1 Experimental Facts

Problem 1.1

Show that every galilean transformation of the space $\mathbb{R} \times \mathbb{R}^3$ can be written in a unique way as the composition of a rotation, a translation, and a uniform motion ($g = g_1 \circ g_2 \circ g_3$)(thus the dimension of the galilean group is equal to $3+4+3 = 10$).

Solution 1.1

Firstly, to preserve the time—a linear mapping $t : \mathbb{R}^4 \to \mathbb{R}$—, every point in the universe should remain in its space of simultaneous events. Or, in other words, planes of simultaneity will remain parallel and gaps between them will not be changed after the transformation. Next, galilean transformations should be affine, i.e., if $x-y = x'-y'$ for some $x, y, x', y' \in A^4$, then $g(x)-g(y) = g(x')-g(y')$. That is, a transformation is an affine if and only if it preserves lines and parallelism. Thus, possible transformations that remained are translations of the origin and shears. Note that rotation is nothing more than a composition of shears. Finally, in one space of simultaneous events—an affine subspace A^3—, to preserve the distance between simultaneous events, all space of spaces of simultaneous events should undergo the same orthogonal transformation.

Problem 1.2

Show that all galilean spaces are isomorphic to each other and, in particular, isomorphic to the coordinate space $\mathbb{R} \times \mathbb{R}^3$.

Solution 1.2

Nothing to show since galilean transformations is a composition of rotations, translations and shears.

Problem 1.3

Is it possible for the trajectory of a differentiable motion on the plane to have the shape drawn in Figure 3? Is it possible for the acceleration vector to have the value shown?

Solution 1.3

Yes. No.

Problem 1.4

Show that if a mechanical system consists of only one point, then its acceleration in an inertial coordinate system is equal to zero ("Newton's first law").

Solution 1.4

By examples 1 and 2 the acceleration vector does not depend on $\mathbf{x}$, $\dot{\mathbf{x}}$, or t, and by Example 3 the vector $\mathbf{F}$ is invariant with respect to rotation. Thus, $\mathbf{F} = \mathbf{0}$ and so the acceleration is equal to zero.

Problem 1.5

A mechanical system consists of two points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points will stay on the line which connected them at the initial moment.

Solution 1.5

At the initial instant, $\mathbf{x}_j - \mathbf{x}_k$ and $\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k$ $(j, k = 1, 2)$ are lying on the line. Since one can choose G as a rotation transformation whose axis is the line in Example 3, $\mathbf{f}_i$ should also be a vector lying on the line:

$$\begin{aligned} G\mathbf{f}_i(\{\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k\}) &= \mathbf{f}_i(\{G(\mathbf{x}_j - \mathbf{x}_k), G(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k)\}) \\ &= \mathbf{f}_i(\{\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k\}). \end{aligned}$$

Problem 1.6

A mechanical system consists of three points. At the initial moment their velocities (in some inertial coordinate system) are equal to zero. Show that the points always remain in the plane which contained them at the initial moment.

Solution 1.6

At the initial instant, $\mathbf{x}_j - \mathbf{x}_k$ and $\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k$ $(j, k = 1, 2, 3)$ are lying on the plane. Since one can choose G as a reflection transformation with respect to the plane, $\mathbf{f}_i$ should also be a vector lying on the plane:

$$\begin{aligned} G\mathbf{f}_i(\{\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k\}) &= \mathbf{f}_i(\{G(\mathbf{x}_j - \mathbf{x}_k), G(\dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k)\}) \\ &= \mathbf{f}_i(\{\mathbf{x}_j - \mathbf{x}_k, \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_k\}). \end{aligned}$$

Problem 1.7

A mechanical system consists of two points. Show that for any initial conditions there exists an inertial coordinate system in which the two points remain in a fixed plane.

Solution 1.7

In the inertial frame that fixes one of the points, the initial state becomes a planar motion. Since one can choose G as a reflection, the statement has been proved.

Problem 1.8

Show that mechanics "through the looking glass" is identical to ours.

Solution 1.8

$$\mathbf{F}(G\mathbf{x}, G\dot{\mathbf{x}}) = G\mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}).$$

In addition, in the case of the magnetic field, its behaviour is not invariant under reflection so the magnetic force behaves well.

Problem 1.9

Is the class of inertial systems unique?

Solution 1.9

No. Other classes can be obtained if one changes the units of length and time or the direction of time.

Problem 1.10

Determine with what velocity a stone must be thrown in order that it fly infinitely far from the surface of the earth.

Solution 1.10

11.2 km/s.
