# 2  Lie Groups

**Definition 2.1** (Lie groups). A **Lie group** is a group that is also a smooth manifold, where the group operations are smooth.

## 2.1  Generators

Suppose $g \in G$ depends smoothly on a set of continuous parameters; $g(\alpha)$ depends on $\alpha_a$ for $a = 1$ to $N$. It is useful to set

$$g(\alpha)\Big|_{\alpha=0} = e; \quad D(\alpha)\Big|_{\alpha=0} = 1. \tag{2.1}$$

In some neighbourhood of the identity, the Taylor expansion yields

$$D(\mathrm{d}\alpha) = 1 + i\mathrm{d}\alpha_a X_a + \cdots . \tag{2.2}$$

**Definition 2.2** (generators). For $a = 1$ to $N$,

$$X_a \equiv -i\frac{\partial}{\partial \alpha_a} D(\alpha)\Big|_{\alpha=0} \tag{2.3}$$

are called the **generators** of the group.

*Remark* 2.1. If $D$ is unitary, $X_a$ are hermitian.

*Remark* 2.2. The generators, $X_a$'s, form a vector space.

**Claim 2.1** (exponential parametrisation).

$$D(\alpha) = \lim_{k\to\infty} \left(1 + \frac{i\alpha_a X_a}{k}\right)^k = e^{i\alpha_a X_a} \tag{2.4}$$

## 2.2  Lie Algebras

**Claim 2.2.**
$$[\alpha_a X_a, \beta_b X_b] \in \mathrm{span}\{X_a\} \tag{2.5}$$

*Proof.*
$$e^{i\alpha_a X a} e^{i\beta_b X_b} = e^{i\delta_a X_a}, \tag{2.6}$$

where $\delta$ is given by the Baker–Campbell–Hausdorff formula:

$$i\delta_a X_a = i\alpha_a X_a + i\beta_a X_A - \frac{1}{2}[\alpha_a X_a.\beta_b X_b] + \cdots . \tag{2.7}$$

Consider a function

$$Z(t) = \ln\big(1 + e^{it\alpha_a X_a} e^{it\beta_b X_b} - 1\big), \quad t \in \mathbb{R}. \tag{2.8}$$

Taylor expansion with respect to $t$ yields

$$Z(t) = t(\alpha_a X_a + \beta_b X_b) + \frac{1}{2}t^2[\alpha_a X_a, \beta_a X_b] + \mathcal{O}(t^3). \tag{2.9}$$

Since

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = D(\alpha)D(\beta) = D(\alpha\beta), \tag{2.10}$$

$Z(t)$ belongs to span$\{X_a\}$. Therefore, each coefficient of $t^k$, including $[\alpha_a X_a, \beta_b X_b]$, in eq. (2.9) must also lie in the vector space. □

**Definition 2.3** (Lie algebras)**.** A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $F$ with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying bilinearity, alternating property, and the Jacobi identity.

*Remark* 2.3. The Lie algebra is independent of the choice of representation, unless the representation is unfaithful.[3]

**Definition 2.4** (structure constants)**.** The **structure constants** $f_{abc}$ of the group defines the commutator algebra

$$[X_a, X_b] = i f_{abc} X_c, \tag{2.11}$$

where $f_{abc} = -f_{bac}$

**Definition 2.5** (Lie-algebra representations)**.** A **representation** of a Lie-algebra $\mathfrak{g}$ is a mapping, $D$, of the elements of $\mathfrak{g}$ onto a set of linear operators with the following properties:

1. $D$ is linear: $D(aX + bY) = aD(X) + bD(Y)$.

2. $D$ preserves the Lie bracket structure:
   $D([X, Y]) = [D(X), D(Y)] = D(X)D(Y) - D(Y)D(X)$.

**Claim 2.3.** *If there is any unitary representation of the algebra, then the $f_{abc}$'s are real.*

*Proof.*

$$i f_{abc} X_c = -i f_{bac} X_c = -[X_b, X_a] = -[X_a, X_b]^\dagger = i f_{abc}^* X_c \tag{2.12}$$

□

## 2.3   The Jacobi Identity

*Remark* 2.4. The matrix generators satisfy the following identity:

$$[X_a, [X_b, X_c]] + [X_c, [X_a, X_b]] + [X_b, [X_c, X_a]] = 0. \tag{2.13}$$

This is called the Jacobi identity.

*Remark* 2.5.
$$[X_a, X_b X_c] = [X_a, X_b]X_c + X_b[X_a, X_c]. \tag{2.14}$$

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[3]There is vanishing $X_a$ if and only if the representation is unfaithful; we see only the quotient algebra.

## 2.4   The Adjoint Representation

**Definition 2.6** (adjoint representations). The **adjoint representation** $T$ of an algebra is the one generated by the structure constants:

$$[T_a]_{bc} \equiv -i f_{abc}. \tag{2.15}$$

*Proof.* The Jacobi identity implies

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0, \tag{2.16}$$

which is equivalent to

$$[iT_a, iT_b] = -f_{abc}iT_c, \tag{2.17}$$

since

$$[X_a, [X_b, X_c]] = i f_{bcd}[X_a, X_d] = -f_{bcd}f_{abe}X_e. \tag{2.18}$$

$\square$

*Remark* 2.6. The dimension of the adjoint representation is the number of independent generators, which is the number of real parameters required to describe a group element.

**Claim 2.4.**

$$\operatorname{tr}(T_a T_b) = k^a \delta_{ab} \quad (\text{no sum}) \tag{2.19}$$

*Proof.* Consider a linear transformation

$$L : X_a \mapsto X_a' = L_{ab}X_b. \tag{2.20}$$

Then,

$$[X_a', X_b'] = i L_{ad}L_{be}f_{deg}L_{gc}^{-1}X_c' \tag{2.21}$$

implies

$$L : f_{abc} \mapsto f_{abc}' = L_{ad}L_{be}f_{deg}L_{gc}^{-1}, \tag{2.22}$$

or equivalently,

$$L : T_a \mapsto T_a' = L_{ad}L T_d L^{-1}. \tag{2.23}$$

Since similarity transformations do not change the trace,

$$L : \operatorname{tr}(T_a T_b) \mapsto \operatorname{tr}(T_a' T_b') = L_{ac}L_{bd}\operatorname{tr}(T_c T_d). \tag{2.24}$$

By choosing an appropriate orthogonal operator $L$, we get eq. (2.19).   $\square$

*Remark* 2.7. Rescaling $L$ appropriately, we could choose all the non-zero $k^a$'s to have absolute value 1.

**Definition 2.7** (compact Lie algebras). A Lie algebra is said to be **compact** if all $k^a$'s are positive.

For now, only compact Lie algebras will be considered:

$$\operatorname{tr}(T_a T_b) = \lambda \delta_{ab}, \quad \lambda > 0. \tag{2.25}$$

**Claim 2.5.**

$$f_{abc} = -i\lambda^{-1}\operatorname{tr}([T_a, T_b]T_c) \tag{2.26}$$

*Proof.*

$$\operatorname{tr}([T_a, T_b]T_c) = \operatorname{tr}(if_{abd}T_dT_c) = if_{abd}\operatorname{tr}(T_dT_c) = if_{abd}\lambda\delta_{dc} \tag{2.27}$$

$\square$

**Claim 2.6.** $f_{abc}$ *is completely antisymmetric.*

*Proof.*

$$f_{abc} = f_{bca} \tag{2.28}$$

since

$$\begin{aligned}
\operatorname{tr}([T_a, T_b]T_c) &= \operatorname{tr}(T_aT_bT_c - T_bT_aT_c) \\
&= \operatorname{tr}(T_bT_cT_a - T_cT_bT_a) = \operatorname{tr}([T_b, T_c]T_a).
\end{aligned} \tag{2.29}$$

$\square$

## 2.5   Simple Algebras and Groups

**Definition 2.8** (invariant subalgebras)**.** An **invariant subalgebra** is some set of generators which goes into itself under commutation with any element of the algebra. I.e., if $X$ is any generator in the invariant subalgebra and $Y$ is any generator in the whole algebra, then $[Y, X]$ is a generator in the invariant subalgebra.

**Claim 2.7.** *An invariant subalgebra generates the corresponding invariant subgroup by exponentiation, i.e.,*

$$e^{-iY}e^{iX}e^{iY} = e^{iX'}, \tag{2.30}$$

*where*

$$X' = e^{-iY}Xe^{iY}. \tag{2.31}$$

*Proof.*

$$e^{-iY}(iX)^n e^{iY} = (e^{-iY}iXe^{iY})^n \tag{2.32}$$

$\square$

**Definition 2.9** (trivial algebras)**.** An algebra which has no nontrivial invariant subalgebra is called **simple**, where the whole algebra and 0 are trivial invariant subalgebra.

*Remark* 2.8. A simple algebra generates a simple group.

**Claim 2.8.** *The adjoint representation of a simple Lie algebra is irreducible.*[4]

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[4]The simplicity of a group does not automatically imply the irreducibility of the adjoint representation.

*Proof.* Assume the contrary. Then there is an invariant subspace spanned by $T_r$ for $x = 1$ to $K$, where the rest of the generators $T_x$ for $x = K+1$ to $N$. Therefore, since

$$[T_a, T_x] = i f_{axr} T_r \tag{2.33}$$

we must have

$$[T_a]_{xr} = -i f_{axr} = 0. \tag{2.34}$$

By virtue of the complete antisymmetry of the structure constants, the nonzero structure constants involve either three $r$'s or three $x$'s, and thus the algebra falls apart into two nontrivial invariant subalgebras, and is not simple. □

**Definition 2.10** (semisimple algebras). Algebras without Abelian invariant subalgebras are called **semisimple**.

## 2.7   Fun with Exponentials

**Claim 2.9.**

$$\frac{\partial}{\partial \alpha_b} e^{i \alpha_a X_a} = \int_0^1 \mathrm{d}s \; e^{i s \alpha_a X_a} i X_b e^{i(1-s)\alpha_c X_c} \tag{2.35}$$

*Proof.* We can always define the exponential as a power series:

$$e^{i \alpha_a X_a} = \sum_{n=0}^{\infty} \frac{1}{n!} (i \alpha_a X_a)^n. \tag{2.36}$$

Referring to eq. (2.4),

$$\frac{\partial}{\partial \alpha_b} e^{i \alpha_a X_a} = \frac{\partial}{\partial \alpha_b} \lim_{k \to \infty} \left( 1 + \frac{i \alpha X_a}{k} \right)^k \tag{2.37}$$

$$= \lim_{k \to \infty} \sum_{l=1}^{k} \left( 1 + \frac{i \alpha X_a}{k} \right)^{l-1} \left( \frac{i X_b}{k} \right) \left( 1 + \frac{i \alpha X_a}{k} \right)^{k-l} \tag{2.38}$$

$$= \int_0^1 \mathrm{d}s \; e^{i s \alpha_a X_a} i X_b e^{i(1-s)\alpha_c X_c}. \tag{2.39}$$

□