

Ricci Notation

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1 Index Notation

1.1 Contravariance and Covariance

A p -contravariant- q -covariant component of a tensor T is denoted as $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$.

1.2 Einstein Summation Convention

Summation over an index appearing once each as upper and lower within a term is denoted without the sum sign. For instance,

$$x_\mu e^\mu = \sum_{\mu=0 \text{ or } 1}^n x_\mu e^\mu$$

Such indices are said to be dummy indices, otherwise free indices.

1.3 Raising and Lowering Indices

For a given metric tensor g , an upper index can be lowered or vice versa:

$$\begin{cases} x_\mu = g_{\mu\nu} x^\nu \\ x^\mu = g^{\mu\nu} x_\nu \end{cases}$$

2 Symmetrisation and Antisymmetrisation

2.1 Parentheses and Square Brackets

Suppose a tensor $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$. One can symmetrise $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$ over the first r indices by summing components about permutated indices divided by $r!$, with σ a permutation of $\{1, \cdots, r\}$, that is,

$$T^{\mu_1 \cdots \mu_r \mu_{r+1} \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \frac{1}{r!} \sum_{\sigma} T^{\mu_{\sigma(1)} \cdots \mu_{\sigma(r)} \mu_{r+1} \cdots \mu_p}_{\nu_1 \cdots \nu_q}$$

Symmetrisation is denoted with parentheses. Similarly, the antisymmetrised component, denoted with square brackets, is given as

$$T_{\nu_1 \dots \nu_q}^{[\mu_1 \dots \mu_r] \mu_{r+1} \dots \mu_p} = \frac{1}{r!} \sum_{\sigma} \text{sgn}(\sigma) T_{\nu_1 \dots \nu_q}^{\mu_{\sigma(1)} \dots \mu_{\sigma(r)} \mu_{r+1} \dots \mu_p}$$

2.2 Parcelling Indices

One can symmetrise or antisymmetrise over discrete indices, using vertical bar notation. Say,

$$T_{\nu_1 \dots \nu_q}^{(\mu_1 \dots \mu_s | \mu_{s+1} \dots \mu_t | \mu_{t+1} \dots \mu_r) \mu_{r+1} \dots \mu_p} = \frac{1}{r!} \sum_{\sigma} T_{\nu_1 \dots \nu_q}^{\mu_{\sigma(1)} \dots \mu_{\sigma(s)} \mu_{s+1} \dots \mu_t \mu_{\sigma(t+1)} \dots \mu_{\sigma(r)} \mu_{r+1} \dots \mu_p}$$

Where σ is a permutation of $\{1, \dots, s, t+1, \dots, r\}$.

3 Kronecker Delta and Levi-Civita Symbol

3.1 The Kronecker Delta

The Kronecker delta δ_{ν}^{μ} is a type $(1, 1)$ tensor whose component is 1 only if $\mu = \nu$, otherwise 0. The Kronecker delta behaves as an ‘index reducer’, say,

$$\delta_{\nu}^{\mu} x^{\nu} = x^{\mu}$$

3.2 The Anti-Symmetric Tensor Density

The anti-symmetric tensor density $\varepsilon^{\mu_1 \dots \mu_p} = \varepsilon_{\mu_1 \dots \mu_p}$ is defined as

$$\varepsilon^{\mu_1 \dots \mu_p} = \varepsilon_{\mu_1 \dots \mu_p} = \begin{cases} \text{sgn}(\mu), & \mu \text{ is a permutation of } \{1, \dots, p\} \\ 0, & \text{otherwise, i.e., } \mu_i = \mu_j \text{ for some } i \neq j \end{cases}$$

3.3 The Generalised Kronecker Delta

The p -dimensional generalised Kronecker delta $\delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p}$ is an (p, p) tensor given as

$$\delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p} = \varepsilon^{\mu_1 \dots \mu_p} \varepsilon_{\nu_1 \dots \nu_p}$$

This behaves as an ‘index antisymmetrizer’, say,

$$\frac{1}{p!} \delta_{\nu_1 \dots \nu_p}^{\mu_1 \dots \mu_p} a^{\nu_1 \dots \nu_p} = a^{[\mu_1 \dots \mu_p]}$$

4 Differentiation

4.1 Partial Derivative

A partial derivative is denoted with ∂ .

$$\partial_{\mu} = \frac{\partial}{\partial \mu}$$

4.2 Covariant Derivative

A covariant derivative is denoted with ∇ . The components of the covariant derivative $(\nabla_\nu Y)^\mu$ have been written as $\nabla_\nu Y^\mu$ for simplicity.

$$\begin{aligned}\nabla_\nu f &= \partial_\nu f \\ \nabla_\nu Y^\mu &= \partial_\nu Y^\mu + \Gamma_{\nu\rho}^\mu Y^\rho \\ \nabla_\nu Y_\mu &= \partial_\nu Y_\mu + \Gamma_{\nu\mu}^\rho Y_\rho\end{aligned}$$