

Part I

Manifolds

1 Elementary Topology

1.1 Topological Spaces

Topology

For a collection of points X , $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfies

1. $\emptyset, X \in \mathcal{T}$
2. $\bigcup_{i \in I} \mathcal{O}_i \in \mathcal{T}$ where $\mathcal{O}_i \in \mathcal{T}$
3. $\bigcap_{i \in I} \mathcal{O}_i \in \mathcal{T}$ where $\mathcal{O}_i \in \mathcal{T}$ and I is finite

called a **topology** on X . A set X endowed with a topology \mathcal{T} is called **topological space**.

Open Sets

Elements of \mathcal{T} are called **open sets**.

1.2 Neighbourhoods and Hausdorff Spaces

Neighbourhood

For a point $P \in X$, a subset $N \subseteq \mathcal{T}$ is called **neighbourhood** of P , if there exists an open set \mathcal{O} such that $P \in \mathcal{O} \subseteq N$.

Hausdorff Space

For arbitrary points p_1, p_2 in topological space (X, \mathcal{T}) , if there exist neighbourhoods $N_1 \ni p_1$ and $N_2 \ni p_2$ such that $N_1 \cap N_2 = \emptyset$, the space is called **Hausdorff space**.

1.3 Homeomorphism

Continuity of Maps

Suppose a map $f : X_1 \rightarrow X_2$ between topological spaces (X_1, \mathcal{T}_1) and (X_2, \mathcal{T}_2) . For an arbitrary open set $\mathcal{O} \in \mathcal{T}_2$, if $f^{-1}(\mathcal{O}) \in \mathcal{T}_1$, f is said to be **continuous**.

Homeomorphism

A map $f : X_1 \rightarrow X_2$ between topological spaces X_1 and X_2 is called **homeomorphism** if it satisfies

1. f is bijective
2. f is continuous
3. Inverse of f is continuous

Topological spaces X_1 and X_2 are said to be **homeomorphic** if there exists homeomorphism between the spaces.

2 Manifolds

Manifold is a *locally euclidean* (i.e., locally homeomorphic to euclidean space) topological space.

Charts

2.1 Definitions

Let $(\mathcal{M}, \mathcal{T})$ be a topological space. For an open set $\mathcal{O} \in \mathcal{M}$ and a homeomorphism $\phi : \mathcal{O} \rightarrow O_i \subseteq \mathbb{R}^n$, a pair (\mathcal{O}, ϕ) is called a **chart** on a topological space \mathcal{M} .

Topological Manifold

For every open set $\mathcal{O}_i \in \mathcal{T}$, if \mathcal{M} covered with $\{\mathcal{O}_i\}$ and provided a family of pairs $\{(\mathcal{O}_i, \phi_i)\}$ where $\phi_i : \mathcal{O}_i \rightarrow O_i$, \mathcal{M} is a homeomorphism from \mathcal{O}_i to $O_i \subseteq \mathbb{R}^n$, the topological space is said to be an n-dimensional **topological manifold**.

2.2 Charts and Coordinates

Charts

The pair (\mathcal{O}_i, ϕ_i) is called a **chart**.

Coordinates

For every point $p \in \mathcal{M}$, a tuple $\phi(p) = (x^1(p), x^2(p), \dots, x^n(p))$ is called a **coordinate** of p .

Atlas

For a manifold \mathcal{M} , an atlas $\{(\mathcal{O}_i, \phi_i)\}$ is a collection of whole charts consisting \mathcal{M} .

Transition Map

Let \mathcal{M} be a manifold whose atlas is $\{(\mathcal{O}_i, \phi_i)\}$. For open sets $\mathcal{O}_i, \mathcal{O}_j \in \mathcal{M}$ if $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$, one can construct coordinate transition from \mathcal{O}_i to \mathcal{O}_j . A map $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ is called **transition map**.

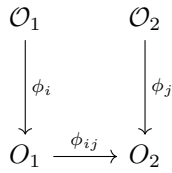


Figure 1: Transition Map

2.3 Differential Structure

Compatiblity of Atlases

If the union of two atlases $\{(\mathcal{O}_i, \phi_i)\} \cup \{(\mathcal{O}'_i, \phi'_i)\}$ is also an atlas on the manifold, the atlases are said to be **compatible**.

Differential Structure

Since the compatibility of atlases is an equivalence equation, it gives an equivalence class, which is called **differential structure**