3 $SU(2)$

The $SU(2)$ algebra is the simplest compact Lie algebra:

$$[J_j, J_k] = i\epsilon_{jkl}J_l. \tag{3.1}$$

3.1 J_3 Eigenstates

Remark 3.1. Since a commutator of generators does not vanish, the generators cannot be diagonalised simultaneously.

Claim 3.1. *We have a set of the highest states*

$$J_3\,|j,\alpha\rangle = j\,|j,\alpha\rangle, \tag{3.2}$$

where

$$\langle j,\alpha|\,j,\beta\rangle = \delta_{\alpha\beta}.^5 \tag{3.3}$$

Proof. We use the assumption that the space is finite-dimensional. □

3.2 Raising and Lowering Operators

Definition 3.1 (raising and lowering operators)**.**

$$J^\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2) \tag{3.4}$$

Proof. Suppose

$$J_3\,|m\rangle = m\,|m\rangle. \tag{3.5}$$

Then,

$$J_3J^\pm\,|m\rangle = J^\pm J_3\,|m\rangle \pm J^\pm\,|m\rangle = (m\pm 1)J^\pm\,|m\rangle. \tag{3.6}$$

□

Remark 3.2.

$$(J^+)^\dagger = J^- \tag{3.7}$$

$$[J_3, J^\pm] = \pm J^\pm \tag{3.8}$$

$$[J^+, J^-] = J_3 \tag{3.9}$$

Remark 3.3. We can use the raising and lowering operators to construct the whole irreducible representation.

Remark 3.4. There is no state with $J_3 = j+1$; $J^+\,|j,\alpha\rangle = 0$.

Definition 3.2 (normalisation factors)**.** The **normalisation factors** N_m are the coefficients that are chosen in order to ensure that the lowered state has unit norm:

$$J^-\,|m\rangle = N_m\,|m-1\rangle. \tag{3.10}$$

[5] The highest state need not be unique.

Claim 3.2.

$$N_j = \sqrt{j} \tag{3.11}$$

Proof. Let

$$J^- |j, \alpha\rangle = N_j(\alpha) |j, \alpha\rangle . \tag{3.12}$$

Then,

$$N_j^*(\beta)N_j(\alpha)\langle j-1, \beta|j-1, \alpha\rangle = \langle j, \beta| \underbrace{J^+ J^-}_{\substack{=[J^+,J^-]\\ +J^- J^+}} |j, \alpha\rangle = j\langle j, \beta|j, \alpha\rangle = j\delta_{\alpha\beta}. \tag{3.13}$$

N's can always be chosen to be real:

$$N_j(\alpha) = N_j = \sqrt{j}. \tag{3.14}$$

□

Claim 3.3.

$$N_m = \sqrt{\frac{(j+m)(j-m+1)}{2}} \tag{3.15}$$

Proof. By definition, we have

$$J^- |j-k, \alpha\rangle = N_{j-k} |j-k-1, \alpha\rangle , \tag{3.16}$$

as well as

$$J^+ |j-k-1\rangle = N_{j-k} |j-k, \alpha\rangle \tag{3.17}$$

since

$$\langle j-k, \alpha| J^+ |j-k-1, \alpha\rangle^* = \langle j-k-1, \alpha| J^- |j-k, \alpha\rangle = N_{j-k}. \tag{3.18}$$

Therefore, the normalisation factors satisfy

$$\begin{aligned} {N_{j-k}}^2 &= \langle j-k, \alpha| \underbrace{J^+ J^-}_{\substack{=[J^+,J^-]\\ +J^- J^+}} |j-k, \alpha\rangle \\ &= (j-k) + {N_{j-k+1}}^2. \end{aligned} \tag{3.19}$$

Telescoping yields

$$\begin{array}{lclcl} N_j^2 & & & = j \\ N_{j-1}^2 & - & N_j^2 & = j-1 \\ \vdots & & & \\ N_{j-k}^2 & - & N_{j-k+1}^2 & = j-k \\ \hline N_{j-k}^2 & & & = (k+1)j - \dfrac{k(k+1)}{2} \\ & & & = \dfrac{1}{2}(k+1)(2j-k), \end{array} \tag{3.20}$$

or setting $k = j - m$,

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}. \tag{3.21}$$

□

Claim 3.4.

$$j = \frac{l}{2}, \quad l \in \mathbb{N}_0. \tag{3.22}$$

Proof. Since we assumed that the representation is finite-dimensional, there exists a non-negative integer l such that

$$J^- |j - l, \alpha\rangle = 0. \tag{3.23}$$

The vanishing norm of $J^- |j - l, \alpha\rangle$ directly implies

$$N_{j-l} = \sqrt{\frac{(2j-l)(l+1)}{2}} = 0, \tag{3.24}$$

or equivalently,

$$l = 2j. \tag{3.25}$$

□

Remark 3.5. The dimension of a representation with the highest J_3 value j is $2j + 1$.

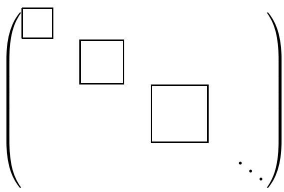

Figure 4: Block-diagonal matrix showing direct sum of the irreducible representations of a Lie group $SU(2)$; the Hilbert spaces of each angular momentum state.

3.3 The Standard Notation

Definition 3.3 (standard notations)**.** The **standard notation** is the labelling of the states of an irreducible representation by its highest J_3 value in the representation and the J_3 value:

$$|j, m\rangle. \tag{3.26}$$

Remark 3.6.

$$\langle j, m'| J_3 |j, m\rangle = m\delta_{m'm} \tag{3.27}$$

$$\langle j, m'| J^+ |j, m\rangle = \sqrt{\frac{(j+m+1)(j-m)}{2}}\delta_{m',m+1} \tag{3.28}$$

$$\langle j, m'| J^- |j, m\rangle = \sqrt{\frac{(j+m)(j-m+1)}{2}}\delta_{m',m-1} \tag{3.29}$$

Definition 3.4 (spin representations)**.** The **spin-j representation** of the $SU(2)$ algebra is defined by the matrix elements

$$[J_a^j]_{kl} = \langle j, j+l-k|\, J_a \,|j, j+1-l\rangle\,, \tag{3.30}$$

often relabelled as

$$[J_a^j]_{m'm} = \langle j, m'|\, J_a \,|j, m\rangle\,, \tag{3.31}$$

where $m = j+1-l$ and $m' = j+1-k$ run from $-j$ to j in steps of 1.

Example 3.1 (the spin-$\frac{1}{2}$ representation)*.*

$$J_1^{1/2} = \frac{1}{2}\underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}_{=\sigma_1}, \quad J_2^{1/2} = \frac{1}{2}\underbrace{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}_{=\sigma_2}, \quad J_3^{1/2} = \frac{1}{2}\underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{=\sigma_3} \tag{3.32}$$

$$J^+ = \begin{pmatrix}0 & 1/\sqrt{2}\\ 0 & 0\end{pmatrix}, \quad J^- = \begin{pmatrix}0 & 0\\ 1/\sqrt{2} & 0\end{pmatrix} \tag{3.33}$$

Remark 3.7*.*

$$\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c \Leftrightarrow [J_a, J_b] = i\epsilon_{abc}J_c \tag{3.34}$$

Claim 3.5 (highest weight construction)**.** *The construction of the irreducible representations above generalises to any compact Lie algebra.*

Proof.

1. Diagonalie J_3; the J_3 values are called **weights**.
2. Find the states with the highest J_3 value, j.
3. For each such state, explicitly construct the states of the irreducible spin-j representation by applying the lowering operator to the states with highest weight.
4. Concentrate on the subspace orthogonal to the subspace in which the spin-j representation acts.
5. Go to step 2 and start again with the states with the next highest J_3 value.

□

Claim 3.6. *The end result of the highest weight construction is a basis for the Hilbert space of the form*

$$|j, m, \alpha\rangle \tag{3.35}$$

where

$$\langle j', m', \alpha'|j, m, \alpha\rangle = \delta_{m'm}\delta_{j'j}\delta_{\alpha'\alpha}. \tag{3.36}$$

Proof. Consider a matrix element

$$\langle j', m', \alpha'|\, J_a \,|j, m, \alpha\rangle\,. \tag{3.37}$$

Insertion of a resolution yields

$$\begin{aligned}\langle j', m', \alpha' | J_a | j, m, \alpha\rangle &= \langle j', m', \alpha' | J_a | j', m'', \alpha'\rangle\langle j', m'', \alpha' | j, m, \alpha\rangle \\ &= [J_a^{j'}]_{m'm''}\langle j', m'', \alpha' | j, m, \alpha\rangle,\end{aligned} \tag{3.38}$$

as well as

$$\begin{aligned}\langle j', m', \alpha' | J_a | j, m, \alpha\rangle &= \langle j', m', \alpha' | j, m'', \alpha\rangle\langle j, m'', \alpha | J_a | j, m, \alpha\rangle \\ &= \langle j', m', \alpha' | j, m'', \alpha\rangle [J_a^{j}]_{m''m}.\end{aligned} \tag{3.39}$$

In other words, $\langle j', m', \alpha' | j, m, \alpha\rangle$ commutes with all the elements of an irreducible representation. Thus, Schur's lemma guarantees that it is proportional to $\delta_{j'j}\delta_{m'm}$. □

3.4 Tensor Products

Definition 3.5 (tensor products)**.** A tensor product of two states $|i\rangle$ and $|x\rangle$ is

$$|i, x\rangle \equiv |i\rangle \otimes |x\rangle = |i\rangle|x\rangle\,. \tag{3.40}$$

Remark 3.8. The tensor product $|i, x\rangle$ is transformed by $D_{1\otimes 2} \equiv D_1 \otimes D_2$:

$$\begin{aligned}D(g)\,|i, x\rangle &= |j, y\rangle\,[D_{1\otimes 2}]_{jyix} \\ &= |j\rangle|y\rangle\,[D_1(g)]_{ji}[D_2(g)]_{yx} \\ &= (|j\rangle\,[D_1(g)]_{ji})(|y\rangle\,[D_2(g)]_{yx}).\end{aligned} \tag{3.41}$$

Claim 3.7.

$$[J_a^{1\otimes 2}(g)]_{jyix} = [J_a^1]_{ji}\delta_{yx} + \delta_{ji}[J_a^2]_{yx} \tag{3.42}$$

Proof. Consider an infinitesimal α_a, i.e., $D = 1 + i\alpha_a J_a$. Then, a tensor product transforms as follows:

$$\begin{aligned}(1 + i\alpha_a J_a)\,|i, x\rangle &= |j, y\rangle\langle j, y|\,(1 + i\alpha_a J_a)\,|i, x\rangle \\ &= |j, y\rangle\left(\delta_{ji}\delta_{yx} + i\alpha[J_a^{1\otimes 2}]_{jyix}\right) \\ &= |j\rangle\left(\delta_{ji} + i\alpha_a[J_a^1]_{ji}\right)|y\rangle\left(\delta_{yx} + i\alpha_a[J_a^2]_{yx}\right).\end{aligned} \tag{3.43}$$

Comparing the first power of α_a, we get eq. (3.42), often simplified as

$$J_a^{1\otimes 2} = J_a^1 + J_a^2. \tag{3.44}$$

□

Remark 3.9.

$$J_a(|j\rangle|x\rangle) = (J_a\,|j\rangle)\,|x\rangle + |j\rangle\,(J_a\,|x\rangle) \tag{3.45}$$

3.5 J_3 Values Add

Claim 3.8. *The J_3 values of tensor product states are just the sums of the J_3 values of the factors:*

$$J_3(|j_1, m_1\rangle|j_2, m_2\rangle) = (m_1 + m_2)(|j_1, m_1\rangle|j_2, m_2\rangle). \tag{3.46}$$

Example 3.2 (spin-$\frac{1}{2}$$\otimes$spin-1)*.* The (unique) highest weight state is

$$|3/2, 3/2\rangle = |1/2, 1/2\rangle|1, 1\rangle. \tag{3.47}$$

Lowering this, we get

$$J^- |3/2, 3/2\rangle = \sqrt{\frac{3}{2}}|3/2, 1/2\rangle, \tag{3.48}$$

as well as

$$J^- (|1/2, 1/2\rangle|1, 1\rangle) = \sqrt{\frac{1}{2}}|1/2, -1/2\rangle|1, 1\rangle + |1/2, 1/2\rangle|1, 0\rangle. \tag{3.49}$$

Continuing the process gives

$$|3/2, -1/2\rangle = \sqrt{\frac{1}{3}}|1/2, -1/2\rangle|1, 1\rangle + \sqrt{\frac{2}{3}}|1/2, 1/2\rangle|1, 0\rangle, \tag{3.50}$$

$$|3/2, -3/2\rangle = \sqrt{\frac{2}{3}}|1/2, -1/2\rangle|1, 0\rangle + \sqrt{\frac{1}{3}}|1/2, 1/2\rangle|1, -1\rangle. \tag{3.51}$$

The two-dimensional subspace orthogonal to these must be the space spanned by $|1/2, 1/2\rangle$ and $|1/2, -1/2\rangle$:

$$|1/2, 1/2\rangle = \sqrt{\frac{2}{3}}|1/2, -1/2\rangle|1, 1\rangle - \sqrt{\frac{1}{3}}|1/2, 1/2\rangle|1, 0\rangle, \tag{3.52}$$

$$|1/2, -1/2\rangle = \sqrt{\frac{1}{3}}|1/2, -1/2\rangle|1, 0\rangle - \sqrt{\frac{2}{3}}|1/2, 1/2\rangle|1, -1\rangle. \tag{3.53}$$

Claim 3.9.

$$\{j\} \otimes \{s\} = \bigoplus_{l=|s-j|}^{s+j} \{l\}, \tag{3.54}$$

where $\{k\}$ denotes the spin-k representation of $SU(2)$.

Proof. The highest weight procedure generates states from $|j + s, j + s\rangle$ down to $|j + s, -(j + s)\rangle$. Meanwhile, since

$$J^- (|j\rangle|s\rangle) \in \operatorname{span}\{|j - 1\rangle|s\rangle, |j\rangle|s - 1\rangle\}, \tag{3.55}$$

there must be two linearly independent states with the J_3 value $j+s-1$; in particular, the presence of $|j + s - 1, j + s - 1\rangle$ is guaranteed. Thus, the highest weight procedure can generate states from $|j + s - 1, j + s - 1\rangle$ down to $|j + s - 1, -(j + s - 1)\rangle$. By induction, the claim is proved. □