Penrose–Terrell Rotation

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1 Introduction

In terms of the Lorentz contraction, it is often said that moving objects appear shortened. However, describing what we actually see is a subtly different problem, since the light travels from the object to the eye takes a nonzero time. Not until 1959, more than 40 years after special relativity had been established, it was published that a moving sphere appears to be also a sphere to any observer.

2 Aberration of Light

The direction of a photon ray changes according to the choice of reference frame:

$$\begin{pmatrix} \gamma & -\gamma v & 0 \\ -\gamma v & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ p\cos\theta \\ p\sin\theta \end{pmatrix} = \begin{pmatrix} p\gamma(1 - v\cos\theta) \\ p\gamma(-v + \cos\theta) \\ p\sin\theta \end{pmatrix}.$$

Thus we obtain the formula for relativistic aberration:

$$\tan\theta' = \frac{\sin\theta}{\gamma(-v + \cos\theta)} \Leftrightarrow \tan\frac{\theta'}{2} = \sqrt{\frac{1-v}{1+v}}\tan\frac{\theta}{2}. \tag{1}$$

3 Stereographic Projection

The stereographic projection is a conformal projection from the unit sphere $S^2 : \{p \in \mathbb{R}^3 : \|p\| = 1\}$ to $\mathbb{R}^2 \cup \{\infty\}$. The projection of a point p on S^2 is defined as the intersection of $z = 0$ and the line through n and p, where $n = (0, 0, 1)$ is the north pole.

It is known that the circle on the sphere not going through n remains a circle, where going through n becomes a straight line.

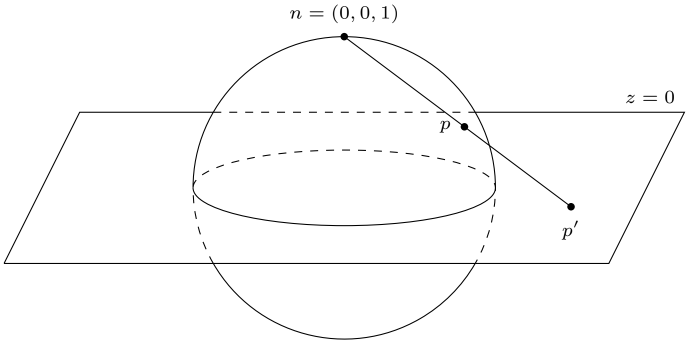

Figure 1: Stereographic projection

4 Penrose–Terrell Rotation

Suppose a sphere $\mathfrak{S}$ that is at rest in the inertial frame $\mathcal{S}$. A stationary observer at the origin of $\mathcal{S}$, sees the sphere as a circular outline. In other words, the intersection of a bunch of rays emitted on $\mathfrak{S}$ toward the origin and the unit sphere located at the origin draws a circle, denoted by C.

Consider another inertial frame $\mathcal{S}'$ that shares the origin and moves in a relative velocity $\mathbf{v}$. When we move the frame from $\mathcal{S}$ to $\mathcal{S}'$, the stereographically-projected image of C, with respect to the plane perpendicular to $\mathbf{v}$, undergoes a homothetic transformation by eq. (1). Thus its inverse image, the image of $\mathfrak{S}$ the observer who rests at $\mathcal{S}'$ sees, is also a circle.

Although the sphere $\mathfrak{S}$ appears to an observer in $\mathcal{S}'$ as a circle, it does not mean that both observers agree with the image. The image inside the circular outline will be distorted, and this is why the phenomenon is named rotation.

References

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[2] Lars Ahlfors. *Complex Analysis*. McGraw-Hill, 1996.