4 Lagrangian Mechanics on Manifolds

4.1 Differentiable Manifolds

Differentiable manifold

Plenty of non-euclidean spaces can be treated as locally-euclidean. To formulate this, one introduces a concept of **charts**; an open set U in the euclidean coordinate space $\mathbf{q} = (q_1, \cdots, q_n)$, together with a one-to-one map $\phi : U \to \phi U \subseteq M$. A **manifold** is a set M provided with a countable collection of charts so that every point in M can be represented in at least one chart. A **neighbourhood** of a point $\mathbf{x}$ on a manifold M is the image of a neighbourhood of $\mathbf{q} = \phi^{-1}\mathbf{x}$ under a map ϕ. A **Hausdorff manifold** is a manifold that for every two distinct points on M, the points have non-intersecting neighbourhoods.

Suppose two points $\mathbf{p}_1 \in U_1$ and $\mathbf{p}_2 \in U_2$ have the same image in manifold M. When there exists $V_1 = {\phi_1}^{-1}(\phi_1 U_1 \cap \phi_2 U_2)$ and $V_2 = {\phi_2}^{-1}(\phi_1 U_1 \cap \phi_2 U_2)$, one can introduce a coordinate transform ${\phi_2}^{-1}\phi_1 : V \to V'$. The charts U and U' are said to be **compatible** if maps ${\phi_2}^{-1}\phi_1$ and ${\phi_1}^{-1}\phi_2$ are differentiable.

An **atlas** is a union of compatible charts. If the union of atlases is also an atlas, they are called equivalent. A differentiable manifold is an equivalent class of **atlases**. Hereafter, a manifold will be understood as a connected Hausdorff manifold. Then the **dimension** of the manifold is naturally defined.

The configuration space of some dynamical systems can be reduced to a manifold, rather than a Euclidean space. The dimension of the configuration space is called the **degrees of freedom**.

Tangent space and tangent vector

Suppose a k-dimensional manifold M embedded in E^n. Then for every point $\mathbf{x} \in M$, there exists the k-dimensional space that tangents to M. One can define this concept in an intrinsic way; without a notion of embedding.

Suppose a curve $\boldsymbol{\phi}(t) : \mathbb{R} \to M$. In extrinsic viewpoint, a vector tangent to $\boldsymbol{\phi}$ at $\mathbf{x} = \boldsymbol{\phi}(0)$ can be written as differentiation:

$$\dot{\mathbf{x}} = \lim_{t \to 0} \frac{\boldsymbol{\phi}(t) - \boldsymbol{\phi}(0)}{t}.$$

Since all curves satisfies $\boldsymbol{\psi}(0) = \mathbf{x}$ and $\boldsymbol{\psi}'(0) = \dot{\mathbf{x}}$ gives the same vector, the intrinsic definition is of **tangent vector** to M at the point at $\mathbf{x}$ is an equivalence class of such curves. The set of all tangent vectors to M at $\mathbf{x}$ is called the **tangent space**, denoted as $TM_{\mathbf{x}}$. A chart gives the components of the tangent vector naturally.

Tangent bundle

The union of all tangent spaces to n-dimensional manifold M, $\bigcup_{\mathbf{x} \in M} TM_{\mathbf{x}}$ forms $2n$-dimensional manifold whose point is a vector tangent to at some point of M. This manifold is called the **tangent bundle** of M.

The map $p : TM \to M$ which takes a tangent vector at some point to the point is called the **natural projection**. The preimage of a point $\mathbf{x} \in M$, $p^{-1}(\mathbf{x}) = TM_{\mathbf{x}}$, is called the **fiber** of the tangent bundle of M over the point $\mathbf{x}$.

Riemannian manifold

When a manifold M is embedded in a euclidean space, a euclidean metric can be used to measure lengths and angles on M. A manifold endowed with a fixed positive definite quadratic form $\langle \cdot, \cdot \rangle$ on every tangent space $TM_{\mathbf{x}}$ is called **riemannian manifold**.

Derivative map

A map $f : M \to N$ where M and N are manifolds, is said to be **differentiable** if it is given by differentiable functions in local coordinates on M and N.

The derivative of $f : M \to N$ at $\mathbf{x} \in M$, $f_{*\mathbf{x}} : TM_{\mathbf{x}} \to TN_{f(\mathbf{x})}$ is defined as

$$f_{*\mathbf{x}}\mathbf{v} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\boldsymbol{\phi}(t))$$

where $\boldsymbol{\phi} : \mathbb{R} \to M$ is a curve which satisfies $\boldsymbol{\phi}(0) = \mathbf{x}$ and $\mathrm{d}\boldsymbol{\phi}/\mathrm{d}t \,\big|_{t=0} = \mathbf{v}$. By definition, $f_{*\mathbf{x}}\mathbf{v}$ is the velocity vector of the curve $f \circ \boldsymbol{\phi} : \mathbb{R} \to N$.

4.2 Lagrangian Dynamical Systems

Lagrangian system

Let manifold M be the configuration space of a given system. Then one can reformulate the action functional with a lagrangian $L : TM \to \mathbb{R}$ and motion $\gamma : \mathbb{R} \to M$:

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\dot{\boldsymbol{\gamma}})\mathrm{d}t.$$

Of course, $\dot{\boldsymbol{\gamma}}(t)$ is a tangent vector on $\boldsymbol{\gamma}(t)$.

The time evolution of local coordinates $\mathbf{q}$ of $\boldsymbol{\gamma}(t)$ satisfies the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}.$$

The lagrangian system on a riemannian manifold is called **natural** if the lagrangian is given as

$$L(\mathbf{x}, \dot{\mathbf{x}}) = T(\dot{\mathbf{x}}) - U(\mathbf{x}) = \frac{1}{2}\dot{\mathbf{x}}^2 - U(\mathbf{x}),$$

where $T : TM \to \mathbb{R}$ is called the **kinetic energy**.

Holonomic Constraint

If a constraint of motion reduces available configuration space to a lower dimensional manifold, the constraint is said to be **holonomic**. This available surface is given by $k = 3n - m$ functionally independent equations $f_i(\mathbf{r}) = 0$, where the system is constituted with n points and m holonomic constraints.

A system with holonomic constraints can be treated as a system with no constraint but has an additional term $Nf(\mathbf{q})^2$ where $N \to \infty$ in potential energy.

Non-autonomous system

A non-autonomous system has an additional dependence of the lagrangian on time:

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) : TM \times \mathbb{R} \to \mathbb{R}.$$

In this case, holonomic constraint conditions also can depend explicitly on the time so that the submanifold of the configuration space now becomes a function of time.

4.3 Noether's Theorem

Noether's theorem

For a lagrangian system and one-parameter group of diffeomorphisms $h^s : M \to M$, $s \in \mathbb{R}$ such that

$$L(h_* \mathbf{v}) = L(\mathbf{v}), \quad \mathbf{v} \in TM,$$

i.e., a differentiable symmetry of coordinate transform, Noether's theorem states that there exists the corresponding first integral.

Let $M = \mathbb{R}^n$ be a coordinate space $\mathbf{q} = \boldsymbol{\phi}(t) : \mathbb{R} \to M$ be a solution to Lagrange's equations. Since h^s preserves L, $h^s \circ \boldsymbol{\phi} : \mathbb{R} \to M$ also be a solution. Let $\mathbf{\Phi}(s, t) = h^s \circ \boldsymbol{\phi}$. By consider $\mathbf{q} = \mathbf{\Phi}(s_0, t)$ for any fixed $s = s_0$,

$$\begin{aligned}
0 &= \left.\frac{\partial L(\mathbf{\Phi}, \dot{\mathbf{\Phi}})}{\partial s}\right|_{s=s_0} \\
&= \left.\left(\frac{\partial L}{\partial \mathbf{\Phi}} \cdot \frac{\partial \mathbf{\Phi}}{\partial s} + \frac{\partial L}{\partial \dot{\mathbf{\Phi}}} \cdot \frac{\partial \dot{\mathbf{\Phi}}}{\partial s} \right)\right|_{s=s_0} \\
&= \left.\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\partial \mathbf{\Phi}}{\partial s} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathbf{\Phi}}{\partial s} \right)\right|_{s=s_0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\partial \mathbf{\Phi}}{\partial s} \right)\right|_{s=s_0}
\end{aligned}$$

Thus, the quantity

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \left.\frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{\mathrm{d}h^s(\mathbf{q})}{\mathrm{d}s}\right|_{s=0}$$

is conserved. Note that $I(\mathbf{v})$ does not depend on the choice of the coordinate system.

4.4 D'Alembert's Principle

Constraint force

For a point on the configuration manifold to remain on the surface under time evolution, there must be an additional term in Newton's equation $m\ddot{\mathbf{x}} + \partial U/\partial \mathbf{x} = 0$ so only the tangential components of $m\ddot{\mathbf{x}}$ not vanishes. This quantity

$$\mathbf{R}(t) = m\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}}$$

is called **constraint force**.

D'Alembert–Lagrange principle

In lagrangian systems, a tangent vector to the configuration manifold is called **virtual variation**.

D'Alembert-Lagrange principle states that

$$\left(m\ddot{\mathbf{x}} + \frac{\partial U}{\partial \mathbf{x}}\right) \cdot \boldsymbol{\xi} = 0$$

for any virtual variation $\boldsymbol{\xi}$. In other words, the virtual work done by constraint forces is zero.

For a system with multiple points, the principle takes the form

$$\sum_i \left(m\ddot{\mathbf{x}}_i + \frac{\partial U}{\partial \mathbf{x}_i}\right) \cdot \boldsymbol{\xi}_i = 0.$$

Constraints that satisfy this property are said to be **ideal**.

Calculus of variations on manifolds

The action functional of a natural lagrangian system is given by

$$\Phi = \int_{t_0}^{t_1} \left(\frac{1}{2}\dot{\mathbf{x}}^2 - U(\mathbf{x})\right) \mathrm{d}t.$$

The curve $\mathbf{x}$ is called **conditional extremal** if $\delta_M \Phi = 0$ when the variation consists of nearby curves in the configuration manifold M.

The fundamental lemma of calculus of variations now states that, if for every continuous tangent vector field $\boldsymbol{\xi}(t) \in TM_{\mathbf{x}(t)}$ with $\boldsymbol{\xi}(t_0) = \boldsymbol{\xi}(t_1) = 0$, the integral

$$\int_{t_0}^{t_1} \mathbf{f}(t)\boldsymbol{\xi}(t)\mathrm{d}t = 0$$

vanishes, then a continuous vector field $\boldsymbol{f}(t)$ is always perpendicular to M where $t \in (t_0, t_1)$.

Equivalence with the variational principle

When $\delta\mathbf{x} = \boldsymbol{\xi}$, the variation of the action functional is

$$\delta\Phi = \int_{t_0}^{t_1} \left(\dot{\mathbf{x}}\dot{\boldsymbol{\xi}} - \frac{\partial U}{\partial \mathbf{x}}\boldsymbol{\xi} \right) \mathrm{d}t.$$

Since $\boldsymbol{\xi}(t_0) = \boldsymbol{\xi}(t_1) = 0$, the condition that $\mathbf{x}$ to be the extremal becomes equivalent to the d'Alembert's principle.

In conclusion, the following definitions of an ideal holonomic constraint are equivalent:

1. The limit of potential energy $U + NU'$ as $N \to \infty$.
2. The lagrangian system (M, L).
3. The d'Alembert's principle.