h_{\mu\nu}$ where*

$$v^h_{\mu\nu}(i,j) = \sin(\mu\pi x_i)\sin(\nu\pi y_j), \tag{1.3}$$

with the corresponding eigenvalues

$$\lambda^h_{\mu\nu} = 4h^{-2}(\sin^2(\mu\pi h/2) + \sin^2(\nu\pi h/2)), \;\; \mu, \nu = 1, \cdots, (n-1). \tag{1.4}$$

The eigenvalues of Laplace equation for the domain $[0,a] \times [0,b]$ are

$$\lambda_{\mu\nu} = (\frac{\mu^2}{a^2} + \frac{\nu^2}{b^2})\pi^2, \;\; \mu, \nu = 1, 2, \cdots . \tag{1.5}$$

Conjugate Gradient Method

The Conjugate Gradient method for solving SPD system $A\mathbf{x} = \mathbf{b}$.

Let $\mathbf{x}^0 = 0$, $\mathbf{d}^0 = \mathbf{r}^0 = \mathbf{b}$, and

$$\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k \tag{1.6}$$

$$\text{Stop if } \|\mathbf{r}^k\| \le Tol \approx 10^{-15}? \tag{1.7}$$

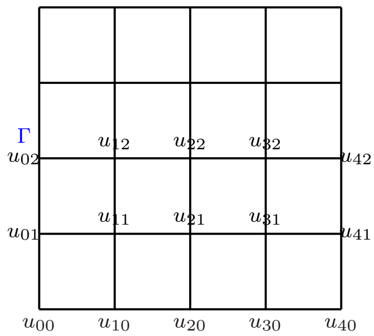
$$\alpha_k = (\mathbf{d}^k, \mathbf{r}^k)/(A\mathbf{d}^k, \mathbf{d}^k) \tag{1.8}$$

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \alpha_k \mathbf{d}^k, \tag{1.9}$$

$$\beta_k = (A\mathbf{d}^k, \mathbf{r}^{k+1})/(A\mathbf{d}^k, \mathbf{d}^k) \tag{1.10}$$

$$\mathbf{d}^{k+1} = \mathbf{r}^{k+1} - \beta_k \mathbf{d}^k. \tag{1.11}$$

Note that since $\mathbf{r}^{k+1} = \mathbf{r}^k - \alpha_k A\mathbf{d}^k$, only one evaluation of A is necessary for each step and there is no need to estimate β_k.



Domain with grid points (x_i, y_j) when $n = 4$

Final homework project.

HW. Set (A) - Paper work. For problems (1) -(4), A is a general $n \times n$ matrix.

(1) Explain briefly (within one page) the Jacobi algorithm to find the eigenvalues of a real symmetric matrix A. (include an explanation of the Jacobi rotation)

(2) Explain briefly (within two page) the Givens-Householder's algorithm to find the eigenvalues of a real symmetric matrix A. (include an explanation of the Givens rotation and Householder transformation)

(3) Explain QR decomposition of A.

(4) Explain QR iterations(with some practical tips to reduce the computations and to speed up the convergence) to find eigenvalues of A. State at least one property about the convergence behavior.

HW. Set (B) - Computer work. Consider the matrix $N \times N$ matrix A in (1.2), where $N = (n-1) \times (n-1)$.

(1) Use Gaussian quadrature with $n = 2, 3, 4, 5$ points to evaluate

$$\int_{-2}^{0} x^2 e^x dx.$$

Check the error by comparing with the exact value. (You can find Gauss points and the weights from the internet and transform appropriately.)

(2) Solve the O.D.E by Heun's method (at $t = 1$ with $h = 1/5, 1/10, 1/20$ resp.)

$$x'(t) = 2x(t) + 0.2 + t, \ x(0) = 0.$$

(3) Solve the PDE (1.1) for $n = 4, 8, 16, 32$ (A is 9×9, 49×49, $225 \times 225, \cdots$ etc), by conjugate gradient method. Report the discrete L^2-error is defined by

$$E_h = h \left(\sum_{i,j=1}^{n-1} |u_{ij}^h - u(x_i, y_j)|^2 \right)^{1/2} \approx \left(\int_\Omega |(u^h - u)(x, y)|^2 dxdy \right)^{1/2}$$

(4) Use the orthogonal iterations of section 6.8.5(Algorithm 4.6-1) without shift to find 10 (largest) eigenvalues of A for $n = 16$ with 225×10 initial matrix X_0. (first ten columns are standard basis vectors $\mathbf{e}_i, i = 1, \cdots, 10$) Note the exact eigenvalues are given by

$$\lambda_{\mu\nu}^h = 4h^{-2}(\sin^2(\mu\pi h/2) + \sin^2(\nu\pi h/2)), \ \mu, \nu = 1, \cdots, (n-1). \quad (1.12)$$

(5) Also use the orthogonal iterations of section 6.8.5(Algorithm 4.6-1) with shift ($\sigma = 50$) to find 10 eigenvalues of A for $n = 16$.

(6) Use the similarity transform by Householder to A to obtain a tridiagonal matrix T for $n = 16$ (A is 225×225). Print out the 5×5 principal submatrices of the final matrix T.

(7) Then apply QR iterations with shift to T to find all the eigenvalues. (However, print only smallest 10 of them) Explain how you choose shift during the QR iterations.

Remark 1.1.2. In your report, include the following (wherever necessary):

(1) Design a suitable stopping criteria.

(2) Report the number of iterations, execution time, discussions etc.