2 Investigation of the Equations of Motion

2.1 Energy

Kinetic energy

The **kinetic energy** is the quadratic form

$$T = \frac{1}{2}\dot{\mathbf{x}}^2.$$

Potential energy

A system is said to be **conservative** if there exists a function $U : E^n \to \mathbb{R}$ such that

$$\mathbf{f} = \nabla U = -\frac{\partial U}{\partial \mathbf{x}}.$$

U is called the **potential energy**. The potential energy defines f, that is, the system of the form eq. (2) can be specified when the potential energy is given. Adding a constant to the potential energy does not change the system.

Total energy

The **total energy** is the sum

$$E(\mathbf{x}, \dot{\mathbf{x}}) = T + U.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + U) = \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}} \cdot \dot{\mathbf{x}} = 0,$$

the total energy of points moving according to the motion eq. (2) is conserved—its time derivative vanishes.

2.2 Phase Space

Degree of freedom

A system with n degree of freedom is a system described by the differential equations

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in E^n. \tag{2}$$

In a system with one degree of freedom, one can always introduce the potential energy

$$U(x) = -\int_{x_0}^{x} f(\xi)\mathrm{d}\xi,$$

since there is no curl in one dimension.

Phase space

eq. (2) is equivalent to the system determined by two equations:

$$\begin{cases} \dot{x_i} = y_i \\ \dot{y_i} = -\dfrac{\partial U}{\partial x_i} \end{cases} \tag{3}$$

The phase space of a system with n degrees of freedom is the $2n$-dimensional space with coordinates x_is and y_is. The points of the phase space are called **phase points**.

Phase curve

A solution of eq. (2) is a motion $\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^n$ of a phase point in the phase space. The image of the motion is called the **phase curve**. In other words, the phase curve is given by parametric equations

$$\begin{cases} \mathbf{x} = \boldsymbol{\phi}(t) \\ \mathbf{y} = \dot{\boldsymbol{\phi}}(t). \end{cases}$$

The time derivative of a phase point on the phase curve is called **phase velocity**. The law of conservation of energy guarantees each phase curve lies entirely in one energy level set $E(\mathbf{x}, \mathbf{y}) = h$. Since the kinetic energy cannot be negative, all points on the trajectory remain inside the potential wall $U(\mathbf{x}) \leq E$ for all time.

Equilibrium position

A phase curve consisting of only one point is called an **equilibrium position**. A point is an equilibrium position if the potential energy is at one of the critical points at the point. And if the number E is not a critical value of the potential energy, the level set on which the energy is equal to E is a smooth curve. Of course, the equilibrium positions must lie on the subspace $\mathbf{y} = \mathbf{0}$.

Phase plane

A two-dimensional phase space is said to be **phase plane**. The time takes to go from x_1 to x_2 in one direction is equal to

$$\begin{aligned} t_2 - t_1 &= \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\mathrm{d}x} \mathrm{d}x \\ &= \pm \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2\left(E - U(x)\right)}}. \end{aligned} \tag{4}$$

Suppose the potential energy is equal to the total energy, i.e. $U(x)$ is at one of its maximum points. One can express the slope of the phase curve at a given

point in general:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\pm\sqrt{2(E-U)}\right)\\ &= \pm\frac{U'}{\sqrt{2(E-U)}}\end{aligned}$$

Using the series expansion in a neighbourhood of the point ξ, $U'(x) \approx U'(\xi) + U''(\xi)(x-\xi)$ and $U(x) \approx U(\xi) + U'(\xi)(x-\xi) + (1/2)U''(\xi)(x-\xi)^2$ is valid. Since $U(x_0) = E$ and $U'(x_0) = 0$, one obtains

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\xi} = \pm\sqrt{-U''(\xi)}. \tag{5}$$

The area of a closed phase curve—of a periodic motion—

$$S = \oint_{\partial D(E)} -y\mathrm{d}x$$

is a function of the total energy where eq. (2) is fixed. Its derivative with respect to the total energy can be written with two terms: a change in the range and an extra term associated with the boundary—It is known as Leibniz integral rule—. Let x_1 and $x_2 (x_1 < x_2)$ are two maximum points of the potential energy. One could write the change of the area of the phase curve with respect to the change of the total energy:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}E} &= \frac{\mathrm{d}}{\mathrm{d}E}\left(\int_{x_1}^{x_2} y\mathrm{d}x - \int_{x_2}^{x_1} y dx\right)\\ &= 2\int_{x_1}^{x_2}\frac{\partial y}{\partial E}\mathrm{d}x + 2\,y\frac{dx}{dE}\bigg|_{x_2} - 2\,y\frac{dx}{dE}\bigg|_{x_1}\\ &= 2\int_{x_1}^{x_2}\frac{\partial}{\partial E}\sqrt{2(E-U)}\mathrm{d}x\\ &= 2\int_{x_1}^{x_2}\frac{\mathrm{d}x}{\sqrt{2\left(E-U(x)\right)}},\end{aligned} \tag{6}$$

which is coincidence to eq. (4).

Suppose a small oscillation—the total energy approaches the minimum potential energy— in a neighbourhood of ξ, the minimum point of the potential function. Then, the equation of the phase curve is given as:

$$\begin{aligned}E &= \frac{1}{2}y^2 + U(x)\\ &\approx \frac{1}{2}y^2 + U(\xi) + U'(\xi)(x-\xi) + \frac{1}{2}U''(\xi)(x-\xi)^2.\end{aligned}$$

Since $U'(\xi) = 0$, the phase curve becomes an ellipse which has the area $2\pi(E - E_0)/\sqrt{U''(\xi)}$. Thus, the period of the small oscillation is equal to

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{2\pi}{\sqrt{U''(\xi)}}. \tag{7}$$

Phase flow

Let M be a phase point. Assuming that any solution of the system defined by eq. (2) can be extended to the whole time axis. The time evolution of M is denoted by

$$M(t) = g^t M,$$

where $g^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a map of the phase space to itself. Since comes from the ordinary differential equation, g^t is a diffeomorphism. The diffeomorphism g^t, $t \in \mathbb{R}$, form a one-parameter group.

2.3 Conservative Force Fields

Work done by a force field along a path

Suppose given a vector field $\mathbf{F}$ and a curve l of finite length. Then the work of the field $\mathbf{F}$ on the path is by definition a line integral

$$A = \int_l \mathbf{F} \cdot d\mathbf{S}.$$

Conservative field

Suppose that the work of a field $\mathbf{F}$ does not depend on the path. Then

$$U(M) = -\int_{M_0}^{M} \mathbf{F} \cdot d\mathbf{S}$$

is well-defined. That is, U is the potential energy and the field is conservative. Of course, the additional constant $U(M_0)$ can be chosen arbitrarily. Conversely, if $\mathbf{F}$ is supposed to be conservative, the work done by $\mathbf{F}$

$$\int_{M_0}^{M} \mathbf{F} \cdot d\mathbf{S} = -U(M) + U(M_0)$$

does not depend on the shape of the path. In conclusion, a vector field $\mathbf{F}$ is conservative if and only if its work along any path M_1M_2 depends only on the endpoints of the path. This condition is also equivalent that $\mathbf{F}$ is curl-less.

2.4 Central-Force Problems

Central fields

A vector field in the space E^3 is called **central** with centre at 0, if it is invariant with respect to the group of motions of the plane which fix 0—rotations and reflections—. Of course, the central field can be written in the form

$$\mathbf{F} = F(r)\frac{\mathbf{r}}{|r|}.$$

Thus, every central field is conservative and its potential energy depends only on the distance to the centre of the field; $U = U(r)$. For a point with unit mass in a central field, eq. (2) becomes

$$\ddot{\mathbf{r}} = \Phi(r)\mathbf{e}_r$$

where $\mathbf{e}_r = \mathbf{r}/|r|$.

Angular Momentum

The **angular momentum** of a material point of unit mass relative to the point 0 is the vector product

$$\mathbf{M} = \mathbf{r} \times \dot{\mathbf{r}}.$$

Using the Leibniz rule, one can write the change of angular momentum with time:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} \\ &= \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \left(\Phi(r)\frac{\mathbf{r}}{|r|}\right) \\ &= 0.\end{aligned}$$

Thus the angular momentum is conserved. Since $\mathbf{M} \times \mathbf{r} = 0$, every motion in a central field is a motion in its position vector perpendicular to constant vector $\mathbf{M}$ or is just rectilinear motion through 0, so the motion is planar.

The conservation of angular momentum can be derived in a slightly different way. One can introduce polar coordinates r and φ on the plane of motion. Then the angular momentum can be expressed in terms of polar coordinates:

$$\begin{aligned}\mathbf{M} &= \mathbf{r} \times \dot{\mathbf{r}} \\ &= \mathbf{r} \times (\dot{r}\mathbf{e}_r) + \mathbf{r} \times (r\dot{\varphi}\mathbf{e}_\varphi) \\ &= r^2\dot{\varphi}\left(\mathbf{e}_r \times \mathbf{e}_\phi\right).\end{aligned}$$

Therefore the quantity $M = r^2\dot{\varphi}$ called **sectorial velocity** is conserved. A geometrical meaning of this quantity is the rate of change of the area $S(t)$ swept out by the radius vector $C = \mathrm{d}S/\mathrm{d}t = \frac{1}{2}r^2\dot{\varphi}$. Historically, the law of conservation of angular velocity was formerly stated by Kepler in terms of sectorial velocity.

Effective Potential Energy

Thanks to the conservation of angular momentum, a motion in a central field can be reduced to a one-dimensional problem. Differentiating the relation $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi$, one find

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\varphi}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\mathbf{e}_\varphi.$$

Since the field is central,

$$\frac{\partial U}{\partial \mathbf{r}} = \frac{\partial U}{\partial r}\mathbf{e}_r.$$

Therefore, the equation of motion takes the form

$$\begin{cases} \ddot{r} - r\dot{\varphi}^2 = -\dfrac{\partial U}{\partial r} \\ 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \end{cases} .$$

Substituting $M = r^2\dot{\varphi}$ into the equation of motion, one obtains

$$\ddot{r} = -\frac{\partial V}{\partial r}, \text{ where } V = U + \frac{M^2}{2r^2}.$$

The quantity $V(r)$ is called the **effective potential energy**. Remark the total energy

$$\begin{aligned} E &= \frac{\dot{\mathbf{r}}^2}{2} + U(r) \\ &= \frac{\dot{r}^2}{2} + \frac{r^2\dot{\varphi}^2}{2} + U(r) \\ &= \frac{\dot{r}^2}{2} + V(r) \end{aligned}$$

is the same as the total energy derived in the one-dimensional problem.

Integration of the equation of motion

The dependence of r on t is defined by the quadrature

$$\int \mathrm{d}t = \int \frac{\mathrm{d}r}{\sqrt{2(E - V(r))}}.$$

Since

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{M}{r^2}\frac{1}{\sqrt{2(E - V(r))}},$$

$$\varphi = \int \frac{M/r^2\mathrm{d}r}{\sqrt{2(E - V(r))}}.$$

Investigation of the orbit

The angle between the successive pericentres and apocentres is given by

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{M/r^2\mathrm{d}r}{\sqrt{2(E - V(r))}}. \tag{8}$$

The orbit is closed if and only if the angle Φ is commensurable with 2π. The substitution $x = M/r$ gives

$$\Phi = \int_{x_{\min}}^{x_{\max}} \frac{\mathrm{d}x}{\sqrt{2\,(E - U(x))}},$$

which is equal to the semi-period of an oscillation in the one-dimensional system with the potential energy $W(x) = U(M/x) + (x^2/2)$(remark eq. (6)). If the orbit

is close to the circle of radius r, the corresponding one-dimensional system is a small oscillation. By eq. (7), in this case, Φ can be written as

$$\Phi \approx \frac{\pi M}{r^2\sqrt{V''(r)}} = \pi\sqrt{\frac{U'}{3U' + rU''}}.$$

Φ is independent of the radius r if the potential energy U takes the form $U(r) = ar^\alpha (\alpha \geq -2, \alpha \neq 0)$ or $U(r) = b\log r$. $U = ar^2$ or $I = -k/r$.

Kepler's problem

Consider a central field whose potential energy is in the form $U = -k/r$. Using eq. (8), one obtains

$$\phi = \arccos \frac{\dfrac{M}{r} - \dfrac{k}{M}}{\sqrt{2E + \dfrac{k^2}{M^2}}},$$

where assumed the constant of integration is equal to zero; this is equivalent to the choice of an origin of reference for the angle ϕ at the pericentre. Here introduce the following notation:

$$p = \frac{M^2}{k} \quad e = \sqrt{1 + \frac{2EM^2}{k^2}}.$$

The number p is called the **parameter**, and e the **eccentricity**. Therefore, one obtains **focal equation** of a conic section

$$r = \frac{p}{1 + e\cos\varphi}.$$

In the case of an elliptical orbit, i.e., $E < 0$ and $e < 1$, according to simple geometry, the parameter and eccentricity are related with the semi-axes by the formula

$$a = \frac{p}{1 - e^2}.$$

Kepler's law

From the result of the above, one gets Kepler's first law—the orbit of a planet is an ellipse with the Sun at one of the two foci. Kepler's second law, which states that the sectorial velocity is constant, is guaranteed by the law of conservation of angular momentum. Kepler's third law can be derived from the fact $2S = MT$, since $M/2$ is the sectorial velocity. Since

$$a = \frac{k}{e|E|} \text{ and } b = \frac{M}{\sqrt{2|E|}},$$

one obtains

$$T = 2\pi\frac{k}{(2|E|)^{3/2}} = 2\pi a^{3/2}k^{-1/2}.$$

Note that the total energy E depends only on the major semi-axis a but not the minor-semi axis b.

2.5 Multi-Body Systems

Closed and open systems

Newton's equations for the motion of a system consisting of n material points, with masses m_i and radius vectors $\mathbf{r}_i$ are the equations

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i, \quad i = 1, 2, \cdots, n.$$

The vector $\mathbf{F}_i$ is called the **force acting on the i-th point**. If forces between two points are equal in magnitude and act in opposite directions along the straight line joining the points, such forces are called **forces of interaction**. A system that all forces are of interaction, is said to be **closed**. By definition, in a closed system, the force acting on the i-th point can be expressed as

$$\mathbf{F}_i = \sum_{j\neq i} \mathbf{F}_{ij}$$

where the vector $\mathbf{F}_{ij}$ is the force with which the j-th point acts on the i-th.

If the system is not closed, extra terms showed in the expression of $\mathbf{F}_i$:

$$\mathbf{F}_i = \sum_{j\neq i} \mathbf{F}_{ij} + \mathbf{F'}_i,$$

where $\mathbf{F'}_i$ is the so-called **external force**.

The law of conservation of momentum

The **momentum of a system** is the vector

$$\mathbf{P} = \sum_i m_i\dot{\mathbf{r}}_i.$$

The rate of change of momentum of a system

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = \sum_i m_i\ddot{\mathbf{r}}_i = \sum_i \mathbf{F'}_i$$

since $\sum_{i,j} \mathbf{F}_{ij} = 0$. Thus, the momentum of a closed system is conserved.

The **centre of mass** of a system is the point

$$\mathbf{r} = \frac{\sum m_i\mathbf{r}_i}{\sum m_i}.$$

Then the momentum of a system can be written as

$$\mathbf{P} = \left(\sum_i m_i\right)\dot{\mathbf{r}}.$$

So, the centre of mass of a closed system moves uniformly and linearly.

The law of conservation of angular momentum

The **angular momentum** of a material point of mass m relative to the point 0 is the vector

$$\mathbf{M} = \mathbf{r} \times m\dot{\mathbf{r}}.$$

The **angular momentum of a system** relative to 0 is the sum of the angular momenta of all the points in the system:

$$\mathbf{M} = \sum_i \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i.$$

The rate of change of the angular momentum of a system

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \sum_i \dot{\mathbf{r}}_i \times m_i \dot{\mathbf{r}}_i + \sum_i \mathbf{r} \times m_i \ddot{\mathbf{r}}_i = \sum_i \mathbf{r}_i \times \mathbf{F'}_i$$

is equal to the sum of the external torques, since the torque caused by forces of interaction is equal to zero. Thus, the angular momentum of a closed system is conserved.

The law of conservation of energy

The **kinetic energy** of a point of mass m is

$$T = \frac{1}{2} m\dot{\mathbf{r}}^2.$$

The **kinetic energy** of a system of mass points is the sum of the kinetic energies of the points

$$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2.$$

The rate of change of the kinetic energy of a system

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i = \sum_i \dot{\mathbf{r}}_i \cdot \mathbf{F}_i.$$

By integrating with respect to the time,

$$T(t_1) - T(t_0) = \int_{t_0}^{t_1} \dot{\mathbf{r}} \cdot \mathbf{F} \mathrm{d}t = \int_{\mathbf{r}(t_0)}^{\mathbf{r}(t_1)} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

the increase in kinetic energy is equal to the work of the force $\mathbf{F}$ on the path $\mathbf{r}(t)$ in configuration space. Since

$$\int_{\mathbf{r}(t_0)}^{\mathbf{r}(t_1)} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = U(\mathbf{r}(t_0)) - U(\mathbf{r}(t_1)),$$

the total energy of a conservative system is preserved under the motion: $E(t_1) = E(t_0)$.

Suppose the forces of interaction depend only on distance, i.e., $\mathbf{F}_{ij} = f_{ij}\mathbf{e}_{ij}$. The potential energy describing this interaction is

$$U_{ij}(\mathbf{r}) = \int_{r_0}^{r} f_{ij}(\rho)\mathrm{d}\rho$$

where $\mathbf{e}_{ij}$ is the unit vector in the direction from the i-th point to the j-th point. Therefore, the total potential energy of the interaction will be

$$U(\mathbf{r}) = \sum_{i>j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|).$$

In a non-conservative system, the total mechanical energy($E = T + U$) is not generally conserved. A decrease in the mechanical energy $E(t_0) - E(t_1)$ is called an increase in the **non-mechanical energy** E':

$$E'(t_1) - E'(t_0) = E(t_0) - E(t_1).$$

By definition, the total energy $H = E + E'$ is conserved.

2.6 Mechanical Similarity

Let the potential energy is a homogeneous function of degree k, i.e.,

$$U(\alpha\mathbf{r}) = \alpha^k U(\mathbf{r}) \quad \text{for any } \alpha > 0.$$

Suppose a transformation $\mathbf{r} \mapsto \alpha\mathbf{r}$, $t \mapsto \beta t$. Then the acceleration vector will be changed by a factor α^2/β^2, where the potential energy is changed by α^k. If α and β satisfies the relation

$$\frac{\alpha^2}{\beta^2} = \alpha^k \Leftrightarrow \beta = \alpha^{1-\frac{1}{2}k},$$

the transformation leaves the equation of motion.