Inadequateness of the Open End Boundary Condition in Acoustic Resonance in Air Columns

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1 INTRODUCTION

Many freshman-level physics textbooks introduce acoustic resonance in air columns. That is, when producing a sound in front of the open end, the column amplifies and sustains the sound if the frequency of the sound matches one of the natural frequencies of the column. However, their explanations are inadequate to describe how actually reflections work. This paper shows a boundary condition about the open end is unusual and is only consistent in the cases of resonance.

2 REFLECTION AT THE CLOSED END

Air particles moving toward the closed end collide with the end and reverse their direction. As a result of this kind of elastic collision, the reflected wave has the same phase angle as the original wave, where the moving direction is opposite. Consequently, a sum of the original and reflected wave indicates that the displacement of air particles at the closed end is always zero.

3 FORMATION OF COMPOSITE WAVE

Typical acoustic resonance problems assume that the air pressure at the open end remains constant at the atmospheric pressure. With this assumption, acoustic waves undergo a free-end reflection at the open end.

Suppose an air column of length L whose open end and closed end are located at $x = 0$ and $x = L$, respectively. Let the original wave be $\Phi_0^R(x, t)$. n^{th} wave moving right is denoted as Φ_n^R, where Φ_n^L denotes moving left. That is, Φ_n^R produces Φ_n^L

at the closed end and Φ_n^L produces Φ_{n+1}^R at the open end. Explicit expressions are

$$\begin{cases} \Phi_n^R = A\sin\left(kx - \omega t + \dfrac{4n\pi L}{\lambda} + n\pi\right) \\ \Phi_n^L = A\sin\left(k(L-x) - \omega t + \dfrac{(4n+2)\pi L}{\lambda} + (n+1)\pi\right) \end{cases} \tag{1}$$

The infinite sum of the reflected waves is

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\Phi_n^R + \Phi_n^L\right) &= 2A\sin\left(kx - \frac{2\pi L}{\lambda}\right)\sum_{n=0}^{\infty}\cos\left(\left(\frac{4L}{\lambda}+1\right)n\pi - \omega t + \frac{2\pi L}{\lambda}\right) \\ &= 2A\sin\left(kx - \varphi\right)\Re\left(\sum_{n=0}^{\infty} e^{i(n\Theta - \omega t + \varphi)}\right) \end{aligned}$$

The series above doesn't converge technically, however, one can find a physically meaningful solution. A partial sum of phasors $\sum_{n=0}^{N} e^{in\Theta}$ always lies on a circle. By taking the limit as energy loss tends to zero, the solution must be the centre of the circle. In a mathematical sense, this technique is called Cesàro sum. Thus, one obtains the composite wave

$$\Psi(x,t) = 2A\sin\left(kx-\varphi\right)\Re\left(\frac{ie^{-i\Theta}}{2\sin\Theta}e^{i(-\omega t+\varphi)}\right) \tag{2}$$

$$= A\sin\left(kx-\varphi\right)\frac{\sin\left(\Theta + \omega t + \varphi\right)}{\sin\Theta} \tag{3}$$

Since this is a standing wave with a wavelength of the original wave, the composite wave contradicts to the boundary condition, i.e., the air pressure at the open end is constant. This result implies that the condition is physically not permitted.