

Dirichlet and Neumann Green's Function

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Published on Feb 24, 2024

1 The Uniqueness Theorem of Poisson's Equation

1.1 Green's Identities

Applying the divergence theorem to a vector field $\phi_1 \nabla \phi_2$, we obtain Green's first identity:

$$\int_{\mathcal{V}} (\phi_1 \nabla^2 \phi_2 + \nabla \phi_1 \cdot \nabla \phi_2) d^3x = \oint_{\partial \mathcal{V}} \phi_1 \nabla \phi_2 \cdot \hat{\mathbf{n}} d^2x. \quad (1)$$

By interchanging ϕ_1 and ϕ_2 and subtracting from the original eq. (1), Green's second identity, also known as Green's theorem arises:

$$\int_{\mathcal{V}} (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) d^3x = \oint_{\partial \mathcal{V}} (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \hat{\mathbf{n}} d^2x. \quad (2)$$

1.2 Statement of the Theorem

Consider a bounded volume $\mathcal{V} \subseteq \mathbb{R}^3$ and its boundary $\partial \mathcal{V}$. Let $\rho(\mathbf{x})$ be an arbitrary charge density in \mathcal{V} . Then, Poisson equation

$$\nabla^2 \phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\epsilon_0} \quad (3)$$

has the unique solution ϕ , under either the following two conditions:

1. Dirichlet condition: $\phi = \psi$ on $\partial \mathcal{V}$.
2. Neumann condition: χ on $\partial \mathcal{V}$ such that $\int_{\partial \mathcal{V}} \chi d^2x = -\frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho d^3x$.

1.3 Proof of the Theorem

Let there exist two solutions ϕ and ϕ' of the Poisson equation eq. (3).

By applying Green's first identity eq. (1) to $\phi_1 = \phi_2 = \Delta\phi = \phi - \phi'$, we get

$$\int_{\mathcal{V}} ((\Delta\phi)\nabla^2(\Delta\phi) + \nabla(\Delta\phi) \cdot \nabla(\Delta\phi)) d^3x = \oint_{\partial\mathcal{V}} (\Delta\phi)\nabla(\Delta\phi) \cdot \hat{\mathbf{n}} d^2x,$$

which reduces to

$$\int_{\mathcal{V}} |\nabla(\Delta\phi)|^2 d^3x = 0.$$

The above result implies that $\nabla(\Delta\phi) = 0$ everywhere in \mathcal{V} . Thus the solution is unique or unique apart from a constant when Dirichlet and Neumann condition is given, respectively. Note that those two types of boundary conditions, of course, also can be used jointly—i.e., use Dirichlet's over some part on $\partial\mathcal{V}$ as well as Neumann's for the remaining part.

2 Dirichlet Green's Function

Green's function in electrostatics is defined as the solution to

$$\nabla_x^2 G(\mathbf{x}, \mathbf{x}') = -\frac{1}{\epsilon_0} \delta(\mathbf{x} - \mathbf{x}'), \quad (4)$$

which implies

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|} + \tilde{G}(\mathbf{x}, \mathbf{x}'), \quad (5)$$

where

$$\nabla_x^2 \tilde{G}(\mathbf{x}, \mathbf{x}') = 0 \text{ in } \mathcal{V}.$$

A continuous function \tilde{G} is generally determined by the shape of the volume.

Let us first consider the Dirichlet boundary condition of a conducting cavity, which is $G(\mathbf{x}, \mathbf{x}') = 0$ on $\partial\mathcal{V}$. The Dirichlet Green's function satisfies

$$G_D(\mathbf{x}, \mathbf{x}') = 0 \text{ on } \partial\mathcal{V}.$$

By the definition, the (partial) solution can be constructed as

$$\phi^{(1)}(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' \quad (6)$$

Subsequently, to treat an arbitrary Dirichlet boundary condition than $\psi = 0$, we would construct the remaining part of $\phi(\mathbf{x})$ due to the boundary. Let $\phi_1(\mathbf{x}) = G_D(\mathbf{x}, \mathbf{x}')$ and $\phi_2(\mathbf{x}) = \phi^{(2)}(\mathbf{x})$. Then Green's theorem eq. (2) yields

$$\phi^{(2)} = -\epsilon_0 \int_{\partial\mathcal{V}} \psi(\mathbf{x}') \nabla_{x'} G_D(\mathbf{x}', \mathbf{x}) \cdot \hat{\mathbf{n}} d^2x'. \quad (7)$$

Thus, we obtain the complete solution by adding $\phi^{(1)}$ and $\phi^{(2)}$:

$$\phi(\mathbf{x}) = \int_{\mathcal{V}} G_D(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' - \epsilon_0 \int_{\partial\mathcal{V}} \psi(\mathbf{x}') \nabla_{x'} G_D(\mathbf{x}', \mathbf{x}) \cdot \hat{\mathbf{n}} d^2x'. \quad (8)$$

3 Neumann Green's Function

In like manner of the previous section, let us consider the Neumann Green's function that satisfies

$$\hat{\mathbf{n}} \cdot \nabla_x G_N(\mathbf{x}, \mathbf{x}') = -\frac{1}{\epsilon A} \text{ on } \partial\mathcal{V} \quad (9)$$

and a constraint

$$\int_{\partial\mathcal{V}} G_N(\mathbf{x}, \mathbf{x}') d^2x = 0, \quad (10)$$

where A is the surface area of \mathcal{V} .

Following a similar procedure, we obtain the solution written in the Neumann Green's function:

$$\phi(\mathbf{x}) = \int_V G_N(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') d^3x' + \epsilon_0 \int_{\partial\mathcal{V}} \chi(\mathbf{x}') G_N(\mathbf{x}', \mathbf{x}) d^2x'. \quad (11)$$

Note that the above results can be extended to an unbounded region with finitely many disconnected volumes naturally.

References

- [1] John D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.
- [2] Robert M. Wald. *Advanced Classical Electrodynamics*. Princeton University Press, 2022.