Vector Calculus Identities

in three-dimensions

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1 Definitions

1.1 Kronecker Delta and Levi-Civita Symbol

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1}$$

Levi-Civita Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

1.2 The Del Operator

Gradient

$$\nabla \varphi = \mathbf{e}_i \partial_i \varphi \tag{3}$$

Divergence

$$\nabla \cdot (F_i \mathbf{e}_i) = \partial_i F_i \tag{4}$$

Curl

$$\nabla \times (F_i \mathbf{e}_i) = \mathbf{e}_k \varepsilon_{ijk} \partial_i F_j \tag{5}$$

1.3 Second Derivatives

Scalar Laplacian

$$\nabla^2\varphi = \nabla \cdot (\nabla\varphi) \tag{6}$$
$$= \partial_i\partial_i\varphi \tag{7}$$

Vector Laplacian

$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \tag{8}$$
$$= \mathbf{e}_j\partial_i\partial_i F_j \tag{9}$$

2 Identities

2.1 Kronecker Delta and Levi-Civita Symbol

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{10}$$
$$\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn} \tag{11}$$
$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{12}$$

2.2 Gradient

$$\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi \tag{13}$$
$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \tag{14}$$
$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) \tag{15}$$

2.3 Divergence

$$\nabla \cdot (\mathbf{F} + \mathbf{G}) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G} \tag{16}$$
$$\nabla \cdot (\varphi\mathbf{F}) = \varphi\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla\varphi \tag{17}$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - (\nabla \times \mathbf{G}) \cdot \mathbf{F} \tag{18}$$

2.4 Curl

$$\nabla \times (\mathbf{F} + \mathbf{G}) = \nabla \times \mathbf{F} + \nabla \times \mathbf{G} \tag{19}$$
$$\nabla \times (\varphi\mathbf{F}) = \varphi(\nabla \times \mathbf{F}) + \nabla\varphi \times \mathbf{F} \tag{20}$$
$$\nabla \times (\varphi\nabla\psi) = \nabla\varphi \times \nabla\psi \tag{21}$$
$$\nabla \times (\mathbf{F} + \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \tag{22}$$

2.5 Second Derivatives

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0 \tag{23}$$

$$\nabla \times (\nabla \varphi) = \mathbf{0} \tag{24}$$

$$\nabla \cdot (\varphi \nabla \psi) = \varphi \nabla^2 \psi + \nabla \varphi \cdot \nabla \psi \tag{25}$$

$$\varphi \nabla^2 \psi - \psi \nabla^2 \varphi = \nabla \cdot (\varphi \nabla \psi - \psi \nabla \varphi) \tag{26}$$

$$\nabla^2 (\varphi \psi) = \varphi \nabla^2 \psi + 2 \nabla \varphi \cdot \nabla \psi + \psi \nabla^2 \varphi \tag{27}$$

$$\nabla^2 (\varphi \mathbf{F}) = \mathbf{F} \nabla^2 \varphi + 2(\nabla \varphi \cdot \nabla) \mathbf{F} + \varphi \nabla^2 \mathbf{F} \tag{28}$$

$$\nabla^2 (\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \cdot \nabla^2 \mathbf{G} - \mathbf{G} \cdot \nabla^2 \mathbf{F} + 2 \nabla \cdot ((\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F})) \tag{29}$$

2.6 Integration

Gradient Theorem

$$\varphi(\mathbf{q}) - \varphi(\mathbf{p}) = \int_C \nabla \varphi(\mathbf{r}) \cdot d\mathbf{r} \tag{30}$$

Divergence Theorem

$$\oiint_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F} dV \tag{31}$$

Curl Theorem

$$\oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{32}$$

Green's First Identity

$$\oiint_{\partial V} \varphi \nabla \psi \cdot d\mathbf{S} = \iiint_V (\varphi \nabla^2 \psi + \nabla \varphi \cdot \nabla \psi) dV \tag{33}$$

Green's Second Identity

$$\oiint_{\partial V} (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\mathbf{S} = \iiint_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dV \tag{34}$$