1 Finite Groups

1.1 Groups and Representations

$$\text{Group } (G,\cdot)\begin{cases}\text{closure} & f\cdot g\in G\\ \text{associativity} & f\cdot(g\cdot h)=(f\cdot g)\cdot h\\ \text{existence of identity} & \exists e \text{ s.t. } \forall f: e\cdot f=f\cdot e=f\\ \text{existence of inverse} & \forall f\in G\ \exists f^{-1} \text{ s.t. } f\cdot f^{-1}=f^{-1}\cdot f=e\end{cases}$$

Example 1.1. $\mathbb{Z}_2 \doteq (\{0,1\},+_{\text{mod2}}) \doteq (\{1,-1\},\times)$

Definition 1.1 (group representations)**.** A **representation** of a group G is a mapping, D, of the elements of G onto a set of linear operators with the following properties:

1. $D(e)=1$, where 1 is the identity operator in the space on which the linear operators act.
2. $D(g_1)D(g_2)=D(g_1g_2)$, in other words, the group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act.

Remark 1.1.

$D(g)=1$: trivial representation

$D(g)=1\to g=e$: non-trivial representation

1.2 Example—$\mathbb{Z}_3$

Definition 1.2 (finite groups)**.** A group is **finite** if it has a finite number of elements. Otherwise, it is infinite.

Definition 1.3 (order)**.** The **order** of a group G, $\operatorname{ord} G$, is the number of elements in a finite group G.

Definition 1.4 (abelian groups)**.** An **abelian group** is one in which the multiplication law is commutative.

Table 1: The Cayley table of $\mathbb{Z}_3$

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Example 1.2. $D(e)=1$, $D(a)=\exp(2\pi i/3)$, $D(b=a^{-1})=\exp(4\pi i/3)$ is a representation of $\mathbb{Z}_3$.

Definition 1.5 (dimension)**.** The **dimension** of a representation is the dimension of the linear space in which it acts.

1.3 The Regular Representation

Definition 1.6 (regular representations)**.** The regular representation D of a group G is a representation which is defined by

$$D(g_1)|g_2\rangle = |g_1 g_2\rangle\,. \tag{1.1}$$

Remark 1.2. For the regular representation D of a group G, $\dim D = \operatorname{ord} G$.

Remark 1.3. Consider the regular representation D.

$$[D(g)]_{ij} = \langle e_i|D(g)|e_j\rangle \tag{1.2}$$

$$\begin{aligned}
[D(g_1g_2)]_{ij} &= \langle e_i|D(g_1)D(g_2)|e_j\rangle \\
&= \sum_k \langle e_i|D(g_1)|e_k\rangle\langle e_k|D(g_2)|e_j\rangle \\
&= \sum_k [D(g_1)]_{ik}[D(g_2)]_{kj} \\
&= [D(g_1)D(g_2)]_{ij}
\end{aligned} \tag{1.3}$$

Example 1.3. The regular representation of $\mathbb{Z}_3$ is

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

1.4 Irreducible Representations

A group representation is powerful since it is in a linear space, such that a basis change is allowed.

Definition 1.7 (equivalent representations)**.** Two representations D' and D are **equivalent** if

$$\exists S \text{ s.t. } \forall g \in G : D'(G) = S^{-1}D(g)S. \tag{1.5}$$

Definition 1.8 (unitary representations)**.** A representation is **unitary** if

$$\forall g \in G : D(g)^\dagger = D(g)^{-1}. \tag{1.6}$$

Definition 1.9 (reducible representations)**.** A representation is reducible if it has an invariant subspace. I.e., there exists a projection operator P onto the subspace so that

$$\forall g \in G : PD(g)P = D(g)P. \tag{1.7}$$

Example 1.4. The regular representation of $\mathbb{Z}_3$ has an invariant subspace projected on by

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{1.8}$$

Definition 1.10 (completely reducible representations)**.** A representation is **completely reducible** if it is equivalent to a representation whose matrix elements have the block diagonal form. I.e.,

$$D(g) \sim \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.9}$$

where $D_j(g)$ is irreducible for all j.

Remark 1.4. A completely reducible representation can be expressed as a direct sum of irreducible representations, i.e.,

$$D(G) = D_1(g) \oplus D_2(g) \oplus \cdots \oplus D_n(g). \tag{1.10}$$

Example 1.5. For ex. 1.3, take

$$S = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad \text{where } \omega = e^{2\pi i/3}. \tag{1.11}$$

Then,

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad D'(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \tag{1.12}$$

i.e.,

$$D'(g_j) = 1^{j+2} \oplus \omega^{j+2} \oplus (\omega^2)^{j+2}, \tag{1.13}$$

where $e = g_1$, $a = g_2$, and $b = g_3$.

1.8 Example: Addition of Integers

Consider a two-dimensional representation

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{1.14}$$

of a group $(\mathbb{Z}, +)$. This is reducible but not completely reducible since

$$D(x)P = P \tag{1.15}$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.16}$$

but

$$D(x)(I - P) \neq (I - P). \tag{1.17}$$

1.9 Useful Theorems

Theorem 1.1. *Every representation of a finite group is equivalent to a unitary representation.*

Proof. Suppose $D(g)$ is a representation of a finite group G. Construct the operator

$$S = \sum_{g \in G} D(g)^\dagger D(g). \tag{1.18}$$

By construction, S is hermitian and positive-semidefinite, i.e.,

$$S = U^{-1} \mathbb{D} U \tag{1.19}$$

where $\mathbb{D} = \mathrm{diag}(d_1, d_2, \cdots)$, where $d_j \geq 0$ for all j.

Assume that $d_j = 0$ for some j. Then for an corresponding eigenvector λ_j of S, i.e., $S\lambda_j = 0$,

$$\lambda_j^{\mathrm{T}} S \lambda_j = \sum_{g \in G} \left\| D(g)\lambda_j \right\|^2 > \left\| D(e)\lambda_j \right\|^2 = \left\| \lambda_j \right\|^2 > 0. \tag{1.20}$$

Thereby we conclude that $d_j > 0$ for all j.

Therefore, we can construct X a square-root of S that is hermitian invertible as follows:

$$X = S^{1/2} \equiv U^{-1} \mathbb{D}^{1/2} U. \tag{1.21}$$

Now we define

$$D'(g) = X D(g) X^{-1}. \tag{1.22}$$

It is unitary since

$$\begin{aligned} D'(g)^\dagger D'(g) &= X^{-1} D(g)^\dagger S D(g) X^{-1} \\ &= X^{-1} \left(\sum_{h \in G} \underbrace{D(g)^\dagger D(h)^\dagger}_{=D(hg)^\dagger} \underbrace{D(h) D(g)}_{=D(hg)} \right) X^{-1} \\ &= X^{-1} S X^{-1} \quad (\because \text{ rearangement thm.}) \\ &= 1. \end{aligned} \tag{1.23}$$

Thus, we have shown that D is equivalent to a unitary representation D'. □

Theorem 1.2. *Every representation of a finite group is completely reducible.*

Proof. The theorem holds if and only if every unitary representation is completely reducible.

Consider a reducible representation D such that one of its invariant subspaces is projected by a projector P, i.e.,

$$\forall g \in G : P D(g) P = D(g) P. \tag{1.24}$$

Taking the adjoint gives

$$P D(g)^\dagger P = P D(g)^\dagger. \tag{1.25}$$

Since $D(g)^\dagger = D(g)^{-1} = D(g^{-1})$,

$$\forall g \in G : PD(g)P = PD(g). \tag{1.26}$$

Therefore

$$\forall g \in G : (1-P)D(g)(1-P) = D(g)(1-P), \tag{1.27}$$

which implies that $D(g)$ is always possible to completely reduced. □

1.10 Subgroups

Definition 1.11 (subgroups)**.** A group H whose elements are all elements of a group G is called a **subgroup** of G.

Remark 1.5. The identity and the group G are said to be trivial subgroups of G.

Definition 1.12 (right-coset)**.** A **right-coset** of a subgroup H in a group G is

$$Hg = \{hg : h \in H\} \tag{1.28}$$

for some fixed $g \in G$. We define a left-coset as well.

Definition 1.13 (coset-space)**.** The group G/H defined by regarding each coset as a single element of the space is the **coset-space**.

Definition 1.14 (invariant/normal subgroup)**.** A subgroup H of G is called an **invariant** or **normal** subgroup if

$$\forall g \in G : gH = Hg \Leftrightarrow gHg^{-1} = H. \tag{1.29}$$

Remark 1.6. For an invariant subgroup,

$$(Hg_1)(Hg_2) = H(g_1 H {g_1}^{-1}) g_1 g_2 = H g_1 g_2. \tag{1.30}$$

Remark 1.7. The coset-space G/H is called the factor group of G by H given that H is an invariant subgroup of G.

Remark 1.8. In general, when there is an invariant subgroup H of G, there are representations of G that are constant on H, forming a representation of the factor group, G/H.

Definition 1.15 (centre)**.** The **centre** of a group G is the set of all elements of G that commute with all other elements of G.

Remark 1.9. The centre is always an abelian invariant subgroup of G.

Definition 1.16 (congujacy classes)**.** A subset S of G satisfying

$$\forall g \in G : gSg^{-1} = S \tag{1.31}$$

is called a conjugacy class.

Example 1.6. The conjugacy classes of S_3 are $\{e\}$, $\{a_1, a_2\}$, and $\{a_4, a_5, a_6\}$.

1.11 Schur's Lemma

Theorem 1.3. *If $D_1(g)A = AD_g(g)$ for all $g \in G$ where D_1 and D_2 are inequivalent irreducible representations, then $A = 0$.*

Proof. We proceed by cases.

Case 1: $\exists|\mu\rangle$ s.t. $A|\mu\rangle = 0$.
Consider a projector P_μ such that $\forall|\alpha\rangle : |\mu\rangle \propto P_\mu|\alpha\rangle$. Then

$$\forall g \in G : A \underbrace{D_2(g)P}_{\substack{\text{reproduces}\\\text{all space}}} = D_1(g) \underbrace{AP}_{=0}. \tag{1.32}$$

In the LHS, $D_2(g)|\mu\rangle$ reproduces all space since D_2 is irreducible. Thus, in order for LHS to vanish, $A = 0$.

Case 2: $\exists\langle\mu|$ s.t. $\langle\mu|A = 0$.
It can be shown by a similar argument.

Case 3: A is invertible. $D_2(g) = A^{-1}D_1(g)A$, which contradicts the assumption. □

Theorem 1.4. *If $D(g)A = AD(g)$ for all $g \in G$ where D is a finite-dimensional irreducible representation, then $A \propto I$.*

Proof. The characteristic equation

$$\det(A - \lambda I) = 0 \tag{1.33}$$

has at least one root. Thus,

$$\forall g \in G : D(g)(A - \lambda I) = (A - \lambda I)D(g) \tag{1.34}$$

By multiplying a corresponding eigenvector $|\mu\rangle$ both sides, we get

$$\forall g \in G : 0 = (A - \lambda I)D(g)|\mu\rangle. \tag{1.35}$$

Since $D(g)|\mu\rangle$ reproduces the whole space, $A = \lambda I$. □

Under the symmetry transformation, the states and operators transforms like

$$|\mu\rangle \mapsto D(g)|\mu\rangle, \quad \langle\mu| \mapsto \langle\mu|D(g)^\dagger, \quad O \mapsto D(g)OD(g)^\dagger \tag{1.36}$$

in order that $\langle\nu|O|\mu\rangle$ remains uncahnged. An invariant operator satisfies

$$\forall g \in G : O \mapsto D(g)OD(g)^\dagger = O \Leftrightarrow [O, D(g)] = 0. \tag{1.37}$$

Consider the orthonormal basis states as

$$|a, j, x\rangle = \begin{cases} a\text{: choice of an irr. rep.} \\ j \in \{1, 2, \cdots, n_a\}\text{: state within the rep.} \\ x\text{: other physical parameters} \end{cases} \tag{1.38}$$

satisfying

$$\langle a,j,x|b,k,y\rangle = \delta_{ab}\delta_{jk}\delta_{xy}, \quad \langle a,j,x|D(g)|b,k,y\rangle = \delta_{ab}\delta_{xy}[D_a(g)]_{jk}. \tag{1.39}$$

Then, we can constrain the matrix element

$$\langle a,j,x|O|b,k,y\rangle \tag{1.40}$$

by arguing as follows:

$$\begin{aligned}
0 &= \langle a,j,x|[O,D(g)]|b,k,y\rangle \\
&= \sum_{k'} \langle a,j,x|O|b,k',y\rangle\langle b,k',y|D(g)|b,k,y\rangle \\
&\quad - \sum_{j'} \langle a,j,x|D(g)|a,j',x\rangle\langle a,j',x|D(g)|b,k,y\rangle \\
&= \sum_{k'} \langle a,j,x|O|b,k',y\rangle\,[D_b(g)]_{k'k} \\
&\quad - \sum_{j'} [D_a(g)]_{jj'}\,\langle a,j',x|D(g)|b,k,y\rangle\,.
\end{aligned} \tag{1.41}$$

According to Schur's lemma, the block elements of O are proportional to I if $a=b$ and 0 if $a \neq b$. Thus,

$$\langle a,j,x|O|b,k,y\rangle = f_a(x,y)\delta_{ab}\delta_{jk}. \tag{1.42}$$

1.12 Orthogonality Relations

Consider a linear operator

$$A^{ab}_{jl} \equiv \sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) \tag{1.43}$$

where D_a and D_b are finite-dimensional irreducible representations of a group G. By using the substitution $g' = g{g_1}^{-1}$ and the rearrangement theorem, we get

$$\begin{aligned}
D_a(g_1)A^{ab}_{jl} &= \sum_{g\in G} D_a(g_1 g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) \\
&= \sum_{g'\in G} D_a({g'}^{-1})\,|a,j\rangle\langle b,l|\,D_b(g')D_b(g_1) \\
&= A^{ab}_{jl} D_b(g_1).
\end{aligned} \tag{1.44}$$

Now, Schur's lemma implies

$$A^{ab}_{jl} = \delta_{ab}\lambda_{jl^a} I. \tag{1.45}$$

λ^a_{jl} can be determined by calculating the trace of A^{ab}_{jl} in two ways. On the one hand,

$$\operatorname{tr} A^{ab}_{jl} = \delta_{ab}\lambda^a_{jl}\operatorname{tr} I = \delta_{ab}\lambda^a_{jl} n_a, \quad n_a = \dim D_a. \tag{1.46}$$

On the other hand,

$$\begin{aligned}\operatorname{tr} A^{ab}_{jl} &= \sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle a,l|\,D_a(g)\\ &= \delta_{ab}\sum_{g\in G}\langle a,l|\,D_a(g)D_a(g^{-1})\,|a,j\rangle\\ &= N\delta_{ab}\delta_{jl}, \quad N = \operatorname{ord} G.\end{aligned} \tag{1.47}$$

Thus,

$$\lambda^a_{jl} = \frac{N}{n_a}\delta_{jl}. \tag{1.48}$$

So,

$$A^{ab}_{jl} = \sum_{g\in G} D_a(g^{-1})\,|a,j\rangle\langle b,l|\,D_b(g) = \frac{N}{n_a}\delta_{ab}\delta_{jl}I. \tag{1.49}$$

Taking the matrix elements of these relations yields orthogonality relations for the matrix elements of irreducible representations:

$$\boxed{\sum_{g\in G}\frac{n_a}{N}[D_a(g^{-1})]_{jk}[D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}.} \tag{1.50}$$

For unitary irreducible representations, we can rewrite it as

$$\sum_{g\in G}\frac{n_a}{N}[D_a(g)]^*_{jk}[D_b(g)]_{lm} = \delta_{ab}\delta_{jl}\delta_{km}. \tag{1.51}$$

With proper normalisation, the matrix elements of the inequivalent unitary irreducible representations take the form

$$\sqrt{\frac{n_a}{N}}[D_a(g)]_{jk}. \tag{1.52}$$

Example 1.7.

$$\frac{1}{N}\sum_{j=0}^{N-1} e^{-2\pi i n' j/N} e^{2\pi i n j/N} = \delta_{n'n} \tag{1.53}$$

Consider a function $F : G \to \mathbb{C}$, i.e.,

$$\langle F| = \sum_{g'\in G} F(g')\,\langle g'|. \tag{1.54}$$

Since $|g\rangle = D_R(g)\,|e\rangle$ for the regular representation D_R,

$$F(g) = \langle F|g\rangle = \sum_{g'\in G} F(g')\,\langle g'|\,D_R(g)\,|e\rangle = F(g')[D_R(g)]_{g'e} \tag{1.55}$$

Theorem 1.5. *The matrix elements of the unitary irreducible representations of G are a complete orthonormal set for the vector space of the regular representation, or alternatively, for functions of $g \in G$.*

Proof. Since D_R is completely reducible, it can be rewritten as a linear combination of the matrix elements of the irreducible representations. □

Corollary 1.6. *The order $N = \operatorname{ord} G$ of the group G is the sum of squares of the dimensions of the irreducible representations $n_j = \dim D_j$, i.e.,*

$$\boxed{N = \sum_j {n_j}^2.} \tag{1.56}$$

1.13 Characters

Definition 1.17 (characters)**.** The **characters** χ_D of a group representation D are the traces of the linear operators of the representation or their matrix elements:

$$\chi_D(g) \equiv \operatorname{tr} D(g) = \sum_j [D(g)]_{jj} \tag{1.57}$$

Remark 1.10. The characters are unchanged by similarity transformations; all equivalent representations have the same characters.

Remark 1.11. The characters are different for each inequivalent irreducible representation.

Claim 1.1. *The characters are orthogonal up to an overall factor of N.*

Proof. By applying eq. (1.51), we get:

$$\sum_{\substack{g\in G\\ j=k\\ l=m}} \frac{1}{N}[D_a(g)]^*_{jk}[D_b(g)]_{lm} = \sum_{\substack{j=k\\ l=m}} \frac{1}{n_a}\delta_{ab}\delta_{jl}\delta_{km} = \delta_{ab}. \tag{1.58}$$

Equivalently,

$$\boxed{\frac{1}{N}\sum_{g\in G} \chi_{D_a}(g)^*\chi_{D_b}(g) = \delta_{ab}.} \tag{1.59}$$

□

Claim 1.2. *The characters are constant on a conjugacy class.*

Proof.

$$\operatorname{tr} D(g^{-1}g_1 g) = \operatorname{tr}\big(D(g^{-1})D(g_1)D(g)\big) = \operatorname{tr} D(g_1), \tag{1.60}$$

thanks to the cyclic property of traces. □

Claim 1.3. *The characters, $\chi_a(g)$, of the independent irreducible representations form a complete orthonormal basis set for the functions that are constant on conjugacy classes.*

Proof. See the textbook. □

Remark 1.12. As a consequence of claim 1.3, the number of irreducible representations is equal to the number of conjugacy classes.

Claim 1.4.

$$\sum_a \chi_{D_a}(g_\alpha)^* \chi_{D_a}(g_\beta) = \frac{N}{k_\alpha}\delta_{\alpha\beta}, \tag{1.61}$$

where k_α is the number of elements in the conjugacy class α.

Proof. Suppose the matrix V with matrix elements

$$V_{\alpha a} = \frac{k_\alpha}{N}\chi_{D_a}(g_\alpha), \tag{1.62}$$

where g_α denotes the conjugacy class α. Then, eq. (1.59) can be rewritten as $V^\dagger V = 1$. Since V is a square matrix ($\because$ remark 1.12), $VV^\dagger = 1$, equivalently, eq. (1.62), also holds. □

Example 1.8 (The characters of S_3). According to remark 1.12, S_3 has three independent irreducible representations. Therefore, eq. (1.56) holds as

$$6 = \operatorname{ord} S_3 = \sum_j {n_j}^2 = 1^2 + 1^2 + 2^2. \tag{1.63}$$

For the trivial representation D_1, $\chi_0(g) = 1$. Remark 1.8 guarantees that there are nontrivial representation of $S_3/\{e, a_1, a_2\}$: the sign representation.

It is clear that $\chi_2(e) = 2$. Let $\chi_2 = (2, x, y)$. Using the orthogonality relation 1.59 with $\alpha = a$, we get

$$\begin{cases} (a,b) = (1,2)\colon\ 1\cdot 2\cdot 1 + 1\cdot x\cdot 2 + 1\cdot y\cdot 3 = 0, \\ (a,b) = (1',2)\colon\ 1\cdot 2\cdot 1 + 1\cdot x\cdot 2 - 1\cdot y\cdot 3 = 0. \end{cases} \tag{1.64}$$

Therefore, we get $x = -1$, $y = 0$.

Table 2: The character table of S_3

	e	$\{a_1, a_2\}$	$\{a_3, a_4, a_5\}$
1	1	1	1
1′	1	1	−1
2	2	−1	0

Claim 1.5. *The projection operator onto the subspace that transforms under the representation D_a takes the form*

$$P_a = \frac{n_a}{N}\sum_{g\in G}\chi_{D_a}(g)^* D(g). \tag{1.65}$$

Proof. By setting $j = k$, eq. (1.51) becomes

$$\frac{n_a}{N}\sum_{g\in G}\chi_{D_a}(g)^*[D_b(g)]_{lm} = \delta_{ab}\delta_{lm}. \tag{1.66}$$

Since D is equivalent to a direct sum of irreducible representations,

$$D \sim \bigoplus_i D_i, \tag{1.67}$$

it follows that

$$P_a \sim \bigoplus_i I_{n_i \times n_i} \delta_{ai}. \tag{1.68}$$

It is now clear that eq. (1.65) functions as the projector onto the required subspace. □

Claim 1.6.

$$\operatorname{tr}(P_a) = n_a m_a^D, \tag{1.69}$$

where m_a is the number of repetitions of irreducible representations in D which are equivalent to D_a.

Example 1.9. Consider the defining representation of S_3:[1]

$$\begin{aligned} D_3(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D_3(a_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad D_3(a_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ D_3(a_3) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D_3(a_4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad D_3(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{aligned}. \tag{1.70}$$

Simple calculation reads

$$P_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{1.71}$$

$\operatorname{tr} P_0 = 1$ and $\operatorname{tr} P_2 = 2$ leads

$$D_3 \sim D_0 \oplus D_2. \tag{1.72}$$

1.20 Conjugacy Classes

In this subsection, only the permutation groups, S_ns, are considered.

Claim 1.7. *Conjugation does not change the cycle structure.*[2]

Proof. If $g_1 \doteq i \mapsto j$, then $g g_1 g^{-1} \doteq g(i) \mapsto g(j)$. □

Claim 1.8. *The conjugacy classes must consist of all possible permutations with a particular cycle structure.*

Proof. Given two permutations $g_1, g_2 \in S_n$ with the same cycle structure. Obviously, it is always possible to relabel i's in order to switch between two permutations, i.e., $g g_1 g^{-1} = g_2$. □

Remark 1.13. The conjugacy classes are the cycle structures.

[1] The defining representation D_{def} of the permutation group S_n is n-dimensional vector space which is defined by $D_{\text{def}}(j \mapsto k)|j\rangle = |k\rangle$, i.e., $\langle l|D_{\text{def}}(j \mapsto k)|j\rangle = \delta_{jk}$

[2] Conjugation resembles similarity transformations.

Claim 1.9. *The number of different permutations in a conjugacy class of S_n which consists of k_j copies of j-cycles is*

$$\frac{n!}{\prod_j j^{k_j} k_j!}. \tag{1.73}$$

Proof. j^{k_j} eliminates k copies of the degrees of freedom($= j$) of cycling indices within the j-cycles, e.g., $(123) = (231)$. $k_j!$ eliminates the degrees of freedom of permutation between the j-cycles, e.g., $(12)(34) = (34)(12)$. □

1.21 Young Tableux

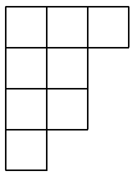
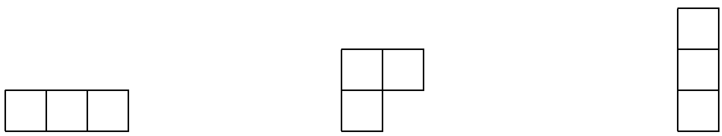


Figure 1: The Young Tableau represents an eight-dimensional permutation with a 4-cycle, a 3-cycle, and a 1-cycle.

A Young tableau can represent a conjugacy class by having k_j copies of a column with length j.

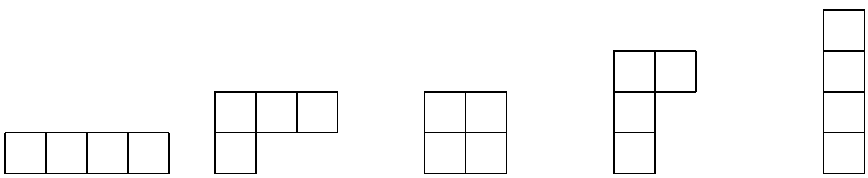
Example 1.10. The permutation group S_3 has three conjugacy classes.



(a) $3!/3! = 1$ (b) $3!/2 = 3$ (c) $3!/3 = 2$

Figure 2: The conjugacy classes of S_3 with the numbers of their elements.

Example 1.11. The permutation group S_4 has five conjugacy classes.



(a) $4!/4! = 1$ (b) $4!/4 = 6$ (c) $4!/8 = 3$ (d) $4!/3 = 8$ (e) $\frac{4!}{4} = 6$

Figure 3: The conjugacy classes of S_4 with the numbers of their elements.

1.14 Eigenstates

Theorem 1.7. *If a hermitian operator, H, commutes with all the elements, $D(g)$, of a representation of the group G, then you can choose the eigenstates of H to transform according to irreducible representations of G. If an irreducible representation appears only once in the Hilbert space, every state in the irreducible representation is an eigenstate of H with the same eigenvalue.*

Proof. Consider a eigenstate $|\psi\rangle$ of H. Since $[H, D(g)] = 0$, $D(g)$ cannot change the H eigenvalue of the state:

$$H(D(g)|\psi\rangle) = D(g)H|\psi\rangle = D(g)E|\psi\rangle = E(D(g)|\psi\rangle). \tag{1.74}$$

In other words, H cannot move a state from one irreducible representation's subspace to another, i.e.,

$$H|a, j, x\rangle = \sum_y c_j\,|a, j, y\rangle. \tag{1.75}$$

According to Schur's lemma, H must be proportional to the identity within the subspace of a given irreducible representation. □

Theorem 1.8. *All of the irreducible representations of a finite abelian group are 1-dimensional.*

Proof. Every element of an abelian group is a conjugacy class by itself. By remark 1.12, the number of irreducible representations is the order of the group; the only way to satisfy eq. (1.56) is $n_j = 1$ for all irreducible representations. □