Part I
NEWTONIAN MECHANICS

1 Experimental Facts

1.1 Affine Space and Euclidean Structure

Affine space

The **affine space** is a vector space forgotten its origin. Therefore, the addition of elements of an affine space has no meaning whereas the subtraction gives an element of the underlying vector space. In this note, only $\mathbb{R}^n$ will be considered as the underlying set hereafter. n-dimensional affine space is denoted as A^n.

Affine transformation

An **affine transformation** is a transformation that preserves collinearity and ratios of distances. Any affine transformation can be written as a composition of translations and linear transformations.

Euclidean structure

A **euclidean structure** is an inner product—positive-definite symmetric bilinear form— on $\mathbb{R}^n$. An affine space A^n endowed with a euclidean structure is said to be a **euclidean space** E^n.

1.2 Galilean Structure

Universe

A **universe** is a four-dimensional affine space A^4. The points of A^4 are said to be **events**. The parallel displacements constitute a vector space $\mathbb{R}^4$.

Time

A **time** t is a linear map from $\mathbb{R}^4$, the vector space of parallel displacements of the universe onto $\mathbb{R}$, a real time-axis. If the time interval from event a to event b $t(b-a)=0$, the events are said to be **simultaneous**. Since t is a linear map, the equivalence class of simultaneity is a three-dimensional affine space A^3. It is called a **space of simultaneous events**

Distance

A **distance** between simultaneous events a and b

$$\rho(a,b) = \|b-a\|$$

is given by a euclidean structure on $\mathbb{R}^3$. Thus, spaces of simultaneous events become a euclidean space E^3.

Galilean structure

A **galilean spacetime structure** consists of universe, time and distance. A four-dimensional affine space equipped with a galilean structure is called a **galilean space**. Same-place-ity has not been defined without a coordinate system although simultaneity is an ingredient of galilean space.

Galilean group

The **galilean group** is the group of all transformations of a galilean space which preserves its galilean structure. Since affine transformations preserve lines and parallelism, all spaces of simultaneous events could be only inclined in the same way. To preserve time, the time axis cannot be stretched or contracted. In each A^3 of simultaneity, only orthogonal transformations are allowed by the preservation of distance. Consequently, every galilean transformation can be written in a unique way as the composition of a uniform motion with velocity $\mathbf{v}$—

$$g_1(t, \mathbf{x}) = (t, \mathbf{x} + \mathbf{v}t) \quad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3,$$

a translation of the origin by parallel displacement $(s, \mathbf{s})$—

$$g_1(t, \mathbf{x}) = (t + s, \mathbf{x} + \mathbf{s}) \quad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3,$$

and a rotation of the coordinate axes by orthogonal transformation $G : \mathbb{R}^3 \to \mathbb{R}^3$—

$$g_3(t, \mathbf{x}) = (t, G\mathbf{x}) \quad \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3.$$

Galilean coordinate system

Let M be a set. A bijection map $\phi : M \to \mathbb{R} \times \mathbb{R}^3$ is called a **galilean coordinate system** on the set M. ϕ gives M a galilean structure.

1.3 Motion

Configuration space

The direct product of n copies of $\mathbb{R}^3$ is called the **configuration space** of the system of n points. It contains information on positions of n points.

Motion of a system of n points

A **motion** $\mathbf{x}(t)$ in $\mathbb{R}^N$ is a differentiable map from the time interval I to the configuration space $\mathbb{R}^{3n} = \mathbb{R}^N$. The motion of the i-th point is denoted as $\mathbf{x}_i$.

Velocity and acceleration

The derivative

$$\dot{\mathbf{x}}(t_0) = \left.\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right|_{t=t_0}$$

is called the **velocity vector** at the point $t_0 \in I$. And the second derivative

$$\ddot{\mathbf{x}}(t_0) = \left.\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right|_{t=t_0}$$

is called the **acceleration vector** at the point t_0. Maps, functions, etc. are understood to be differentiable maps, functions, etc. hereafter.

World line

The graph of the motion of a point, the subset of the direct product $\mathbb{R} \times \mathbb{R}^3$ consisting of all pairs $(t, \mathbf{x}_i(t))$ with $t \in I$, is called a world line.

Newton's principle of determinancy

All motions of systems are uniquely determined by their initial state—positions and velocities. In particular, at each time, the position and velocity at that moment determine the acceleration in such a way as to satisfy **Newton's equation**:

$$\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) \tag{1}$$

x The theorem of existence and uniqueness of solutions to ordinary differential equations guarantees the function $\mathbf{F} : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ and the initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ uniquely determine a motion.

1.4 Homogeneity and Isotropy of Spacetime

Galileo's principle of relativity

Galileo's principle of relativity states that in physical spacetime there exists a selected galilean structure which makes the right-hand side of eq. (1) to be invariant with respect to the galilean group. Coordinate systems that give such galilean structure are called **inertial frames**.

Homogeneity of time

By virtue of the homogeneity of time, if $\mathbf{x} = \boldsymbol{\phi}(t)$ is a solution to eq. (1), then $\mathbf{x} = \boldsymbol{\phi}(t + s)$ also be a solution. Thus, the right-hand side of eq. (1) does not depend explicitly on the time:

$$\ddot{\mathbf{x}} = \boldsymbol{\Phi}(\mathbf{x}, \dot{\mathbf{x}}).$$

Homogeneity of space

By virtue of the space is homogeneous, if $\mathbf{x}_i = \boldsymbol{\phi}_i(t)$ is a solution to eq. (1), then $\mathbf{x}_i = \boldsymbol{\phi}_i(t) + \mathbf{r}$ also be a solution. Since uniform motions also are elements of the galilean group, the right-hand side of eq. (1) depends only on the relative coordinates and velocities:

$$\ddot{x}_i = \mathbf{f}_i(\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k), \quad i, j, k \in \{1, \cdots, n\}.$$

Isotropy of space

The equation $\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$ is invariant under an orthogonal transformation means $G\ddot{\mathbf{x}} = \mathbf{F}(G\mathbf{x}, \dot{G\mathbf{x}})$. Thus,

$$\mathbf{F}(G\mathbf{x}, \dot{G\mathbf{x}}) = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}).$$