

1 Finite Groups

1.1 Groups and Representations

$$\text{Group } (G, \cdot) \begin{cases} \text{closure} & f \cdot g \in G \\ \text{associativity} & f \cdot (g \cdot h) = (f \cdot g) \cdot h \\ \text{existence of identity} & \exists e \text{ s.t. } \forall f : e \cdot f = f \cdot e = f \\ \text{existence of inverse} & \forall f \in G \exists f^{-1} \text{ s.t. } f \cdot f^{-1} = f^{-1} \cdot f = e \end{cases}$$

Example 1.1. $\mathbb{Z}_2 = (\{0, 1\}, +_{\text{mod}2}) = (\{1, -1\}, \times)$

Definition 1.1 (group representations). A **representation** of a group G is a mapping, D , of the elements of G onto a set of linear operators with the following properties:

1. $D(e) = 1$, where 1 is the identity operator in the space on which the linear operators act.
2. $D(g_1)D(g_2) = D(g_1g_2)$, in other words, the group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act.

Remark 1.1.

$D(g) = 1$: trivial representation

$D(g) = 1 \rightarrow g = e$: non-trivial representation

1.2 Example— \mathbb{Z}_3

Definition 1.2 (finite groups). A group is **finite** if it has a finite number of elements. Otherwise, it is infinite.

Definition 1.3 (order). The **order** of a group G , $\text{ord } G$, is the number of elements in a finite group G .

Definition 1.4 (abelian groups). An **abelian group** is one in which the multiplication law is commutative.

Table 1: The Cayley table of \mathbb{Z}_3

| \cdot | e | a | b |
|---------|-----|-----|-----|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

Example 1.2. $D(e) = 1$, $D(a) = \exp(2\pi i/3)$, $D(b = a^{-1}) = \exp(4\pi i/3)$ is a representation of \mathbb{Z}_3 .

Definition 1.5 (dimension). The **dimension** of a representation is the dimension of the linear space in which it acts.

1.3 The Regular Representation

Definition 1.6 (regular representations). The regular representation D of a group G is a representation which is defined by

$$D(g_1)|g_2\rangle = |g_1g_2\rangle. \quad (1.1)$$

Remark 1.2. For the regular representation D of a group G , $\dim D = \text{ord } G$.

Remark 1.3. Consider the regular representation D .

$$[D(g)]_{ij} = \langle e_i | D(g) | e_j \rangle \quad (1.2)$$

$$[D(g_1g_2)]_{ij} = \langle e_i | D(g_1)D(g_2) | e_j \rangle \quad (1.3)$$

$$= \sum_k \langle e_i | D(g_1) | e_k \rangle \langle e_k | D(g_2) | e_j \rangle \quad (1.4)$$

$$= \sum_k [D(g_1)]_{ik} [D(g_2)]_{kj} \quad (1.5)$$

$$= [D(g_1)D(g_2)]_{ij} \quad (1.6)$$

Example 1.3. The regular representation of \mathbb{Z}_3 is

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1.7)$$

1.4 Irreducible Representations

A group representation is powerful since it is in a linear space, such that a basis change is allowed.

Definition 1.7 (equivalent representations). Two representations D' and D are **equivalent** if

$$\exists S \text{ s.t. } \forall g \in G : D'(g) = S^{-1}D(g)S. \quad (1.8)$$

Definition 1.8 (unitary representations). A representation is **unitary** if

$$\forall g \in G : D(g)^\dagger = D(g)^{-1}. \quad (1.9)$$

Definition 1.9 (reducible representations). A representation is **reducible** if it has an invariant subspace. I.e., there exists a projection operator P onto the subspace so that

$$\forall g \in G : PD(g)P = D(g)P. \quad (1.10)$$

Example 1.4. The regular representation of \mathbb{Z}_3 has an invariant subspace projected on by

$$P = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (1.11)$$

Definition 1.10 (completely reducible representations). A representation is **completely reducible** if it is equivalent to a representation whose matrix elements have the block diagonal form. I.e.,

$$D(g) \sim \begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad (1.12)$$

where $D_j(g)$ is irreducible for all j .

Remark 1.4. A completely reducible representation can be expressed as a direct sum of irreducible representations, i.e.,

$$D(G) = D_1(g) \oplus D_2(g) \oplus \cdots \oplus D_n(g). \quad (1.13)$$

Example 1.5. For ex. 1.3, take

$$S = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad \text{where } \omega = e^{2\pi i/3}. \quad (1.14)$$

Then,

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad D'(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad (1.15)$$

i.e.,

$$D'(g_j) = 1^{j+2} \oplus \omega^{j+2} \oplus (\omega^2)^{j+2}, \quad (1.16)$$

where $e = g_1$, $a = g_2$, and $b = g_3$.

1.8 Example: Addition of Integers

Consider a two-dimensional representation

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad (1.17)$$

of a group $(\mathbb{Z}, +)$. This is reducible but not completely reducible since

$$D(x)P = P \quad (1.18)$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (1.19)$$

but

$$D(x)(I - P) \neq (I - P). \quad (1.20)$$

1.9 Useful Theorems

Theorem 1.1. *Every representation of a finite group is equivalent to a unitary representation.*

Proof. Suppose $D(g)$ is a representation of a finite group G . Construct the operator

$$S = \sum_{g \in G} D(g)^\dagger D(g). \quad (1.21)$$

By construction, S is hermitian and positive-semidefinite, i.e.,

$$S = U^{-1} \mathbb{D} U \quad (1.22)$$

where $\mathbb{D} = \text{diag}(d_1, d_2, \dots)$, where $d_j \geq 0$ for all j .

Assume that $d_j = 0$ for some j . Then for an corresponding eigenvector λ_j of S , i.e., $S\lambda_j = 0$,

$$\lambda_j^\top S \lambda_j = \sum_{g \in G} \|D(g)\lambda_j\|^2 > \|D(e)\lambda_j\|^2 = \|\lambda_j\|^2 > 0. \quad (1.23)$$

Thereby we conclude that $d_j > 0$ for all j .

Therefore, we can construct X a square-root of S that is hermitian invertible as follows:

$$X = S^{1/2} \equiv U^{-1} \mathbb{D}^{1/2} U. \quad (1.24)$$

Now we define

$$D'(g) = X D(g) X^{-1}. \quad (1.25)$$

It is unitary since

$$D'(g)^\dagger D'(g) = X^{-1} D(g)^\dagger S D(g) X^{-1} \quad (1.26)$$

$$= X^{-1} \left(\sum_{h \in G} \underbrace{D(g)^\dagger D(h)^\dagger}_{=D(hg)^\dagger} \underbrace{D(h) D(g)}_{=D(hg)} \right) X^{-1} \quad (1.27)$$

$$= X^{-1} S X^{-1} \quad (\because \text{rearrangement thm.}) \quad (1.28)$$

$$= 1. \quad (1.29)$$

Thus, we have shown that D is equivalent to a unitary representation D' . \square

Theorem 1.2. *Every representation of a finite group is completely reducible.*

Proof. The theorem holds if and only if every unitary representation is completely reducible.

Consider a reducible representation D such that one of its invariant subspaces is projected by a projector P , i.e.,

$$\forall g \in G : PD(g)P = D(g)P. \quad (1.30)$$

Taking the adjoint gives

$$PD(g)^\dagger P = PD(g)^\dagger. \quad (1.31)$$

Since $D(g)^\dagger = D(g)^{-1} = D(g^{-1})$,

$$\forall g \in G : PD(g)P = PD(g). \quad (1.32)$$

Therefore

$$\forall g \in G : (1 - P)D(g)(1 - P) = D(g)(1 - P), \quad (1.33)$$

which implies that $D(g)$ is always possible to completely reduced. \square

1.10 Subgroups

Definition 1.11 (subgroups). A group H whose elements are all elements of a group G is called a **subgroup** of G .

Remark 1.5. The identity and the group G are said to be trivial subgroups of G .

Definition 1.12 (right-coset). A **right-coset** of a subgroup H in a group G is

$$Hg = \{hg : h \in H\} \quad (1.34)$$

for some fixed $g \in G$. We define a left-coset as well.

Definition 1.13 (coset-space). The group G/H defined by regarding each coset as a single element of the space is the **coset-space**.

Definition 1.14 (invariant/normal subgroup). A subgroup H of G is called an **invariant** or **normal** subgroup if

$$\forall g \in G : gH = Hg \Leftrightarrow gHg^{-1} = H. \quad (1.35)$$

Remark 1.6. For an invariant subgroup,

$$(Hg_1)(Hg_2) = H(g_1Hg_1^{-1})g_1g_2 = Hg_1g_2. \quad (1.36)$$

Remark 1.7. The coset-space G/H is called the factor group of G by H given that H is an invariant subgroup of G .

Remark 1.8. In general, when there is an invariant subgroup H of G , there are representations of G that are constant on H , forming a representation of the factor group, G/H .

Definition 1.15 (centre). The **centre** of a group G is the set of all elements of G that commute with all other elements of G .

Remark 1.9. The centre is always an abelian invariant subgroup of G .

Definition 1.16 (conjugacy classes). A subset S of G satisfying

$$\forall g \in G : gSg^{-1} = S \quad (1.37)$$

is called a conjugacy class.

Example 1.6. The conjugacy classes of S_3 are $\{e\}$, $\{a_1, a_2\}$, and $\{a_4, a_5, a_6\}$.

1.11 Schur's Lemma

Theorem 1.3. *If $D_1(g)A = AD_2(g)$ for all $g \in G$ where D_1 and D_2 are inequivalent irreducible representations, then $A = 0$.*

Proof. We proceed by cases.

Case 1: $\exists |\mu\rangle$ s.t. $A|\mu\rangle = 0$.

Consider a projector P_μ such that $\forall |\alpha\rangle : |\mu\rangle \propto P_\mu |\alpha\rangle$. Then

$$\forall g \in G : A \underbrace{D_2(g)P}_{\substack{\text{reproduces} \\ \text{all space}}} = D_1(g) \underbrace{AP}_{=0}. \quad (1.38)$$

In the LHS, $D_2(g)|\mu\rangle$ reproduces all space since D_2 is irreducible. Thus, in order for LHS to vanish, $A = 0$.

Case 2: $\exists \langle\mu|$ s.t. $\langle\mu|A = 0$.

It can be shown by a similar argument.

Case 3: A is invertible. $D_2(g) = A^{-1}D_1(g)A$, which contradicts the assumption. \square

Theorem 1.4. *If $D(g)A = AD(g)$ for all $g \in G$ where D is a finite-dimensional irreducible representation, then $A \propto I$.*

Proof. The characteristic equation

$$\det(A - \lambda I) = 0 \quad (1.39)$$

has at least one root. Thus,

$$\forall g \in G : D(g)(A - \lambda I) = (A - \lambda I)D(g) \quad (1.40)$$

By multiplying a corresponding eigenvector $|\mu\rangle$ both sides, we get

$$\forall g \in G : 0 = (A - \lambda I)D(g)|\mu\rangle. \quad (1.41)$$

Since $D(g)|\mu\rangle$ reproduces the whole space, $A = \lambda I$. \square

Under the symmetry transformation, the states and operators transforms like

$$|\mu\rangle \mapsto D(g)|\mu\rangle, \quad \langle\mu| \mapsto \langle\mu|D(g)^\dagger, \quad O \mapsto D(g)OD(g)^\dagger \quad (1.42)$$

in order that $\langle\nu|O|\mu\rangle$ remains unchanged. An invariant operator satisfies

$$\forall g \in G : O \mapsto D(g)OD(g)^\dagger = O \Leftrightarrow [O, D(g)] = 0. \quad (1.43)$$

Consider the orthonormal basis states as

$$|a, j, x\rangle = \begin{cases} a: \text{choice of an irr. rep.} \\ j \in \{1, 2, \dots, n_a\}: \text{state within the rep.} \\ x: \text{other physical parameters} \end{cases} \quad (1.44)$$

satisfying

$$\langle a, j, x | b, k, y \rangle = \delta_{ab} \delta_{jk} \delta_{xy}, \quad \langle a, j, x | D(g) | b, k, y \rangle = \delta_{ab} \delta_{xy} [D_a(g)]_{jk}. \quad (1.45)$$

Then, we can constrain the matrix element

$$\langle a, j, x | O | b, k, y \rangle \quad (1.46)$$

by arguing as follows:

$$0 = \langle a, j, x | [O, D(g)] | b, k, y \rangle \quad (1.47)$$

$$= \sum_{k'} \langle a, j, x | O | b, k', y \rangle \langle b, k', y | D(g) | b, k, y \rangle \quad (1.48)$$

$$- \sum_{j'} \langle a, j, x | D(g) | a, j', x \rangle \langle a, j', x | D(g) | b, k, y \rangle \quad (1.49)$$

$$= \sum_{k'} \langle a, j, x | O | b, k', y \rangle [D_b(g)]_{k'k} \quad (1.50)$$

$$- \sum_{j'} [D_a(g)]_{jj'} \langle a, j', x | D(g) | b, k, y \rangle. \quad (1.51)$$

According to Schur's lemma, the block elements of O are proportional to I if $a = b$ and 0 if $a \neq b$. Thus,

$$\langle a, j, x | O | b, k, y \rangle = f_a(x, y) \delta_{ab} \delta_{jk}. \quad (1.52)$$

1.12 Orthogonality Relations

Consider a linear operator

$$A_{jl}^{ab} \equiv \sum_{g \in G} D_a(g^{-1}) |a, j\rangle \langle b, l| D_b(g) \quad (1.53)$$

where D_a and D_b are finite-dimensional irreducible representations of a group G . By using the substitution $g' = gg_1^{-1}$ and the rearrangement theorem, we get

$$D_a(g_1) A_{jl}^{ab} = \sum_{g \in G} D_a(g_1 g^{-1}) |a, j\rangle \langle b, l| D_b(g) \quad (1.54)$$

$$= \sum_{g' \in G} D_a(g'^{-1}) |a, j\rangle \langle b, l| D_b(g') D_b(g_1) \quad (1.55)$$

$$= A_{jl}^{ab} D_b(g_1). \quad (1.56)$$

Now, Schur's lemma implies

$$A_{jl}^{ab} = \delta_{ab} \lambda_{jl}^a I. \quad (1.57)$$

λ_{jl}^a can be determined by calculating the trace of A_{jl}^{ab} in two ways. On the one hand,

$$\text{tr } A_{jl}^{ab} = \delta_{ab} \lambda_{jl}^a \text{tr } I = \delta_{ab} \lambda_{jl}^a n_a, \quad n_a = \dim D_a. \quad (1.58)$$

On the other hand,

$$\text{tr } A_{jl}^{ab} = \sum_{g \in G} D_a(g^{-1}) |a, j\rangle \langle a, l| D_a(g) \quad (1.59)$$

$$= \delta_{ab} \sum_{g \in G} \langle a, l| D_a(g) D_a(g^{-1}) |a, j\rangle \quad (1.60)$$

$$= N \delta_{ab} \delta_{jl}, \quad N = \text{ord } G. \quad (1.61)$$

Thus,

$$\lambda_{jl}^a = \frac{N}{n_a} \delta_{jl}. \quad (1.62)$$

So,

$$A_{jl}^{ab} = \sum_{g \in G} D_a(g^{-1}) |a, j\rangle \langle b, l| D_b(g) = \frac{N}{n_a} \delta_{ab} \delta_{jl} I. \quad (1.63)$$

Taking the matrix elements of these relations yields orthogonality relations for the matrix elements of irreducible representations:

$$\boxed{\sum_{g \in G} \frac{n_a}{N} [D_a(g^{-1})]_{jk} [D_b(g)]_{lm} = \delta_{ab} \delta_{jl} \delta_{km}.} \quad (1.64)$$

For unitary irreducible representations, we can rewrite it as

$$\sum_{g \in G} \frac{n_a}{N} [D_a(g)]_{jk}^* [D_b(g)]_{lm} = \delta_{ab} \delta_{jl} \delta_{km}. \quad (1.65)$$

With proper normalisation, the matrix elements of the inequivalent unitary irreducible representations take the form

$$\sqrt{\frac{n_a}{N}} [D_a(g)]_{jk}. \quad (1.66)$$

Example 1.7.

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi i n' j/N} e^{2\pi i n j/N} = \delta_{n'n} \quad (1.67)$$

Consider a function $F : G \rightarrow \mathbb{C}$, i.e.,

$$\langle F| = \sum_{g' \in G} F(g') \langle g'|. \quad (1.68)$$

Since $|g\rangle = D_R(g)|e\rangle$ for the regular representation D_R ,

$$F(g) = \langle F|g\rangle = \sum_{g' \in G} F(g') \langle g'| D_R(g)|e\rangle = F(g') [D_R(g)]_{g'e} \quad (1.69)$$

Theorem 1.5. *The matrix elements of the unitary irreducible representations of G are a complete orthonormal set for the vector space of the regular representation, or alternatively, for functions of $g \in G$.*

Proof. Since D_R is completely reducible, it can be rewritten as a linear combination of the matrix elements of the irreducible representations. \square

Corollary 1.6. *The order $N = \text{ord } G$ of the group G is the sum of squares of the dimensions of the irreducible representations $n_j = \dim D_j$, i.e.,*

$$\boxed{N = \sum_j n_j^2.} \quad (1.70)$$

1.13 Characters

Definition 1.17 (characters). The **characters** χ_D of a group representation D are the traces of the linear operators of the representation or their matrix elements:

$$\chi_D(g) \equiv \text{tr } D(g) = \sum_j [D(g)]_{jj} \quad (1.71)$$

Remark 1.10. The characters are unchanged by similarity transformations; all equivalent representations have the same characters.

Remark 1.11. The characters are different for each inequivalent irreducible representation.

Claim 1.1. *The characters are orthogonal up to an overall factor of N .*

Proof. By applying eq. (1.65), we get:

$$\sum_{\substack{g \in G \\ j=k \\ l=m}} \frac{1}{N} [D_a(g)]_{jk}^* [D_b(g)]_{lm} = \sum_{\substack{j=k \\ l=m}} \frac{1}{n_a} \delta_{ab} \delta_{jl} \delta_{km} = \delta_{ab}. \quad (1.72)$$

Equivalently,

$$\boxed{\frac{1}{N} \sum_{g \in G} \chi_{D_a}(g)^* \chi_{D_b}(g) = \delta_{ab}.} \quad (1.73)$$

\square

Claim 1.2. *The characters are constant on a conjugacy class.*

Proof.

$$\text{tr } D(g^{-1}g_1g) = \text{tr}(D(g^{-1})D(g_1)D(g)) = \text{tr } D(g_1), \quad (1.74)$$

thanks to the cyclic property of traces. \square

Claim 1.3. *The characters, $\chi_a(g)$, of the independent irreducible representations form a complete orthonormal basis set for the functions that are constant on conjugacy classes.*

Proof. See the textbook. \square

Remark 1.12. As a consequence of claim 1.3, the number of irreducible representations is equal to the number of conjugacy classes.

Claim 1.4.

$$\sum_a \chi_{D_a}(g_\alpha)^* \chi_{D_a}(g_\beta) = \frac{N}{k_\alpha} \delta_{\alpha\beta}, \quad (1.75)$$

where k_α is the number of elements in the conjugacy class α .

Proof. Suppose the matrix V with matrix elements

$$V_{\alpha a} = \frac{k_\alpha}{N} \chi_{D_a}(g_\alpha), \quad (1.76)$$

where g_α denotes the conjugacy class α . Then, eq. (1.73) can be rewritten as $V^\dagger V = 1$. Since V is a square matrix (\because remark 1.12), $VV^\dagger = 1$, equivalently, eq. (1.76), also holds. \square

Example 1.8 (The characters of S_3). According to remark 1.12, S_3 has three independent irreducible representations. Therefore, eq. (1.70) holds as

$$6 = \text{ord } S_3 = \sum_j n_j^2 = 1^2 + 1^2 + 2^2. \quad (1.77)$$

For the trivial representation D_1 , $\chi_0(g) = 1$. Remark 1.8 guarantees that there are nontrivial representation of $S_3/\{e, a_1, a_2\}$: the sign representation.

It is clear that $\chi_2(e) = 2$. Let $\chi_2 = (2, x, y)$. Using the orthogonality relation 1.73 with $\alpha = a$, we get

$$\begin{cases} (a, b) = (1, 2): 1 \cdot 2 \cdot 1 + 1 \cdot x \cdot 2 + 1 \cdot y \cdot 3 = 0, \\ (a, b) = (1', 2): 1 \cdot 2 \cdot 1 + 1 \cdot x \cdot 2 - 1 \cdot y \cdot 3 = 0. \end{cases} \quad (1.78)$$

Therefore, we get $x = -1, y = 0$.

Table 2: The character table of S_3

| | e | $\{a_1, a_2\}$ | $\{a_3, a_4, a_5\}$ |
|----|-----|----------------|---------------------|
| 1 | 1 | 1 | 1 |
| 1' | 1 | 1 | -1 |
| 2 | 2 | -1 | 0 |

Claim 1.5. *The projection operator onto the subspace that transforms under the representation D_a takes the form*

$$P_a = \frac{n_a}{N} \sum_{g \in G} \chi_{D_a}(g)^* D(g). \quad (1.79)$$

Proof. By setting $j = k$, eq. (1.65) becomes

$$\frac{n_a}{N} \sum_{g \in G} \chi_{D_a}(g)^* [D_b(g)]_{lm} = \delta_{ab} \delta_{lm}. \quad (1.80)$$

Since D is equivalent to a direct sum of irreducible representations,

$$D \sim \bigoplus_i D_i, \quad (1.81)$$

it follows that

$$P_a \sim \bigoplus_i I_{n_i \times n_i} \delta_{ai}. \quad (1.82)$$

It is now clear that eq. (1.79) functions as the projector onto the required subspace. \square

Claim 1.6.

$$\text{tr}(P_a) = n_a m_a^D, \quad (1.83)$$

where m_a is the number of repetitions of irreducible representations in D which are equivalent to D_a .

Example 1.9. Consider the defining representation of S_3 :¹

$$\begin{aligned} D_3(e) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & D_3(a_1) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & D_3(a_2) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \\ D_3(a_3) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & D_3(a_4) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & D_3(a_5) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (1.84)$$

Simple calculation reads

$$P_0 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \quad (1.85)$$

$\text{tr } P_0 = 1$ and $\text{tr } P_2 = 2$ leads

$$D_3 \sim D_0 \oplus D_2. \quad (1.86)$$

1.20 Conjugacy Classes

In this subsection, only the permutation groups, S_n s, are considered.

Claim 1.7. *Conjugation does not change the cycle structure.*²

Proof. If $g_1 \doteq i \mapsto j$, then $gg_1g^{-1} \doteq g(i) \mapsto g(j)$. \square

Claim 1.8. *The conjugacy classes must consist of all possible permutations with a particular cycle structure.*

Proof. Given two permutations $g_1, g_2 \in S_n$ with the same cycle structure. Obviously, it is always possible to relabel i 's in order to switch between two permutations, i.e., $gg_1g^{-1} = g_2$. \square

Remark 1.13. The conjugacy classes are the cycle structures.

¹The defining representation D_{def} of the permutation group S_n is n -dimensional vector space which is defined by $D_{\text{def}}(j \mapsto k)|j\rangle = |k\rangle$, i.e., $\langle l|D_{\text{def}}(j \mapsto k)|j\rangle = \delta_{jk}$

²Conjugation resembles similarity transformations.

Claim 1.9. *The number of different permutations in a conjugacy class of S_n which consists of k_j copies of j -cycles is*

$$\frac{n!}{\prod_j j^{k_j} k_j!}. \quad (1.87)$$

Proof. j^{k_j} eliminates k copies of the degrees of freedom(= j) of cycling indices within the j -cycles, e.g., $(123) = (231)$. $k_j!$ eliminates the degrees of freedom of permutation between the j -cycles, e.g., $(12)(34) = (34)(12)$. \square

1.21 Young Tableaux

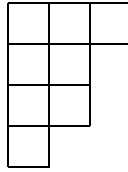


Figure 1: The Young Tableau represents an eight-dimensional permutation with a 4-cycle, a 3-cycle, and a 1-cycle.

A Young tableau can represent a conjugacy class by having k_j copies of a column with length j .

Example 1.10. The permutation group S_3 has three conjugacy classes.

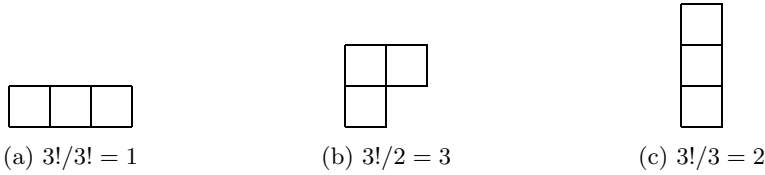


Figure 2: The conjugacy classes of S_3 with the numbers of their elements.

Example 1.11. The permutation group S_4 has five conjugacy classes.

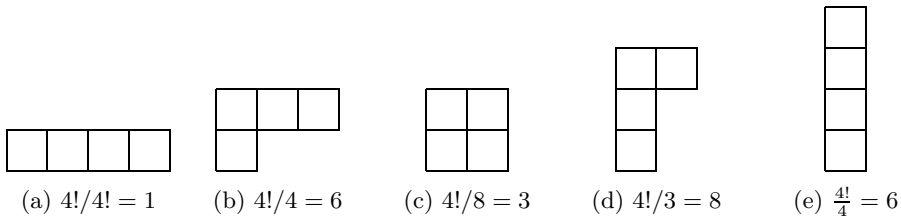


Figure 3: The conjugacy classes of S_4 with the numbers of their elements.

1.14 Eigenstates

Theorem 1.7. *If a hermitian operator, H , commutes with all the elements, $D(g)$, of a representation of the group G , then you can choose the eigenstates of H to transform according to irreducible representations of G . If an irreducible representation appears only once in the Hilbert space, every state in the irreducible representation is an eigenstate of H with the same eigenvalue.*

Proof. Consider a eigenstate $|\psi\rangle$ of H . Since $[H, D(g)] = 0$, $D(g)$ cannot change the H eigenvalue of the state:

$$H(D(g)|\psi\rangle) = D(g)H|\psi\rangle = D(g)E|\psi\rangle = E(D(g)|\psi\rangle). \quad (1.88)$$

In other words, H cannot move a state from one irreducible representation's subspace to another, i.e.,

$$H|a, j, x\rangle = \sum_y c_j |a, j, y\rangle. \quad (1.89)$$

According to Schur's lemma, H must be proportional to the identity within the subspace of a given irreducible representation. \square

Theorem 1.8. *All of the irreducible representations of a finite abelian group are 1-dimensional.*

Proof. Every element of an abelian group is a conjugacy class by itself. By remark 1.12, the number of irreducible representations is the order of the group; the only way to satisfy eq. (1.70) is $n_j = 1$ for all irreducible representations. \square