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헬레-쇼 방정식의 풀이를 위한 경계함유 유한 요소법

Immersed finite element method for solving Hele-Shaw equations

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최 윤 정 (崔 允 禎 Choi, Yoonjeong)

한 국 과 학 기 술 원

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Immersed finite element method for solving Hele-Shaw equations

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The study was conducted in accordance with Code of Research Ethics[1].

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초 록

이 논문에서는 경계가 움직이는 문제 중 하나인 헬레 쇼 방정식을 풀기 위한 새로운 수치적 방법을 제안한다. 성질이 다른 두 개의 유체 사이에 생기는 경계는 유체의 다르시 속도에 따라서 움직이기 때문에 유체가 받는 압력을 통해 정확한 다르시 속도를 구하는 것이 중요하다. 정확한 다르시 속도를 구하기 위해 경계 함유 유한 요소 공간에 불연속 상수를 추가한다. 다르시 속도를 구한 뒤에는 WENO 방법을 사용하여 움직이는 경계를 예측한다.

핵 심 낱 말 헬레 쇼 방정식, 경계 함유 유한 요소법

Abstract

We propose a new numerical method to solve Hele Shaw equation which is an moving interface problem. Since the interface between two different fluids moves along the velocity of the fluids, it is important to find the Darcy velocity from the pressure. Thus we enrich the immersed finite element space to find the pressure and Darcy velocity of the fluid. Then we apply weighted essentially non-oscillatory(WENO) schemes to update the interface.

Keywords Hele Shaw equation, Immersed finite element methods, enriched Galerkin methods, weighted essentially non-oscillatory schemes

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Chapter 1. Introduction

Hele-Shaw equation is a special case of Stokes equation, which describes flows between two parallel plates with small gap. We assume the situation when one fluid is injected to the other fluid so the injected fluid pushes away the other fluid. Moreover, we assume that these two fluids are not mixed. Then there exists an interface between the fluids and the pressure of the fluids is discontinuous along the interface. Thus the Hele-Shaw equation is a moving interface problem. By ignoring the depth of the gap, we obtain two dimensional problems.

The problem for Hele-Shaw flows consists of two parts. One is to describe the pressure and velocity of the fluids with the interface where the velocity is Darcy velocity obtained from the pressure. The other is to describe the location of the interface. Since the interface moves along the velocity field, an accurate approximation of the velocity is important. Thus the approximation of the pressure and the location of the interface are related to each other.

For finite difference methods(FDM), immersed interface methods(IIM) approximate Hele-Shaw flow in [1]. For finite element methods(FEM), on the other hand, immersed finite element methods(IFEM) introduced in [2] can be applied to find the discontinuous pressure solution of the Hele-Shaw equation when time is fixed. However, the standard conforming-based IFEM may leads to violation of local conservation of the velocity field. This violation makes the approximation of this moving interface problem.

In this paper, we give a new finite element method to approximate the pressure and velocity of the Hele-Shaw flows. The method is mainly based on IFEM and discontinuous bubble schemes in [2]. To obtain the accurate velocity, we enrich the conforming space using the concept of enriched Galerkin(EG) in [5, 6]. The approximation of the interface is based on [1] where we use the weighted essentially non-oscillatory(WENO) scheme to update the interface.

The rest of the paper is organized as follows. In chapter 2, we describe the governing equations and the overall procedure to solve the equations. In chapter 3, we review the standard IFEM and then propose enriched-IFEM and velocity recovery. Chapter 4 describes the level set method to approximate the interface and WENO schemes to update the interface. We presents numerical simulations in chapter 5. Conclusion is given in chapter 6.

Chapter 2. Hele-Shaw Equations and Discretization

2.1 Governing Equation

In this section, we introduce the governing equations of Hele-Shaw flows described in [1]. Let Ω be the domain which is separated into $\Omega^+(t)$ and $\Omega^-(t)$ by an interface $\Gamma(t)$. Then, for fixed time t, the pressure p and the velocity field $\mathbf{u}$ are determined by

$$\mathrm{div}\mathbf{u}^s = f^s, \quad \text{in } \Omega^s(t), \quad s = -, + \tag{2.1}$$

$$\mathbf{u}^s = -\beta^s \nabla p^s, \quad \text{in } \Omega^s(t), \quad s = -, + \tag{2.2}$$

$$[p]_{\Gamma(t)} = \tau\kappa, \tag{2.3}$$

$$[\beta \nabla p \cdot \mathbf{n}]_{\Gamma(t)} = 0, \tag{2.4}$$

$$p = g, \quad \text{in } \partial\Omega, \tag{2.5}$$

where the coefficient β^s is a positive constant determined by the viscosity of the fluid in Ω^s, τ is the surface tension and κ is the curvature of the interface $\Gamma(t)$. Here, $\mathbf{n}$ is a outer unit normal vector to $\Gamma(t)$ and the bracket $[\cdot]_\Gamma$ means the jump across the interface, i.e., $[p]_\Gamma = p|_{\Omega^-} - p|_{\Omega^+}$. We assume the source term f is zero in Ω^+. We define the amalgamated surface tension d_0 as

$$d_0 = \frac{2\tau\pi\beta^+}{IR},$$

where $\mathrm{IR} := \int_\Omega f$ is the total injection rate. Thus, for the high injection rate, the surface tension increases leading to the high jump of pressure across the interfaces.

To describe the moving interface, we use the level set method in [1]. Let $\Phi(t, X) = \Phi(t, x, y),\ X = (x, y)$ be a function whose zero level set describes the interface $\Gamma(t)$ at time t i.e

$$\Phi(t, X) > 0 \quad \text{for} \quad \mathrm{X} \in \Omega^-(\mathrm{t})$$

$$\Phi(t, X) = 0 \quad \text{for} \quad \mathrm{X} \in \Gamma(\mathrm{t})$$

$$\Phi(t, X) < 0 \quad \text{for} \quad \mathrm{X} \in \Omega^+(\mathrm{t})$$

Assume that the level set functions move along the velocity fields, i.e $\Phi(t, X(t)) = constant$ where $X(t)$ is any trajectory which describes the position of a particle at time t. By differentiating $\Phi(t, X(t)) = constant$ with respect to time t, we obtain the equation of the level set function

$$\Gamma(t) = \{\Phi(t, X) = 0\} \tag{2.6}$$

$$\frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \nabla\Phi = 0 \tag{2.7}$$

2.2 Discretization of Domain and Time

Now we discretize the spatial domain Ω and the time domain $[0, T]$ where T is a target time. For numerical simulations, we assume that the domain Ω is a rectangular domain. We discretize the spatial domain by uniform structured grids. Let h denote the mesh size of the structured grid. Then we choose $\Delta t = \mathcal{O}(h^2)$ and divide the time domain $[0, T]$ uniformly with Δt. Denote $t_n = n\Delta t$ and $p^n(x, y) = p(t_n, x, y), \mathbf{u}^n(x, y) = \mathbf{u}(t_n, x, y)$ and $\Phi^n(x, y) = \Phi(t_n, x, y)$.

We propose the overall procedures. We solve the pressure equations (2.1)-(2.5) and the level set equation (2.7) sequentially. Suppose that we obtain $p^n, \mathbf{u}^n$ and Φ^n at the time t_n.

1. Solve for p^{n+1} and $\mathbf{u}^{n+1}$:

$$
\begin{aligned}
\mathrm{div}\mathbf{u}^{n+1} &= f, \quad \text{in } \Omega^{\pm}(t_n), \\
\mathbf{u}^{n+1} &= -\beta \nabla p^{n+1}, \quad \text{in } \Omega^{\pm}(t), \\
[p^{n+1}]_{\Gamma(t_n)} &= \tau\kappa, \\
[\beta \nabla p^{n+1} \cdot \mathbf{n}]_{\Gamma(t_n)} &= 0,
\end{aligned}
$$

2. Solve for Φ^{n+1} :

$$
\frac{\Phi^{n+1} - \Phi^n}{\partial t} + \mathbf{u}^{n+1} \cdot \nabla \Phi^n = 0.
$$

3. Update interface location $\Gamma(t^{n+1}) = \{X | \Phi^{n+1}(X) = 0\}$.

Chapter 3. Numerical Methods for pressure and velocity

In this chapter, we propose new numerical methods to solve for pressure and velocity equations. We fix the time $t = t_n$. For simplicity, we write $p = p^n$, $\mathbf{u} = \mathbf{u}^n$, $\Gamma = \Gamma(t_n)$ and $\Omega^\pm = \Omega^\pm(t_n)$. Moreover, in this chapter, we consider the following elliptic interface problem which is a general case of (2.1)-(2.5)

$$-\nabla \cdot \beta \nabla p = f, \quad \text{in } \Omega^s, \quad s = -, + \tag{3.1}$$

$$[p]_\Gamma = J_1, \tag{3.2}$$

$$[\beta \nabla p \cdot \mathbf{n}]_\Gamma = J_2, \tag{3.3}$$

$$p = g, \quad \text{in } \partial\Omega, \tag{3.4}$$

We introduce some notations for function spaces and their norms. For any bounded subdomain D, we denote $D^+ = D \cap \Omega^+$, $D^- = D \cap \Omega^-$ and define $H^m(D)$, $H_0^1(D)$, $H^m(\partial D)$ to be the ordinary Sobolev spaces of order m with the norm $||\cdot||_{m,D}$ and the semi-norm $|\cdot|_{m,D}$. For $m = 1, 2$, the space $\tilde{H}^m(D)$ is defined as

$$\tilde{H}^m(D) := H^m(D^+) \cap H^m(D^-),$$

with norms (semi-norms)

$$||u||^2_{\tilde{H}^m(D)} := ||u||^2_{H^m(D^+)} + ||u||^2_{H^m(D^-)}.$$

The subspace $\tilde{H}_0^1(D)$ is defined as

$$\tilde{H}_0^1(D) := \{u \in \tilde{H}^1(D) \,|\, u = 0 \text{ on } \partial D\}.$$

Since the equation (3.1)-(3.4) has jump conditions, we need subspaces of $\tilde{H}^m(\Omega)$ where jump conditions are imposed. We define $\mathcal{U}(\Omega)$ and $\mathcal{U}_0(\Omega)$ as follows:

$$\begin{aligned}
\mathcal{U}(\Omega) \quad &:= \{p \in \tilde{H}^2(\Omega) \,|\, [p]_\Gamma = J_1,\ [\beta \nabla p \cdot \mathbf{n}]_\Gamma = J_2\} \\
\mathcal{U}_0(\Omega) \quad &:= \{p \in \mathcal{U}(\Omega) \,|\, p = 0 \text{ on } \partial D\ \}
\end{aligned}$$

3.1 Variational Formula with jump conditions

We review the weak formulation introduced in [2] for the nonhomogeneous conditions along the interface, (3.1)-(3.4). Without loss of generality, we may assume that $g = 0$. To get a weak formulation, we multiply $q \in \tilde{H}_0^1(\Omega)$ on both side of (3.1) and apply Green's theorem. Then we obtain

$$-\int_{\partial\Omega^s} \beta(\nabla p \cdot \mathbf{n}) q ds + \int_{\Omega^s} \beta \nabla p \cdot \nabla q d\mathbf{x} = \int_{\Omega^s} f q d\mathbf{x}, \quad \forall v \in H_0^1(\Omega),$$

where $\mathbf{n}$ is a unit outer normal vector to $\Omega^s (s = +, -)$. Adding these two equations, we obtain

$$\begin{aligned}
\int_{\Omega^+} \beta \nabla p \cdot \nabla q d\mathbf{x} &+ \int_{\Omega^-} \beta \nabla p \cdot \nabla q d\mathbf{x} \\
&= \int_\Gamma [\beta \nabla p \cdot \mathbf{n}_\Gamma]\, q ds + \int_\Omega f q d\mathbf{x} \\
&= \int_\Gamma J_2 q ds + \int_\Omega f q d\mathbf{x}, \quad \forall v \in H_0^1(\Omega)
\end{aligned}$$

where $\mathbf{n}_\Gamma$ is a unit outer normal vector to Ω^-. Here, we define a bilinear form $a(\cdot,\cdot)$ as

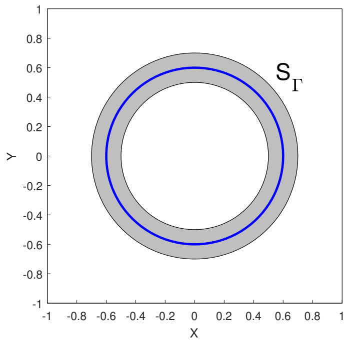
$$a(p,q) := \int_{\Omega^+} \beta\nabla p\cdot\nabla q d\mathbf{x} + \int_{\Omega^-} \beta\nabla p\cdot\nabla q d\mathbf{x} \quad \forall p,q\in H^1(\Omega^+)\cap H^1(\Omega^-). \tag{3.5}$$

Then a weak problem of the problem (3.1)-(3.4) is given as follow: Find $u\in\tilde{H}_0^1(\Omega)$ satisfying the jump condition $[p]_\Gamma = J_1$ and

$$a(p,q) =< J_2, q >_\Gamma + (f,q),\ \forall q\in\tilde{H}_0^1(\Omega), \tag{3.6}$$

where $(\cdot,\cdot)$ denotes the usual inner product inΩ and $<\cdot,\cdot>_\Gamma$ denotes the $L^2(\Gamma)$ inner product.

Since $\mathcal{U}(\Omega)$ is not a vector space, we need to subtract the nonhomogeneous jump conditions of the pressure. To do this, we introduce a *bubble function* p^* so that we can divide the pressure into $p = p^0+p^*$ where $p^0\in H_0^1(\Omega)$ and p^* satisfies the jump conditions (3.2)-(3.3). Note that there are infinitely many ways to choose p^*. For numerical computation, we need to find a specific p^*. To specify p^*, we let S_Γ be a thin tube containing the interface Γ in its interior(see Figure 3.1).



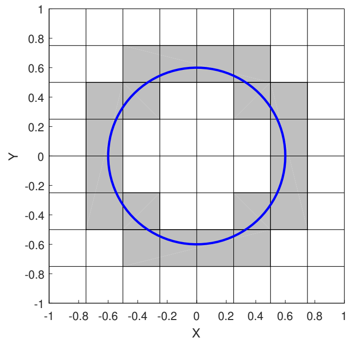


Figure 3.1: Example of thin strip S_Γ (left) and discretized strip S_Γ^h (right) for the case of circular interface (blue curve) with radius $r = 0.6$.

We define p^* on S_Γ such that p^* satisfies the jump conditions, i.e

$$[p^*]_\Gamma = J_1, \tag{3.7}$$

$$[\beta\nabla p^*\cdot\mathbf{n}]_\Gamma = J_2 \tag{3.8}$$

$$p^*(\mathbf{x}) = 0 \quad \text{for } \mathbf{x}\notin S_\Gamma. \tag{3.9}$$

Then we can rewrite the equation (3.1) as follow:

$$-\nabla\cdot\beta\nabla p^0 \quad = f + \nabla\cdot\beta\nabla p^*, \quad \text{in } \Omega^\pm$$

Thus we obtain a new problem for (3.1)-(3.4) as :

$$\begin{aligned} -\nabla\cdot\beta\nabla p^0 &= f+\nabla\cdot\beta\nabla p^*, && \text{in } \Omega\setminus\Gamma, && (3.10)\\ [p^0]_\Gamma &= 0, && \text{on } \Gamma, && (3.11)\\ \left[\beta\nabla p^0\cdot\mathbf{n}\right]_\Gamma &= 0, && \text{on } \Gamma, && (3.12)\\ p^0 &= g, && \text{on } \partial\Omega. && (3.13)\end{aligned}$$

where $p^0\in H_0^1(\Omega)$. Since $a(p,q) = a(p^0+p^*,q) = a(p^0,q)+a(p^*,q)$, we obtain a new weak problem given as following: Find $p_0\in H_0^1(\Omega)$ satisfying

$$a(p^0,q) =< J_2, q>_\Gamma + (f,q) - a(p^*,q),\ \forall q\in\tilde{H}_0^1(\Omega), \tag{3.14}$$

3.2 Discontinuous bubble IFEM

In this section, we describe how to discretize p^0 and p^*. From this section on, we assume that the domain Ω is a rectangle. Note that we discretize the domain Ω by structured grids. Let $\mathcal{T}_h$ be a triangulation of Ω consisting of rectangles. Let $T \in \mathcal{T}_h$ be an *interface element* if T is cut by the interface Γ and be a *non-interface element* otherwise. We introduce some notations:

$\mathcal{T}_h^* =$ the set of all interface elements in $\mathcal{T}_h$

$\mathcal{T}_h^N =$ the set of all non-interface elements in $\mathcal{T}_h$

$\mathcal{S}_h = \bigcup_{T\in\mathcal{T}_h^*} T =$ the union of all interface elements

For discretizing p^0, we review the Q1-conforming based IFEMs introduced in [2], [4]. We let $S_h(T)$ be a space of standard bilinear functions on T with nodal d.o.f(degree of freedom) for each element $T \in \mathcal{T}_h$. For every non-interface element $T \in \mathcal{T}_h^N$, we just consider basis functions in $S_h(T)$ as same as the standard Q1-conforming methods. For an interface element $T \in \mathcal{T}_h^*$, on the other hand, we modify basis functions in $S_h(T)$. Suppose that A_i, $i = 1, 2, 3, 4$ be the nodes of the element T and that the interface Γ cuts the element T through two edges e_1 and e_2 at points E_1 and E_2 respectively so that the element T is divided into T^+ and T^- (see Figure 3.2). Now we modify a function $\phi \in S_h(T)$ as a new piecewise bilinear function $\hat{\phi}$ of the form

$$\hat{\phi}(x,y) := \begin{cases} \hat{\phi}^+(x,y) = a^+ + b^+x + c^+y + d^+xy, & (x,y) \in T^+, \\ \hat{\phi}^-(x,y) = a^- + b^-x + c^-y + d^-xy, & (x,y) \in T^- \end{cases} \tag{3.15}$$

The coefficients in (3.15) are determined by nodal conditions and jump conditions:

$$\hat{\phi}(A_i) = V_i, \quad i = 1, 2, 3, 4 \tag{3.16a}$$

$$\hat{\phi}^+(E_i) = \hat{\phi}^-(E_i), \quad i = 1, 2 \tag{3.16b}$$

$$d^+ = d^- \tag{3.16c}$$

$$\int_{\overline{E_1E_2}} \beta^+ \nabla\hat{\phi}^+ \cdot \mathbf{n}_{\overline{E_1E_2}} = \int_{\overline{E_1E_2}} \beta^- \nabla\hat{\phi}^- \cdot \mathbf{n}_{\overline{E_1E_2}} \tag{3.16d}$$

where V_i, $i = 1, 2, 3, 4$ are nodal values. It is well-known that the coefficients of $\hat{\phi}$ are uniquely determined by the conditions (3.16a)-(3.16d)(see [3]). We denote $\hat{S}_h(T)$ as the space of modified functions $\hat{\phi}$. For approximation of p^0, the space $\hat{S}_h(\Omega)$ of Q1-conforming based IFEM is defined as follow:

$$\begin{cases} \phi|_T \in S_h(T) & \text{if } T \text{ is a noninterface element,} \\ \phi|_T \in \widehat{S}_h(T) & \text{if } T \text{ is an interface element,} \\ \phi|_{T_1}(X) = \phi|_{T_2}(X) & \text{if } T_1 \text{ and } T_2 \text{ are adjacent elements} \\ & \text{and } X \text{ is a common node of } T_1 \text{ and } T_2, \\ \phi(X) = 0 & \text{if } X \text{ is a node on the boundary edges.} \end{cases}$$

For discretizing p^*, we use the conditions (3.7)-(3.9). For a non-interface element T, we let $p_h^*(T) = 0$. For an interface element T, we define $p_h^*|_T$ as a piecewise bilinear function of the form given in (3.15) satisfying

$$p_h^*(A_i) = 0, \quad i = 1, 2, 3, 4 \tag{3.17a}$$

$$[p_h^*(E_i)]_{\overline{E_1E_2}} = J(E_i), \quad i = 1, 2, \tag{3.17b}$$

$$d^+ = d^- \tag{3.17c}$$

$$\int_{\overline{E_1E_2}} \beta^+ \nabla p_h^*|_{T^+} \cdot \mathbf{n}_{\overline{E_1E_2}} = \int_{\overline{E_1E_2}} \beta^- \nabla p_h^*|_{T^-} \cdot \mathbf{n}_{\overline{E_1E_2}} \tag{3.17d}$$

Once we determine p_h^*, we discretize the weak form in (3.14) : Find $p_h^0 \in \hat{S_h}(\Omega)$ such that for every $q_h \in \hat{S_h}(\Omega)$,

$$\sum_{T\in\mathcal{T}_h}\left(\int_{T^+}\beta\nabla p_h^0\cdot\nabla q_h \mathrm{d}\mathbf{x}+\int_{T^-}\beta\nabla p_h^0\cdot\nabla q_h \mathrm{d}\mathbf{x}\right)=\sum_{T\in\mathcal{T}_h}\int_T f q_h \mathrm{d}\mathbf{x}+\sum_{T\in\mathcal{T}_h^*}\int_{\Gamma^*} J_2 q_h \mathrm{d}\mathbf{x}$$
$$-\sum_{T\in\mathcal{T}_h}\left(\int_{T^+}\beta\nabla p_h^*\cdot\nabla q_h \mathrm{d}\mathbf{x}+\int_{T^-}\beta\nabla p_h^*\cdot\nabla q_h \mathrm{d}\mathbf{x}\right)$$

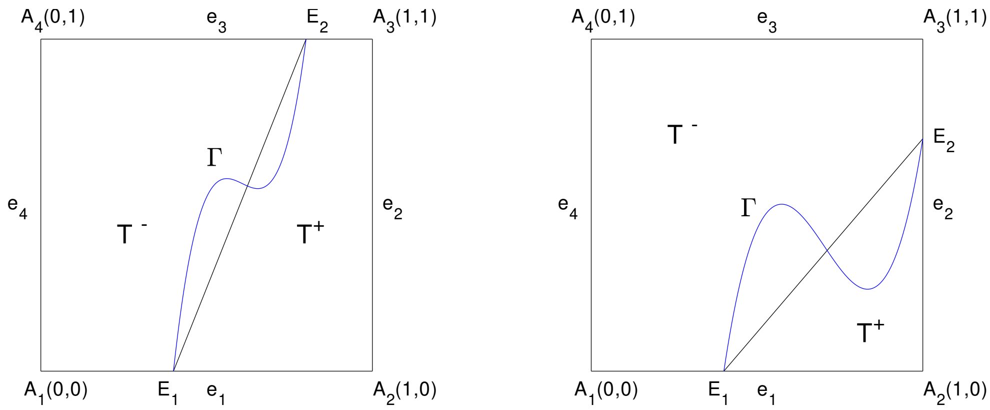


Figure 3.2: Examples of interface elements cut by interfaces

3.3 Enriched IFEM and Velocity recovery

Although the discontinuous bubble IFEM scheme can solve the problem (3.1)-(3.4) for pressure, this scheme has restriction to compute the Darcy velocity accurately. Thus we use the concept of enriched Galerkin methods [5] for computation of Darcy velocity and local conservation of the velocity field. The function space $\hat{S_h}(\Omega)$ is enriched by bubble functions. We define the enriched IFEM space as

$$\hat{E_h}(\Omega) := \hat{S_h}(\Omega) \oplus P_h^0(\Omega)$$

where $P_h^0(\Omega)$ is the space of piecewise constant functions on T for $T \in \mathcal{T}_h$.

Note that functions in $E_h(\Omega)$ are discontinuous across the edges. Thus we need to modify the weak form in (3.18). Before we design the associated weak form $a_h(p_h, q_h) = l(q_h)$, we introduce some notations:

$\mathcal{E}_h^0$ = set of interior edges of the elements

$\mathcal{E}_h^\partial$ = set of boundary edges of the elements

$\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\partial$ = set of all edges

Let $e = \partial T_1 \cap \partial T_2$ be the common edge of two adjacent elements T_1 and T_2. We define the jump $[\cdot]_e$ and the average $\{\cdot\}_e$ along the edge e as

$$[\phi]_e(x) := \lim_{\delta\to 0+}(\phi(x-\delta\mathbf{n_e})-\phi(x+\delta\mathbf{n_e})) \tag{3.18}$$

$$\{\phi\}_e(x) := \frac{1}{2}\lim_{\delta\to 0+}(\phi(x-\delta\mathbf{n_e})+\phi(x+\delta\mathbf{n_e})) \tag{3.19}$$

Simply, we denote $\{\phi\}_e = \phi_1 - \phi_2$ and $[\phi]_e = \frac{\phi_1+\phi_2}{2}$ where $\phi_i = \phi|_{T_i}$ for $i = 1, 2$.

If the edge e is on the boundary, we assign the jump and the average the boundary value. That is, the jump and average along $e \in \mathcal{E}_h$ are given as

$$[\phi]_e = \begin{cases} \phi_1 - \phi_2 & \text{if } e \in \mathcal{E}_h^0 \\ \phi|_e & \text{if } e \in \mathcal{E}_h^\partial \end{cases} \quad , \quad \{\phi\}_e = \begin{cases} \frac{\phi_1+\phi_2}{2} & \text{if } e \in \mathcal{E}_h^0 \\ \phi|_e & \text{if } e \in \mathcal{E}_h^\partial \end{cases} \tag{3.20}$$

Also, we have the following equation :

$$a_1b_1 - a_2b_2 = (\frac{a_1+a_2}{2})(b_1-b_2) + (a_1-a_2)(\frac{b_1+b_2}{2}) = \{a\}[b] + [a]\{b\} \tag{3.21}$$

Finally, we let $\mathbf{n_e}$ be a specified normal vector of the edge e.

Suppose $q_h \in \hat{E}_h(\Omega)$. Multiplying (3.1) by q_h, we obtain

$$\sum_{T\in\mathcal{T}_h} \left(\int_{T^+} \beta\nabla p \cdot \nabla q_h + \int_{T^-} \beta\nabla p \cdot \nabla q_h \right) - \sum_{e\in\mathcal{E}_h} \int_e [\beta\nabla p \cdot \mathbf{n_e}]_e q_h = \sum_{T\in\mathcal{T}_h} \int_T f q_h + \int_\Gamma J_2 q_h$$

by considering the line integral part including the interface Γ. Here, $[\beta\nabla p \cdot \mathbf{n_e}]q_h = \{\beta\nabla p \cdot \mathbf{n_e}\}_e[q_h]_e + [\beta\nabla p \cdot \mathbf{n_e}]_e\{q_h\}_e$ for $e \in \mathcal{E}_h^0$ by (3.21). Thus we obtain

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} \left(\int_{T^+} \beta\nabla p \cdot \nabla q_h + \int_{T^-} \beta\nabla p \cdot \nabla q_h \right) - \sum_{e\in\mathcal{E}_h^0} \left(\int_e \{\beta\nabla p \cdot \mathbf{n_e}\}_e[q_h]_e + [\beta\nabla p \cdot \mathbf{n_e}]_e\{q_h\}_e \right) \\ - \sum_{e\in\mathcal{E}_h^\partial} \int_e [\beta\nabla p \cdot \mathbf{n_e}]_e q_h = \sum_{T\in\mathcal{T}_h} \int_T f q_h + \int_\Gamma J_2 q_h \end{aligned}$$

Since $p \in \tilde{H}^2(\Omega)$, $[\beta\nabla p \cdot \mathbf{n_e}]_e = 0$ a.e for $e \in \mathcal{E}_h^0$. With (3.20), we obtain

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} \left(\int_{T^+} \beta\nabla p \cdot \nabla q_h + \int_{T^-} \beta\nabla p \cdot \nabla q_h \right) - \sum_{e\in\mathcal{E}_h} \int_e \{\beta\nabla p \cdot \mathbf{n_e}\}_e[q_h]_e \\ = \sum_{T\in\mathcal{T}_h} \int_T f q_h + \int_\Gamma J_2 q_h \end{aligned}$$

We add boundary penalty terms to handle the boundary condition (3.4). Moreover, since $[p]_e = 0$ a.e for $e \in \mathcal{E}_h^0$, we add interior penalty terms to make the bilinear form coercive and add $-\sum_{e\in\mathcal{E}_h}\int_e [p]_e\{\beta\nabla q_h \cdot \mathbf{n_e}\}_e$ to make the form symmetric, i.e

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} \left(\int_{T^+} \beta\nabla p \cdot \nabla q_h + \int_{T^-} \beta\nabla p \cdot \nabla q_h \right) - \sum_{e\in\mathcal{E}_h} \int_e \{\beta\nabla p \cdot \mathbf{n_e}\}_e[q_h]_e \\ + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e [p][q_h] - \sum_{e\in\mathcal{E}_h} \int_e [p]_e\{\beta\nabla q_h \cdot \mathbf{n_e}\}_e \\ = (f, q_h) + \int_\Gamma J_2 q_h + \sum_{e\in\mathcal{E}_h^\partial} \frac{\sigma}{|e|} \int_e g q_h - \sum_{e\in\mathcal{E}_h^\partial} \int_e [p]_e\{\beta\nabla q_h \cdot \mathbf{n_e}\}_e \end{aligned}$$

where σ is a penalty parameter for stability.

Now we propose our enriched DB-IFEM scheme : Find $p_h^0 \in \hat{E}_h(\Omega)$ such that

$$a_h(p_h^0, q_h) = l(q_h) - a_h(p_h^*) \quad \text{for every } q_h \in \hat{E}_h(\Omega) \tag{3.22}$$

where

$$a_h(p_h, q_h) = \sum_{T\in\mathcal{T}_h} \int_T \beta\nabla p_h \cdot \nabla q_h - \sum_{e\in\mathcal{E}_h} \int_e \{\beta\nabla p_h \cdot \mathbf{n_e}\}_e[q_h]_e + \sum_{e\in\mathcal{E}_h} \frac{\sigma}{|e|} \int_e [p_h][q_h] - \sum_{e\in\mathcal{E}_h} \int_e [p_h]\{\beta\nabla q_h \cdot \mathbf{n_e}\}$$

$$l(q_h) = (f, q_h) + \int_\Gamma J_2 q_h + \sum_{e\in\mathcal{E}_h^\partial} \frac{\sigma}{|e|}\int_e g q_h - \sum_{e\in\mathcal{E}_h^\partial}\int_e g(\beta\nabla q_h \cdot \mathbf{n_e})$$

Since the bilinear form a_h in (3.22) is continuous and coercive, there exists a unique solution p_h^0 for the problem (3.22) by Lax-Milgram theorem.

To find the velocity $\mathbf{u}_h$, we let $V_h(\Omega)$ be the lowest order of Raviart-Thomas element [7]. We construct $\mathbf{u}_h \in V_h(\Omega)$ locally [5] as

$$(u_n, n)_e = \left(-\{\beta\nabla p_h^0 \cdot \mathbf{n}\}_e + \frac{\sigma}{|e|}[p_h^0]\right) + \left(-\{\beta\nabla p_h^* \cdot \mathbf{n}\}_e + \frac{\sigma}{|e|}[p_h^*]\right) \tag{3.23}$$

for each edge e of the elements in $\mathcal{T}_h$. Then the velocity $\mathbf{u}_h$ is locally conservative by following Propositions.

Theorem 3.3.1 *(Local mass conservation) For every $T \in \mathcal{T}_h$,*

$$\int_{\partial T} \mathbf{u}_h \cdot \mathbf{n} = \int_T f$$

Proof Taking a test function $\phi_h = \mathcal{X}_T$ in (3.22), we obtain

$$\begin{aligned} &-\sum_{e\in\partial T}\int_e \{\beta\nabla p_h^0 \cdot \mathbf{n}_e\} + \sum_{e\in\partial T}\frac{\sigma}{|e|}\int_e [p_h^0] \\ &= \int_T f + \sum_{e\in\partial T}\int_e \{\beta\nabla p_h^* \cdot \mathbf{n}_e\} - \sum_{e\in\partial T}\frac{\sigma}{|e|}\int_e [p_h^*] \end{aligned}$$

By the definition of $\mathbf{u}_h$ in (3.23) and the previous identity,

$$\begin{aligned} \int_{\partial T} \mathbf{u}_h \cdot \mathbf{n} &= -\sum_{e\in\partial T}\int_e \{\beta\nabla p_h^0 \cdot \mathbf{n}\}_e + \sum_{e\in\partial T}\int_e \frac{\sigma}{|e|}[p_h^0] \\ &- \sum_{e\in\partial T}\int_e \{\beta\nabla p_h^* \cdot \mathbf{n}\}_e + \sum_{e\in\partial T}\int_e \frac{\sigma}{|e|}[p_h^*] = \int_T f. \quad \square \end{aligned}$$

By summation over $\mathcal{T}_h$, we have the following theorem.

Theorem 3.3.2 *(Global mass conservation)*

$$\int_{\partial\Omega} \mathbf{u}_h \cdot \mathbf{n} = \int_\Omega f$$

Chapter 4. Numerical Methods for the interface

Note that we assume that the domain Ω is a rectangle and that the domain is discretized by structured grid. For simplicity, we assume that the grid is uniform i.e $\Delta x = \Delta x_i$ and $\Delta y = \Delta y_j$ for all i, j. Let $\Phi(t, x, y)$ be a function whose zero level set describes the interface $\Gamma(t)$ at time t. We approximate $\Phi_h(t, x, y)$ at center points of each element $T \in \mathcal{T}_h$ at discretized time $t^n = n\Delta t$ (see Figure 4.1). We let $\Phi_{i,j}^n$ be a numerical approximation to the solution $\Phi(t^n, x_i, y_j) = \Phi(n\Delta t, (i + \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta y)$.

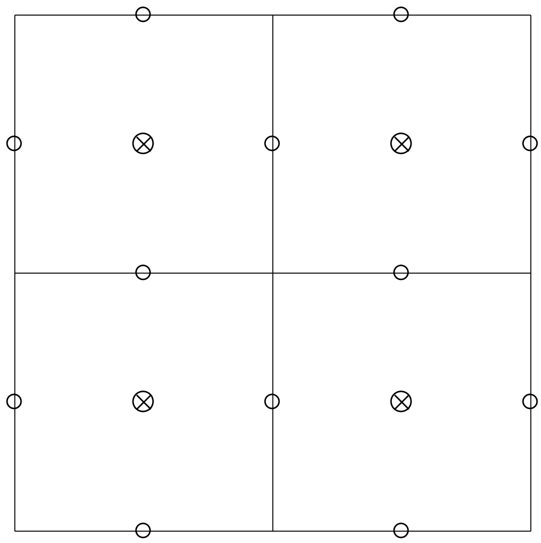


Figure 4.1: DOFs of Φ_h located at cell center $\otimes$ and DOFs of $\mathbf{u}_h$ located at edges $\circ$

4.1 Interface Reconstruction

Since only values of level set functions evaluated at centers of elements are given, we need to find points on the interface with these values so that we can apply the enriched-DB IFEM introduced in the previous chapter. Note that these points are contained in interface elements.

We propose how to reconstruct the interface from the values $\Phi_{i,j}^n$ when the time $t = t_n$ is fixed, which is introduced in [1]. We say that a center point $X = (x_i, y_j)$ of an element is a *control point* if $\Phi^n(X) < 0$ and the number of adjacent elements whose centers have positive values of Φ^n is 1,2, or 3. That is, all center points of elements whose centers have positive values of Φ^n are encircled by the set of the control points.

We reconstruct the interface by projecting control points on the interface. We let a *interface point* be a approximated point on the interface obtained by projection of a control point. For simple notation, we write $\Phi = \Phi^n$ in this section. Let $X = (x_i, y_j)$ be a control point. We let $\mathbf{n}$ be the unit gradient vector of Φ:

$$\mathbf{n} = \frac{\nabla\Phi}{||\nabla\Phi||}$$

Then we can find the interface point X^*, which is the projection of X along $\mathbf{n}$:

$$X^* = X + t\mathbf{n}$$

Here, t is the positive solution of the quadratic equation:

$$\frac{1}{2}(\mathbf{n}^T He(\Phi)\mathbf{n})t^2 + ||\nabla\Phi(X)||t + \Phi(X) = 0$$

where $He(\Phi)$ is the Hessian matrix evaluated at the point X,

$$He(\Phi) = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{yx} & \Phi_{yy} \end{bmatrix}$$

Moreover, we approximate the curvature κ of the interface as

$$\kappa = \frac{\Phi_{xx}\Phi_y^2 - 2\Phi_{xy}\Phi_x\Phi_y + \Phi_{yy}\Phi_x^2}{(\Phi_x^2 + \Phi_y^2)^{3/2}}$$

which is used in the jump condition (2.3) of pressure equations. Note that we only compute the curvature κ in interface elements.

In numerical simulations, we approximate all partial derivatives of Φ by the central difference schemes.

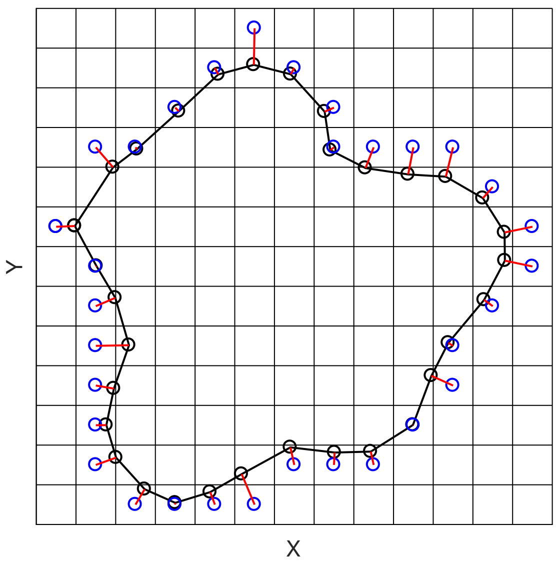


Figure 4.2: Blue circles are control points and black circles are result of the projection(red lines)

4.2 Evolution of Interface with WENO

Once we obtain the velocity $\mathbf{u}$ from the Hele Shaw equations at time t, we can update the interface $\Gamma(t)$ with (2.7). This equation is a Hamilton-Jacobi equation, so we can rewrite above equation as

$$\Phi_t + H(\Phi_x, \Phi_y) = 0 \tag{4.1}$$

where $H(\phi, \psi) = u\phi + v\psi$ and $\mathbf{u} = (u, v)$

To solve the equation, we apply the fifth-order WENO scheme with the local Lax-Friedrichs(LLF) flux in [9] and [10]. Now suppose that $\Phi^n_{i,j}$ is a numerical approximation of (4.1) at $t = t^n$. Then the WENO scheme given in [8] is

$$\Phi^{n+1}_{i,j} = \Phi^n_{i,j} - \Delta t \hat{H}(\Phi^+_{x,i,j}, \Phi^-_{x,i,j}, \Phi^+_{y,i,j}, \Phi^-_{y,i,j}) \tag{4.2}$$

where $\hat{H}$ is the LLF flux and $\Phi^{\pm}_{x,i,j}$ and $\Phi^{\pm}_{y,i,j}$ are fifth-order WENO approximations in [9] of the partial derivatives Φ_x and Φ_y at (x_i, y_j) respectively.

The LLF flux $\hat{H}$ is defined as

$$\begin{aligned}\hat{H}(\phi^+, \phi^-, \psi^+, \psi^-) = H(&\frac{\phi^+ + \phi^-}{2}, \frac{\psi^+ + \psi^-}{2}) \\ &- \alpha(\phi^+, \phi^-)\frac{\phi^+ - \phi^-}{2} - \beta(\psi^+, \psi^-)(\frac{\psi^+ - \psi^-}{2})\end{aligned}$$

where

$$\alpha(\phi^+, \phi^-) = \max_{\phi \in I(\phi^-, \phi^+), \psi \in [C,D]} |H_1(\phi, \psi)|$$

$$\beta(\psi^+, \psi^-) = \max_{\psi \in I(\psi^-, \psi^+), \phi \in [A,B]} |H_2(\phi, \psi)| .$$

Here, H_1 and H_2 are the partial derivatives of H with respect to Φ_x and Φ_y respectively. Also $[A, B]$ (and $[C, D]$) stands for the value range of $\phi^{\pm}$ (and that of $\psi^{\pm}$) and $I(a, b) = [min(a, b), max(a, b)]$. Since $H(\phi, \psi) = u\phi + v\psi$, $H_1 = u$ and $H_2 = v$. Thus we can rewrite the LLF flux as

$$\hat{H}(\phi^+, \phi^-, \psi^+, \psi^-) = (u - |u|)\frac{\phi^+}{2} + (u + |u|)\frac{\phi^-}{2} + (v - |v|)\frac{\psi^+}{2} + (v + |v|)\frac{\psi^-}{2}$$

For the fifth-order WENO approximations $\Phi^{\pm}_{x,i,j}$ and $\Phi^{\pm}_{y,i,j}$, we only describe WENO approximations $\Phi^{\pm}_{x,i,j}$ of x-direction partial derivative. One can find $\Phi^{\pm}_{y,i,j}$ in similar way. For simplicity, we denote $\phi(x) = \Phi_x(t^n, x, y)$ when $y = y_j$ and $t = t_n$ are fixed. We define the cell averages of ϕ as

$$\overline{\phi}_i = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \phi(x)\mathrm{d}x \tag{4.3}$$

We denote 3 candidate stencils of the point x_i by $S_k(i)$, $k = 0, 1, 2$ where

$$S_k(i) = \{x_{i-\frac{1}{2}-k}, x_{i-\frac{1}{2}-k+1}, x_{i-\frac{1}{2}-k+2}\}$$

i.e $S_0(i) = \{x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}+1}, x_{i-\frac{1}{2}+2}\}, S_1(i) = \{x_{i-\frac{1}{2}-1}, x_{i-\frac{1}{2}}, x_{i-\frac{1}{2}+1}\}, S_2(i) = \{x_{i-\frac{1}{2}-2}, x_{i-\frac{1}{2}-1}, x_{i-\frac{1}{2}}\}$. Then there exist a interpolation polynomial $p_k(x)$ of degree 2 on each stencil $S_k(i)$ such that

$$\overline{\phi}_j = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_k(x)\mathrm{d}x, \qquad j = i - \frac{1}{2} - k, i - \frac{1}{2} - k + 1, i - \frac{1}{2} - k + 2$$

Let $\phi^{(k)-}_i = p_k(x_i)$ and $\phi^{(k)+}_{i-1} = p_k(x_{i-1})$. Then there exist coefficients $c_{k,j}$ in [9] so that

$$\phi^{(k)-}_i = \sum_{l=0}^{2} c_{k,l}\overline{\phi}_{i-k+j}, \quad \phi^{(k)+}_{i-1} = \sum_{l=0}^{2} c_{k-1,l}\overline{\phi}_{i-k+j}$$

The key idea of WENO is the choice of the weights ω_k and $\tilde{\omega}_k, k = 0, 1, 2$ s.t

$$\phi^-_i = \sum_{k=0}^{2}\omega_k\phi^{(k)-}_i, \phi^+_{i-1} = \sum_{k=0}^{2}\tilde{\omega}_k\phi^{(k)+}_{i-1} \quad and \quad \sum_{k=0}^{2}\omega_k = \sum_{k=0}^{2}\tilde{\omega}_k = 1$$

where ϕ_i^- and ϕ_{i-1}^+ are our WENO approximations of $\Phi_{x,i,j}^-$ and $\Phi_{x,i-1,j}^+$

Note that WENO schemes are based on essentially non-oscillating(ENO) schemes where the basic idea of ENO is to avoid discontinuous cells in the stencils which yield oscillations [8]. Thus, to emulate ENO schemes, the weights ω_k and $\tilde{\omega}_k$ would be chosen so that they are assigned zero when the candidate stencil has a discontinuity. If the function ϕ is smooth in all candidate stencils $S_k(i), k = 0, 1, 2$, then there exist constants d_k and $\tilde{d}_k, k = 0, 1, 2$ such that

$$\phi_i^- = \sum_{k=0}^{2} d_k \phi_i^{(k)-}, \quad \phi_{i-1}^+ = \sum_{k=0}^{2} \tilde{d}_k \phi_{i-1}^{(k)+}$$

where

$$d_0 = \frac{3}{10},\ d_1 = \frac{3}{5},\ d_2 = \frac{1}{10}$$
$$\tilde{d}_k = d_{2-k}$$

by symmetry. Then the weights are given as follow:

$$\omega_k = \frac{\alpha_k}{\sum_{s=0}^{2} \alpha_s}, \quad \tilde{\omega}_k = \frac{\tilde{\alpha}_k}{\sum_{s=0}^{2} \tilde{\alpha}_s}, \quad k = 0, 1, 2$$

where

$$\alpha_k = \frac{d_k}{(\epsilon + \beta_k)^2}, \quad \tilde{\alpha}_k = \frac{\tilde{d}_k}{(\epsilon + \beta_k)^2}$$

Here, $\epsilon > 0$ is a small positive number to avoid overflow when β_k is closed to zero and we will use $\epsilon = 10^{-6}$ in our numerical simulations. Also, β_k would be a measurement for smoothness of the function ϕ in the stencil $S_k(i)$. Since total variation measures smoothness of a function, β_k would be chosen so that the total variation of the interpolation polynomial p_k in the stencil $S_k(i)$ is minimized. Thus we define

$$\beta_k = \sum_{s=1}^{2} \int_{x_i}^{x_{i+1}} \Delta x^{2s-1} \left(\frac{\partial^s p_k(x)}{\partial^s x}\right)^2 \mathrm{d}x$$

i.e β_k is a sum of the square of L^2 norms of all derivatives of p_k over $[x_i, x_{i+1}]$ Thus $\beta_k, k = 0, 1, 2$ are given as follow:

$$\beta_0 = \frac{13}{12}(\overline{\phi}_i - 2\overline{\phi}_{i+1} + \overline{\phi}_{i+2})^2 + \frac{1}{4}(3\overline{\phi}_i - 4\overline{\phi}_{i+1} + \overline{\phi}_{i+2})^2$$
$$\beta_1 = \frac{13}{12}(\overline{\phi}_{i-1} - 2\overline{\phi}_i + \overline{\phi}_{i+1})^2 + \frac{1}{4}(\overline{\phi}_{i-1} - \overline{\phi}_{i+1})^2$$
$$\beta_2 = \frac{13}{12}(\overline{\phi}_{i-2} - 2\overline{\phi}_{i-1} + \overline{\phi}_i)^2 + \frac{1}{4}(\overline{\phi}_{i-2} - 4\overline{\phi}_{i-1} + 3\overline{\phi}_i)^2$$

Finally, if ϕ is continuous then (4.3) can be rewritten as

$$\overline{\phi}_i = \frac{1}{\Delta x}(\Phi_{i+1,j} - \Phi_{i,j}) \tag{4.4}$$

Thus if we approximate $\overline{\phi}_i$ as (4.4), we obtain the WENO approximations $\Phi_{x,i,j}^{\pm}$.

Chapter 5. Numerical results

In this chapter, we provide some numerical experiments for Hele-Shaw problems with various interface Γ and coefficients β and d_0. In our experiments, we assume that the domain is $\Omega = [-2, 2]^2$ and we consider a uniform triangulation $\mathcal{T}_h$ by rectangles whose size is h.

We use the similar settings for the simulations as in [1]. The source term is given as

$$f = \begin{cases} \frac{6V_0}{\alpha^2}(\alpha - r), & \text{if } r \le \alpha \\ 0, & \text{otherwise} \end{cases}$$

where V_0 is a controlling parameter and α is the radius for the source. In Example 1, we consider the problem with an analytic solution whose initial interface is a circle. We provide the L^2 and H^1 errors for the pressure and L^2 errors for the velocity at the specific target time. In Example 2 and Example 3, we consider more complex interface conditions.

5.1 Example1 : Benchmark Problem

In this example, we apply our method to a benchmark problem [1] given as

$$p(r) = \begin{cases} \frac{V_0}{\beta^-}\left(\frac{2r^3}{3\alpha^2} - \frac{3r^2}{2\alpha}\right) + C_1, & \text{if } 0 \le r \le \alpha \\ -\frac{V_0\alpha}{\beta^-}\log(r) + C_0 & \text{if } \alpha < r \le r_\Gamma \\ -\frac{V_0\alpha}{\beta^+}\log(r), & \text{otherwise,} \end{cases}$$

where $r_\Gamma = \sqrt{2\alpha V_0 t + r_0^2}$ and r_0 is the radius of the initial interface. The boundary condition is given as

$$p = -\frac{V_0\alpha}{\beta^+}\log(r) + 0.1, \quad \text{on } \partial\Omega,$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.1$, $d_0 = 2.5 \cdot 10^{-3}$. Target time is $T = 10$ with time-step $\Delta t = 8h^2$. We set the initial interface as circle $x^2 + y^2 = 0.4^2$.

Table 5.1: Error table for a benchmark problem

Element	$\|\|p - p_h\|\|_{L^2(\Omega)}$	order	$\|\|p - p_h\|\|_{1,h}$	order	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega)}$	order
16^2	$1.844E-2$		$2.731E-2$		$7.984E-2$	
32^2	$2.632E-3$	2.808	$4.215E-3$	2.696	$3.236E-2$	1.303
64^2	$1.093E-3$	1.268	$1.820E-3$	1.211	$1.183E-2$	1.451
128^2	$9.564E-5$	3.514	$6.675E-4$	1.447	$5.318E-3$	1.154

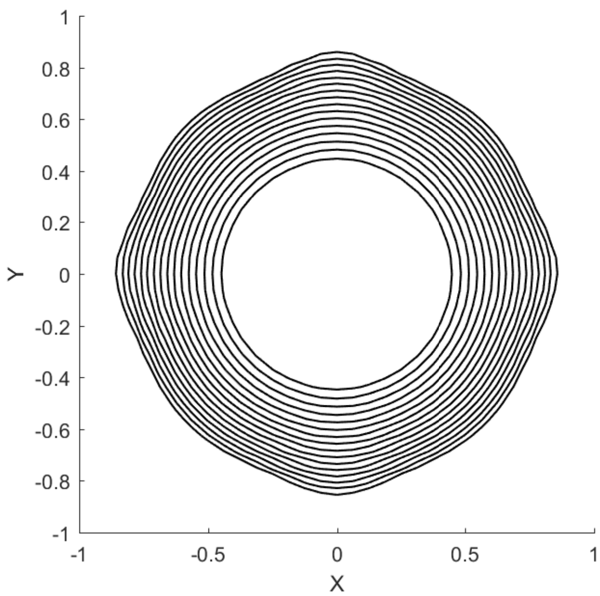


Figure 5.1: Interfaces

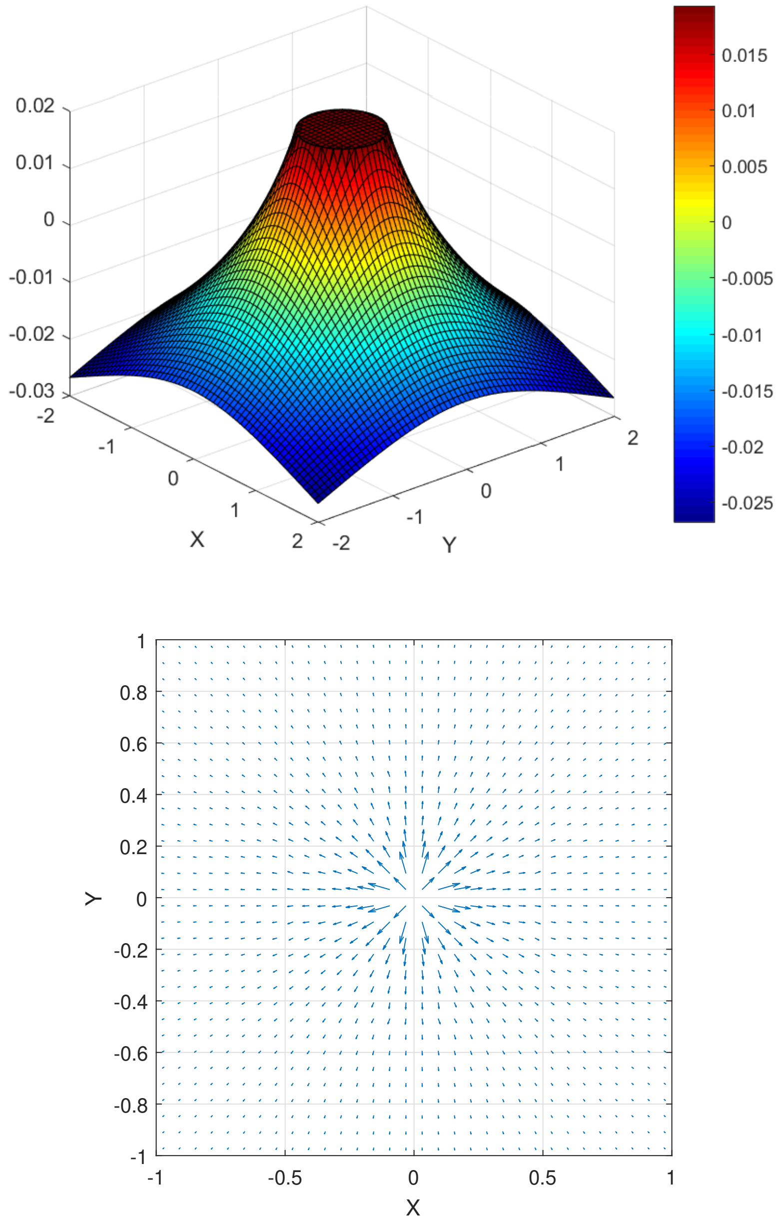


Figure 5.2: Pressure and Velocity field at time t=0

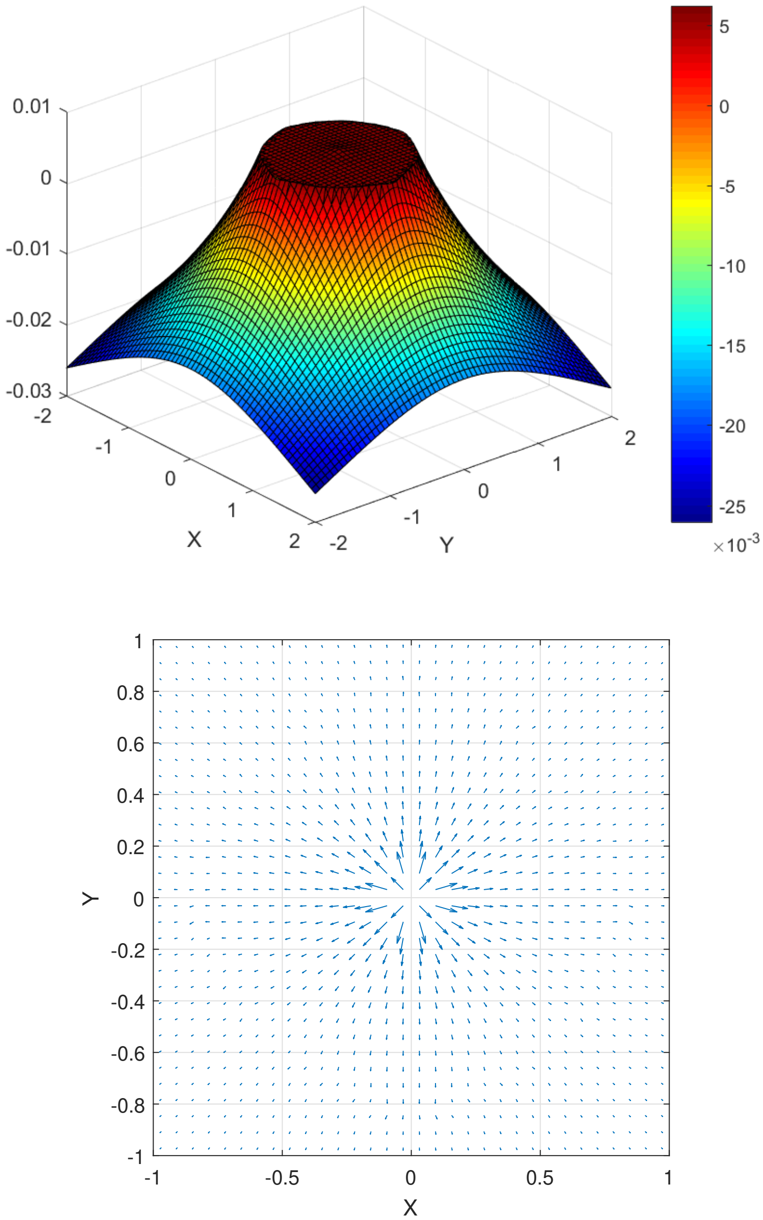


Figure 5.3: Pressure and Velocity field at time t=10

5.2 Example2 : Symmetric Interfaces

Two initial interfaces are given by

$$r1 = r_0 + 0.1\sin(3\theta)$$

$$r2 = r_0 + 0.1\sin(5\theta)$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.3$, $d_0 = 2.3 \cdot 10^{-3}$ and $r_0 = 0.7$. Target time is $T = 8$

The boundary condition is given as

$$p = -\frac{V_0\alpha}{\beta^+}\log(r) + 0.1, \quad \text{on } \partial\Omega,$$

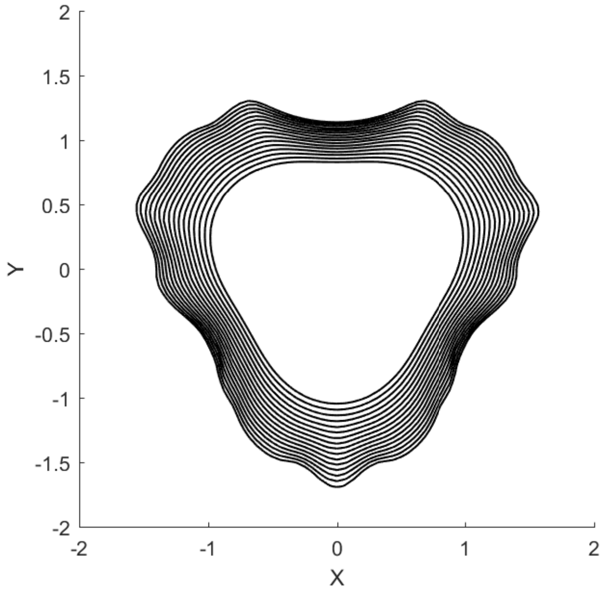


Figure 5.4: Interfaces for r_1

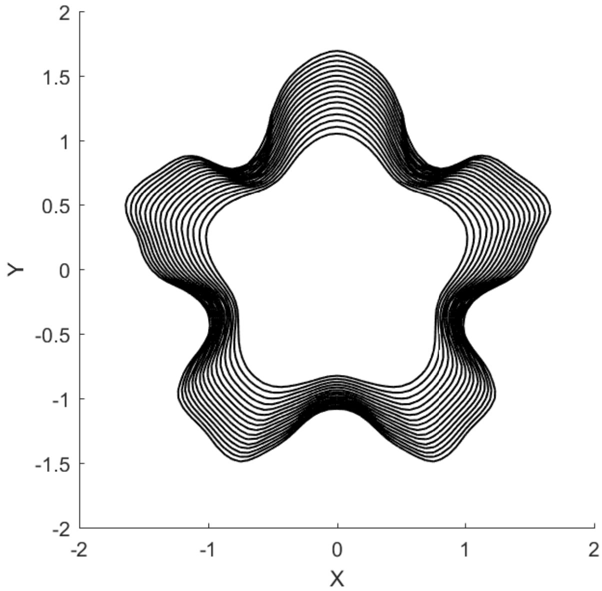


Figure 5.5: Interfaces for r_2

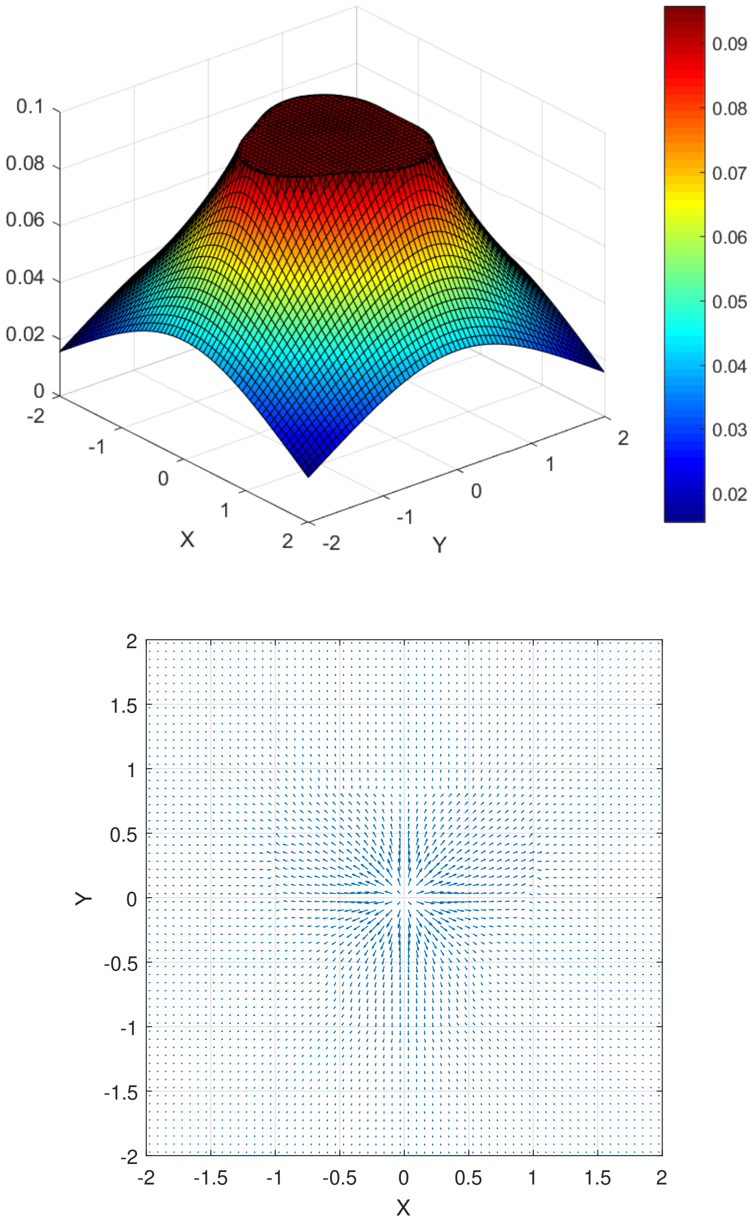


Figure 5.6: Pressure and Velocity field for r_1 at time t=0

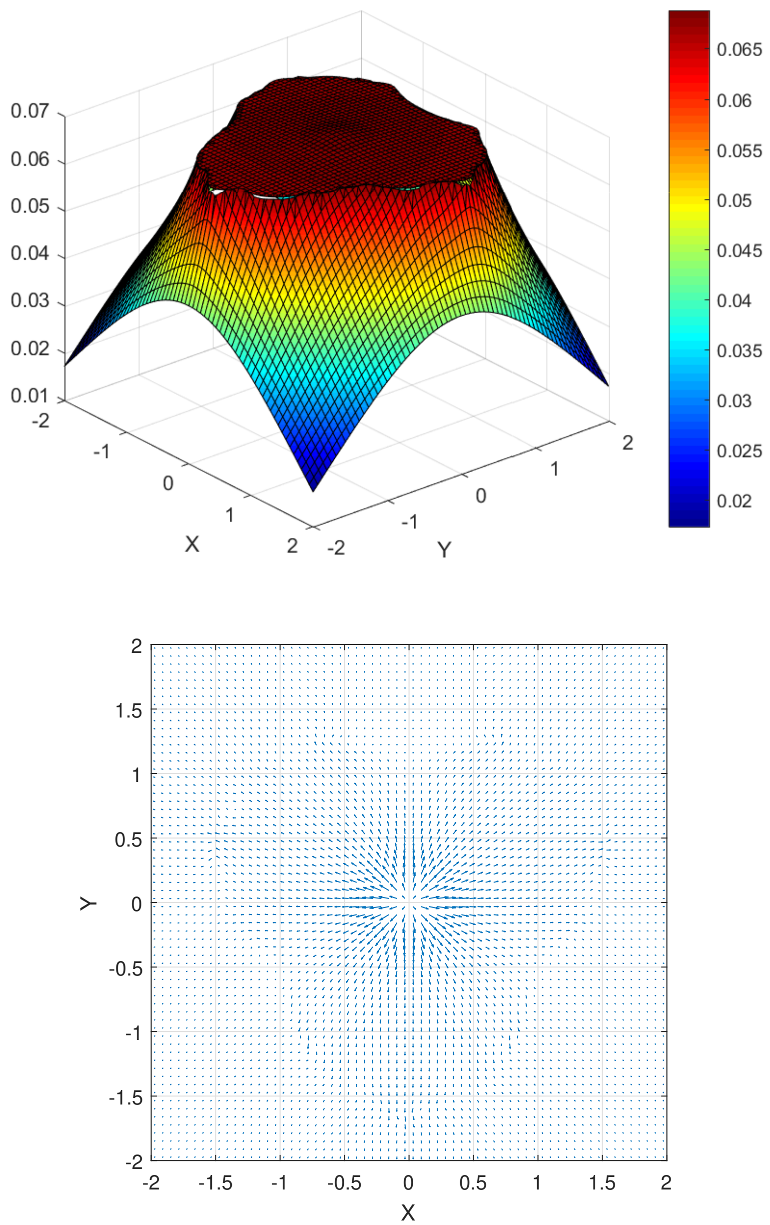


Figure 5.7: Pressure and Velocity field for r_1 at time t=8

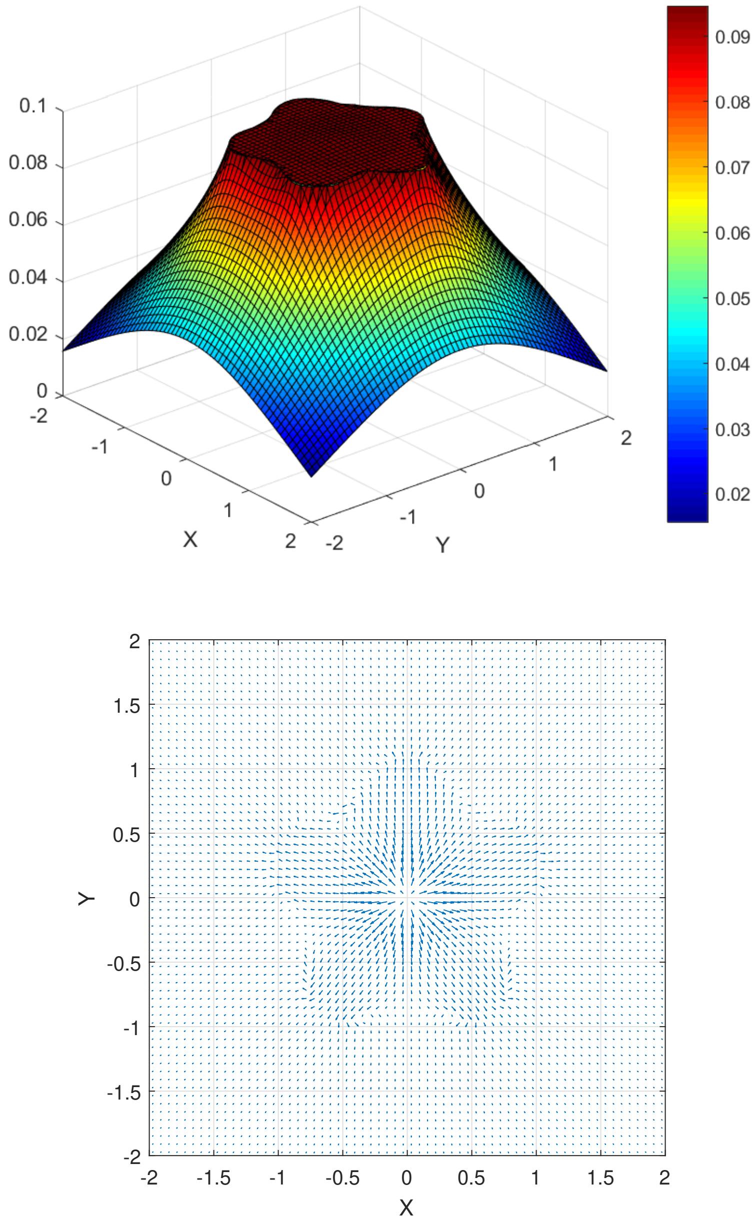


Figure 5.8: Pressure and Velocity field for r_2 at time t=0

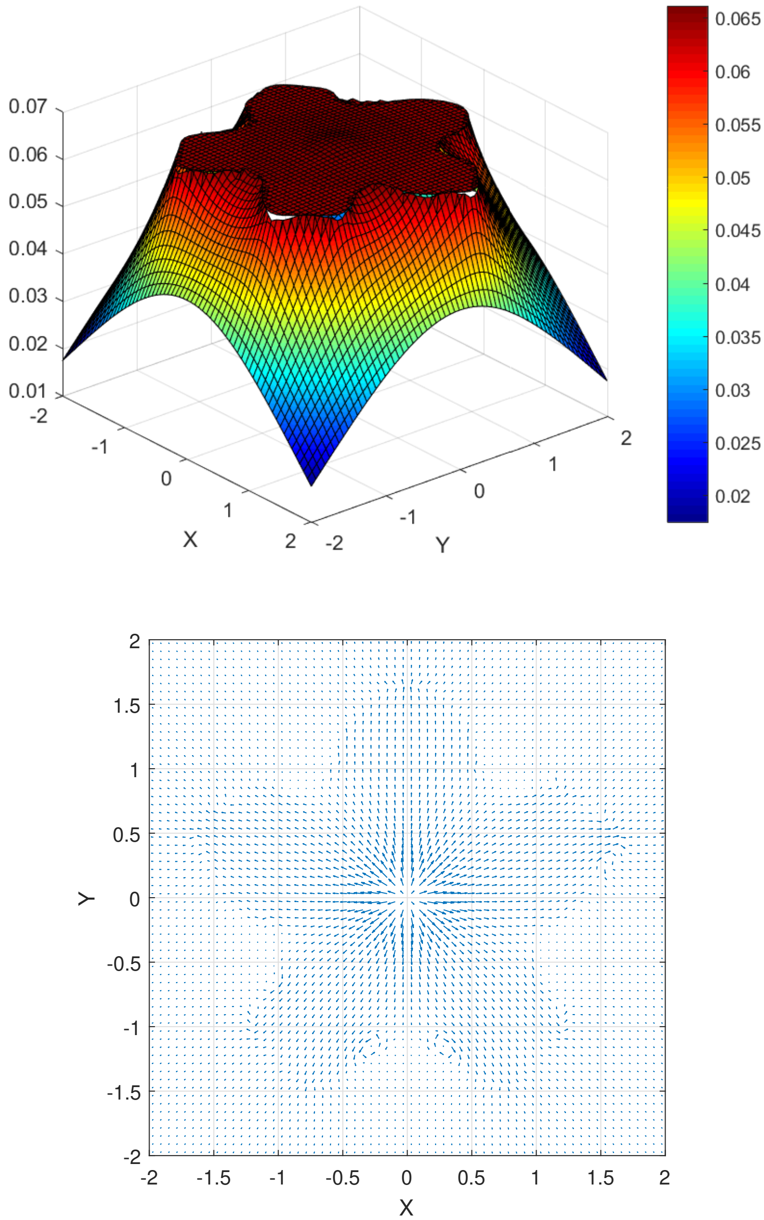


Figure 5.9: Pressure and Velocity field for r_2 at time t=8

5.3 Example3 : Asymmetric Interfaces

Two initial interfaces are given by

$$r1 = r_0 + 0.05(\sin(2\theta) + \cos(3\theta))$$
$$r2 = r_0 + 0.05(\sin(5\theta) + \cos(3\theta))$$

Parameters are $V_0 = 0.25$, $\beta^+ = 1$, $\beta^- = 100$, $\alpha = 0.3$, $d_0 = 2.1 \cdot 10^{-3}$ and $r_0 = 0.55$. Target time is $T = 8$

The boundary condition is given as

$$p = -\frac{V_0\alpha}{\beta^+}\log(r) + 0.1, \quad \text{on } \partial\Omega,$$

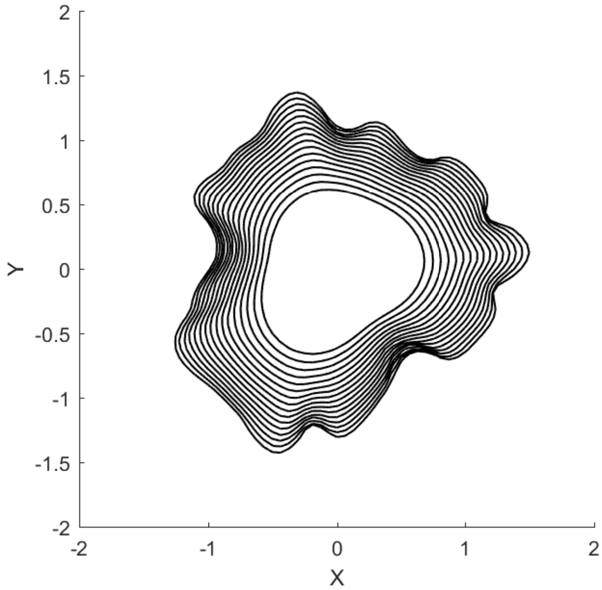


Figure 5.10: Interfaces for r_1

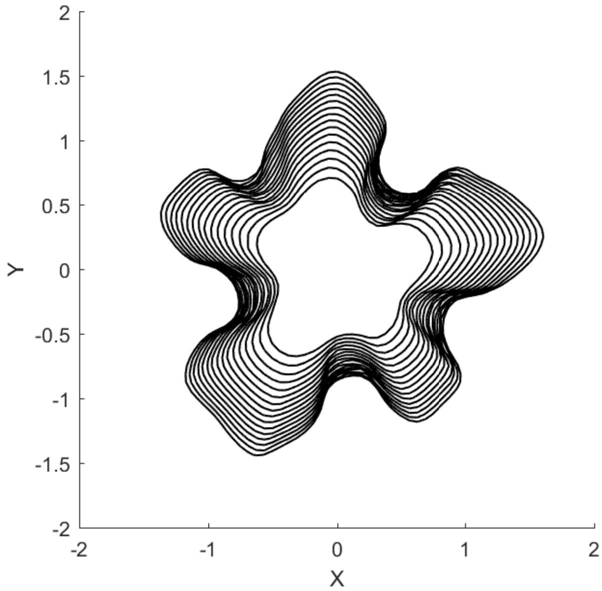


Figure 5.11: Interfaces for r_2

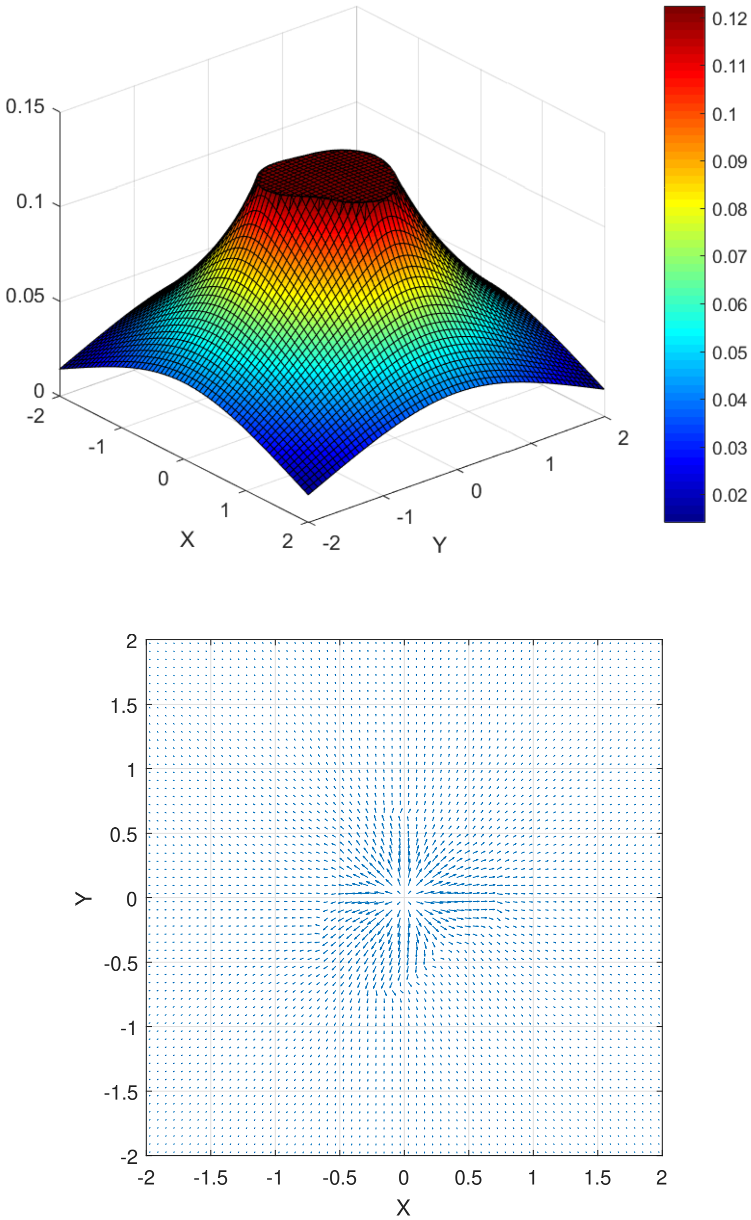


Figure 5.12: Pressure and Velocity field for r_1 at time t=0

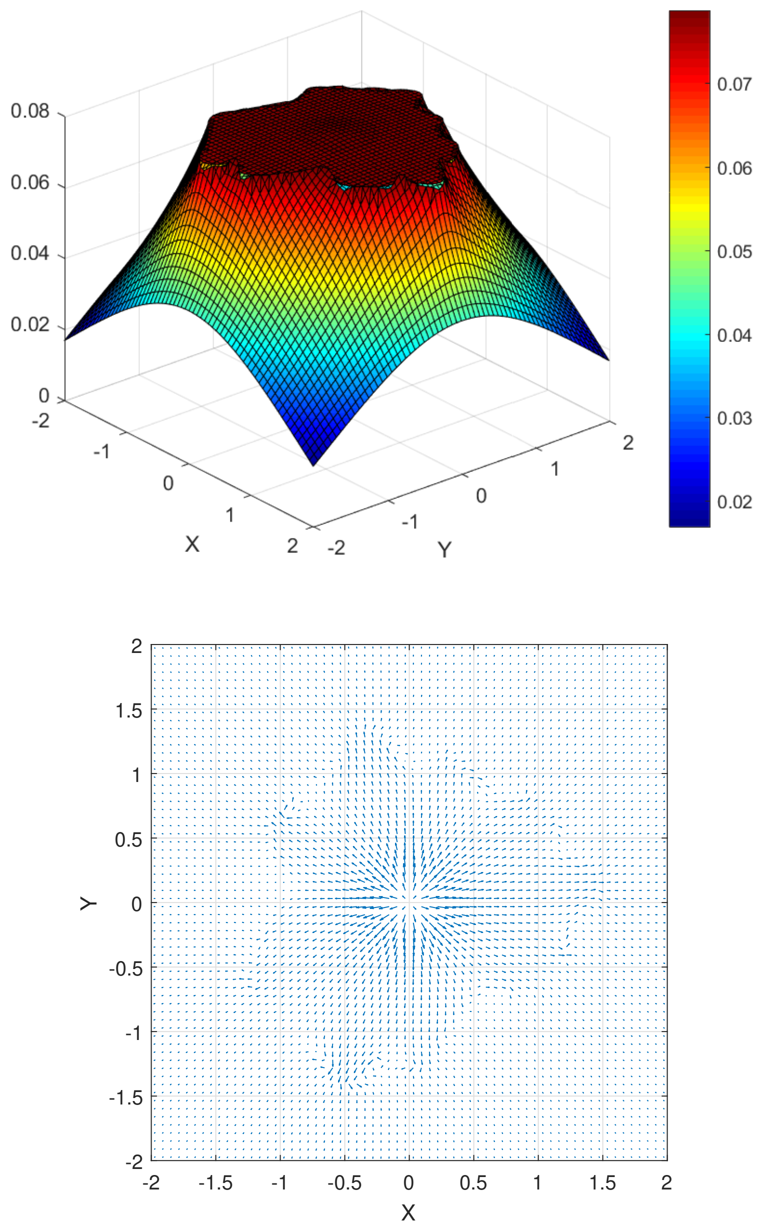
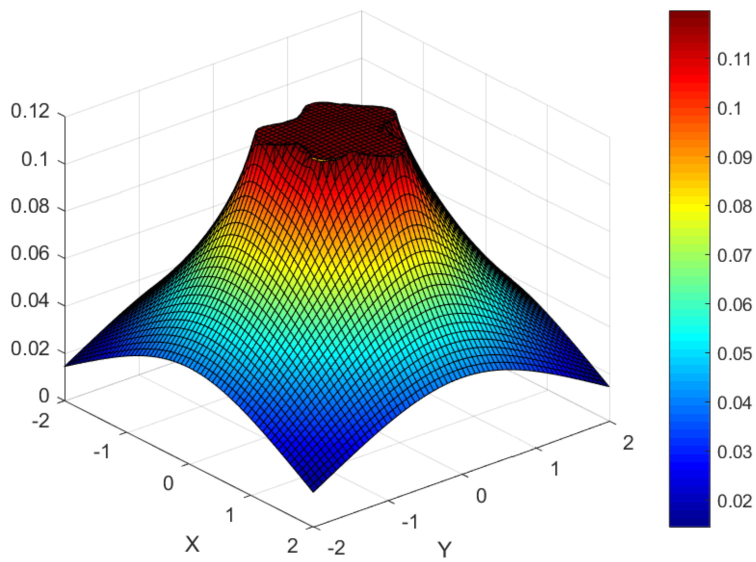


Figure 5.13: Pressure and Velocity field for r_1 at time t=8



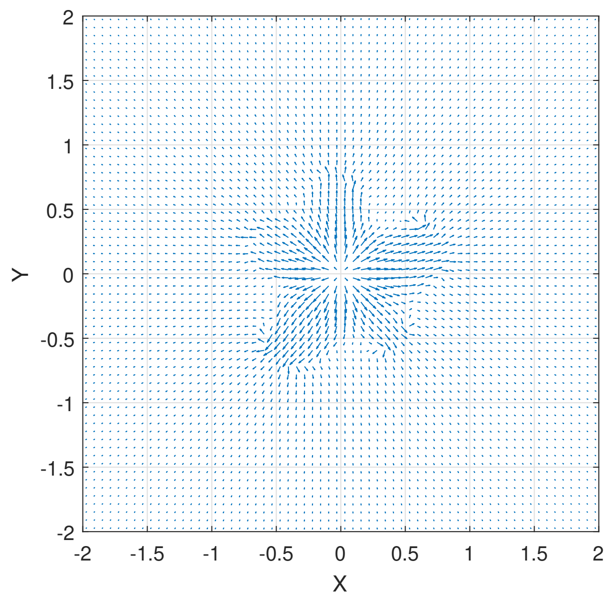


Figure 5.14: Pressure and Velocity field for r_2 at time t=0

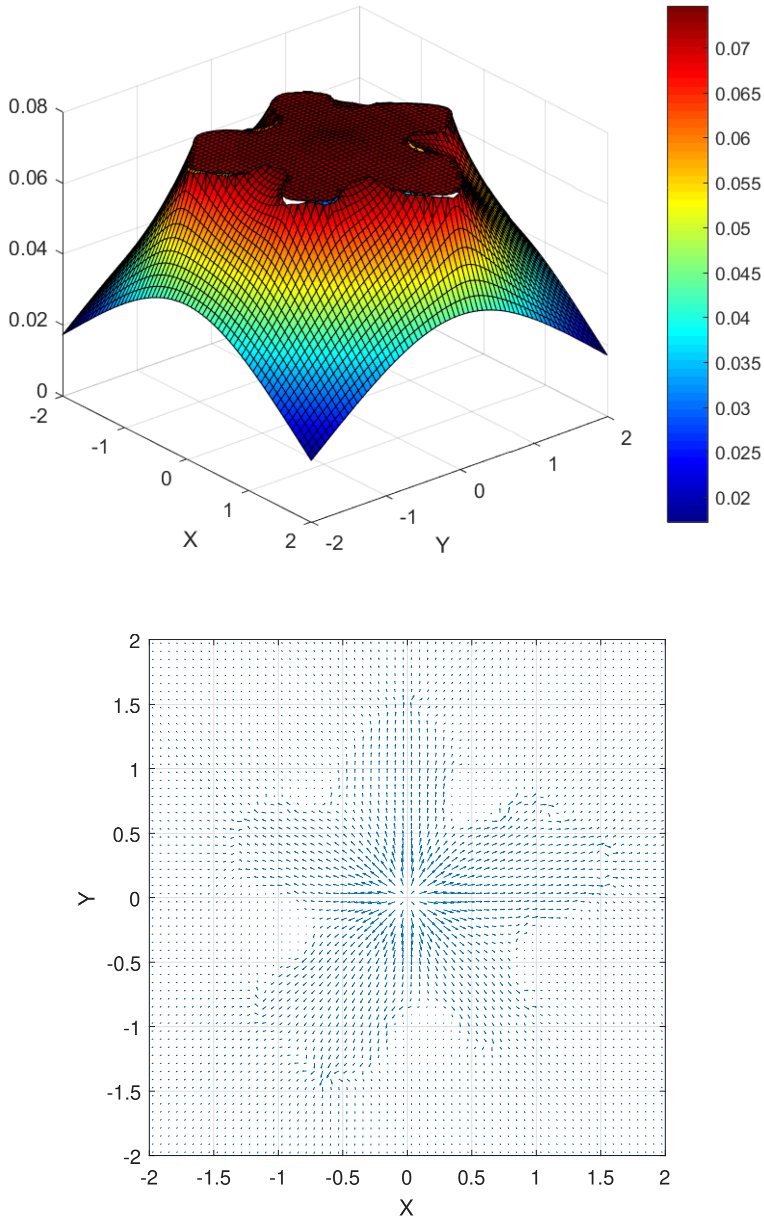


Figure 5.15: Pressure and Velocity field for r_2 at time t=8

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