Boundary-Value Problems in Electrostatics

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1 Uniqueness Theorem

1.1 Green's Identities

Applying the divergence theorem to a vector field $\phi\nabla\psi$, Green's first identity

$$\int_{\mathcal{V}} (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\mathrm{d}^3x = \oint_{\partial\mathcal{V}} \phi\nabla\psi\cdot\mathbf{n}\mathrm{d}^2x \tag{1}$$

holds. By interchanging ϕ and ψ and subtracting from eq. (1), we obtain Green's second identity, also known as Green's theorem:

$$\int_{\mathcal{V}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\mathrm{d}^3x = \oint_{\partial\mathcal{V}} (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\mathrm{d}^2x. \tag{2}$$

1.2 Boundary Conditions

Following two boundary conditions on the surface of a volume uniquely determine the solution of the Laplace equation inside the volume.
Dirichlet boundary condition: Specification of the potential
Neumann boundary Condition: Specification of the normal derivative

1.3 Uniqueness Theorem

Consider the Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0} \tag{3}$$

Let there exist two solutions ϕ_1 and ϕ_2; inside a volume $\mathcal{V}$ under the same boundary condition. Then for $\psi = \phi_1 - \phi_2$,

$$\nabla^2\psi = 0 \text{ in } \mathcal{V} \text{ and } \psi = 0 \text{ on } \partial\mathcal{V} \quad \text{(Dirichlet condition)} \tag{4}$$

or

$$\nabla^2\psi = 0 \text{ in } \mathcal{V} \text{ and } \nabla\psi\cdot\mathbf{n} = 0 \text{ on } \partial\mathcal{V} \quad \text{(Neumann condition).} \tag{5}$$

By applying Green's first identity eq. (1), we can find

$$\int_{\mathcal{V}} (\psi\nabla^2\psi + \nabla\psi\cdot\nabla\psi)\mathrm{d}^3x = \oint_{\partial\mathcal{V}} \psi\nabla\psi\cdot\mathbf{n}\mathrm{d}^2x \tag{6}$$

which reduces to

$$\int_{\mathcal{V}} |\nabla\psi|^2\mathrm{d}^3x = 0, \tag{7}$$

so that $\nabla\psi$ be everywhere zero; ψ is a constant. For Dirichlet boundary conditions, $\psi = 0$ on $\partial\mathcal{V}$ thus $\phi_1 = \phi_2$ inside $\mathcal{V}$. Those two types of boundary conditions, of course, also can be used together—Dirichelt's on somewhere $\partial\mathcal{V}$ and Neumann's for the remaining part.

2 Orthonormal Function Expansion

Consider an interval $\mathcal{I}$. A set of square-integrable functions defined in $\mathcal{I}$ is said to be orthonormal if a condition

$$\int_{\mathcal{I}} U_n{}^*(\xi)U_m(\xi)\mathrm{d}\xi = \delta_{nm} \tag{8}$$

holds. When the set of orthonormal functions $U_n(\xi)$ is complete, any function $f(\xi) : \mathcal{I} \to \mathbb{C}$ can be expanded in a series of $U_n(\xi)$:

$$f(\xi) = \sum_{n=1}^{\infty} c_n U_n(\xi). \tag{9}$$

By virtue of the orthogonality, the coefficients are given by integrating both sides of eq. (9). This technique is called Fourier's trick:

$$c_n = \int_{\mathcal{I}} U_n{}^*(\xi)f(\xi)\mathrm{d}\xi.$$

Thus, eq. (9) can be rewritten as

$$f(\xi) = \sum_{n=1}^{\infty}\left(\int_{\mathcal{I}} U_n{}^*(\xi')f(\xi')\mathrm{d}\xi'\right)U_n(\xi) = \int_{\mathcal{I}}\left[\sum_{n=1}^{\infty} U_n{}^*(\xi')U_n(\xi)\right]f(\xi')\mathrm{d}\xi'. \tag{10}$$

Note that the sum in a square bracket is simply the Dirac delta:

$$\sum_{n=1}^{\infty} U_n{}^*(\xi')U_n(\xi) = \delta(\xi' - \xi). \tag{11}$$

If eq. (11) holds, by reverse steps, the relation implies eq. (9); completeness.

The orthonormal function expansion can be generalised in a natural way:

$$f(\xi_1, \cdots \xi_N) = \sum_{n_1,\cdots,n_N} c_{n_1\cdots n_N} U_{1,n_1}(\xi_1)\cdots U_{N,n_N}(\xi_N) \tag{12}$$

where

$$c_{n_1\cdots n_N} = \int_{\mathcal{I}_N} \cdots \int_{\mathcal{I}_1} U_{1,n_1}{}^*(\xi_1)\cdots U_{N,n_N}{}^*(\xi_N) f(\xi_1,\cdots,\xi_N)\mathrm{d}\xi_1\cdots\mathrm{d}\xi_N \tag{13}$$

3 Separation of Variables in Rectangular Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(x,y,z) = X(x)Y(y)Z(z). \tag{14}$$

Now the Laplace equation in rectangular coordinates becomes

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0 \tag{15}$$

Since the equation holds for independent variables, each term must be constant.

3.1 Fourier Series

A Fourier series is one of the most famous examples of expansion by a complete orthogonal set; sines and cosines. Let us consider an interval $\mathcal{I} = (-a/2, a/2)$. The orthonormal functions are

$$\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi nx}{a}\right), \quad \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi nx}{a}\right) \tag{16}$$

where m is a non-negative integer. With this set of functions, eq. (9) becomes

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\sin\left(\frac{2\pi nx}{a}\right) + b_n\cos\left(\frac{2\pi nx}{a}\right)\right] \tag{17}$$

where

$$a_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi nx}{a}\right)\mathrm{d}x, \quad b_n = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi nx}{a}\right)\mathrm{d}x. \tag{18}$$

3.2 Example

Suppose a rectangular box where the boundary condition is given as

$$\begin{cases} \phi(0,*,*) = 0, & \phi(*,0,*) = 0, & \phi(*,*,0) = 0 \\ \phi(a,*,*) = 0, & \phi(*,b,*) = 0, & \phi(*,*,c) = V_0. \end{cases} \tag{19}$$

Suppose each term in the Laplace equation takes the form[1]

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\alpha^2, \quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = -\beta^2, \quad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \gamma^2 \tag{20}$$

[1] To be honest, the determination of signs is slightly intuitive. Though, if our choice of the sign was wrong, α becomes an imaginary number and so gets the correct answer.

where $\alpha^2 + \beta^2 = \gamma^2$. Starting with the conditions at $x = 0$, $y = 0$ or $z = 0$, X, Y and Z must takes the form

$$X = \sin \alpha x, \quad Y = \sin \beta y, \quad Z = \sinh \gamma z. \tag{21}$$

To have remaining conditions for x and y, α, β and γ must takes the form

$$\alpha_n = \frac{n\pi}{a}, \quad \beta_n = \frac{m\pi}{b}, \quad \gamma_{nm} = \sqrt{{\alpha_n}^2 + {\beta_n}^2} = \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\pi.$$

So the general solution can be written as a series

$$\phi(x, y, z) = \sum_{n,m=1}^{\infty} C_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \tag{22}$$

From the last condition

$$\phi(x, y, c) = \sum_{n,m=1}^{\infty} C_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) = V_0, \tag{23}$$

we can obtain the explicit expression of C_{nm}:

$$\begin{aligned} C_{nm} &= \frac{4V_0}{ab \sinh(\gamma_{nm} c)} \int_0^a \int_0^b \sin(\alpha_n x) \sin(\beta_m y) \mathrm{d}x \mathrm{d}y \\ &= \begin{cases} \dfrac{16V_0}{\pi^2 nm \sinh(\gamma_{nm} c)}, & \text{if } n \text{ and } m \text{ are odd,} \\ 0, & \text{if } n \text{ or } m \text{ is even.} \end{cases} \end{aligned}$$

4 Separation of Variables in Spherical Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi). \tag{24}$$

Now the Laplace equation in spherical coordinates becomes

$$\frac{\sin^2 \theta}{R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r} \right) + \frac{\sin \theta}{\Theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin \theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} \right) + \frac{1}{\Phi} \frac{\mathrm{d}^2 \Phi}{\mathrm{d}\varphi^2} = 0. \tag{25}$$

Since only the last term depends on φ, it must be a constant; let us call it $-m^2$—$-m^2$ cannot be positive, since Φ is periodic. Then eq. (25) reduces to

$$\frac{1}{R} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r} \right) + \frac{1}{\Theta \sin \theta} \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sin \theta \frac{\mathrm{d}\Theta}{\mathrm{d}\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0, \tag{26}$$

where r and θ are now separable:

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r} \right) - l(l+1)R = 0, \tag{27}$$

It is known that l can only be non-negative integers.

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\Theta=0. \tag{28}$$

As we've done before, the solutions Φ are straightforward:

$$\Phi=e^{\pm im\varphi}. \tag{29}$$

Since the potential is single-valued, m must be an integer. From the power series expansion of $R(r)$, to satisfy eq. (27), it must takes the form

$$R(r)=A_l r^l+B_l r^{-(l+1)}. \tag{30}$$

4.1 Legendre Polynomials

If the problem has azimuthal symmetry—$\Phi(\varphi)=1$, in other words, $m=0$—, eq. (24) become

$$\phi(r,\theta)=\sum_{l=0}^{\infty}\left[A_l r^l+B_l r^{-(l+1)}\right]P_l(\cos\theta) \tag{31}$$

and eq. (28) reduces to an equation called Legendre equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P(x)}{\mathrm{d}x}\right]+l(l+1)P(x)=0 \tag{32}$$

where $\Theta(\theta)=P(\cos\theta)=P(x)$. Since it is a second-order ODE, it will have two linearly independent solutions. It is known that the solution of eq. (32) can be decomposed by the polynomial part and the non-polynomial part when l is an integer.

The polynomial part normalised to have the value 1 at $x=1$ is called the Legendre polynomial of order l, denoted as $P_l(x)$[2]. It is possible to obtain an explicit expression of Legendre polynomials, known as Rodrigues' formula:

$$P_l(x)=\frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l. \tag{33}$$

The Legendre polynomials form a complete orthogonal set of functions on the interval $\mathcal{I}=(-1,1)$. Its orthogonality provides:

$$\int_{-1}^{1}P_{l'}(x)P_l(x)\mathrm{d}x=\int_0^{\pi}P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta\mathrm{d}\theta=\frac{2}{2l+1}\delta_{l'l}. \tag{34}$$

The orthonormal basis in the sense of eq. (8) is

$$U_l(x)=\sqrt{\frac{2l+1}{2}}P_l(x). \tag{35}$$

[2]The other part is called Legendre function of the second kind, denoted as $Q_l(x)$. Its singularity at $x=\pm1$ is known so that makes this physically useless in many problems.

4.2 Example: Azimuthal Symmetry

Suppose a sphere with radius r_0 whose surface charge density at the surface is given as

$$\sigma(\theta) = k\cos^3\theta.$$

In order to make the potential not to blow at the origin and the infinity, it takes the form:

$$\phi(r,\theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l U_l(\cos\theta), & r < r_0 \\ \sum_{l=0}^{\infty} B_l r^{-(l+1)} U_l(\cos\theta), & r > r_0 \end{cases}. \tag{36}$$

By the continuity of the potential at $r = r_0$ and orthonormality of U_l, it follows that

$$B_l = A_l {r_0}^{2l+1}. \tag{37}$$

Applying Gauss' law for an infinitesimal cylinder, we obtain

$$-\left.\frac{\partial\phi(r,\theta)}{\partial r}\right|_{r=r_0+\epsilon} + \left.\frac{\partial\phi(r,\theta))}{\partial r}\right|_{r=r_0-\epsilon} = \frac{1}{\epsilon_0}\sigma(\theta), \tag{38}$$

equivalently,

$$\sum_{l=0}^{\infty}(2l+1)A_l {r_0}^{l-1} U_l(\cos\theta) = \frac{k}{\epsilon_0}\left(\frac{3}{5}\sqrt{\frac{2}{3}}U_1(\cos\theta) + \frac{2}{5}\sqrt{\frac{2}{7}}P_3(\cos\theta)\right). \tag{39}$$

Using Fourier's trick, the coefficient is now determined:

$$A_1 = \frac{1}{5}\sqrt{\frac{2}{3}}\frac{k}{\epsilon_0}, \quad A_3 = \frac{2}{35{r_0}^2}\sqrt{\frac{2}{7}}\frac{k}{\epsilon_0}. \tag{40}$$

Thus, the solution is

$$\phi(r,\theta) = \begin{cases} \left[\frac{r}{5}\cos\theta + \frac{r^3}{35{r_0}^2}\left(5\cos^3\theta - 3\cos\theta\right)\right]\frac{k}{\epsilon_0}, & r < r_0 \\ \left[\frac{{r_0}^3}{5r^2}\cos\theta + \frac{{r_0}^5}{35r^4}\left(5\cos^3\theta - 3\cos\theta\right)\right]\frac{k}{\epsilon_0}, & r > r_0. \end{cases} \tag{41}$$

4.3 Associated Legendre Polynomials

If there is no azimuthal symmetry in the problem, eq. (28) cannot be reduced to the Legendre equation. In such case, the polynomial $P_l^m(x)$ satisfying the general Legendre equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}P(x)}{\mathrm{d}x}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P(x) = 0 \tag{42}$$

is called the associated Legendre polynomial. It is known that m can only be integers in an interval $[-l, l]$. An explicit expression of $P_l^m(x)$ is given as:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}} (x^2-1)^l. \tag{43}$$

Analogous to the Legendre polynomials, the associated Legendre polynomials also satisfy orthogonality:

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x) \mathrm{d}x = \int_0^\pi P_{l'}^m(\cos\theta) P_l^m(\cos\theta) \sin\theta \mathrm{d}\theta = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l}. \tag{44}$$

4.4 Spherical Harmonics

We've separated θ and φ in general problems. By considering both θ and φ, we can construct a complete orthogonal set of functions on the whole surface of the unit sphere. With normalisation in the sense of eq. (8), the functions called spherical harmonics are given as

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}. \tag{45}$$

By definition, its orthogonality relation

$$\int_0^{2\pi} \int_0^\pi Y_{l'm'}{}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = \delta_{l'l} \delta_{m'm}. \tag{46}$$

holds. Also, spherical harmonics satisfy the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}{}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \delta(\varphi' - \varphi) \delta(\cos\theta' - \cos\theta) \tag{47}$$

so that every function $\phi(r, \theta, \varphi)$ can be expanded in spherical harmonics:

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_l r^l + B_l r^{-(l+1)} \right] Y_{lm}(\theta, \varphi). \tag{48}$$

5 Separation of Variables; Cylindrical Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(s, \varphi, z) = S(s) \Phi(\varphi) Z(z). \tag{49}$$

Now the Laplace equation in cylindrical coordinates becomes

$$\frac{1}{sS} \frac{\mathrm{d}S}{\mathrm{d}s} \left(s \frac{\mathrm{d}S}{\mathrm{d}s} \right) + \frac{1}{s^2 \Phi} \frac{\mathrm{d}^2 \Phi}{\mathrm{d}\varphi^2} + \frac{1}{Z} \frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = 0. \tag{50}$$

We can separating sequentially from φ to s:

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - k^2 Z = 0, \tag{51}$$

$$\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\varphi^2} + \nu^2 \Phi = 0, \tag{52}$$

$$\frac{\mathrm{d}^2 S}{\mathrm{d}s^2} + \frac{1}{s}\frac{\mathrm{d}S}{\mathrm{d}s} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) S = 0. \tag{53}$$

Note that ν must be an integer.

5.1 Bessel Functions

By assuming k is real, eq. (53) can be rewritten as the form called the Bessel equation:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}R}{\mathrm{d}x} + \left(1 - \frac{\nu^2}{x^2}\right) R = 0, \tag{54}$$

where $S(s) = R(ks) = R(x)$. The solutions of this equation are called Bessel functions of order $\pm\nu$. Since the Bessel equation is second-order ODE, the general solution must be written in a linear combination of two linearly independent functions. The part which has no singularity at the origin is called the Bessel function of the first kind $J_\nu(x)$, and the remaining part is so-called the Bessel function of the second kind $Y_\nu(x)$[3].

For the Bessel function of the first kind, $\sqrt{s}J_\nu(x_{\nu n}s/a)$ forms a complete orthogonal set on the interval $\mathcal{I} = [0, a]$. $J_\nu(x_{\nu n}s/a)$ satisfies the orthogonality relation

$$\int_0^a sJ_\nu\left(\frac{x_{\nu n'}s}{a}\right) J_\nu\left(\frac{x_{\nu n}s}{a}\right) \mathrm{d}s = \frac{a^2}{2} {J_{\nu+1}}^2(x_{\nu n})\delta_{n'n}. \tag{55}$$

The Bessel function has an infinite number of roots. A consideration of the roots is important in various situations. One of them can be an equipotential cylinder. The nth root of $J_\nu(x)$ is denoted as $x_{\nu n}$.

Hankel functions are defined as linear combinations of the two kinds of Bessel functions:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{56}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x). \tag{57}$$

If the constant k^2 has been taken as $-k^2$, eq. (53) become

$$\frac{\mathrm{d}^2 R}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}R}{\mathrm{d}x} - \left(1 + \frac{\nu^2}{x^2}\right) R = 0. \tag{58}$$

The solutions of this equation are called modified Bessel functions, given by

$$I_\nu = i^{-\nu} J_\nu(ix) \tag{59}$$

$$K_\nu = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix). \tag{60}$$

[3] It is valid only if ν is an integer. If not, J_ν and $J_{-\nu}$ form a linearly independent pair.

5.2 Example

Suppose a cylinder between $z = 0$ and $z = L$ has a radius r and the axis is the z-axis. Only the top($z = L$) surface of the cylinder has a potential $\phi(s, \varphi, L) = V_0$, and $\phi = 0$ everywhere else on the surface of the cylinder.

In order for the potential not to blow at the axis, the solution $S(s)$ cannot have the second kind component. Since the potential vanishes at $s = r$, the general solution takes the form

$$\phi(s, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}s) \sinh(k_{mn}z)(A_{mn} \sin m\varphi + B_{mn} \cos m\varphi). \tag{61}$$

where $k_{mn}r = x_{mn}$. Using the condition $\phi(s, \varphi, L) = V_0$, we can obtain the coefficients:

$$A_{mn} = \frac{2V_0 \operatorname{csch}(k_{mn}L)}{\pi a^2 {J_{m+1}}^2(k_{mn}r)} \int_0^{2\pi} \int_0^a sJ_m(k_{mn}s) \sin(m\varphi) \mathrm{d}s\mathrm{d}\varphi \tag{62}$$

$$B_{mn} = \frac{2V_0 \operatorname{csch}(k_{mn}L)}{\pi a^2 {J_{m+1}}^2(k_{mn}r)} \int_0^{2\pi} \int_0^a sJ_m(k_{mn}s) \cos(m\varphi) \mathrm{d}s\mathrm{d}\varphi. \tag{63}$$

References

[1] David J. Griffiths. *Introduction to Electrodynamics*. Pearson Education, 4th edition, 2013.

[2] John D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.