1 Mathematical Introduction

1.1 Hilbert Space

Inner Product

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)\mathrm{d}x \tag{1}$$

Orthogonality Relation

$$\langle i|j\rangle = \delta(i-j) \tag{2}$$

Completeness Relation

$$\int_{-\infty}^{\infty} |x\rangle\langle x|\mathrm{d}x = \mathbb{I} \tag{3}$$

1.2 Dirac Delta Function

By inserting the identity at the adequate place:

$$\int_{-\infty}^{\infty} \langle x|x'\rangle\langle x'|f\rangle\mathrm{d}x' = \langle x|\mathbb{I}|f\rangle = \langle x|f\rangle,$$

we obtain a Dirac delta function

$$\delta(x-x') = \langle x|x'\rangle \tag{4}$$

that gives us the normalisation of the basis. The Dirac delta function is defined with the properties:

$$\delta(x-x') = 0, \quad x \neq x'$$
$$\int_{-\infty}^{\infty} \delta(x-x')\mathrm{d}x' = 1, \quad a < x < b.$$

Integrating by parts, we can write the derivatives of the Dirac delta:

$$\int_{-\infty}^{\infty} \delta'(x-x')f(x')\mathrm{d}x' = -\,\delta(x-x')f(x')\big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \delta(x-x')f'(x')\mathrm{d}x' = f'(x);$$

$$\delta'(x-x') = \delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x'}. \tag{5}$$

The Fourier transform of a function $f(x)$ is given as

$$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x)\mathrm{d}x.$$

Feeding the inverse into $f(x)$, the expression becomes

$$f(x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx'}f(k)\mathrm{d}k = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{ik(x'-x)}\mathrm{d}k\right]f(x)\mathrm{d}x.$$

Thus we get the Fourier transform of Dirac delta:

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\mathrm{d}k. \tag{6}$$

1.3 Differentiation Operator

1.3.1 Hermitianisation

The operator D such that

$$D|f\rangle = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right\rangle \tag{7}$$

is called the differentiation operator. We can rewrite its action using the identity:

$$\int \langle x|D|x'\rangle\langle x'|f\rangle \mathrm{d}x' = \frac{\mathrm{d}f}{\mathrm{d}x}.$$

Thus, we obtain

$$D_{xx'} = \langle x|D|x'\rangle = \delta(x-x')\frac{\mathrm{d}}{\mathrm{d}x'} = \delta'(x-x').$$

The differentiation operator D itself is not a Hermitian operator but

$$K = -iD \tag{8}$$

could be. Unlike in finite dimensions, the Hermicity of an operator is not guaranteed by the self-adjoint property of components. We have to check whether the definition:

$$\langle g|K|f\rangle = \langle f|K|g\rangle^*. \tag{9}$$

Since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle g|x\rangle\langle x|K|x'\rangle\langle x'|f\rangle \mathrm{d}x\mathrm{d}x' = \int_{-\infty}^{\infty} g^*(x)\left(-i\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)\mathrm{d}x,$$

the Hermicity condition holds if and only if

$$-i\int_{-\infty}^{\infty} g^*(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\mathrm{d}x = i\int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}g^*(x)}{\mathrm{d}x}\mathrm{d}x.$$

Using integration by parts we can conclude K is Hermitian if and only if the boundary term

$$ig^*(x)f(x)\Big|_{-\infty}^{\infty}$$

vanishes.

1.3.2 Eigenket of Differential Operator

Consider an eigenket $|k\rangle$ of operator K corresponds to eigenvalue k. Taking bra to the eigenequation, we get

$$\langle x|K|k\rangle = k\langle x|k\rangle.$$

We can rewrite the eigenequation by expanding the identity:

$$k\psi_k(x) = k\langle x|k\rangle = \langle x|K|k\rangle = \int \langle x|K|x'\rangle\langle x'|k\rangle \mathrm{d}x' = -i\frac{\mathrm{d}}{\mathrm{d}x}\psi_k(x).$$

Thus the eigenkets, the solutions to the above differential equation is

$$\langle x|k\rangle = \psi_k(x) = \frac{1}{\sqrt{2\pi}}e^{ikx}, \tag{10}$$

where the constant $1/\sqrt{2\pi}$ was set to normalise the solution; $\langle k|k'\rangle = \delta(k-k')$. Note that k must be a real number to satisfy the boundary condition.

1.4 Relating Two Basis

The image of a ket $|f\rangle$ in the X basis is an ordinary function:

$$f(x) = \langle x|f\rangle.$$

By expanding the identity at an appropriate place, we obtain

$$f(x) = \langle x|f\rangle = \int_{-\infty}^{\infty} \langle x|k\rangle\langle k|f\rangle \mathrm{d}k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(k)\mathrm{d}k; \tag{11}$$

the Fourier transform of $f(x)$, and vice versa. Thus, in the sense of change of basis, the Fourier transform is just a passage from bases $|x\rangle$ and $|k\rangle$.

In the previous section, we investigated the action of K in the X basis. The action of X is also clear:

$$\langle x|(X|f\rangle) = \int_{-\infty}^{\infty} \langle x|X|x'\rangle\langle x'|f\rangle \mathrm{d}x' = \int_{-\infty}^{\infty} x'\delta(x-x')f(x')\mathrm{d}x' = xf(x).$$

Using this result, we can calculate matrix entries of X in the K basis:

$$\langle k|X|k'\rangle = \int_{-\infty}^{\infty} \langle k|x\rangle\langle x|X|k'\rangle \mathrm{d}x = i\frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k'-k)x}\mathrm{d}x\right] = i\delta'(k-k').$$

Thus, we obtain the canonical commute relation (in the Plank units):

$$[X, K] = i\mathbb{I}. \tag{12}$$

1.5 Time-Evolution Operator

Suppose a differential equation

$$|\ddot{x}(t)\rangle = \Omega|x(t)\rangle \tag{13}$$

describing a linear oscillating system where Ω is a Hermitian operator. It is known that the eigenvectors of Hermitian operators span the whole space. Therefore the general solution can be written as

$$|x(t)\rangle = \int_0^{\infty} |\omega\rangle c_\omega(t)\mathrm{d}(\omega).$$

where $|\omega\rangle$ is the eigenvector corresponds to eigenvalue $-\omega^2$. Then the equation becomes

$$|0\rangle = \int_0^{\infty} |\omega\rangle \left(\ddot{c}_\omega(t) + \omega^2 c_\omega(t)\right) \mathrm{d}\omega.$$

If we assume that initial velocities vanish, the general solution takes the form

$$|x(t)\rangle = \int_0^\infty |\omega\rangle c_\omega(0) \cos\omega t \mathrm{d}\omega = \int_0^\infty |\omega\rangle\langle\omega|x(0)\rangle \cos\omega t \mathrm{d}\omega \tag{14}$$

the last equating is guaranteed by the orthogonality of eigenvectors. Consequently, we can introduce a unitary operator

$$U(t) = \int_0^\infty |\omega\rangle\langle\omega| \cos\omega t \mathrm{d}\omega,$$

thus we can write the general solution using $U(t)$:

$$|x(t)\rangle = U(t)\,|x(0)\rangle\,. \tag{15}$$

1.6 Theorems

Theorem 1. Any vector $|V\rangle$ in an n-dimensional space can be written as a linear combination of linearly independent vectors $|1\rangle, \cdots, |n\rangle$.

Theorem 2. The expansion

$$|V\rangle = \sum_{i=0}^{n} v_i\,|i\rangle \tag{16}$$

is unique.

Theorem 3 (Gram–Schmidt). Given a linearly independent basis we can form linear combinations of the basis vectors to obtain an orthonormal basis.

Theorem 4. The dimensionality of a space equals $n_\perp$, the maximum number of mutually orthogonal vectors in it.

Theorem 5 (Cauchy–Schwartz Inequality).

$$|\langle V|W\rangle| \leq \|V\|\,\|W\| \tag{17}$$

Theorem 6 (Triangle Inequality).

$$\|V+W\| \leq \|V\| + \|W\| \tag{18}$$

Theorem 7. Unitary operators preserve the inner product between the vectors they act on.

Theorem 8. If one treats the columns of an $n \times n$ unitary matrix as components of n vectors, these vectors are orthonormal. In the same way, the rows may be interpreted as components of n orthonormal vectors.

Theorem 9. The eigenvalues of a Hermitian operator are real.

Theorem 10. Hermitian operators are orthogonally diagonalisable.

Theorem 11. The eigenvalues of a unitary operator are complex numbers of the unit modulus.

Theorem 12. The eigenvectors of a unitary operator are mutually orthogonal. (We assume there is no degeneracy.)

Theorem 13. If Ω and Λ are two commuting Hermitian operators, there exists (at least) a basis of common eigenvectors that diagonalises them both.