

6 Roots and Weights

6.1 Weights

Definition 6.1 (Cartan subalgebras). A subset of commuting hermitian generators which is as large as possible is called a **Cartan subalgebra**.

Definition 6.2 (Cartan generators). The **Cartan generators** are the elements of a Cartan subalgebra of a Lie algebra.

Remark 6.1. Cartan generators can be simultaneously diagonalised.

Definition 6.3 (rank). The **rank** of the algebra is the number of independent Cartan generators.

Remark 6.2. For all $i, j = 1, 2, \dots, m (= \text{rank } \mathfrak{g})$,

$$H_i = H_i^\dagger \quad (6.1)$$

$$[H_i, H_j] = 0 \quad (6.2)$$

Remark 6.3. We can choose a basis in which the generators satisfy

$$\text{tr}(H_i H_j) = \lambda_D \delta_{ij} \quad (6.3)$$

for $i, j = 1$ to m , where k_D depends on the representation D and the normalisation of the generators (see eq. (2.25)).

Definition 6.4 (weights). The eigenvalues μ_i of Cartan generators are called **weights**:

$$H_i |\mu, x, D\rangle = \mu_i |\mu, x, D\rangle. \quad (6.4)$$

The m -component vector with components μ_i is the **weight vector**.

6.2 More on the Adjoint Representation

Claim 6.1. Each generator X_a corresponds to a basis vector $|X_a\rangle$ in the inner product space on which the adjoint representation, defined by eq. (2.15), acts.

Proof. The inner product is defined as

$$\langle X_a | X_b \rangle = \lambda^{-1} \text{tr}(X_a^\dagger X_b), \quad (6.5)$$

where $\lambda = k_{D_{\text{adj}}}$. □

Claim 6.2.

$$X_a |X_b\rangle = |[X_a, X_b]\rangle \quad (6.6)$$

Proof.

$$X_a |X_b\rangle = |X_c\rangle \langle X_c | X_a |X_b\rangle = |X_c\rangle [X_a]_{cb} = |if_{abc} X_c\rangle = |[X_a, X_b]\rangle \quad (6.7)$$

□

6.3 Roots

Definition 6.5 (roots). The **roots** are the weights of the adjoint representation.

Claim 6.3. *The states corresponding to the Cartan generators have zero weight vectors.*

Proof.

$$H_i |H_j\rangle = [[H_i, H_j]] = 0 \quad (6.8)$$

□

Claim 6.4. *The Cartan states are orthonormal.*

Proof.

$$\langle H_i | H_j \rangle = \lambda^{-1} \text{tr}(H_i H_j) = \delta_{ij} \quad (6.9)$$

□

Claim 6.5. *For a state E_α of the adjoint representation that does not correspond to the Cartan generators,*

$$[H_i, E_\alpha] = \alpha_i E_\alpha \quad (6.10)$$

and

$$E_\alpha^\dagger = E_{-\alpha}, \quad (6.11)$$

where α is the non-zero weight vector.

Proof.

$$|[H_i, E_\alpha]\rangle = H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle = |\alpha_i E_\alpha\rangle \quad (6.12)$$

and

$$[H_i, E_\alpha^\dagger] = -[H_i, E_\alpha]^\dagger = -\alpha_i E_\alpha^\dagger \quad (6.13)$$

□

Example 6.1.

$$(J^+)^\dagger = J^- \quad (6.14)$$

Claim 6.6. *States corresponding to different weights are orthogonal.*

Proof. They have different eigenvalues of at least one of the Cartan generators.

$$\langle E_\alpha | H_i | E_\beta \rangle = \alpha_i \langle E_\alpha | E_\beta \rangle = \beta_i \langle E_\alpha | E_\beta \rangle \quad (6.15)$$

implies the claim.

We choose the normalisation

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \text{tr}(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta}. \quad (6.16)$$

□

6.4 Raising and Lowering

Claim 6.7. *The $E_{\pm\alpha}$ are raising and lowering operators for the weights.*

Proof.

$$H_i E_{\pm\alpha} |\mu, D\rangle = \underbrace{[H_i, E_{\pm\alpha}]}_{=\pm\alpha_i E_{\pm\alpha}} |\mu, D\rangle + E_{\pm\alpha} H_i |\mu, D\rangle = (\pm\alpha + \mu) E_{\pm\alpha} |\mu, D\rangle \quad (6.17)$$

□

Claim 6.8.

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H = \alpha_i H_i \quad (6.18)$$

Proof. Since the state $E_\alpha |E_{-\alpha}\rangle$ has weight $\alpha - \alpha = 0$, it can be written as a linear combination of the Cartan states:

$$\beta_i |H_i\rangle = E_\alpha |E_{-\alpha}\rangle = |[E_\alpha, E_{-\alpha}]\rangle. \quad (6.19)$$

Direct computation of β_i yields

$$\begin{aligned} \beta_i &= \langle H_i | [E_\alpha, E_{-\alpha}] \rangle = \lambda^{-1} \text{tr}(H_i [E_\alpha, E_{-\alpha}]) \\ &= \lambda^{-1} \text{tr}(E_{-\alpha} [H_i, E_\alpha]) \\ &= \lambda^{-1} \text{tr}(E_{-\alpha} \alpha_i E_\alpha) \\ &= \alpha_i. \end{aligned} \quad (6.20)$$

□

Example 6.2.

$$[J^+, J^-] = J_3 \quad (6.21)$$

6.5 Lots of $SU(2)$ s

For each non-zero pair of root vectors $\pm\alpha$, there is an $SU(2)$ subalgebra of the group, with generators

$$E^\pm \equiv |\alpha|^{-1} E_{\pm\alpha} \quad (6.22)$$

$$E_3 \equiv |\alpha|^{-2} \alpha \cdot H \quad (6.23)$$

Claim 6.9. *Remark 3.2 also holds for E^\pm and E_3 .*

Proof.

$$[E_3, E^\pm] = |\alpha|^{-3} [\alpha \cdot H, E_{\pm\alpha}] = |\alpha|^{-3} \alpha \cdot (\pm\alpha E_{\pm\alpha}) = \pm E^\pm \quad (6.24)$$

$$[E^+, E^-] = |\alpha|^{-2} [E_\alpha, E_{-\alpha}] = |\alpha|^{-2} \alpha \cdot H = E_3 \quad (6.25)$$

□

Claim 6.10. *A root vector corresponds to the unique generator.*

Proof. Suppose the contrary: E_α and E'_α both correspond to α . They can be chosen to be orthogonal. That is,

$$\langle E_\alpha | E'_\alpha \rangle = \lambda^{-1} \operatorname{tr}(E_\alpha^\dagger E'_\alpha) = \lambda^{-1} \operatorname{tr}(E_{-\alpha} E'_\alpha). \quad (6.26)$$

By lowering $|E'_\alpha\rangle$, we have $E^-|E'_\alpha\rangle = \beta_i|H_i\rangle$. But,

$$\begin{aligned} \beta_i &= \langle H_i | E^- | E'_\alpha \rangle = \lambda^{-1} \operatorname{tr}(H_i [E^-, E'_\alpha]) \\ &= -\lambda^{-1} \operatorname{tr}(E^- [H_i, E'_\alpha]) \\ &= -\alpha_i \lambda^{-1} \operatorname{tr}(E'_\alpha E^-) = 0, \end{aligned} \quad (6.27)$$

which implies that such an E'_α cannot exist. \square

Claim 6.11. *If α is a root, then no non-zero multiple of α can be a root except for $\pm\alpha$.*

Proof. Suppose $k\alpha$ a root for $k \neq \pm 1$. Under the $SU(2)$ subalgebra corresponding to α , $\mathfrak{su}_\alpha(2)$,

$$[E_3, E_{k\alpha}] = kE_{k\alpha}, \quad (6.28)$$

so its E_3 eigenvalue is k . Since it is a magnetic quantum number of $SU(2)$, k must be an integer or a half-integer.

Case 1: k is an integer other than ± 1 .

$E_{k\alpha}$ sits in a spin- j representation where $j \geq |k|$. Since an $SU(2)$ multiplet with $m = k$ also contains the state with $m = 1$, which is another root- α state.

Case 2: k is a half-integer.

Likewise, there will be a state with the E_3 eigenvalue $1/2$, which corresponds to a vector with root $\alpha/2$. This is clearly a contradiction. \square

6.6 Watch Carefully—This is Important!

Remark 6.4.

$$E_3 |\mu, x, D\rangle = \frac{\alpha \cdot \mu}{\alpha^2} |\mu, x, D\rangle, \quad (6.29)$$

where

$$\frac{2\alpha \cdot \mu}{\alpha^2} \text{ is an integer.} \quad (6.30)$$

Remark 6.5. The general state $|\mu, x, D\rangle$ can always be written as a linear combination of states transforming according to definite representations of the $SU(2)$.

Theorem 6.1 (the master formula).

$$\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2}(p - q), \quad (6.31)$$

where p and q are the maximum possible numbers of raising and lowering, respectively.

Proof. Since

$$(E^+)^p |\mu, x, D\rangle \neq 0 \text{ whereas } (E^+)^{p+1} |\mu, x, D\rangle = 0, \quad (6.32)$$

we have

$$\frac{\alpha \cdot (\mu + p\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + p = j. \quad (6.33)$$

Likewise,

$$\frac{\alpha \cdot (\mu - q\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} - q = -j. \quad (6.34)$$

Adding eq. (6.33) and eq. (6.34), we get the formula. \square

Claim 6.12. *For any pair of roots α and β , the only possible cosine-squared values of the angles $\cos^2 \theta_{\alpha\beta}$ between them are 0, 1/4, 2/4, 3/4, or 4/4.*

Proof.

$$\cos^2 \theta_{\alpha\beta} = \frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} = \frac{(p - q)(p' - q')}{4}. \quad (6.35)$$

\square