Part I
Manifolds

1 Elementary Topology

1.1 Topological Spaces

Topology

For a collection of points X, $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfies

1. $\emptyset$, $X \in \mathcal{T}$
2. $\bigcup_{i \in I} \mathcal{O}_i \in \mathcal{T}$ where $\mathcal{O}_i \in \mathcal{T}$
3. $\bigcap_{i \in I} \mathcal{O}_i \in \mathcal{T}$ where $\mathcal{O}_i \in \mathcal{T}$ and I is finite

called a **topology** on X. A set X endowed with a topology $\mathcal{T}$ is called **topological space**.

Open Sets

Elements of $\mathcal{T}$ are called **open sets**.

1.2 Neighbourhoods and Hausdorff Spaces

Neighbourhood

For a point $P \in X$, a subset $N \subseteq \mathcal{T}$ is called **neighbourhood** of P, if there exists an open set $\mathcal{O}$ such that $P \in \mathcal{O} \subseteq N$.

Hausdorff Space

For arbitrary points p_1, p_2 in topological space $(X, \mathcal{T}$, if there exist neighbourhoods $N_1 \ni p_1$ and $N_2 \ni p_2$ such that $N_1 \cap N_2 = \emptyset$, the space is called **Hausdorff space**.

1.3 Homeomorphism

Continuity of Maps

Suppose a map $f : X_1 \to X_2$ between topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. For an arbitrary open set $\mathcal{O} \in X_2$, if $f^{-1}(\mathcal{O}) \in \mathcal{T}_1$, f is said to be **continuous**.

Homeomorphism

A map $f : X_1 \to X_2$ between topological spaces X_1 and X_2 is called **homeomorphism** if it satisfies

1. f is bijective
2. f is continuous
3. Inverse of f is continuous

Topological spaces X_1 and X_2 are said to be **homeomorphic** if there exists homeomorphism between the spaces.

2 Manifolds

Manifold is a *locally euclidean*(i.e., locally homeomorphic to euclidean space) topological space.

Charts

2.1 Definitions

Let $(\mathcal{M}, \mathcal{T})$ be a topological space. For an open set $\mathcal{O} \in \mathcal{M}$ and a homeomorphism $\phi : \mathcal{O} \to O_i \subseteq \mathbb{R}^n$, a pair $(\mathcal{O}, \phi)$ is called a **chart** on a topological space $\mathcal{M}$.

Topological Manifold

For every open set $\mathcal{O}_i \in \mathcal{T}$, if $\mathcal{M}$ covered with $\{\mathcal{O}_i\}$ and provided a family of pairs $\{(\mathcal{O}_i, \phi_i)\}$ where $\phi_i : \mathcal{O}_i \to O_i$, $\mathcal{M}$ is a homeomorphism from $\mathcal{O}_i$ to $O_i \subseteq \mathbb{R}^n$, the topological space is said to be an n-dimensional **topological manifold**.

2.2 Charts and Coordinates

Charts

The pair (O_i, ϕ_i) is called a **chart**.

Coordinates

For every point $p \in \mathcal{M}$, a tuple $\phi(p) = (x^1(p), x^2(p), \cdots, x^n(p))$ is called a **coordinate** of p.

Atlas

For a manifold $\mathcal{M}$, an atlas $\{(\mathcal{O}_i, \phi_i)\}$ is a collection of whole charts consisting $\mathcal{M}$.

Transition Map

Let $\mathcal{M}$ be a manifold whose atlas is $\{(\mathcal{O}_i, \phi_i)\}$. For open sets $\mathcal{O}_i.\mathcal{O}_j \in M$ if $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$, one can construct coordinate transition from O_i to O_j. A map $\phi_{ij} = \phi_j \circ {\phi_i}^{-1}$ is called **transition map**.

$$\begin{tikzcd}
\mathcal{O}_1 \arrow[d, "\phi_i"] & \mathcal{O}_2 \arrow[d, "\phi_j"] \\
O_1 \arrow[r, "\phi_{ij}"] & O_2
\end{tikzcd}$$

Figure 1: Transition Map

2.3 Differential Structure

Compatiblity of Atlases

If the union of two atlases $\{(\mathcal{O}_i, \phi_i)\} \cup \{(\mathcal{O}'_i, \phi'_i)\}$ is also an atlas on the manifold, the atlases are said to be **compatible**.

Differential Structure

Since the compatibility of atlases is an equivalence equation, it gives an equivalence class, which is called **differential structure**