

# Boundary-Value Problems in Electrostatics

M. Yoon

Published on July 17, 2023

---

## 1 Uniqueness Theorem

### 1.1 Green's Identities

Applying the divergence theorem to a vector field  $\phi \nabla \psi$ , Green's first identity

$$\int_{\mathcal{V}} (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3x = \oint_{\partial \mathcal{V}} \phi \nabla \psi \cdot \mathbf{n} d^2x \quad (1)$$

holds. By interchanging  $\phi$  and  $\psi$  and subtracting from eq. (1), we obtain Green's second identity, also known as Green's theorem:

$$\int_{\mathcal{V}} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3x = \oint_{\partial \mathcal{V}} (\phi \nabla \psi - \psi \nabla \phi) \cdot \mathbf{n} d^2x. \quad (2)$$

### 1.2 Boundary Conditions

Following two boundary conditions on the surface of a volume uniquely determine the solution of the Laplace equation inside the volume.

**Dirichlet boundary condition:** Specification of the potential

**Neumann boundary Condition:** Specification of the normal derivative

### 1.3 Uniqueness Theorem

Consider the Poisson equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \quad (3)$$

Let there exist two solutions  $\phi_1$  and  $\phi_2$ ; inside a volume  $\mathcal{V}$  under the same boundary condition. Then for  $\psi = \phi_1 - \phi_2$ ,

$$\nabla^2 \psi = 0 \text{ in } \mathcal{V} \text{ and } \psi = 0 \text{ on } \partial \mathcal{V} \quad (\text{Dirichlet condition}) \quad (4)$$

or

$$\nabla^2\psi = 0 \text{ in } \mathcal{V} \text{ and } \nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial\mathcal{V} \quad (\text{Neumann condition}). \quad (5)$$

By applying Green's first identity eq. (1), we can find

$$\int_{\mathcal{V}} (\psi \nabla^2\psi + \nabla\psi \cdot \nabla\psi) d^3x = \oint_{\partial\mathcal{V}} \psi \nabla\psi \cdot \mathbf{n} d^2x \quad (6)$$

which reduces to

$$\int_{\mathcal{V}} |\nabla\psi|^2 d^3x = 0, \quad (7)$$

so that  $\nabla\psi$  be everywhere zero;  $\psi$  is a constant. For Dirichlet boundary conditions,  $\psi = 0$  on  $\partial\mathcal{V}$  thus  $\phi_1 = \phi_2$  inside  $\mathcal{V}$ . Those two types of boundary conditions, of course, also can be used together—Dirichlet's on somewhere  $\partial\mathcal{V}$  and Neumann's for the remaining part.

## 2 Orthonormal Function Expansion

Consider an interval  $\mathcal{I}$ . A set of square-integrable functions defined in  $\mathcal{I}$  is said to be orthonormal if a condition

$$\int_{\mathcal{I}} U_n^*(\xi) U_m(\xi) d\xi = \delta_{nm} \quad (8)$$

holds. When the set of orthonormal functions  $U_n(\xi)$  is complete, any function  $f(\xi) : \mathcal{I} \rightarrow \mathbb{C}$  can be expanded in a series of  $U_n(\xi)$ :

$$f(\xi) = \sum_{n=1}^{\infty} c_n U_n(\xi). \quad (9)$$

By virtue of the orthogonality, the coefficients are given by integrating both sides of eq. (9). This technique is called Fourier's trick:

$$c_n = \int_{\mathcal{I}} U_n^*(\xi) f(\xi) d\xi.$$

Thus, eq. (9) can be rewritten as

$$f(\xi) = \sum_{n=1}^{\infty} \left( \int_{\mathcal{I}} U_n^*(\xi') f(\xi') d\xi' \right) U_n(\xi) = \int_{\mathcal{I}} \left[ \sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) \right] f(\xi') d\xi'. \quad (10)$$

Note that the sum in a square bracket is simply the Dirac delta:

$$\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi). \quad (11)$$

If eq. (11) holds, by reverse steps, the relation implies eq. (9).

The orthonormal function expansion can be generalised in a natural way:

$$f(\xi_1, \dots, \xi_N) = \sum_{n_1, \dots, n_N} c_{n_1 \dots n_N} U_{1, n_1}(\xi_1) \dots U_{N, n_N}(\xi_N) \quad (12)$$

where

$$c_{n_1 \dots n_N} = \int_{\mathcal{I}_N} \cdots \int_{\mathcal{I}_1} U_{1,n_1}^*(\xi_1) \cdots U_{N,n_N}^*(\xi_N) f(\xi_1, \dots, \xi_N) d\xi_1 \cdots d\xi_N \quad (13)$$

### 3 Separation of Variables in Rectangular Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(x, y, z) = X(x)Y(y)Z(z). \quad (14)$$

Now the Laplace equation in rectangular coordinates becomes

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad (15)$$

Since the equation holds for independent variables, each term must be constant.

#### 3.1 Fourier Series

A Fourier series is one of the most famous examples of expansion by a complete orthogonal set; sines and cosines. Let us consider an interval  $\mathcal{I} = (-a/2, a/2)$ . The orthonormal functions are

$$\sqrt{\frac{2}{a}} \sin\left(\frac{2\pi n x}{a}\right), \quad \sqrt{\frac{2}{a}} \cos\left(\frac{2\pi n x}{a}\right) \quad (16)$$

where  $m$  is a non-negative integer. With this set of functions, eq. (9) becomes

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[ a_n \sin\left(\frac{2\pi n x}{a}\right) + b_n \cos\left(\frac{2\pi n x}{a}\right) \right] \quad (17)$$

where

$$a_n = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos\left(\frac{2\pi n x}{a}\right) dx, \quad b_n = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin\left(\frac{2\pi n x}{a}\right) dx. \quad (18)$$

#### 3.2 Example

Suppose a rectangular box where the boundary condition is given as

$$\begin{cases} \phi(0, *, *) = 0, & \phi(*, 0, *) = 0, & \phi(*, *, 0) = 0 \\ \phi(a, *, *) = 0, & \phi(*, b, *) = 0, & \phi(*, *, c) = V_0. \end{cases} \quad (19)$$

Suppose each term in the Laplace equation takes the form<sup>1</sup>

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\alpha^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\beta^2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = \gamma^2 \quad (20)$$

---

<sup>1</sup>To be honest, the determination of signs is slightly intuitive. Though, if our choice of the sign was wrong,  $\alpha$  becomes an imaginary number and so gets the correct answer.

where  $\alpha^2 + \beta^2 = \gamma^2$ . Starting with the conditions at  $x = 0$ ,  $y = 0$  or  $z = 0$ ,  $X$ ,  $Y$  and  $Z$  must takes the form

$$X = \sin \alpha x, \quad Y = \sin \beta y, \quad Z = \sinh \gamma z. \quad (21)$$

To have remaining conditions for  $x$  and  $y$ ,  $\alpha$ ,  $\beta$  and  $\gamma$  must takes the form

$$\alpha_n = \frac{n\pi}{a}, \quad \beta_n = \frac{m\pi}{b}, \quad \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_n^2} = \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}\pi.$$

So the general solution can be written as a series

$$\phi(x, y, z) = \sum_{n,m=1}^{\infty} C_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \quad (22)$$

From the last condition

$$\phi(x, y, c) = \sum_{n,m=1}^{\infty} C_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) = V_0, \quad (23)$$

we can obtain the explicit expression of  $C_{nm}$ :

$$\begin{aligned} C_{nm} &= \frac{4V_0}{ab \sinh(\gamma_{nm} c)} \int_0^a \int_0^b \sin(\alpha_n x) \sin(\beta_m y) dx dy \\ &= \begin{cases} \frac{16V_0}{\pi^2 nm \sinh(\gamma_{nm} c)}, & \text{if } n \text{ and } m \text{ are odd,} \\ 0, & \text{if } n \text{ or } m \text{ is even.} \end{cases} \end{aligned}$$

## 4 Separation of Variables in Spherical Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi). \quad (24)$$

Now the Laplace equation in spherical coordinates becomes

$$\frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\varphi^2} = 0. \quad (25)$$

Since only the last term depends on  $\varphi$ , it must be a constant; let us call it  $-m^2$ — $-m^2$  cannot be positive, since  $\Phi$  is periodic. Then eq. (25) reduces to

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = 0, \quad (26)$$

where  $r$  and  $\theta$  are now separable:

$$\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - l(l+1)R = 0, \quad (27)$$

It is known that  $l$  can only be non-negative integers.

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left[ l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0. \quad (28)$$

As we've done before, the solutions  $\Phi$  are straightforward:

$$\Phi = e^{\pm im\varphi}. \quad (29)$$

Since the potential is single-valued,  $m$  must be an integer. From the power series expansion of  $R(r)$ , to satisfy eq. (27), it must take the form

$$R(r) = A_l r^l + B_l r^{-(l+1)}. \quad (30)$$

## 4.1 Legendre Polynomials

If the problem has azimuthal symmetry— $\Phi(\varphi) = 1$ , in other words,  $m = 0$ —, eq. (24) become

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos \theta) \quad (31)$$

and eq. (28) reduces to an equation called Legendre equation:

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP(x)}{dx} \right] + l(l+1)P(x) = 0 \quad (32)$$

where  $\Theta(\theta) = P(\cos \theta) = P(x)$ . Since it is a second-order ODE, it will have two linearly independent solutions. It is known that the solution of eq. (32) can be decomposed by the polynomial part and the non-polynomial part when  $l$  is an integer.

The polynomial part normalised to have the value 1 at  $x = 1$  is called the Legendre polynomial of order  $l$ , denoted as  $P_l(x)$ <sup>2</sup>. It is possible to obtain an explicit expression of Legendre polynomials, known as Rodrigues' formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \quad (33)$$

The Legendre polynomials form a complete orthogonal set of functions on the interval  $\mathcal{I} = (-1, 1)$ . Its orthogonality provides:

$$\int_{-1}^1 P_{l'}(x) P_l(x) dx = \int_0^\pi P_{l'}(\cos \theta) P_l(\cos \theta) \sin \theta d\theta = \frac{2}{2l+1} \delta_{l'l}. \quad (34)$$

The orthonormal basis in the sense of eq. (8) is

$$U_l(x) = \sqrt{\frac{2l+1}{2}} P_l(x). \quad (35)$$

---

<sup>2</sup>The other part is called Legendre function of the second kind, denoted as  $Q_l(x)$ . Its singularity at  $x = \pm 1$  is known so that makes this physically useless in many problems.

## 4.2 Example: Azimuthal Symmetry

Suppose a sphere with radius  $r_0$  whose surface charge density at the surface is given as

$$\sigma(\theta) = k \cos^3 \theta.$$

In order to make the potential not to blow at the origin and the infinity, it takes the form:

$$\phi(r, \theta) = \begin{cases} \sum_{l=0}^{\infty} A_l r^l U_l(\cos \theta), & r < r_0 \\ \sum_{l=0}^{\infty} B_l r^{-(l+1)} U_l(\cos \theta), & r > r_0 \end{cases}. \quad (36)$$

By the continuity of the potential at  $r = r_0$  and orthonormality of  $U_l$ , it follows that

$$B_l = A_l r_0^{2l+1}. \quad (37)$$

Applying Gauss' law for an infinitesimal cylinder, we obtain

$$-\left. \frac{\partial \phi(r, \theta)}{\partial r} \right|_{r=r_0+\epsilon} + \left. \frac{\partial \phi(r, \theta)}{\partial r} \right|_{r=r_0-\epsilon} = \frac{1}{\epsilon_0} \sigma(\theta), \quad (38)$$

equivalently,

$$\sum_{l=0}^{\infty} (2l+1) A_l r_0^{l-1} U_l(\cos \theta) = \frac{k}{\epsilon_0} \left( \frac{3}{5} \sqrt{\frac{2}{3}} U_1(\cos \theta) + \frac{2}{5} \sqrt{\frac{2}{7}} P_3(\cos \theta) \right). \quad (39)$$

Using Fourier's trick, the coefficient is now determined:

$$A_1 = \frac{1}{5} \sqrt{\frac{2}{3}} \frac{k}{\epsilon_0}, \quad A_3 = \frac{2}{35 r_0^2} \sqrt{\frac{2}{7}} \frac{k}{\epsilon_0}. \quad (40)$$

Thus, the solution is

$$\phi(r, \theta) = \begin{cases} \left[ \frac{r}{5} \cos \theta + \frac{r^3}{35 r_0^2} (5 \cos^3 \theta - 3 \cos \theta) \right] \frac{k}{\epsilon_0}, & r < r_0 \\ \left[ \frac{r_0^3}{5 r^2} \cos \theta + \frac{r_0^5}{35 r^4} (5 \cos^3 \theta - 3 \cos \theta) \right] \frac{k}{\epsilon_0}, & r > r_0. \end{cases} \quad (41)$$

## 4.3 Associated Legendre Polynomials

If there is no azimuthal symmetry in the problem, eq. (28) cannot be reduced to the Legendre equation. In such case, the polynomial  $P_l^m(x)$  satisfying the general Legendre equation

$$\frac{d}{dx} \left[ (1-x^2) \frac{dP(x)}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] P(x) = 0 \quad (42)$$

is called the associated Legendre polynomial. It is known that  $m$  can only be integers in an interval  $[-l, l]$ . An explicit expression of  $P_l^m(x)$  is given as:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l. \quad (43)$$

Analogous to the Legendre polynomials, the associated Legendre polynomials also satisfy orthogonality:

$$\int_{-1}^1 P_{l'}^m(x) P_l^m(x) dx = \int_0^\pi P_{l'}^m(\cos \theta) P_l^m(\cos \theta) \sin \theta d\theta = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{l'l}. \quad (44)$$

#### 4.4 Spherical Harmonics

We've separated  $\theta$  and  $\varphi$  in general problems. By considering both  $\theta$  and  $\varphi$ , we can construct a complete orthogonal set of functions on the whole surface of the unit sphere. With normalisation in the sense of eq. (8), the functions called spherical harmonics are given as

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}. \quad (45)$$

By definition, its orthogonality relation

$$\int_0^{2\pi} \int_0^\pi Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) \sin \theta d\theta d\varphi = \delta_{l'l} \delta_{m'm}. \quad (46)$$

holds. Also, spherical harmonics satisfy the completeness relation

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \delta(\varphi' - \varphi) \delta(\cos \theta' - \cos \theta) \quad (47)$$

so that every function  $\phi(r, \theta, \varphi)$  can be expanded in spherical harmonics:

$$\phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left[ A_l r^l + B_l r^{-(l+1)} \right] Y_{lm}(\theta, \varphi). \quad (48)$$

## 5 Separation of Variables; Cylindrical Coordinates

By the assumption of separation of variables, the solution can be written in the form:

$$\phi(s, \varphi, z) = S(s) \Phi(\varphi) Z(z). \quad (49)$$

Now the Laplace equation in cylindrical coordinates becomes

$$\frac{1}{sS} \frac{dS}{ds} \left( s \frac{dS}{ds} \right) + \frac{1}{s^2 \Phi} \frac{d^2 \Phi}{d\varphi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (50)$$

We can separating sequentially from  $\varphi$  to  $s$ :

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0, \quad (51)$$

$$\frac{d^2 \Phi}{d\varphi^2} + \nu^2 \Phi = 0, \quad (52)$$

$$\frac{d^2 S}{ds^2} + \frac{1}{s} \frac{dS}{ds} + \left( k^2 - \frac{\nu^2}{\rho^2} \right) S = 0. \quad (53)$$

Note that  $\nu$  must be an integer.

## 5.1 Bessel Functions

By assuming  $k$  is real, eq. (53) can be rewritten as the form called the Bessel equation:

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left( 1 - \frac{\nu^2}{x^2} \right) R = 0, \quad (54)$$

where  $S(s) = R(ks) = R(x)$ . The solutions of this equation are called Bessel functions of order  $\pm\nu$ . Since the Bessel equation is second-order ODE, the general solution must be written in a linear combination of two linearly independent functions. The part which has no singularity at the origin is called the Bessel function of the first kind  $J_\nu(x)$ , and the remaining part is so-called the Bessel function of the second kind  $Y_\nu(x)$ <sup>3</sup>.

For the Bessel function of the first kind,  $\sqrt{s}J_\nu(x_{\nu n}s/a)$  forms a complete orthogonal set on the interval  $\mathcal{I} = [0, a]$ .  $J_\nu(x_{\nu n}s/a)$  satisfies the orthogonality relation

$$\int_0^a s J_\nu \left( \frac{x_{\nu n'} s}{a} \right) J_\nu \left( \frac{x_{\nu n} s}{a} \right) ds = \frac{a^2}{2} J_{\nu+1}^2(x_{\nu n}) \delta_{n'n}. \quad (55)$$

The Bessel function has an infinite number of roots. A consideration of the roots is important in various situations. One of them can be an equipotential cylinder. The  $n$ th root of  $J_\nu(x)$  is denoted as  $x_{\nu n}$ .

Hankel functions are defined as linear combinations of the two kinds of Bessel functions:

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \quad (56)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x). \quad (57)$$

If the constant  $k^2$  has been taken as  $-k^2$ , eq. (53) become

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left( 1 + \frac{\nu^2}{x^2} \right) R = 0. \quad (58)$$

The solutions of this equation are called modified Bessel functions, given by

$$I_\nu = i^{-\nu} J_\nu(ix) \quad (59)$$

$$K_\nu = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix). \quad (60)$$

---

<sup>3</sup>It is valid only if  $\nu$  is an integer. If not,  $J_\nu$  and  $J_{-\nu}$  form a linearly independent pair.



## 5.2 Example

Suppose a cylinder between  $z = 0$  and  $z = L$  has a radius  $r$  and the axis is the  $z$ -axis. Only the top( $z = L$ ) surface of the cylinder has a potential  $\phi(s, \varphi, L) = V_0$ , and  $\phi = 0$  everywhere else on the surface of the cylinder.

In order for the potential not to blow at the axis, the solution  $S(s)$  cannot have the second kind component. Since the potential vanishes at  $s = r$ , the general solution takes the form

$$\phi(s, \varphi, z) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m(k_{mn}s) \sinh(k_{mn}z) (A_{mn} \sin m\varphi + B_{mn} \cos m\varphi). \quad (61)$$

where  $k_{mn}r = x_{mn}$ . Using the condition  $\phi(s, \varphi, L) = V_0$ , we can obtain the coefficients:

$$A_{mn} = \frac{2V_0 \operatorname{csch}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}r)} \int_0^{2\pi} \int_0^a s J_m(k_{mn}s) \sin(m\varphi) ds d\varphi \quad (62)$$

$$B_{mn} = \frac{2V_0 \operatorname{csch}(k_{mn}L)}{\pi a^2 J_{m+1}^2(k_{mn}r)} \int_0^{2\pi} \int_0^a s J_m(k_{mn}s) \cos(m\varphi) ds d\varphi. \quad (63)$$

## References

- [1] David J. Griffiths. *Introduction to Electrodynamics*. Pearson Education, 4th edition, 2013.
- [2] John D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.