Vector Calculus Identities

in three-dimensions

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1 Definitions

1.1 Kronecker Delta and Levi-Civita Symbol

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$

Levi-Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

1.2 The Del Operator

Gradient

$$\nabla \varphi = \mathbf{e}_i \partial_i \varphi \tag{3}$$

Divergence

$$\nabla \cdot (F_i \mathbf{e}_i) = \partial_i F_i \tag{4}$$

Curl

$$\nabla \times (F_i \mathbf{e}_i) = \mathbf{e}_k \varepsilon_{ijk} \partial_i F_j \tag{5}$$

1.3 Second Derivatives

Scalar Laplacian

$$\nabla^2\varphi = \nabla\cdot(\nabla\varphi) \tag{6}$$
$$= \partial_i\partial_i\varphi \tag{7}$$

Vector Laplacian

$$\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) - \nabla\times(\nabla\times\mathbf{F}) \tag{8}$$
$$= \mathbf{e}_j\partial_i\partial_i F_j \tag{9}$$

2 Identities

2.1 Kronecker Delta and Levi-Civita Symbol

Kronecekr Delta

$$\delta_{ij}a_i = a_j \tag{10}$$
$$\delta_{ij}\delta_{ik} = \delta_{jk} \tag{11}$$
$$\delta_{ij}\delta_{ij} = 3 \tag{12}$$

Levi-Civita Symbol

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{13}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \tag{14}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{15}$$

2.2 Gradient

$$\nabla(\varphi+\psi) = \nabla\varphi + \nabla\psi \tag{16}$$
$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \tag{17}$$
$$\nabla(\mathbf{F}\cdot\mathbf{G}) = (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) \tag{18}$$

2.3 Divergence

$$\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G} \tag{19}$$
$$\nabla\cdot(\varphi\mathbf{F}) = \varphi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\varphi \tag{20}$$
$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - (\nabla\times\mathbf{G})\cdot\mathbf{F} \tag{21}$$

2.4 Curl

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G} \tag{22}$$
$$\nabla \times (\varphi \mathbf{F}) = \varphi(\nabla \times \mathbf{F}) + \nabla\varphi \times \mathbf{F} \tag{23}$$
$$\nabla \times (\varphi \nabla \psi) = \nabla\varphi \times \nabla\psi \tag{24}$$
$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \tag{25}$$

2.5 Second Derivatives

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{26}$$
$$\nabla \times (\nabla \varphi) = \mathbf{0} \tag{27}$$
$$\nabla \cdot (\varphi \nabla \psi) = \varphi \nabla^2 \psi + \nabla\varphi \cdot \nabla\psi \tag{28}$$
$$\varphi \nabla^2 \psi - \psi \nabla^2 \varphi = \nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) \tag{29}$$
$$\nabla^2 (\varphi \psi) = \varphi \nabla^2 \psi + 2\nabla\varphi \cdot \nabla\psi + \psi \nabla^2 \varphi \tag{30}$$
$$\nabla^2 (\varphi \mathbf{F}) = \mathbf{F} \nabla^2 \varphi + 2(\nabla\varphi \cdot \nabla)\mathbf{F} + \varphi \nabla^2 \mathbf{F} \tag{31}$$
$$\nabla^2 (\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \nabla^2 \mathbf{G} - \mathbf{G} \cdot \nabla^2 \mathbf{F} + 2\nabla \cdot ((\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F})) \tag{32}$$

2.6 Integration

Gradient Theorem

$$\varphi(\mathbf{q}) - \varphi(\mathbf{p}) = \int_C \nabla\varphi(\mathbf{r}) \cdot d\mathbf{r} \tag{33}$$

Divergence Theorem

$$\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV \tag{34}$$

Curl Theorem

$$\oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{35}$$

Green's First Identity

$$\oiint_{\partial V} \varphi \nabla \psi \cdot d\mathbf{S} = \iiint_V (\varphi \nabla^2 \psi + \nabla\varphi \cdot \nabla\psi) dV \tag{36}$$

Green's Second Identity

$$\oiint_{\partial V} (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\mathbf{S} = \iiint_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dV \tag{37}$$

3 Proofs of Identities

Identity 18

$$\begin{aligned}(\mathbf{F}\cdot\nabla)\mathbf{G} &= (F_j\partial_j)(G_i\mathbf{e}_i)\\ &= \mathbf{e}_i F_j\partial_j G_i\end{aligned}$$

$$\begin{aligned}\mathbf{F}\times(\nabla\times\mathbf{G}) &= \mathbf{F}\times(\mathbf{e}_n\varepsilon_{lmn}\partial_l G_m)\\ &= \mathbf{e}_k\varepsilon_{ijk}F_i\varepsilon_{lmj}\partial_l G_m\\ &= \mathbf{e}_k\varepsilon_{jki}\varepsilon_{jlm}F_i\partial_l G_m\\ &= \mathbf{e}_k(\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})F_i\partial_l G_m\\ &= \mathbf{e}_k(F_m\partial_k G_m - F_i\partial_i G_k)\\ &= \mathbf{e}_i(F_j\partial_i G_j - F_j\partial_j G_i)\end{aligned}$$

$$\begin{aligned}\therefore\nabla(\mathbf{F}\cdot\mathbf{G}) &= \mathbf{e}_i\partial_i(F_jG_j)\\ &= \mathbf{e}_i(G_j\partial_i F_j + F_j\partial_i G_j)\\ &= \mathbf{e}_i(F_j\partial_j G_i + G_j\partial_j F_i + G_j\partial_i F_j - G_j\partial_j F_i + F_j\partial_i G_j - F_j\partial_j G_i)\\ &= (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F})\end{aligned}$$

Identity 21

$$\begin{aligned}\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \nabla\cdot(\mathbf{e}_k\varepsilon_{ijk}F_iG_j)\\ &= \partial_k\varepsilon_{ijk}F_iG_j\\ &= \varepsilon_{ijk}(F_i\partial_k G_j + G_j\partial_k F_i)\\ &= G_k\varepsilon_{ijk}\partial_i F_j - F_k\varepsilon_{ijk}\partial_i G_j\\ &= (\nabla\times\mathbf{F})\cdot\mathbf{G} - (\nabla\times\mathbf{G})\cdot\mathbf{F}\end{aligned}$$

Identity 25

$$\begin{aligned}\mathbf{F}(\nabla\cdot\mathbf{G}) &= (F_i\mathbf{e}_i)(\partial_j G_j)\\ &= \mathbf{e}_i F_i\partial_j G_j\end{aligned}$$

$$\begin{aligned}(\mathbf{G}\cdot\nabla)\mathbf{F} &= (G_j\partial_j)(F_i\mathbf{e}_i)\\ &= \mathbf{e}_i G_j\partial_j F_i\end{aligned}$$

$$\begin{aligned}\therefore\nabla\times(\mathbf{F}\times\mathbf{G}) &= \nabla\times(\mathbf{e}_n\varepsilon_{lmn}F_lG_m)\\ &= \mathbf{e}_k\varepsilon_{ijk}\partial_i(\varepsilon_{lmj}F_lG_m)\\ &= \mathbf{e}_k\varepsilon_{jki}\varepsilon_{jlm}(F_l\partial_i G_m + G_m\partial_i F_l)\\ &= \mathbf{e}_k(\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})(F_l\partial_i G_m + G_m\partial_i F_l)\\ &= \mathbf{e}_k(F_k\partial_m G_m - F_i\partial_i G_k + G_i\partial_i F_k - G_k\partial_i F_i)\\ &= \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}\end{aligned}$$