The Explanation of Acoustic Resonance in Air Columns

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1 INTRODUCTION

Many freshman-level physics textbooks introduce acoustic resonance in air columns. That is, when producing a sound in front of the open end, the column amplifies and sustains the sound only if the frequency of the sound matches one of the natural frequencies of the column. However, their explanations are insufficient to describe how actually reflections work. Crucially, they were, in principle, wrong. A standing wave can be formed and the column solely sustains an oscillation regardless of frequency considerably. This paper shows standing waves of natural frequencies are not the only possible way acoustic waves can be formed in half-closed air columns, and reveals how much the column amplifies the sound for a specific frequency.

2 REFLECTION OF ACOUSTIC WAVE

2.1 AT THE CLOSED ENDS

Air particles moving toward the closed end collide with the end and reverse their direction. As a result of this kind of elastic collision, the reflected wave has the same phase angle as the original wave, where the moving direction is opposite. Consequently, a sum of the original and reflected wave indicates that the displacement of air particles at the closed end is always zero.

2.2 AT THE OPEN ENDS

In the first instance, typical acoustic resonance problems assume that the air pressure at the open end remains constant at atmospheric pressure. The adequate explanation starts with focusing on the last air parcel just at the open end. When a travelling compressed air parcel arrives at the open end, since there are no more air particles that can be compressed or rarefied, the last parcel behaves as if it were attached to the travelling parcel. This movement starts the reflected wave,

whose phase angle is the same as the original wave. The case that the parcel is rarefied could be explained almost the same way.

3 FORMATION OF COMPOSITE WAVE

As the original wave reflects continually as discussed above, reflected waves eventually form a standing wave. Contrary to common explanations, the composite wave has a non-negligible amplitude even if the frequency doesn't match one of the resonance frequencies.

Suppose an air column of length L whose open end and closed end are located at $x = 0$ and $x = L$, respectively. Let the original wave be $\Phi_0^R(x,t)$. n^{th} wave moving right is denoted as Φ_n^R, where Φ_n^L denotes moving left. That is, Φ_n^R produces Φ_n^L at the closed end and Φ_n^L produces Φ_{n+1}^R at the open end. Explicit expressions are

$$\begin{cases} \Phi_n^R = A\sin\left(kx - \omega t + \dfrac{4n\pi L}{\lambda} + n\pi\right) \\ \Phi_n^L = A\sin\left(k(L-x) - \omega t + \dfrac{(4n+2)\pi L}{\lambda} + (n+1)\pi\right) \end{cases} \tag{1}$$

The infinite sum of the reflected waves is

$$\begin{aligned} \sum_{n=0}^{\infty}\left(\Phi_n^R + \Phi_n^L\right) &= 2A\sin\left(kx - \frac{2\pi L}{\lambda}\right)\sum_{n=0}^{\infty}\cos\left(\left(\frac{4L}{\lambda}+1\right)n\pi - \omega t + \frac{2\pi L}{\lambda}\right) \\ &= 2A\sin\left(kx-\varphi\right)\Re\left(\sum_{n=0}^{\infty} e^{i(n\Theta - \omega t + \varphi)}\right) \end{aligned}$$

The series above doesn't converge technically, however, one can find a physically meaningful solution. A partial sum of phasors $\sum_{n=0}^{N} e^{in\Theta}$ always lies on a circle. By taking the limit as energy loss tends to zero, the solution must be the centre of the circle. In a mathematical sense, this technique is called Cesàro sum. Thus, one obtains the composite wave

$$\Psi(x,t) = 2A\sin\left(kx-\varphi\right)\Re\left(\frac{ie^{-i\Theta}}{2\sin\Theta}e^{i(-\omega t+\varphi)}\right) \tag{2}$$

$$= A\sin\left(kx-\varphi\right)\frac{\sin\left(\Theta+\omega t+\varphi\right)}{\sin\Theta} \tag{3}$$

This is also a standing wave with a node at the closed end, whose maximum displacement is $A/|\sin\Theta|$. Ideally, i.e., in absence of energy loss, the maximum displacement blows up when the initial wave satisfies the ordinary resonance condition.