2 Lie Groups

Definition 2.1 (Lie groups)**.** A **Lie group** is a group that is also a smooth manifold, where the group operations are smooth.

2.1 Generators

Suppose $g \in G$ depends smoothly on a set of continuous parameters; $g(\alpha)$ depends on α_a for $a = 1$ to N. It is useful to set

$$g(\alpha)\Big|_{\alpha=0} = e; \quad D(\alpha)\Big|_{\alpha=0} = 1. \tag{2.1}$$

In some neighbourhood of the identity, the Taylor expansion yields

$$D(\mathrm{d}\alpha) = 1 + i\mathrm{d}\alpha_a X_a + \cdots . \tag{2.2}$$

Definition 2.2 (generators)**.** For $a = 1$ to N,

$$X_a \equiv -i\frac{\partial}{\partial \alpha_a} D(\alpha)\Big|_{\alpha=0} \tag{2.3}$$

are called the **generators** of the group.

Remark 2.1. If D is unitary, X_a are hermitian.

Remark 2.2. The generators, X_a's, form a vector space.

Claim 2.1 (exponential parametrisation)**.**

$$D(\alpha) = \lim_{k\to\infty}\left(1 + \frac{i\alpha_a X_a}{k}\right)^k = e^{i\alpha_a X_a} \tag{2.4}$$

2.2 Lie Algebras

Claim 2.2.

$$[\alpha_a X_a, \beta_b X_b] \in \mathrm{span}\{X_a\} \tag{2.5}$$

Proof.

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}, \tag{2.6}$$

where δ is given by the Baker–Campbell–Hausdorff formula:

$$i\delta_a X_a = i\alpha_a X_a + i\beta_a X_A - \frac{1}{2}[\alpha_a X_a.\beta_b X_b] + \cdots . \tag{2.7}$$

Consider a function

$$Z(t) = \ln\big(1 + e^{it\alpha_a X_a} e^{it\beta_b X_b} - 1\big), \quad t \in \mathbb{R}. \tag{2.8}$$

Taylor expansion with respect to t yields

$$Z(t) = t(\alpha_a X_a + \beta_b X_b) + \frac{1}{2}t^2[\alpha_a X_a, \beta_a X_b] + \mathcal{O}(t^3). \tag{2.9}$$

Since

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = D(\alpha)D(\beta) = D(\alpha\beta), \tag{2.10}$$

$Z(t)$ belongs to $\mathrm{span}\{X_a\}$. Therefore, each coefficient of t^k, including $[\alpha_a X_a, \beta_b X_b]$, in eq. (2.9) must also lie in the vector space. □

Definition 2.3 (Lie algebras)**.** A **Lie algebra** is a vector space $\mathfrak{g}$ over a field F with a binary operation $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying bilinearity, alternating property, and the Jacobi identity.

Remark 2.3. The Lie algebra is independent of the choice of representation, unless the representation is unfaithful.[3]

Definition 2.4 (structure constants)**.** The **structure constants** f_{abc} of the group defines the commutator algebra

$$[X_a, X_b] = i f_{abc} X_c, \tag{2.11}$$

where $f_{abc} = -f_{bac}$

Definition 2.5 (Lie-algebra representations)**.** A **representation** of a Lie-algebra $\mathfrak{g}$ is a mapping, D, of the elements of $\mathfrak{g}$ onto a set of linear operators with the following properties:

1. D is linear: $D(aX + bY) = aD(X) + bD(Y)$.
2. D preserves the Lie bracket structure:
   $D([X, Y]) = [D(X), D(Y)] = D(X)D(Y) - D(Y)D(X)$.

Claim 2.3. *If there is any unitary representation of the algebra, then the f_{abc}'s are real.*

Proof.

$$i f_{abc} X_c = -i f_{bac} X_c = -[X_b, X_a] = -[X_a, X_b]^\dagger = i f_{abc}{}^* X_c \tag{2.12}$$

□

2.3 The Jacobi Identity

Remark 2.4. The matrix generators satisfy the following identity:

$$[X_a, [X_b, X_c]] + [X_c, [X_a, X_b]] + [X_b, [X_c, X_a]] = 0. \tag{2.13}$$

This is called the Jacobi identity.

Remark 2.5.

$$[X_a, X_b X_c] = [X_a, X_b]X_c + X_b[X_a, X_c]. \tag{2.14}$$

[3] There is vanishing X_a if and only if the representation is unfaithful; we see only the quotient algebra.

2.4 The Adjoint Representation

Definition 2.6 (adjoint representations)**.** The **adjoint representation** T of an algebra is the one generated by the structure constants:

$$[T_a]_{bc} \equiv -if_{abc}. \tag{2.15}$$

Proof. The Jacobi identity implies

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0, \tag{2.16}$$

which is equivalent to

$$[iT_a, iT_b] = -f_{abc}iT_c, \tag{2.17}$$

since

$$[X_a, [X_b, X_c]] = if_{bcd}[X_a, X_d] = -f_{bcd}f_{abe}X_e. \tag{2.18}$$

□

Remark 2.6. The dimension of the adjoint representation is the number of independent generators, which is the number of real parameters required to describe a group element.

Claim 2.4.

$$\operatorname{tr}(T_aT_b) = k^a\delta_{ab} \quad (\textit{no sum}) \tag{2.19}$$

Proof. Consider a linear transformation

$$L : X_a \mapsto X'_a = L_{ab}X_b. \tag{2.20}$$

Then,

$$[X'_a, X'_b] = iL_{ad}L_{be}f_{deg}L^{-1}_{gc}X'_c \tag{2.21}$$

implies

$$L : f_{abc} \mapsto f'_{abc} = L_{ad}L_{be}f_{deg}L^{-1}_{gc}, \tag{2.22}$$

or equivalently,

$$L : T_a \mapsto T'_a = L_{ad}LT_dL^{-1}. \tag{2.23}$$

Since similarity transformations do not change the trace,

$$L : \operatorname{tr}(T_aT_b) \mapsto \operatorname{tr}(T'_aT'_b) = L_{ac}L_{bd}\operatorname{tr}(T_cT_d). \tag{2.24}$$

By choosing an appropriate orthogonal operator L, we get eq. (2.19). □

Remark 2.7. Rescaling L appropriately, we could choose all the non-zero k^a's to have absolute value 1.

Definition 2.7 (compact Lie algebras)**.** A Lie algebra is said to be **compact** if all k^a's are positive.

For now, only compact Lie algebras will be considered:

$$\operatorname{tr}(T_aT_b) = \lambda\delta_{ab}, \quad \lambda > 0. \tag{2.25}$$

Claim 2.5.

$$f_{abc} = -i\lambda^{-1}\operatorname{tr}([T_a, T_b]T_c) \tag{2.26}$$

Proof.

$$\operatorname{tr}([T_a, T_b]T_c) = \operatorname{tr}(if_{abd}T_dT_c) = if_{abd}\operatorname{tr}(T_dT_c) = if_{abd}\lambda\delta_{dc} \tag{2.27}$$

□

Claim 2.6. *f_{abc} is completely antisymmetric.*

Proof.

$$f_{abc} = f_{bca} \tag{2.28}$$

since

$$\begin{aligned}\operatorname{tr}([T_a, T_b]T_c) &= \operatorname{tr}(T_aT_bT_c - T_bT_aT_c) \\ &= \operatorname{tr}(T_bT_cT_a - T_cT_bT_a) = \operatorname{tr}([T_b, T_c]T_a).\end{aligned} \tag{2.29}$$

□

2.5 Simple Algebras and Groups

Definition 2.8 (invariant subalgebras)**.** An **invariant subalgebra** is some set of generators which goes into itself under commutation with any element of the algebra. I.e., if X is any generator in the invariant subalgebra and Y is any generator in the whole algebra, then $[Y, X]$ is a generator in the invariant subalgebra.

Claim 2.7. *An invariant subalgebra generates the corresponding invariant subgroup by exponentiation, i.e.,*

$$e^{-iY}e^{iX}e^{iY} = e^{iX'}, \tag{2.30}$$

where

$$X' = e^{-iY}Xe^{iY}. \tag{2.31}$$

Proof.

$$e^{-iY}(iX)^n e^{iY} = (e^{-iY}iXe^{iY})^n \tag{2.32}$$

□

Definition 2.9 (trivial algebras)**.** An algebra which has no nontrivial invariant subalgebra is called **simple**, where the whole algebra and 0 are trivial invariant subalgebra.

Remark 2.8. A simple algebra generates a simple group.

Claim 2.8. *The adjoint representation of a simple Lie algebra is irreducible.*[4]

[4] The simplicity of a group does not automatically imply the irreducibility of the adjoint representation.

Proof. Assume the contrary. Then there is an invariant subspace spanned by T_r for $x = 1$ to K, where the rest of the generators T_x for $x = K + 1$ to N. Therefore, since

$$[T_a, T_x] = i f_{axr} T_r \tag{2.33}$$

we must have

$$[T_a]_{xr} = -i f_{axr} = 0. \tag{2.34}$$

By virtue of the complete antisymmetry of the structure constants, the nonzero structure constants involve either three r's or three x's, and thus the algebra falls apart into two nontrivial invariant subalgebras, and is not simple. □

Definition 2.10 (semisimple algebras)**.** Algebras without Abelian invariant subalgebras are called **semisimple**.

2.7 Fun with Exponentials

Claim 2.9.

$$\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} = \int_0^1 \mathrm{d}s \; e^{is\alpha_a X_a} i X_b e^{i(1-s)\alpha_c X_c} \tag{2.35}$$

Proof. We can always define the exponential as a power series:

$$e^{i\alpha_a X_a} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\alpha_a X_a)^n. \tag{2.36}$$

Referring to eq. (2.4),

$$\begin{aligned}
\frac{\partial}{\partial \alpha_b} e^{i\alpha_a X_a} &= \frac{\partial}{\partial \alpha_b} \lim_{k\to\infty} \left(1 + \frac{i\alpha X_a}{k}\right)^k \\
&= \lim_{k\to\infty} \sum_{l=1}^{k} \left(1 + \frac{i\alpha X_a}{k}\right)^{l-1} \left(\frac{iX_b}{k}\right) \left(1 + \frac{i\alpha X_a}{k}\right)^{k-l} \\
&= \int_0^1 \mathrm{d}s \; e^{is\alpha_a X_a} i X_b e^{i(1-s)\alpha_c X_c}.
\end{aligned} \tag{2.37}$$

□