6 Roots and Weights

6.1 Weights

Definition 6.1 (Cartan subalgebras)**.** A subset of commuting hermitian generators which is as large as possible is called a **Cartan subalgebra**.

Definition 6.2 (Cartan generators)**.** The **Cartan generators** are the elements of a Cartan subalgebra of a Lie algebra.

Remark 6.1. Cartan generators can be simultaneously diagonalised.

Definition 6.3 (rank)**.** The **rank** of the algebra is the number of independent Cartan generators.

Remark 6.2. For all $i, j = 1, 2, \cdots, m (= \operatorname{rank}\mathfrak{g})$,

$$H_i = H_i^\dagger \tag{6.1}$$

$$[H_i, H_j] = 0 \tag{6.2}$$

Remark 6.3. We can choose a basis in which the generators satisfy

$$\operatorname{tr}(H_i H_j) = \lambda_D \delta_{ij} \tag{6.3}$$

for $i, j = 1$ to m, where k_D depends on the representation D and the normalisation of the generators (see eq. (2.25)).

Definition 6.4 (weights)**.** The eigenvalues μ_i of Cartan generators are called **weights**:

$$H_i \, |\mu, x, D\rangle = \mu_i \, |\mu, x, D\rangle \,. \tag{6.4}$$

The m-component vector with components μ_i is the **weight vector**.

6.2 More on the Adjoint Representation

Claim 6.1. *Each generator X_a corresponds to a basis vector $|X_a\rangle$ in the inner product space on which the adjoint representation, defined by eq.* (2.15)*, acts.*

Proof. The inner product is defined as

$$\langle X_a | X_b \rangle = \lambda^{-1} \operatorname{tr}\big(X_a^\dagger X_b\big), \tag{6.5}$$

where $\lambda = k_{D_{\text{adj}}}$. □

Claim 6.2.

$$X_a \, |X_b\rangle = |[X_a, X_b]\rangle \tag{6.6}$$

Proof.

$$X_a \, |X_b\rangle = |X_c\rangle\langle X_c| X_a \, |X_b\rangle = |X_c\rangle \, [X_a]_{cb} = |i f_{abc} X_c\rangle = |[X_a, X_b]\rangle \tag{6.7}$$

□

6.3 Roots

Definition 6.5 (roots)**.** The **roots** are the weights of the adjoint representation.

Claim 6.3. *The states corresponding to the Cartan generators have zero weight vectors.*

Proof.

$$H_i \left|H_j\right\rangle = \left|[H_i, H_j]\right\rangle = 0 \tag{6.8}$$

□

Claim 6.4. *The Cartan states are orthonormal.*

Proof.

$$\langle H_i | H_j \rangle = \lambda^{-1} \operatorname{tr}(H_i H_j) = \delta_{ij} \tag{6.9}$$

□

Claim 6.5. *For a state E_α of the adjoint representation that does not correspond to the Cartan generators,*

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{6.10}$$

and

$$E_\alpha{}^\dagger = E_{-\alpha}, \tag{6.11}$$

where α is the non-zero weight vector.

Proof.

$$\left|[H_i, E_\alpha]\right\rangle = H_i \left|E_\alpha\right\rangle = \alpha_i \left|E_\alpha\right\rangle = \left|\alpha_i E_\alpha\right\rangle \tag{6.12}$$

and

$$[H_i, E_\alpha{}^\dagger] = -[H_i, E_\alpha]^\dagger = -\alpha_i E_\alpha{}^\dagger \tag{6.13}$$

□

Example 6.1.

$$\left(J^+\right)^\dagger = J^- \tag{6.14}$$

Claim 6.6. *States corresponding to different weights are orthogonal.*

Proof. They have different eigenvalues of at least one of the Cartan generators.

$$\langle E_\alpha | H_i | E_\beta \rangle = \alpha_i \langle E_\alpha | E_\beta \rangle = \beta_i \langle E_\alpha | E_\beta \rangle \tag{6.15}$$

implies the claim.

We choose the normalisation

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \operatorname{tr}\left(E_\alpha{}^\dagger E_\beta\right) = \delta_{\alpha\beta}. \tag{6.16}$$

□

6.4 Raising and Lowering

Claim 6.7. *The $E_{\pm\alpha}$ are raising and lowering operators for the weights.*

Proof.

$$H_i E_{\pm\alpha}|\mu, D\rangle = \underbrace{[H_i, E_{\pm\alpha}]}_{=\pm\alpha_i E_{\pm\alpha}}|\mu, D\rangle + E_{\pm\alpha} H_i |\mu, D\rangle = (\pm\alpha + \mu) E_{\pm\alpha}|\mu, D\rangle \tag{6.17}$$

□

Claim 6.8.

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H = \alpha_i H_i \tag{6.18}$$

Proof. Since the state $E_\alpha|E_{-\alpha}\rangle$ has weight $\alpha - \alpha = 0$, it can be written as a linear combination of the Cartan states:

$$\beta_i |H_i\rangle = E_\alpha |E_{-\alpha}\rangle = |[E_\alpha, E_{-\alpha}]\rangle. \tag{6.19}$$

Direct computation of β_i yields

$$\begin{aligned}\beta_i = \langle H_i|[E_\alpha, E_{-\alpha}]\rangle &= \lambda^{-1} \operatorname{tr}(H_i[E_\alpha, E_{-\alpha}]) \\ &= \lambda^{-1} \operatorname{tr}(E_{-\alpha}[H_i, E_\alpha]) \\ &= \lambda^{-1} \operatorname{tr}(E_{-\alpha}\alpha_i E_\alpha) \\ &= \alpha_i.\end{aligned} \tag{6.20}$$

□

Example 6.2.

$$[J^+, J^-] = J_3 \tag{6.21}$$

6.5 Lots of $SU(2)$s

For each non-zero pair of root vectors $\pm\alpha$, there is an $SU(2)$ subalgebra of the group, with generators

$$E^\pm \equiv |\alpha|^{-1} E_{\pm\alpha} \tag{6.22}$$

$$E_3 \equiv |\alpha|^{-2} \alpha \cdot H \tag{6.23}$$

Claim 6.9. *Remark 3.2 also holds for $E^\pm$ and E_3.*

Proof.

$$[E_3, E^\pm] = |\alpha|^{-3}[\alpha \cdot H, E_{\pm\alpha}] = |\alpha|^{-3}\alpha \cdot (\pm\alpha E_{\pm\alpha}) = \pm E^\pm \tag{6.24}$$

$$[E^+, E^-] = |\alpha|^{-2}[E_\alpha, E_{-\alpha}] = |\alpha|^{-2}\alpha \cdot H = E_3 \tag{6.25}$$

□

Claim 6.10. *A root vector corresponds to the unique generator.*

Proof. Suppose the contrary: E_α and E'_α both correspond to α. They can be chosen to be orthogonal. That is,

$$\langle E_\alpha | E'_\alpha \rangle = \lambda^{-1} \operatorname{tr}\left({E_\alpha}^\dagger E'_\alpha \right) = \lambda^{-1} \operatorname{tr}(E_{-\alpha} E'_\alpha). \tag{6.26}$$

By lowering $|E'_\alpha\rangle$, we have $E^- |E'_\alpha\rangle = \beta_i |H_i\rangle$. But,

$$\begin{aligned} \beta_i = \langle H_i | E^- | E'_\alpha \rangle &= \lambda^{-1} \operatorname{tr}\big(H_i [E^-, E'_\alpha]\big) \\ &= -\lambda^{-1} \operatorname{tr}\big(E^- [H_i, E'_\alpha]\big) \\ &= -\alpha_i \lambda^{-1} \operatorname{tr}\big(E'_\alpha E^-\big) = 0, \end{aligned} \tag{6.27}$$

which implies that such an E'_α cannot exist. □

Claim 6.11. *If α is a root, then no non-zero multiple of α can be a root except for $\pm\alpha$.*

Proof. Suppose $k\alpha$ a root for $k \neq \pm 1$. Under the $SU(2)$ subalgebra corresponding to α, $\mathfrak{su}_\alpha(2)$,

$$[E_3, E_{k\alpha}] = k E_{k\alpha}, \tag{6.28}$$

so its E_3 eigenvalue is k. Since it is a magnetic quantum number of $SU(2)$, k must be an integer or a half-integer.

Case 1: k is an integer other than ± 1.

$E_{k\alpha}$ sits in a spin-j representation where $j \geq |k|$. Since an $SU(2)$ multiplet with $m = k$ also contains the state with $m = 1$, which is another root-α state.

Case 2: k is a half-integer.

Likewise, there will be a state with the E_3 eigenvalue $1/2$, which corresponds to a vector with root $\alpha/2$. This is clearly a contradiction. □

6.6 Watch Carefully—This is Important!

Remark 6.4.

$$E_3 |\mu, x, D\rangle = \frac{\alpha \cdot \mu}{\alpha^2} |\mu, x, D\rangle, \tag{6.29}$$

where

$$\frac{2\alpha \cdot \mu}{\alpha^2} \text{ is an integer.} \tag{6.30}$$

Remark 6.5. The general state $|\mu, x, D\rangle$ can always be written as a linear combination of states transforming according to definite representations of the $SU(2)$.

Theorem 6.1 (the master formula)**.**

$$\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2}(p - q), \tag{6.31}$$

where p and q are the maximum possible numbers of raising and lowering, respectively.

Proof. Since

$$\left(E^{+}\right)^{p}|\mu, x, D\rangle \neq 0 \text{ whereas } \left(E^{+}\right)^{p+1}|\mu, x, D\rangle = 0, \tag{6.32}$$

we have

$$\frac{\alpha\cdot(\mu+p\alpha)}{\alpha^2} = \frac{\alpha\cdot\mu}{\alpha^2} + p = j. \tag{6.33}$$

Likewise,

$$\frac{\alpha\cdot(\mu-q\alpha)}{\alpha^2} = \frac{\alpha\cdot\mu}{\alpha^2} - q = -j. \tag{6.34}$$

Adding eq. (6.33) and eq. (6.34), we get the formula. □

Claim 6.12. *For anly pair of roots α and β, the only possible cosine-squared values of the angles $\cos^2\theta_{\alpha\beta}$ between them are 0, 1/4, 2/4, 3/4, or 4/4.*

Proof.

$$\cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4}. \tag{6.35}$$

□

7 $SU(3)$

7.1 The Gell-Mann Matrices

Definition 7.1 (Gell-Mann matrices)**.** The standard basis for the hermitian 3×3 matrices in terms of a generalisation of the Pauli matrices, called the Gell-Mann matrices:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, & \lambda_2 &= \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},\\
\lambda_3 &= \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, & \lambda_4 &= \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\\
\lambda_5 &= \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, & \lambda_6 &= \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\\
\lambda_7 &= \begin{pmatrix}0&0&0\\0&0&-i\\0&-i&0\end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.
\end{aligned} \tag{7.1}$$

Remark 7.1. For $a = 1$ to 3,

$$\lambda_a = \begin{pmatrix}\sigma_a & 0\\ 0 & 0\end{pmatrix}, \tag{7.2}$$

where σ_a are the Pauli matrices.

Remark 7.2. The $SU(3)$ generators are conventionally defined by

$$T_a = \frac{1}{2}\lambda_a \tag{7.3}$$

in order to satisfy

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}. \tag{7.4}$$

Remark 7.3. T_a for $a = 1$ to 3 generate an $SU(2)$ subgroup of $SU(3)$.

Remark 7.4. In the chosen basis, $H_1 = T_3$ and $H_2 = T_8$ form the Cartan subalgebra.

7.2 Weights and Roots of $SU(3)$

Remark 7.5. The eigenvectors and the associated weights are

$$\begin{aligned} \begin{pmatrix}1 & 0 & 0\end{pmatrix}^{\mathrm{T}} &\longleftrightarrow (1/2, \sqrt{3}/6), \\ \begin{pmatrix}0 & 1 & 0\end{pmatrix}^{\mathrm{T}} &\longleftrightarrow (-1/2, \sqrt{3}/6), \\ \begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathrm{T}} &\longleftrightarrow (0, -\sqrt{3}/3). \end{aligned} \tag{7.5}$$

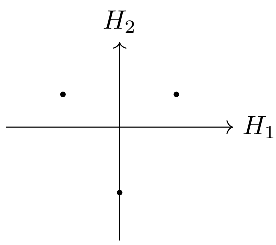

Figure 5: The weights of eq. (7.5).

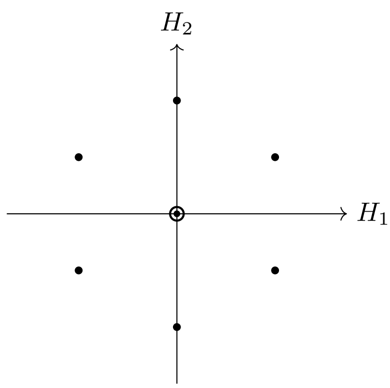

Figure 6: The roots of the Lie algebra $\mathfrak{su}(3)$.