

# Penrose–Terrell Rotation

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## 1 Introduction

In the sense of Lorentz contraction, it is commonly used that the expression that a moving object appears shortened. However, describing what we actually see is a subtly different problem, since the light travels from the object to the eye takes a nonzero time. Not until 1959, more than 40 years after special relativity had been established, it was published that a moving sphere appears to be also a sphere to any observer.

## 2 Aberration of Light

The direction of a photon ray changes according to the choice of reference frame:

$$\begin{pmatrix} \gamma & -\gamma v & 0 \\ -\gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ p \cos \theta \\ p \sin \theta \end{pmatrix} = \begin{pmatrix} p\gamma(1 - v \cos \theta) \\ p\gamma(-v + \cos \theta) \\ p \sin \theta \end{pmatrix};$$

thus we obtain

$$\tan \theta' = \frac{\sin \theta}{\gamma(-v + \cos \theta)} \Leftrightarrow \tan \frac{\theta'}{2} = \sqrt{\frac{1-v}{1+v}} \tan \frac{\theta}{2}. \quad (1)$$

## 3 Stereographic Projection

The stereographic projection is a conformal projection from the unit sphere  $S^2 : \{p \in \mathbb{R}^3 : \|p\| = 1\}$  to  $\mathbb{R}^2 \cup \{\infty\}$ . The projection of a point  $p$  on  $S^2$  is defined as the intersection of  $z = 0$  and the line through  $n$  and  $p$ , where  $n = (0, 0, 1)$  is the north pole.

It is known that the circle on the sphere not going through  $n$  remains a circle, where going through  $n$  becomes a straight line.

## 4 Penrose–Terrell Rotation

Suppose a sphere  $\mathfrak{S}$  that is at rest in the inertial frame  $\mathcal{S}$ . A stationary observer at the origin of  $\mathcal{S}$ , sees the sphere as a circular outline. In other words, the intersection of

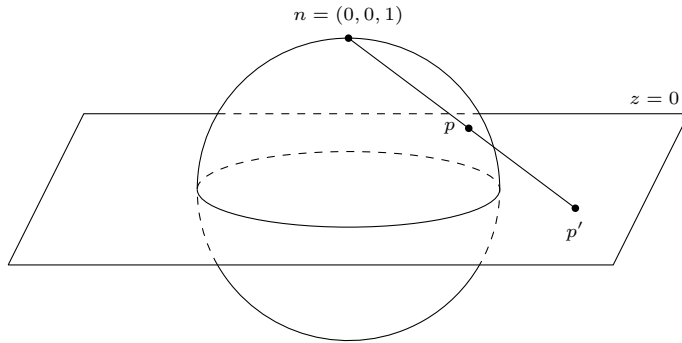


Figure 1: Stereographic projection

a bunch of rays emitted on  $\mathfrak{S}$  toward the origin and the unit sphere located at the origin draws a circle, denoted by  $C$ .

Consider another inertial frame  $\mathcal{S}'$  that shares the origin and moves in a relative velocity  $\mathbf{v}$ . When we move the frame from  $\mathcal{S}$  to  $\mathcal{S}'$ , the stereographically-projected image of  $C$ , with respect to the plane perpendicular to  $\mathbf{v}$ , undergoes a homothetic transformation by eq. (1). Thus its inverse image, the image of  $\mathfrak{S}$  the observer who rests at  $\mathcal{S}'$  sees, is also a circle.

Although the sphere  $\mathfrak{S}$  appears to an observer in  $\mathcal{S}'$  as a circle, it does not mean that both observers agree with the image. The image inside the circular outline will be distorted, and this is why the phenomenon is named rotation.

## References

- [1] R. Penrose. The apparent shape of a relativistically moving sphere. *Mathematical Proceedings of the Cambridge Philosophical Society*, 55(1):137–139, 1959.
- [2] Lars Ahlfors. *Complex Analysis*. McGraw-Hill, 1996.