2 Investigation of the Equations of Motion

Problem 2.1

Show that through every phase point there is one and only one phase curve.

Solution 2.1

The existence and uniqueness theorem for ordinary differential equations guarantees the statement.

Problem 2.2

Prove that the local maximum points of the potential energy are unstable, but the minimum points are stable equilibrium positions.

Solution 2.2

Under a small perturbation, a phase point that is at the minimum point of the potential energy remains in its neighbourhood satisfying the law of conservation of energy, whereas a point at the maximum point 'falls' downward the potential wall.

Problem 2.3

How many phase curves make up the separatrix (Figure 10) curve, corresponding to the level E_2?

Solution 2.3

Three.

Problem 2.4

Determine the duration of motion along the separatrix.

Solution 2.4

It follows from the uniqueness theorem that the time is infinite since also an equilibrium position constitutes the separatrix.

Problem 2.5

Show that the time it takes to go from x_1 to x_2 (in one direction) is equal to

$$t_2 - t_1 = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E - U(x))}}.$$

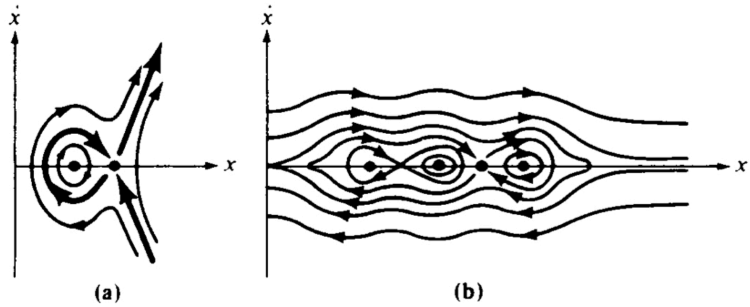
Solution 2.5

$$\begin{aligned} t_2 - t_1 &= \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\mathrm{d}x} \mathrm{d}x \\ &= \pm \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2\left(E - U(x)\right)}}. \end{aligned}$$

Problem 2.6

Draw the phase curves, given the potential energy graph in Figure 11.

Solution 2.6



Problem 2.7

Draw the phase curves for the "equation of an ideal planar pendulum": $\ddot{x} = -\sin x$.

Solution 2.7

Problem 2.8

Draw the phase curves for the "equation of a pendulum on a rotating axis": $\ddot{x} = -\sin x + M$.

Solution 2.8

Problem 2.9

Find the tangent lines to the branches of the critical level corresponding to maximal potential energy $E = U(\xi)$ (Figure 13).

Solution 2.9

One can express the slope of the phase curve at a given point in general:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\pm\sqrt{2(E-U)}\right)\\ &= \pm\frac{U'}{\sqrt{2(E-U)}}\end{aligned}$$

Using the series expansion in a neighbourhood of the point ξ,

$$U'(x) \approx U'(\xi) + U''(\xi)(x-\xi)$$

and

$$U(x) \approx U(\xi) + U'(\xi)(x-\xi) + (1/2)U''(\xi)(x-\xi)^2$$

is valid. Since $U(x_0) = E$ and $U'(x_0) = 0$, one obtains

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\xi} = \pm\sqrt{-U''(\xi)}.$$

Problem 2.10

Let $S(E)$ be the area enclosed by the closed phase curve corresponding to the energy level E. Show that the period of motion along this curve is equal to

$$T = \frac{dS}{dE}.$$

Solution 2.10

The area of the closed phase curve—of the periodic motion—

$$S = \oint_{\partial D(E)} -y\mathrm{d}x$$

is a function of the total energy where the equation of motion is fixed. Its derivative with respect to the total energy can be written with two terms: a change in the range and an extra term associated with the boundary—It is known as Leibniz integral rule—. Let x_1 and $x_2(x_1 < x_2)$ are two maximum points of the potential energy. Thus one could introduce

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}E} &= \frac{\mathrm{d}}{\mathrm{d}E}\left(\int_{x_1}^{x_2} y\mathrm{d}x - \int_{x_2}^{x_1} ydx\right)\\ &= 2\int_{x_1}^{x_2} \frac{\partial y}{\partial E}\mathrm{d}x + 2\, y\frac{dx}{dE}\bigg|_{x_2} - 2\, y\frac{dx}{dE}\bigg|_{x_1}\\ &= 2\int_{x_1}^{x_2} \frac{\partial}{\partial E}\sqrt{2(E-U)}\mathrm{d}x\\ &= 2\int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{2\left(E-U(x)\right)}}.\end{aligned}$$

Problem 2.11

Let E_0 be the value of the potential function at a minimum point ξ. Find the period T_0 of small oscillations in a neighborhood of the point ξ, where $T_0 = \lim_{E\to E_0} T(E)$.

Solution 2.11

The equation of the phase curve is given as:

$$\begin{aligned}E &= \frac{1}{2}y^2 + U(x)\\ &\approx \frac{1}{2}y^2 + U(\xi) + U'(\xi)(x-\xi) + \frac{1}{2}U''(\xi)(x-\xi)^2.\end{aligned}$$

Since $U'(\xi) = 0$, the equation becomes

$$U''(\xi)(x-\xi)^2 + y^2 = 2(E-E_0)$$

, an ellipse which has the area $2\pi(E-E_0)/\sqrt{U''(\xi)}$. Thus, the period of the small oscillation is equal to

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{2\pi}{\sqrt{U''(\xi)}}.$$

Problem 2.12

Consider a periodic motion along the closed phase curve corresponding to the energy level E. Is it stable in the sense of Liapunov?

Solution 2.12

An equilibrium position $\mathbf{x} = 0$ *of equation* $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}),\ \mathbf{x} \in U \subset \mathbb{R}^n$ *is said to be* ***stable (in Lyapunov's sense)*** *if given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *(depending only on* ε *and not on* t*) such that for every* $\mathbf{x}_0$ *for which* $\|\mathbf{x}_0\| < \delta$, *the solution* $\boldsymbol{\varphi}$ *of the equation with initial condition* $\boldsymbol{\varphi}(0) = \mathbf{x}_0$ *can be extended onto the whole half-line* $t > 0$ *and satisfies the inequality* $\|\boldsymbol{\varphi}(t)\| < \varepsilon$ *for all* $t > 0$.

No. Since a phase curve can be extended without bound as $\|\mathbf{v}(\mathbf{x}_0)\|$ increases.

Problem 2.13

Show that the system with potential energy $U = -x^4$ does not define a phase flow.

Solution 2.13

Let M be a point on a line $x = 0$. Since $y = \pm\sqrt{2(E + x^4)}$, the time it takes for x to diverge to infinity is

$$t = \int_0^\infty \frac{\mathrm{d}x}{\sqrt{2(E + x^4)}} = 2\sqrt{\frac{2}{\pi\sqrt{E}}}\Gamma\left(\frac{5}{4}\right)^2 < \infty,$$

that is, the point goes to infinity in finite time. Thus, g^t is not defined.

Problem 2.14

Show that if the potential energy is positive, then there is a phase flow.

Solution 2.14

Since

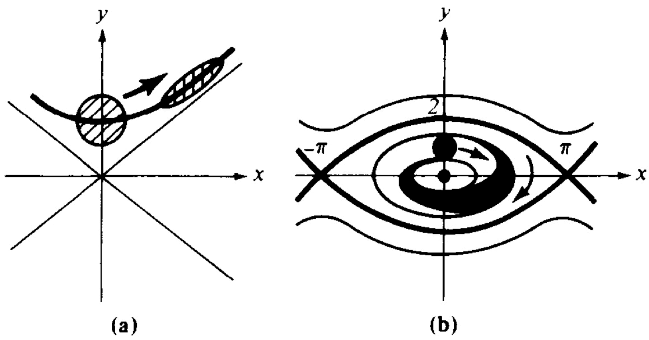
$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{2(E - U)}} > \int_0^\infty \frac{\mathrm{d}x}{\sqrt{2E}} \to \infty,$$

There not exists a phase point whose spatial coordinate diverges in finite time. Thus, the existence and uniqueness theorem for ordinary differential equations guarantees that the phase flow can be defined.

Problem 2.15

Draw the image of the circle $x^2+(y-1)^2 < \frac{1}{4}$ under the action of the transformation of the phase flow for the equations (a) of the "inverse pendulum," $\ddot{x} = x$ and (b) of the "nonlinear pendulum," $\ddot{x} = -\sin x$.

Solution 2.15



Problem 2.16

Find an example of a system of the form $\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x} \in E^2$, which is not conservative.

Solution 2.16

$$\mathbf{f} = -x_2\mathbf{e}_1 + x_1\mathbf{e}_2.$$

Problem 2.17

Show that the phase curves are great circles of this sphere. (A great circle is the intersection of a sphere with a two-dimensional plane passing through its center.)

Solution 2.17

It is sufficient that suggest two linearly independent constant vectors that are always orthogonal to the solution vector:

$$\mathbf{v}_1 = \begin{pmatrix} ({c_3}^2 + {c_4}^2)c_1 & -({c_1}^2 + {c_2}^2)c_3 & ({c_3}^2 + {c_4}^2)c_2 & -({c_1}^2 + {c_2}^2)c_4 \end{pmatrix}^\top$$
$$\mathbf{v}_2 = \begin{pmatrix} ({c_3}^2 + {c_4}^2)c_2 & -({c_1}^2 + {c_2}^2)c_4 & -({c_3}^2 + {c_4}^2)c_1 & ({c_1}^2 + {c_2}^2)c_3 \end{pmatrix}^\top$$

where the solution vector is given by

$$\mathbf{v}_s = \begin{pmatrix} x_1 & x_2 & y_1 & y_2 \end{pmatrix}^\top .$$

Problem 2.18

Show that the set of phase curves on the surface π_{E_0} forms a two-dimensional sphere. The formula $w = (x_1 + iy_1)/(x_2 + iy_2)$ gives the "Hopf map" from the three sphere π_{E_0} to the two sphere (the complex plane of w completed by the point at infinity). Our phase curves are the pre-images of points under the Hopf map.

Solution 2.18

By substituting the solution

$$x_1 = c_1 \cos t + c_2 \sin t \quad x_2 = c_3 \cos t + c_4 \sin t$$
$$y_1 = -c_1 \sin t + c_2 \cos t \quad y_2 = -c_3 \sin t + c_4 \cos t$$

into x_is and y_is, using Euler's formula, one can introduce

$$w = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{c_1 + ic_2}{c_3 + ic_4}.$$

Thus w is a point in $\mathbb{C} \cup \{\infty\}$.

Problem 2.19

Find the projection of the phase curves on the x_1, x_2 plane. (i.e., draw the orbits of the motion of a point).

Solution 2.19

Since the phase curve is a two-circle, the projection must be an ellipse. Its equation can be written in a quadratic form of x_1 and x_2, and its coefficients can be determined to eliminate time-dependent terms:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} {c_1}^2 & {c_3}^2 & c_1c_3 \\ 2c_1c_2 & 2c_3c_4 & c_2c_3 \\ {c_2}^2 & {c_4}^2 & c_2c_4 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

where the equation of the curve is

$$A{x_1}^2 + Bx_1x_2 + C{x_2}^2 = 1.$$

Problem 2.20

Show that this rectangle is inscribed in the ellipse $U \leq E$.

Solution 2.20

Because of the law of conservation of energy.

Problem 2.21

Show that this curve is a parabola. By increasing the phase shift $\varphi_2 - \varphi_1$ we get in turn the curves in Fig. 23.

Solution 2.21

If we choose the solution

$$\begin{cases} x_1 = A_1 \sin\left(t + \dfrac{\pi}{4}\right) = \dfrac{A_1}{\sqrt{2}}(\sin t + \cos t) \\ x_2 = A_2 \sin 2t \end{cases},$$

then x_1 and x_2 satisfies the quadratic relation

$$\left(\frac{\sqrt{2}}{A}x_1\right)^2 = \frac{x_2}{A_2} + 1.$$

Problem 2.22

Show that if $\omega = m/n$, then the Lissajous figure is a closed algebraic curve; but if ω is irrational, then the Lissajous figure fills the rectangle everywhere densely. What does the corresponding phase trajectory fill out?

Solution 2.22

Trivial.

Problem 2.23

Show that the vector field $F_1 = x_2$, $F_2 = -x_1$ is not conservative (Figure 27).

Solution 2.23

Its work along the unit circle whose centre is the origin is not zero.

Problem 2.24

Is the field in the plane minus the origin given by $F_1 = x_2/({x_1}^2 + {x_2}^2)$, $F_2 = -x_1/({x_1}^2 + {x_2}^2)$ conservative? Show that a field is conservative if and only if its work along any closed contour is equal to zero.

Solution 2.24

It is not conservative since its work along the unit circle whose centre is the origin is not zero.

If a field is conservative, its work along a closed path should be equal to the work along a single point, that is, 0.

Suppose a field whose work along any closed contour is equal to zero. Since it is always possible to make a path a closed path by joining the fixed path connecting the endpoints, its work along any path whose endpoints are given is equal to the minus of the work along the fixed path. Thus the field is conservative.

Problem 2.25

Show that all vectors of a central field lie on rays through 0, and that the magnitude of the vector field at a point depends only on the distance from the point to the center of the field.

Solution 2.25

Central fields should be invariant under rotations.

Problem 2.26

Compute the potential energy of a Newtonian field.

Solution 2.26

$$
\begin{aligned}
U &= -\int \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_r^\infty \frac{k}{r^2} \mathrm{d}r \\
&= -\frac{k}{r}.
\end{aligned}
$$

Problem 2.27

For which values of α is motion along a circular orbit in the field with potential energy $U = r^\alpha$, $-2 \le \alpha < \infty$, Liapunov stable?

Solution 2.27

To be stable in the Liapunov sense, the motion explained in the polar coordinates should be stable with respect to r and its period should be at the extremum.

Let r_0 be the radius and V be the effective potential energy of the circular motion. Since the motion is circular, its partial derivative with respect to r vanishes at the radius:

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0} = \alpha {r_0}^{\alpha-1} - M^2 {r_0}^{-3} = 0$$

Therefore, the partial derivative of the second order can be written as

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_0} = M^2 {r_0}^{-4}(\alpha + 2).$$

Thus, the motion is stable only if $\alpha > -2$.

The period $T = 2\pi/\omega$ of the circular motion satisfies the centripetal force condition

$$m r_0 \left(\frac{2\pi}{T}\right)^2 = \alpha {r_0}^{\alpha-1}.$$

Since T is independent of r if and only if $\alpha = 2$, the motion is Liapunov stable only if $\alpha = 2$.

Problem 2.28

Find the angle Φ for an orbit close to the circle of radius r.

Solution 2.28

See Solution 2.31.

Problem 2.29

Examine the shape of an orbit in the case when the total energy is equal to the value of the effective energy V at a local maximum point.

Solution 2.29

A circular orbit, since $\dot{r} = 0$.

Problem 2.30

(Problem 1) Show that the angle Φ between the pericenter and apocenter is equal to the semiperiod of an oscillation in the one-dimensional system with potential energy $W(x) = U(M/x) + (x^2/2)$.

Solution 2.30

Using integration by substitution,

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{M/r^2 \mathrm{d}r}{\sqrt{2(E - V(r))}} = \int_{x_{\min}}^{x_{\max}} \frac{\mathrm{d}x}{\sqrt{2(E - V(r))}}.$$

Problem 2.31

(Problem 2) Find the angle Φ for an orbit close to the circle of radius r.

Solution 2.31

By the results of problem 2.30 and 2.11,

$$\begin{aligned}\Phi &\approx \frac{\pi M}{r^2\sqrt{V''(r)}} \\ &= \pi\sqrt{\frac{U'}{3U' + rU''}}\end{aligned}$$

since $U' = M^2/r^3$.

Problem 2.32

(Problem 3) For which values of U is the magnitude of Φ_{cir} independent of the radius r?

Solution 2.32

The condition for $\mathrm{d}\Phi/\mathrm{d}r = 0$ is equivalent to

$$\frac{U''}{U'} = (\ln U')' \sim \frac{1}{r},$$

which implies

$$U' \sim r^{\alpha-1}.$$

It follows that $\Phi_{\text{cir}} = \pi/\sqrt{\alpha+2}$. If $\alpha < -2$, since they are unstable, Φ_{cir} cannot be defined.

Problem 2.33

(Problem 4) Let in the situation of problem 3 $U(r) \to \infty$ as $r \to \infty$. Find $\lim_{E\to\infty} \Phi(E, M)$.

Solution 2.33

Using the result of problem 1 and substitution $x = yx_{\max}$, Φ can be written as

$$\begin{aligned}
\Phi &= \int_{x=x_{\min}}^{x=x_{\max}} \frac{\mathrm{d}y}{\sqrt{2(E-V)}} \frac{\mathrm{d}x}{\mathrm{d}y} \\
&= x_{\max} \int_{y_{\min}}^{1} \frac{\mathrm{d}y}{\sqrt{2(V(x_{\max}) - V(x))}} \\
&= \int_{y_{\min}}^{1} \frac{\mathrm{d}y}{\sqrt{2(V^*(1) - V^*(y))}},
\end{aligned}$$

where

$$V^*(y) = \frac{1}{{x_{\max}}^2} U\left(\frac{M}{yx_{\max}}\right) + \frac{y^2}{2}.$$

Since the potential energy satisfies the stability condition $\alpha \geq -2$, as $E \to \infty$, $x_{\max} \to \infty$ and $x_{\min} \to 0$, so $y_{\min} \to 0$. Then the first term in $V^*(y)$ can be discarded. Thus,

$$\lim_{E\to\infty} \Phi(E, M) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \frac{\pi}{2}.$$

Problem 2.34

(Problem 5) Let $U(r) = -kr^\beta$, $0 < \beta < 2$. Find $\Phi_0 = \lim_{E\to-\infty} \Phi$.

Solution 2.34

The effective potential

$$V = -kr^{-\beta} + \frac{M^2}{2r^2}$$

is the total energy at the furthest point. Since $V \to -0$ as $r \to \infty$, the limit $E \to -0$ is equivalent to $x_{\max} \to \infty$. Thus, using the same substitution as Solution 2.33,

$$\lim_{E\to -0} \Phi = \int_0^1 \frac{\mathrm{d}y}{\sqrt{y^\beta - y^2}} = \frac{\pi}{2-\beta}.$$

Problem 2.35

(Problem 6) Find all central fields in which bounded orbits exist and are closed.

Solution 2.35

If all bounded orbits are closed, then, in particular, $\Phi_{\text{cir}} = 2\pi(m/n) = \text{const.}$ According to Problem 3, $U = ar^\alpha(\alpha \geq -2, \alpha \neq 0)$, or $U = b \ln r$. In both cases $\Phi_{\text{cir}} = \pi/\sqrt{\alpha+2}(\alpha = 0 \text{ for } U = b\ln r)$. If $\alpha > 0$, then according to Problem 5, $\lim_{E\to -0} \Phi(E, M) = \pi/(2+\alpha)$. Therefore, $\pi/(2+\alpha) = \pi/\sqrt{2+\alpha}$, $\alpha = -1$. In the case $\alpha = 0$ we find $\Phi_{\text{cir}} = \pi/\sqrt{2}$, which is not commensurable with 2π. Therefore, all bounded orbits can be closed only in fields where $U = ar^2$ or $U = -k/r$. In the field $U = ar^2$, $a > 0$ or $U = -k/r$ all bounded orbits are closed and elliptical.

Problem 2.36

At the entry of a satellite into a circular orbit at a distance 300 km from the earth the direction of its velocity deviates from the intended direction by 1° towards the earth. How is the perigee changed?

Solution 2.36

The height of the perigee is less by approximately 110 km.

Problem 2.37

How does the height of the perigee change if the actual velocity is 1 m/s less that intended?

Solution 2.37

Skkiped.

Problem 2.38

The *first cosmic velocity* is the velocity of motion on a circular orbit of radius close to the radius of the earth. Find the magnitude of the first cosmic velocity v_1 and show that $v_2 = \sqrt{2}v_1$.

Solution 2.38

$$\frac{m{v_1}^2}{R_\oplus} = \frac{GM_\oplus m}{{R_\oplus}^2}$$

$$\frac{1}{2}m{v_2}^2 - \frac{GM_\oplus m}{R_\oplus} = 0$$

Problem 2.39

During his walk in outer space, the cosmonaut A. Leonov threw the lens cap of his movie camera towards the earth. Describe the motion of the lens cap with respect to the spaceship, taking the velocity of the throw as 10 m/sec.

Solution 2.39

Skipped.

Problem 2.40

Show that if a field is axially symmetric and conservative, then its potential energy has the form $U = U(r, z)$, where r, φ, and z are cylindrical coordinates.

Solution 2.40

Since a force whose force field is axially symmetric is independent of φ, so the potential energy is.

Problem 2.41

Show that the center of mass is well defined, i.e., does not depend on the choice of the origin of reference for radius vectors.

Solution 2.41

Trivial.

Problem 2.42

Determine the major semi-axis of the ellipse which the center of the earth describes around the common center of mass of the earth and the moon. Where is this center of mass, inside the earth or outside? (The mass of the moon is 1/81 times the mass of the earth.)

Solution 2.42

Skipped.

Problem 2.43

If the radius of a planet is α times the radius of the earth and its mass β times that of the earth, find the ratio of the acceleration of the force of gravity and the first and second cosmic velocities to the corresponding quantities for the earth.

Solution 2.43

The acceleration of the force of gravity g will be changed by ratio β/α^2. Since the first and second cosmic velocities are proportional to $\sqrt{gr}$, they will be changed by ratio $\sqrt{\beta/\alpha}$.

Problem 2.44

A desert animal has to cover great distances between sources of water. How does the maximal time the animal can run depend on the size L of the animal?

Solution 2.44

The store of water is proportional to the volume of the body, i.e., L^3; the evaporation is proportional to the surface area, i.e., L^2. Therefore, the maximal time of a run from one source to another is directly proportional to L.

Problem 2.45

How does the running velocity of an animal on level ground and uphill depend on the size L of the animal?

Solution 2.45

The power developed is proportional to L^2. Since the drag is proportional to L^2 but the gravity is proportional to L^3, the size of an animal on the level ground $\sim L_0$, where on uphill L^{-1}.

Problem 2.46

How does the height of an animal's jump depend on its size?

Solution 2.46

For a jump of height h one needs energy proportional to L^3h, and the work accomplished by muscular strength F is proportional to FL. The force F is proportional to L^2 (since the strength of bones is proportional to their section). Therefore, $L^3h \sim L^2L$, i.e., the height of a jump does not depend on the size of the animal. In fact, a jerboa and a kangaroo can jump to approximately the same height.
