Part II
Fields on Manifolds

3 Vectors

3.1 Functions and Curves

Function

A smooth map $f : \mathcal{M} \to \mathbb{R}$ is called a **function**. The set of all functions on M is denoted as $\mathfrak{F}(\mathcal{M})$

Curves

A **curve** is the image of a continuous map $\gamma : I \to \mathcal{M}$. For the sake of convenience, we will not distinguish γ and the image of γ.

Simple Curves

If a map γ is injective, the curve is said to be **simple**.

Openness and Closedness of Curves

A curve is called an **open curve** if I is an open interval. Likewise, a curve is said to be **closed** if I is a closed interval.

3.2 Vectors

Tangent Vectors

Suppose a manifold $\mathcal{M}$ and a function $f : \mathcal{M} \to \mathbb{R}$. The directional derivative of f along a curve $\gamma(t)$ that passes through a point p on the point is

$$\left.\frac{df(\gamma(t))}{dt}\right|_p = \left.\frac{\partial(f \circ \phi^{-1})}{\partial x^\mu}\frac{dx^\mu(\gamma(t))}{dt}\right|_p$$

For the sake of simplicity, $\partial f/\partial x^\mu$ is understood as $\partial(f \circ \phi^{-1})/\partial x^\mu$. Such differential operators at a given point p form a vector space $T_p\mathcal{M}$ called **tangent space** at p. Elements of tangent spaces are called **tangent vectors**, or simply just **vectors**.

Coordinate Bases

There are natural basis of a tangent space, that is,

$$X_p = X^\mu \partial_\mu|_p$$

A basis $\{\partial_\mu\}$ is said to be **coordinate basis**.

Vector Fields

A **vector field** X is a smooth map that assigns a tangent vector $X|_p$ to each point $p \in \mathcal{M}$. The set of all vector fields on $\mathcal{M}$ is denoted as $\mathfrak{X}(\mathcal{M})$

4 Tensors

4.1 One-Forms

Cotangent Vectors

The dual space of $T_p\mathcal{M}$ is called the **cotangent space** at p, denoted by $T_p^*\mathcal{M}$. The dual basis $\{dx^\mu\}$ is defined as

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu$$

Elements of the dual space $T_p^*\mathcal{M}$ are called **cotangent vectors**, or simply just **covectors**. In other words, covector$\omega|_p = \omega_\mu dx^\mu|_p$ is a linear map $X^\mu\partial_\mu|_p \mapsto X^\mu\omega_\mu$.

One-Form

A **one-form** ω is a covector field. That is, a smooth map that assigns a one-form $\omega|_p$ to each point $p \in \mathcal{M}$. The set of all one-forms on $\mathcal{M}$ is denoted as $\Omega^1(\mathcal{M})$.

4.2 Tensors

Tensors

A **tensor** of type (p, q) or simply (q, r)-tensor is a multilinear map $T_p^*\mathcal{M}^r \times T_p\mathcal{M}^s \to \mathbb{R}$. In terms of components, a tensor can be written as

$$T = T^{\mu_1\cdots\mu_q}_{\nu_1\cdots\nu_r}\partial_{\mu_1}\cdots\partial_{\mu_q}dx^{\nu_1}\cdots dx^{\nu_1}$$

This reads

$$T^{\mu_1\cdots\mu_q}_{\nu_1\cdots\nu_r} = T(dx^{\mu_1}, \cdots, dx^{\mu_q}, \partial_{\nu_1}, \cdots, \partial_{\nu_r})$$

Tensor Product

Given a (q, r)-tensor T and a (s, t)-tensor S, the **tensor product** of two tensors $T \otimes S$ is a $(q + s, r + t)$-tensor whose components are defined by

$$(T \otimes S)^{\mu_1\cdots\mu_q\xi_1\cdots\xi_s}_{\nu_1\cdots\nu_r\pi_1\cdots\pi_t} = T^{\mu_1\cdots\mu_q}_{\nu_1\cdots\nu_r}S^{\xi_1\cdots\xi_q}_{\pi_1\cdots\pi_r}$$

Contrtaction

Given a (q, r)-tensor T, one can construct $(q - 1, r - 1)$-tensor S by taking s-th $T_p^*\mathcal{M}$ entry of T as a basis vector dx^s, and t-th $T_p\mathcal{M}$ entry as ∂_t. This operation is nothing but taking one of upper and lower indices respectively and replacing them with the same index. That is,

$$S^{\mu_1\cdots\mu_{s-1}\mu_{s+1}\cdots\mu_q}_{\nu_1\cdots\nu_{t-1}\nu_{t+1}\cdots\nu_r} = T^{\mu_1\cdots\mu_{s-1}\xi\mu_{s+1}\cdots\mu_q}_{\nu_1\cdots\nu_{t-1}\xi\nu_{t+1}\cdots\nu_r}$$

This operation is called **contraction**.

Tensor Fields

A (q,r)-**tensor field** is a smooth map that assigns a (q,r)-tensor $T|_p$ to each point $p \in \mathcal{M}$. The set of all (q,r)-tensor fields on $\mathcal{M}$ is denoted as $\mathfrak{T}^q_r(M)$ where the set of all (q,r)-tensor at $p \in \mathcal{M}$ is denoted as $\mathfrak{T}^q_{r,p}(\mathcal{M})$.

5 Lie Derivative

5.1 Pushforward and Pullback

Pushforward of Vector

Let $\varphi : \mathcal{M} \to \mathcal{N}$ be a smooth map between two manifolds. For $f \in \mathfrak{F}(\mathcal{N})$ and $X_p \in T_p\mathcal{M}$, φ naturally induces a map $\varphi_* : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$ is defined by

$$(\varphi_* X_p)_{\varphi(p)}(f) = X_p(f \circ \varphi)$$

Note that the LHS maps a point on $\mathcal{N}$ to $\mathbb{R}$, whereas RHS maps a point on $\mathcal{M}$ to $\mathbb{R}$. $\varphi_* X_p$ is called the **pushforward** of X_p by φ.

Pullback of Covariant Tensors

Just like pushing an object, one can also pull an object by constructing a map $\varphi^* : \mathfrak{T}^0_{r,\varphi(p)}(\mathcal{N}) \to \mathfrak{T}^0_{r,p}(\mathcal{M})$. Precisely, for $T_{\varphi(p)} \in \mathfrak{T}^0_{r,\varphi(p)}(\mathcal{N})$, φ^* is defined as

$$(\varphi^* T_{\varphi(p)})_p(X_1, \cdots, X_r) = T_{\varphi(p)}(\varphi_* X_1, \cdots \varphi_* X_r)$$

where $X_i \in T_p\mathcal{M}$. $\varphi^* T_{\varphi(p)}$ is called the **pullback** of $T_{\varphi(p)}$ by φ. Of course, the pullback of a function $g \in \mathfrak{F}(\mathcal{N})$ is nothing but $\varphi^* f = f \circ \varphi$.

The concept of the pullback can be extended to contravariant tensors naturally if φ is bijective. The pullback of vector $X_{\varphi(p)} \in T_{\varphi(p)}\mathcal{N}$ by φ is given by $(\varphi^* X_{\varphi(p)})_p = (\varphi^{-1})_* X_{\varphi(p)}$.

5.2 Flows

Integral Curves

An **integral curve** $\gamma(t)$ of $X \in \mathfrak{X}(\mathcal{M})$ is the solution of a following ODE

$$\gamma(0) = p_0$$
$$\frac{dx^\mu(t)}{dt} = X^\mu(\gamma(t))$$

for given initial contdition $\gamma(0) = p_0$, where $x^\mu(t)$ is the μth component of $\gamma(t)$.

Flows

Suppose a map $\sigma : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$. If σ satisfies

$$\begin{aligned} \sigma_0(p_0) &= p_0 \\ \frac{d}{dt}\sigma_t^\mu(p) &= X^\mu(\sigma_t(p)) \end{aligned}$$

for all $t, s \in \mathbb{R}$, the map is said to be a **flow** generated by X. Of course, σ^μ denotes the μth coordinate. The uniqueness of OESs guarantees that $\sigma_t(p_0)$ is nothing but γ when $\gamma(0) = p_0$ since $\sigma_t(p)$ satisfies the same ODE.

Exponentation

Since the Taylor series of $f(x)$ is given by

$$f(x+t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d}{dx}\right)^n f(x) t^n$$

, one can use $\exp(t\frac{d}{dx})$ as a shift operator. Thus,

$$\sigma_t(p) = e^{tX} p$$

is reasonable. To be precise, the above expression holds for their coordinates.

5.3 Lie Derivatives

Ordinary differentiation $df/dx = \lim_{t\to 0} (f(x+t) - f(x))/\epsilon$ is not valid in manifolds since differential operators at different points belong to different vector spaces. To differentiate objects in the direction of some vector X, one should pull or push objects.

Lie Derivative of Functions

The Lie derivative of a function $f \in \mathfrak{F}(\mathcal{M})$ with respect to vector field $X \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned} \mathcal{L}_X f &= \lim_{t\to 0} \frac{\sigma_t^* f - f}{t} \\ &= \left.\frac{d(\sigma_t^* f)}{dt}\right|_{t=0} \\ &= \left.\frac{\partial f}{\partial \sigma_t^\mu} \frac{d\sigma_t^\mu}{dt}\right|_{t=0} \end{aligned}$$

Thus, $\mathcal{L}_X f = X(f)$.

Lie Derivative of Vector Fields

The Lie derivative of a vector field $Y \in \mathfrak{X}(\mathcal{M})$ with respect to vector field $X \in \mathfrak{X}(\mathcal{M})$ is defined by

$$\begin{aligned}(\mathcal{L}_X Y)(f) &= \lim_{t\to 0} \frac{\sigma_t^* Y - Y}{t} f \\ &= \lim_{t\to 0} \left\{ \frac{\sigma_t^* Y f - Y f}{t} - \frac{(\sigma_t^* Y)(\sigma_t^* f) - (\sigma_t^* Y) f}{t} \right\} \\ &= (XY - YX) f\end{aligned}$$

where $f \in \mathfrak{F}(\mathcal{M})$. Note that $\sigma_t^* = (\sigma_{-t})_*$ pushes $Y_{\sigma_t(p)}$ to vector in $T_p\mathcal{M}$. There used identity $(\sigma_t^* Y f)_p = (\sigma_{-t*} Y)_{\sigma_t(p)} (\sigma_t^* f)$. It is easy to show since $(Yf \circ \sigma_t)_p = Y_{\sigma_t(p)} f$. Thus, $\mathcal{L}_X Y = [X, Y]$.