# Ricci Notation

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# 1 Index Notation

## 1.1 Contravariance and Covariance

A $p$-contravariant–$q$-covariant component of a tensor $T$ is denoted as $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$.

## 1.2 Einstein Summation Convention

Summation over an index appearing once each as upper and lower within a term is denoted without the sum sign. For instance,

$$x_\mu e^\mu = \sum_{\mu=0 \text{ or } 1}^{n} x_\mu e^\mu$$

Such indices are said to be dummy indices, otherwise free indices.

## 1.3 Rasing and Lowering Indices

For a given metric tensor $g$, an upper index can be lowered or vice versa:

$$\begin{cases} x_\mu = g_{\mu\nu} x^\nu \\ x^\mu = g^{\mu\nu} x_\nu \end{cases}$$

# 2 Symmetrisation and Antisymmetrisation

## 2.1 Parentheses and Square Brackets

Suppose a tensor $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$. One can symmetrise $T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q}$ over the first $r$ indices by summing components about permutated indices divided by $r!$, with $\sigma$ a permutation of $\{1, \cdots, r\}$, that is,

$$T^{(\mu_1 \cdots \mu_r) \mu_{r+1} \cdots \mu_p}_{\nu_1 \cdots \nu_q} = \frac{1}{r!} \sum_{\sigma} T^{\mu_{\sigma(1)} \cdots \mu_{\sigma(r)} \mu_{r+1} \cdots \mu_p}_{\nu_1 \cdots \nu_q}$$

Symmetrisation is denoted with parentheses. Similarly, the antisymmetrised component, denoted with square brackets, is given as

$$T^{[\mu_1\cdots\mu_r]\mu_{r+1}\cdots\mu_p}_{\nu_1\cdots\nu_q} = \frac{1}{r!}\sum_\sigma \operatorname{sgn}(\sigma) T^{\mu_{\sigma(1)}\cdots\mu_{\sigma(r)}\mu_{r+1}\cdots\mu_p}_{\nu_1\cdots\nu_q}$$

Of course, symmetric and antisymmetric part of $T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}$ are denoted as $T^{(\mu_1\cdots\mu_p)}_{\nu_1\cdots\nu_q}$ and $T^{\mu_1\cdots\mu_p}_{(\nu_1\cdots\nu_q)}$, respectively.

### 2.2 Parcelling Indices

One can symmetrise or antisymmetrise over discrete indices, using vertical bar notation. Say,

$$T^{(\mu_1\cdots\mu_s|\mu_{s+1}\cdots\mu_t|\mu_{t+1}\cdots\mu_r)\mu_{r+1}\cdots\mu_p}_{\nu_1\cdots\nu_q} = T^{\mu_{\sigma(1)}\cdots\mu_{\sigma(s)}\mu_{s+1}\cdots\mu_t\mu_{\sigma(t+1)}\cdots\mu_{\sigma(r)}\mu_{r+1}\cdots\mu_p}_{\nu_1\cdots\nu_q}$$

Where $\sigma$ is a permutation of $\{1,\cdots,s,t+1,\cdots r\}$.

## 3 Kronecker Delta and Levi-Civita Symbol

### 3.1 The Kronecker Delta

The Kronecker delta $\delta^\mu_\nu$ is a type (1, 1) tensor whose component is 1 only if $\mu=\nu$, otherwise 0. The Kronecker delta behaves as an 'index reducer', say,

$$\delta^\mu_\nu x^\nu = x^\mu$$

### 3.2 The Anti-Symmetric Tensor Density

The anti-symmetric tensor density $\varepsilon^{\mu_1\cdots\mu_p} = \varepsilon_{\mu_1\cdots\mu_p}$ is defined as

$$\varepsilon^{\mu_1\cdots\mu_p} = \varepsilon_{\mu_1\cdots\mu_p} = \begin{cases} \operatorname{sgn}(\mu), & \mu \text{ is a permutation of } \{1,\cdots p\} \\ 0, & \text{otherwise, i.e., } \mu_i=\mu_j \text{ for some } i\neq j \end{cases}$$

### 3.3 The Generalised Kronecker Delta

The $p$-dimensional generalised Kronecker delta $\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}$ is an $(p,p)$ tensor given as

$$\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} = \varepsilon^{\mu_1\cdots\mu_p}\varepsilon_{\nu_1\cdots\nu_p}$$

This behaves as an 'index antisymmetrizer', say,

$$\frac{1}{p!}\delta^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} a^{[\nu_1\cdots\nu_p]} = a^{[\mu_1\cdots\mu_p]}$$

## 4 Differentiation

### 4.1 Partial Derivative

A partial derivative is simply denoted with $\partial$.

$$\partial_\mu = \frac{\partial}{\partial\mu}$$

## 4.2 Covariant Derivative

A covariant derivative

$$\begin{aligned}\nabla_\nu f &= \partial_\nu f \\ \nabla_\nu Y^\mu &= \partial_\nu Y^\mu + \Gamma^\mu_{\nu\rho} Y^\rho \\ \nabla_\nu Y_\mu &= \partial_\nu Y_\mu + \Gamma^\rho_{\nu\mu} Y_\rho\end{aligned}$$