Dirichlet and Neumann Green's Function

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1 The Uniqueness Theorem of Poisson's Equation

1.1 Green's Identities

Applying the divergence theorem to a vector field $\phi_1\nabla\phi_2$, we obtain Green's first identity:

$$\int_{\mathcal{V}} (\phi_1\nabla^2\phi_2 + \nabla\phi_1 \cdot \nabla\phi_2)\mathrm{d}^3x = \oint_{\partial\mathcal{V}} \phi_! \nabla\phi_2 \cdot \hat{\mathbf{n}}\mathrm{d}^2x. \tag{1}$$

By interchanging ϕ_1 and ϕ_2 and subtracting from the original eq. (1), Green's second identity, also known as Green's theorem arises:

$$\int_{\mathcal{V}} (\phi_1\nabla^2\phi_2 - \phi_2\nabla^2\phi_1)\mathrm{d}^3x = \oint_{\partial\mathcal{V}} (\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1) \cdot \hat{\mathbf{n}}\mathrm{d}^2x. \tag{2}$$

1.2 Statement of the Theorem

Consider a bounded volume $\mathcal{V} \subseteq \mathbb{R}^3$ and its boundary $\partial\mathcal{V}$. Let $\rho(\mathbf{x})$ be an arbitrary charge density in $\mathcal{V}$. Then, Poisson equation

$$\nabla^2\phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\epsilon_0} \tag{3}$$

has the unique solution ϕ, under either the following two conditions:

1. Dirichlet condition: $\phi = \psi$ on $\partial\mathcal{V}$.
2. Neumann condition: χ on ∂V such that $\displaystyle\int_{\partial V} \chi\mathrm{d}^2x = -\frac{1}{\epsilon_0}\int_{\mathcal{V}} \rho\mathrm{d}^3x$.

1.3 Proof of the Theorem

Let there exist two solutions ϕ and ϕ' of the Poisson equation eq. (3).

By applying Green's first identity eq. (1) to $\phi_1 = \phi_2 = \Delta\phi = \phi - \phi'$, we get

$$\int_{\mathcal{V}} ((\Delta\phi)\nabla^2(\Delta\phi) + \nabla(\Delta\phi)\cdot\nabla(\Delta\phi))\mathrm{d}^3x = \oint_{\partial\mathcal{V}} (\Delta\phi)\nabla(\Delta\phi)\cdot\hat{\mathbf{n}}\mathrm{d}^2x,$$

which reduces to

$$\int_{\mathcal{V}} |\nabla(\Delta\phi)|^2\mathrm{d}^3x = 0.$$

The above result implies that $\nabla(\Delta\phi) = 0$ everywhere in $\mathcal{V}$. Thus the solution is unique or unique apart from a constant when Dirichlet and Neumann condition is given, respectively. Note that those two types of boundary conditions, of course, also can be used jointly—i.e., use Dirichlet's over some part on $\partial\mathcal{V}$ as well as Neumann's for the remaining part.

2 Dirichlet Green's Function

Green's function in electrostatics is defined as the solution to

$$\nabla_x{}^2 G(\mathbf{x},\mathbf{x}') = -\frac{1}{\epsilon_0}\delta(\mathbf{x}-\mathbf{x}'), \tag{4}$$

which implies

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'|} + \tilde{G}(\mathbf{x},\mathbf{x}'), \tag{5}$$

where

$$\nabla_x{}^2\tilde{G}(\mathbf{x},\mathbf{x}') = 0 \text{ in } \mathcal{V}.$$

A continuous function $\tilde{G}$ is generally determined by the shape of the volume.

Let us first consider the Dirichlet boundary condition of a conducting cavity, which is $G(\mathbf{x},\mathbf{x}') = 0$ on $\partial\mathcal{V}$. The Dirichlet Green's function satisfies

$$G_D(\mathbf{x},\mathbf{x}') = 0 \text{ on } \partial\mathcal{V}.$$

By the definition, the (partial) solution can be constructed as

$$\phi^{(1)}(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}^3x' \tag{6}$$

Subsequently, to treat an arbitrary Dirichlet boundary condition than $\psi = 0$, we would construct the remaining part of $\phi(\mathbf{x})$ due to the boundary. Let $\phi_1(\mathbf{x}) = G_D(\mathbf{x},\mathbf{x}')$ and $\phi_2(\mathbf{x}) = \phi^{(2)}(\mathbf{x})$. Then Green's theorem eq. (2) yields

$$\phi^{(2)} = -\epsilon_0\int_{\partial\mathcal{V}} \psi(\mathbf{x}')\nabla_{x'}G_D(\mathbf{x}',\mathbf{x})\cdot\hat{\mathbf{n}}\mathrm{d}^2x'. \tag{7}$$

Thus, we obtain the complete solution by adding $\phi^{(1)}$ and $\phi^{(2)}$:

$$\phi(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')\mathrm{d}^3x' - \epsilon_0\int_{\partial\mathcal{V}} \psi(\mathbf{x}')\nabla_{x'}G_D(\mathbf{x}',\mathbf{x})\cdot\hat{\mathbf{n}}\mathrm{d}^2x'. \tag{8}$$

3 Neumann Green's Function

In like manner of the previous section, let us consider the Neumann Green's function that satisfies

$$\hat{\mathbf{n}} \cdot \nabla_x G_N(\mathbf{x}, \mathbf{x}') = -\frac{1}{\epsilon A} \text{ on } \partial\mathcal{V} \tag{9}$$

and a constraint

$$\int_{\partial\mathcal{V}} G_N(\mathbf{x}, \mathbf{x}') \mathrm{d}^2 x = 0, \tag{10}$$

where A is the surface area of $\mathcal{V}$.

Following a similar procedure, we obtain the solution in terms of Neumann Green's function:

$$\phi(\mathbf{x}) = \int_V G_N(\mathbf{x}, \mathbf{x}')\rho(\mathbf{x}')\mathrm{d}^3 x' + \epsilon_0 \int_{\partial\mathcal{V}} \chi(\mathbf{x}') G_N(\mathbf{x}', \mathbf{x}) \mathrm{d}^2 x'. \tag{11}$$

Note that the above results can be extended to an unbounded region with finitely many disconnected volumes naturally.

References

[1] John D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.

[2] Robert M. Wald. *Advanced Classical Electrodynamics*. Princeton University Press, 2022.