Enhanced Euler Method

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INTRODUCTION

This research paper will describe and implement the Euler, Midpoint, Modified, and Enhanced Euler Method, based on the published journal "Enhanced Euler's Method to Solve First Order Ordinary Differential Equations with Better Accuracy" by Md Nurujjaman from Jagannath University in March 2020. Although the Euler method is one of the most basic fundamentals of this course, it may not be the fastest or most accurate method to solve first-order differential equations. Hence, I will start by going over the Euler method, then other improved methods such as Midpoint and Modified, and then finally the Enhanced Euler method which is faster and more accurate. This report also recreates some of the graphs found in the journal, although there may be slight differences due to different equations used for clearer graphs. I thought it would be interesting to see the evolution of this fundamental method over time more in depth.

EULER METHOD (FORWARD)

The (forward) Euler method is a tool that uses an initial value and approximations to approximate the exact solution to a differential equation. The process is to find the slope at the initial value, add the slope multiplied by the step "h" (also known as Δx), and continue this process.

We can also write this numerically where $y' = f(x, y)$ is the slope, (x_0, y_0) is the initial condition where $y(x_0) = y_0$, and $n = 0, 1, 2, 3, ..., n$ represents the iteration.

$y_1 = y_0 + h * f(x_0, y_0)$

$y_2 = y_1 + h * f(x_1, y_1)$

$y_3 = y_2 + h * f(x_3, y_3)$

$\dots$

$\therefore\ y_{n+1} = y_n + h * f(x_n, y_n)$

We can also represent this graphically.

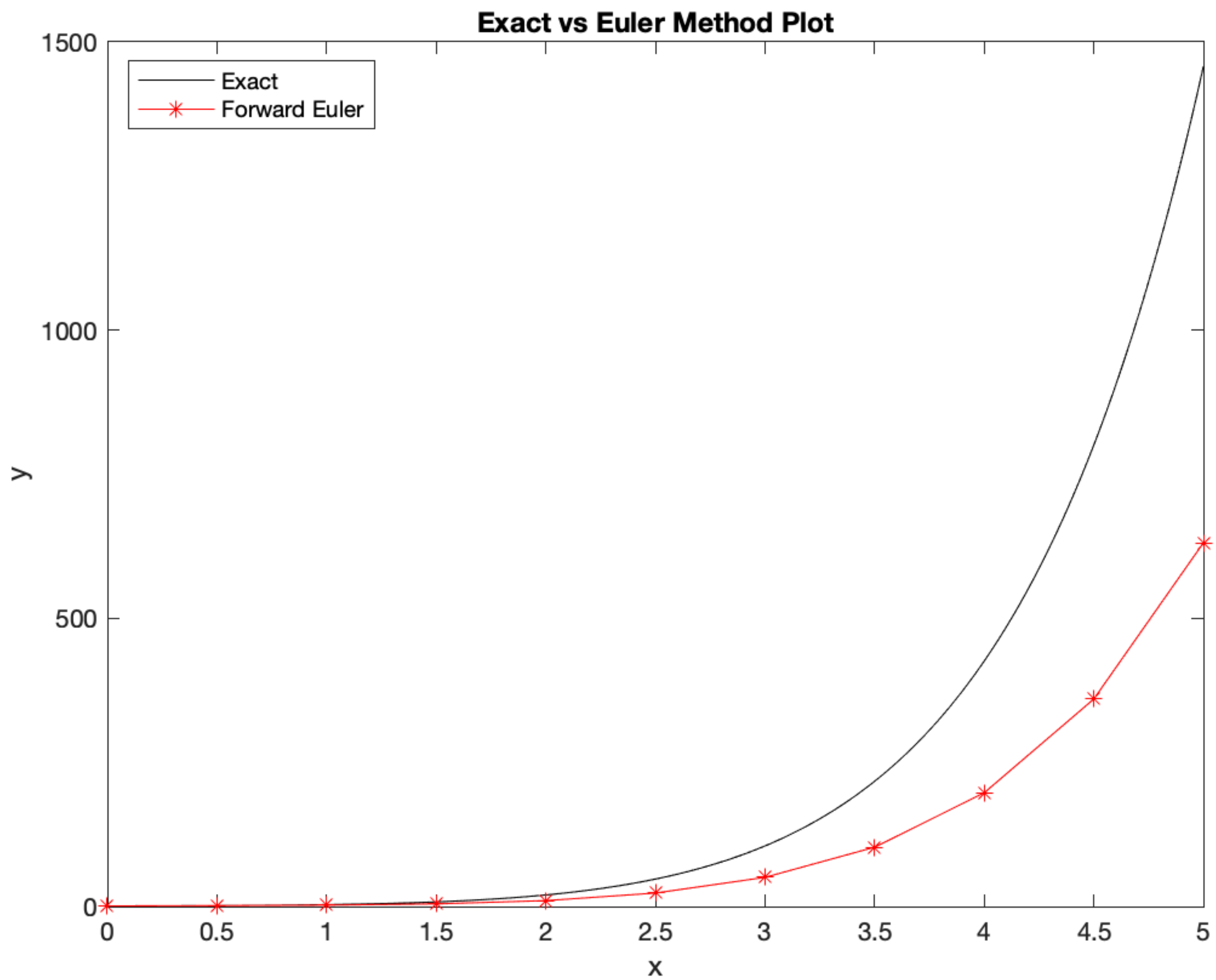

Figure 1: Geometrical Interpretation of Euler's Method

The (x_{n+1}, y_{n+1}) depends on the slope and x, y values at point (x_n, y_n). However, we can see that as $x \to \infty$, the error, or the absolute difference between the exact solution and approximation, becomes bigger and bigger. Methods such as the Midpoint method was developed to lessen the error.

MIDPOINT METHOD

The Midpoint method is very similar to the Euler method, except we use the slope at $(x_{1/2}, y_{1/2})$ instead of (x_0, y_0), as we observed in the last figure that the error grows as x gets larger.

In order to mitigate this, we can use a larger slope to stay close to the exact solution, and this slope will be found in between (x_0, y_0) and (x_1, y_1), which is $(x_{1/2}, y_{1/2})$. We will do this for all timesteps as such.

$y_1 = y_0 + h * f(x_{0+1/2}, y_{0+1/2}) = y_0 + h * f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$

$y_2 = y_1 + h * f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1))$

$y_3 = y_2 + h * f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2))$

…

$\therefore\ y_{n+1} = y_n + h * f(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))$

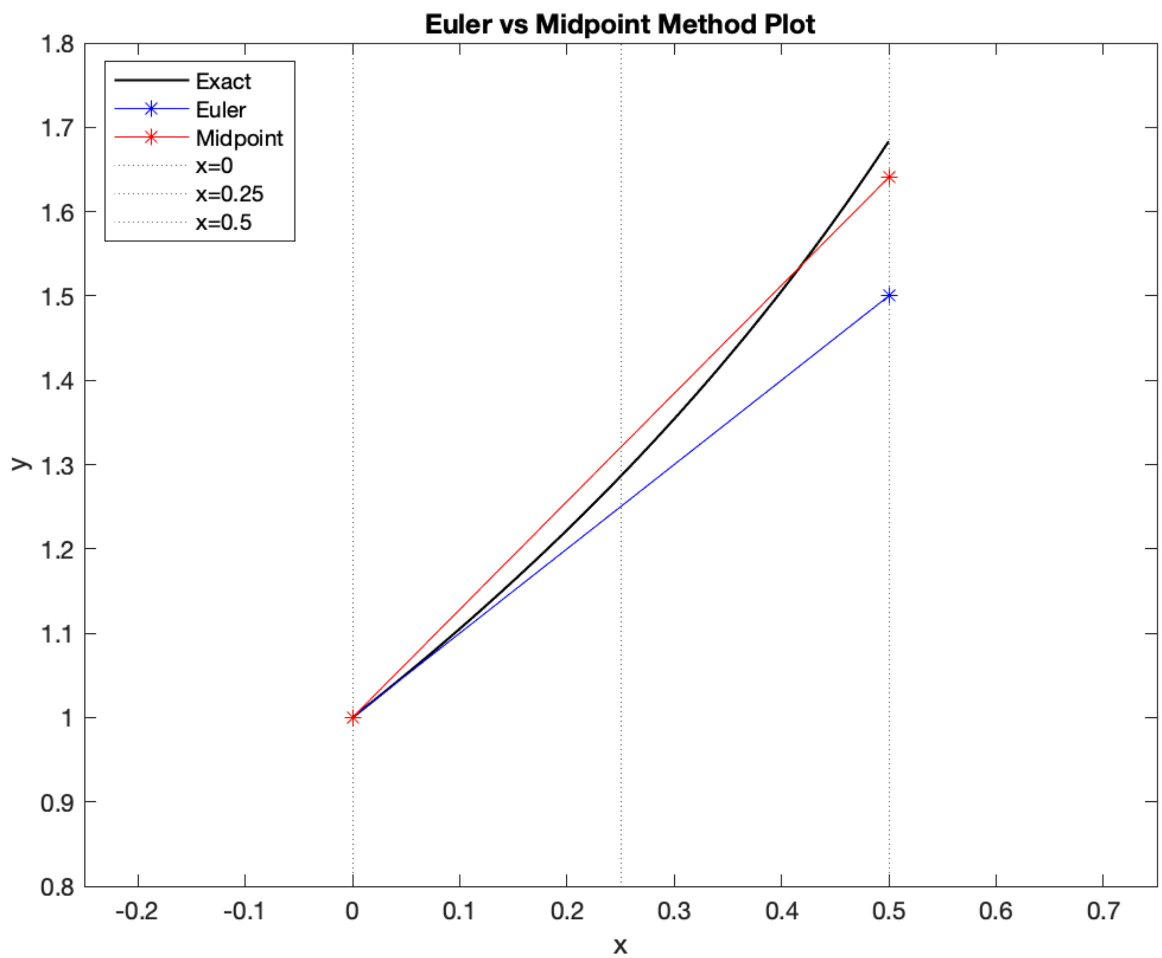

Figure 2: Difference between Euler and Midpoint Method for 1 timestep

We can see from this figure the difference between the Euler and Midpoint method. The slope of the Midpoint method is larger, so it gets a closer approximation to the exact solution on the next iteration. Looking at the points at $x = 0.5$, the Midpoint method is way closer to the exact point than the euler method, so we can see that it is working.

We can see the approximations for the Midpoint method below. Again, the approximations are closer to the exact than Euler.

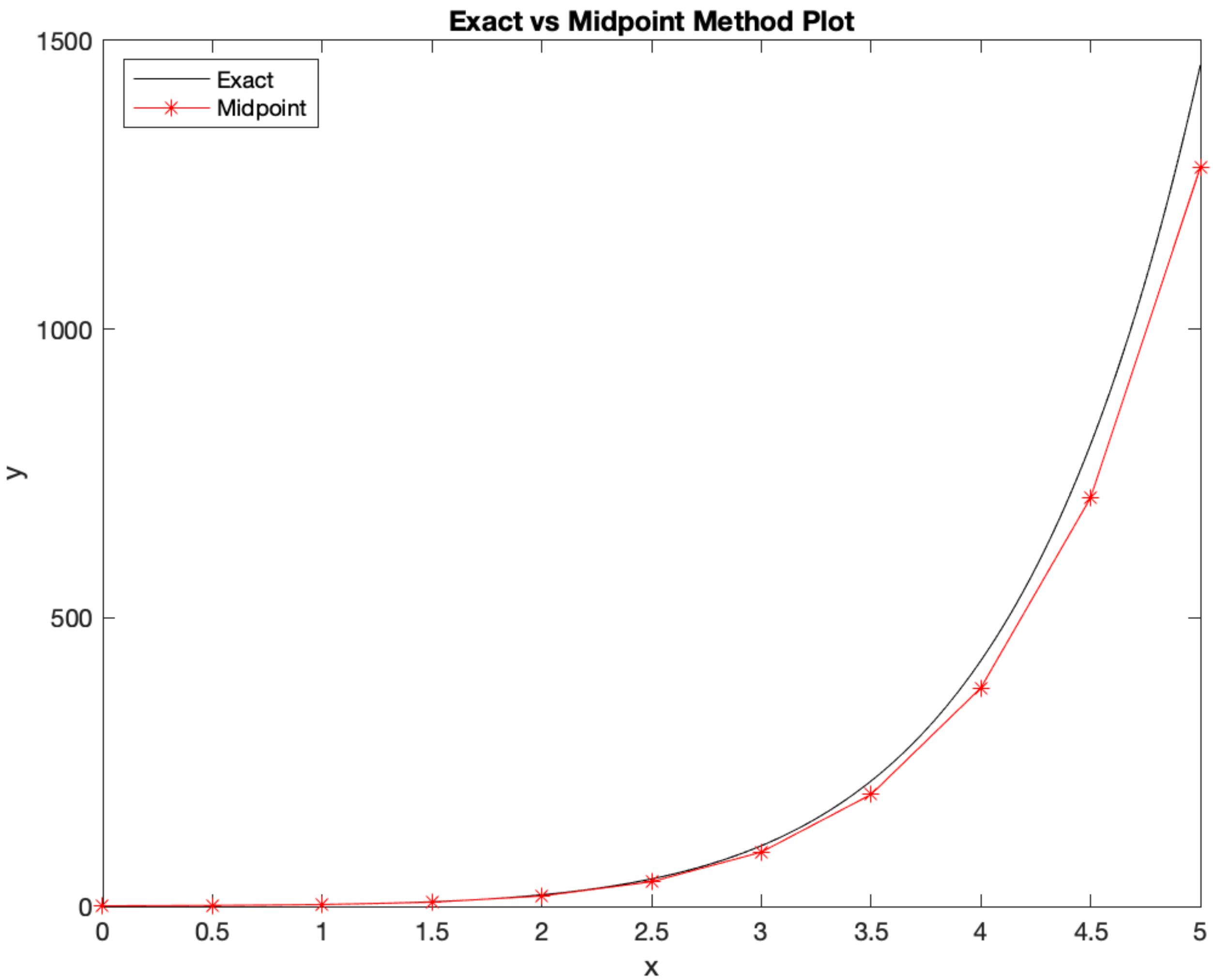

Figure 3: Geometrical Interpretation of Midpoint Method

MODIFIED EULER'S METHOD

The Modified Euler method is a predictor-corrector method that is more advanced and built on the other two methods. It takes the average of the slope at (x_0, y_0) and (x_1, y_1) instead of the slope at the midpoint, since the slope at (x_0, y_0), (x_1, y_1), and (x_2, y_2) may be drastically different. Since this is not a linear system, taking the average is more appropriate than using the midpoint between points.

We can graph one step as such.

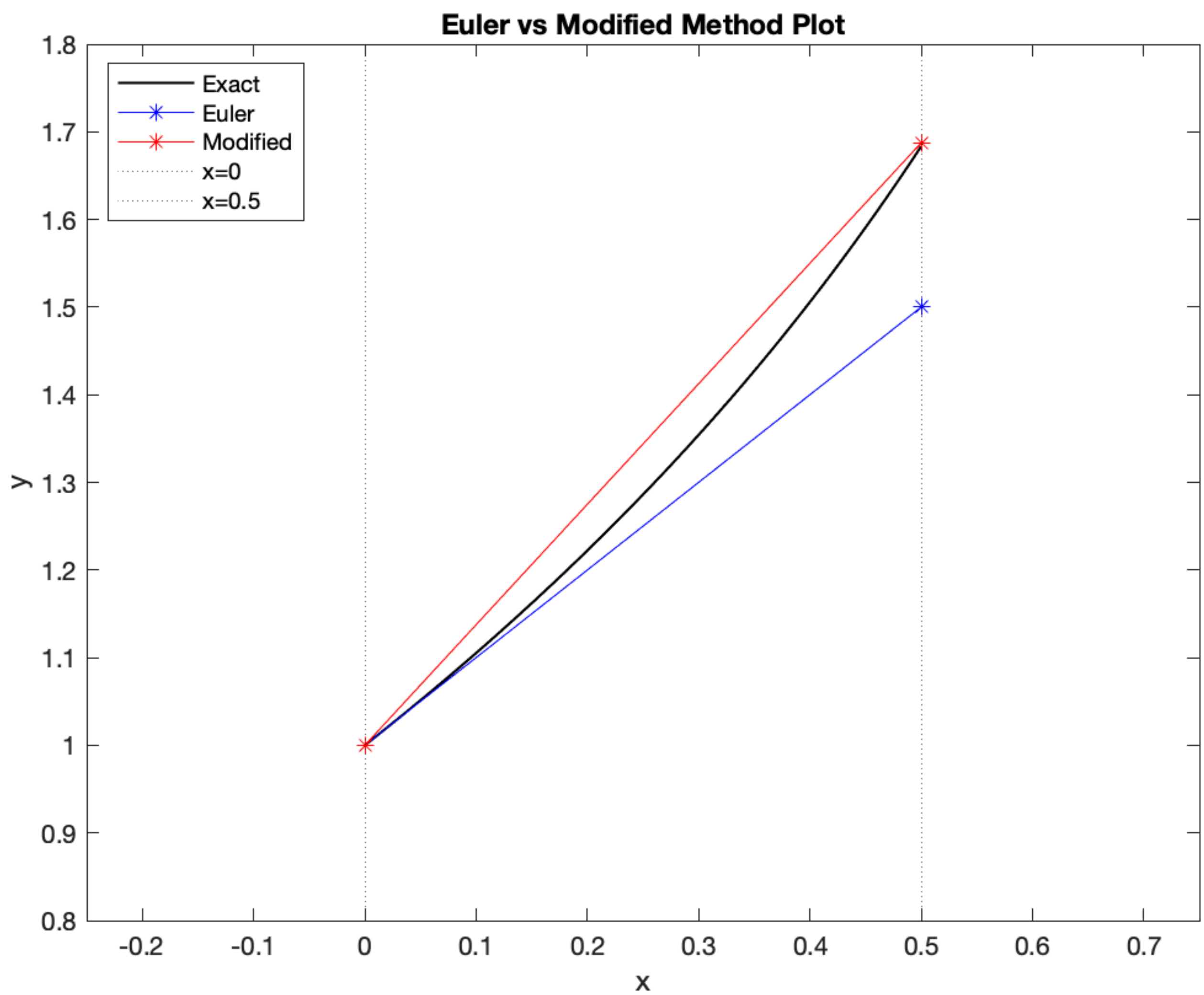

Figure 4: Difference between Euler and Modified Method for 1 timestep

We can see that now the slope used at our first iteration on (x_0, y_0) is not too low like the Euler method and the Midpoint method (this pattern can change based on the formula or timestep h), but closer to the exact value. Just from this graph, it seems it is almost the exact value at $x = 0.5$. We repeat this for each iteration. Using this algorithm, we get this formula.

$$y_1 = y_0 + \frac{h}{2} * (f(x_0, y_0) + f(x_1, y_1))$$
$$y_2 = y_1 + \frac{h}{2} * (f(x_1, y_1) + f(x_2, y_2))$$
$$y_3 = y_2 + \frac{h}{2} * (f(x_2, y_2) + f(x_3, y_3))$$
$$\dots$$
$$\therefore\ y_{n+1} = y_n + \frac{h}{2} * (f(x_n, y_n) + f(x_{n+1}, y_{n+1}))$$

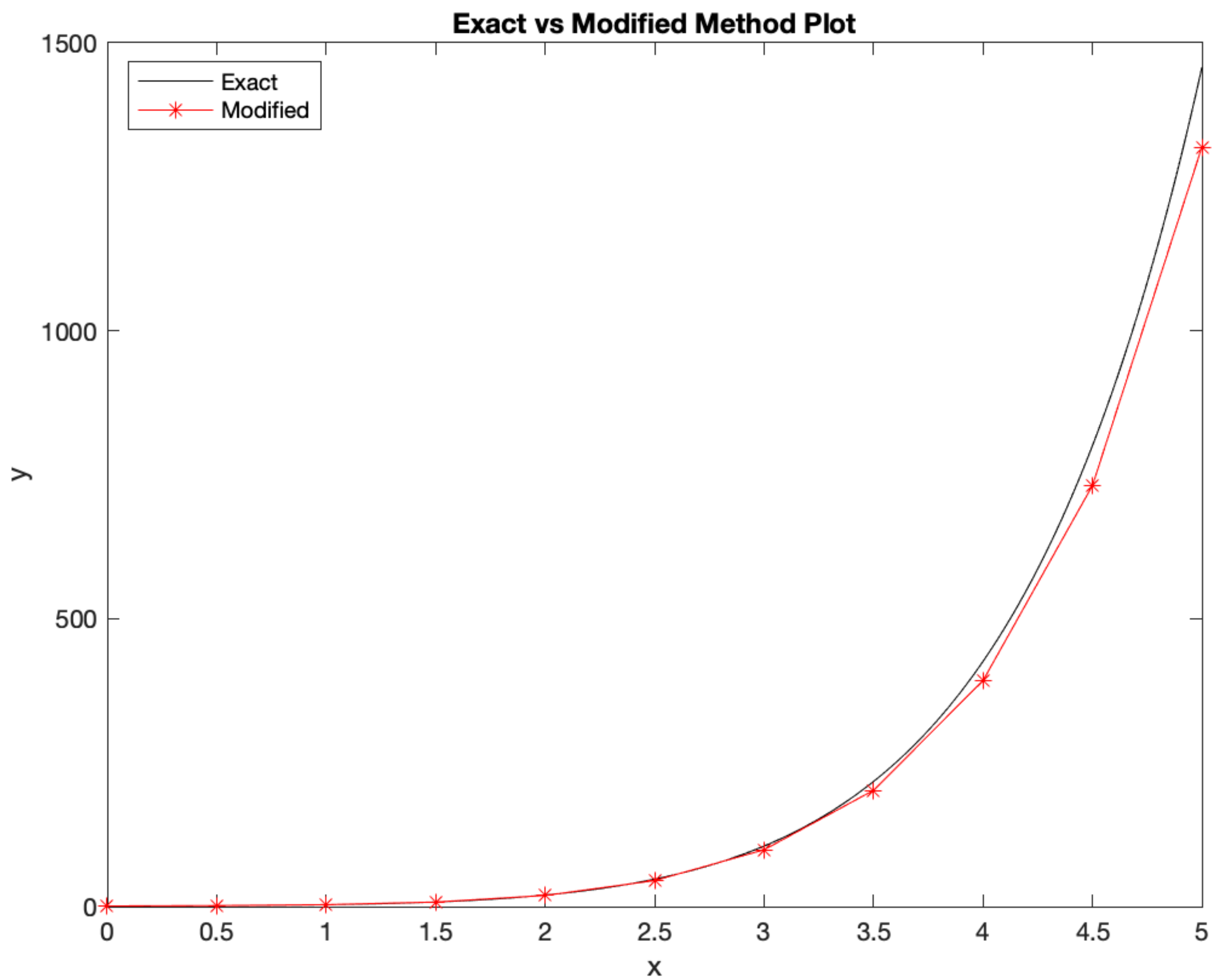

Figure 5: Geometrical Interpretation of Modified Method

We can see that this method seems to give a more approximate estimation to the exact solution.

ENHANCED EULER'S METHOD

Now finally for the Enhanced Euler's method, we can even more improve the approximation by combining the Midpoint and Modified methods. Let's first get the Midpoint equation at $n = 0$.

$$y_1 = y_0 + h * f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(1)}$$

We can also get the Modified Euler method at $n = 0$.

$$y_1 = y_0 + \frac{h}{2} * (f(x_0, y_0) + f(x_1, y_1)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(2)}$$

In the Modified Euler's method, we took the average slopes where y_1 was calculated using the Euler method, but in the Enhanced Euler's method we will use the midpoint method to calculate the y_1. This can be done by substituting equation **(1)** in equation **(2)** like this.

Set up x_1 and y_1 like this.

$x_1 \Rightarrow x_0 + h$

$y_1 \Rightarrow y_0 + h * f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$

Now we can substitute.

$y_1 = y_0 + \frac{h}{2} * (f(x_0, y_0) + f(x_0 + h, y_0 + h * f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))))$

We can now do this for each timestep, so we get this.

$y_1 = y_0 + \frac{h}{2} * (f(x_0, y_0) + f(x_0 + h, y_0 + h * f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))))$

$y_2 = y_1 + \frac{h}{2} * (f(x_1, y_1) + f(x_1 + h, y_1 + h * f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1))))$

$y_3 = y_2 + \frac{h}{2} * (f(x_2, y_2) + f(x_2 + h, y_2 + h * f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2))))$

$\ldots$

$\therefore y_{n+1} = y_n + \frac{h}{2} * (f(x_n, y_n) + f(x_n + h, y_n + h * f(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))))$

If we graph using this method, we can get this.

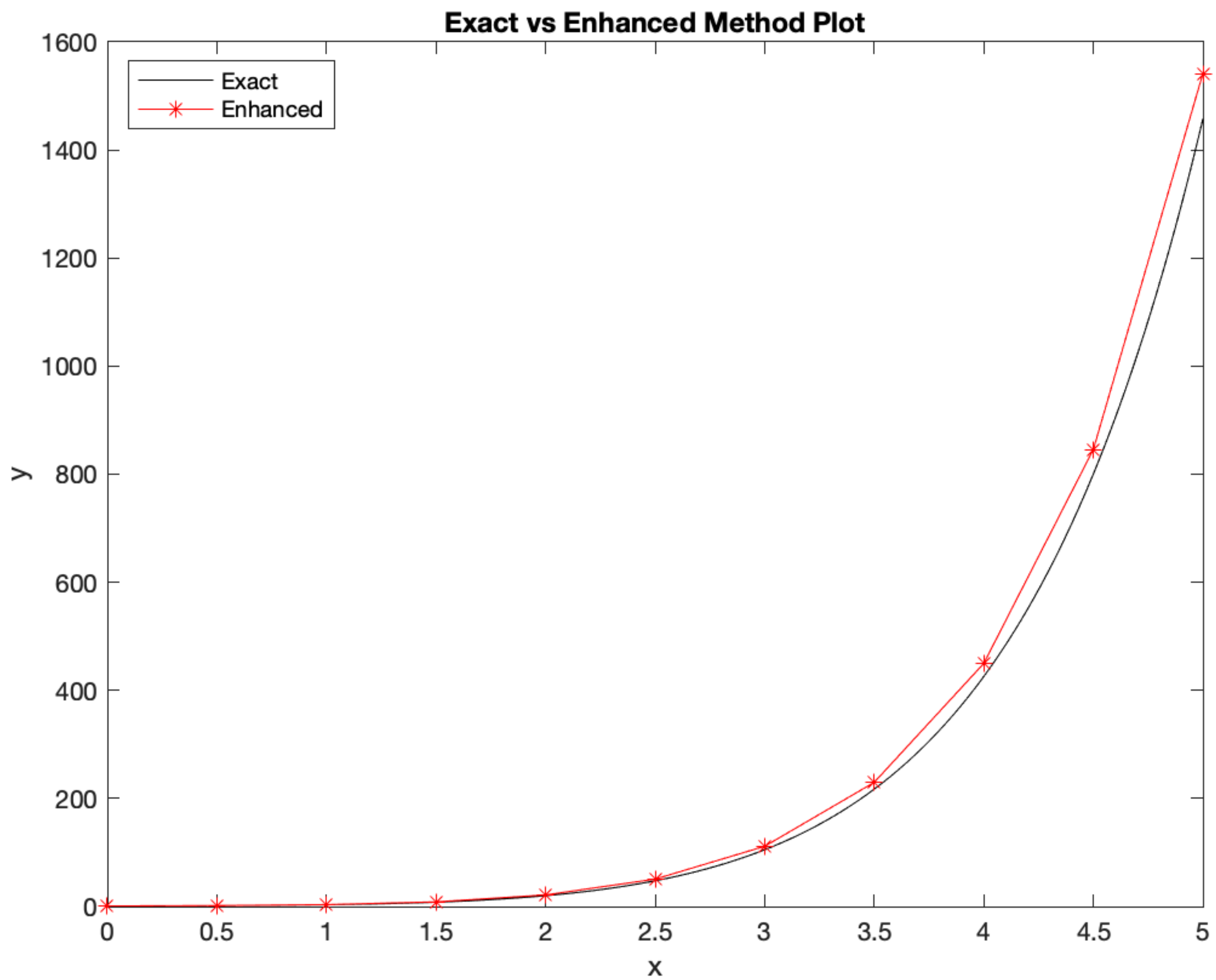

FIgure 6: Geometrical Interpretation of Enhanced Method

We can see that even though the approximation is higher than the exact, the error is still smaller than when using the Modified method. Figure 8 will compare the errors more in depth.

CONCLUSION

These were the descriptions and implementations of the basic (forward) Euler, Midpoint, Modified, and Enhanced methods. Let's compare how each performs compared to the exact solution. For this visualization, we will use

$$y' = 2x^3 + y$$
$$y(x) = y(0) = 1$$

If we solve this, we get $y = 13e^x - 2(x^3 + 3x^2 + 6x + 6)$, which is what is graphed.

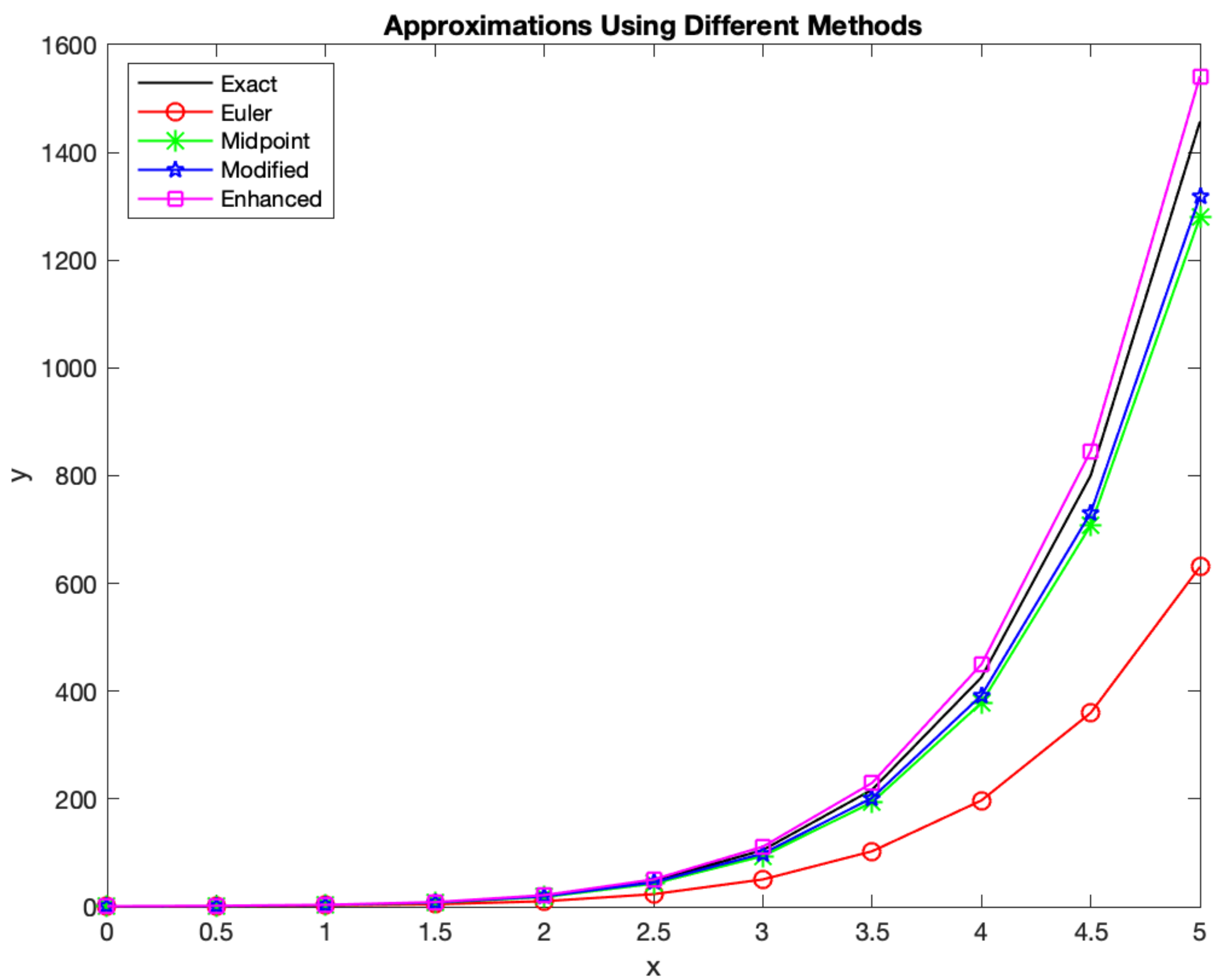

Figure 7: Graphical Comparison Among the Solutions with Different Methods

As expected, the Euler method is the least accurate, Midpoint is better, Modified is even better than that, and Enhanced is the best estimate. Just to make sure this is correct, we can graph the error for each method at each timestep like below. We see that this is consistent with what we have found.

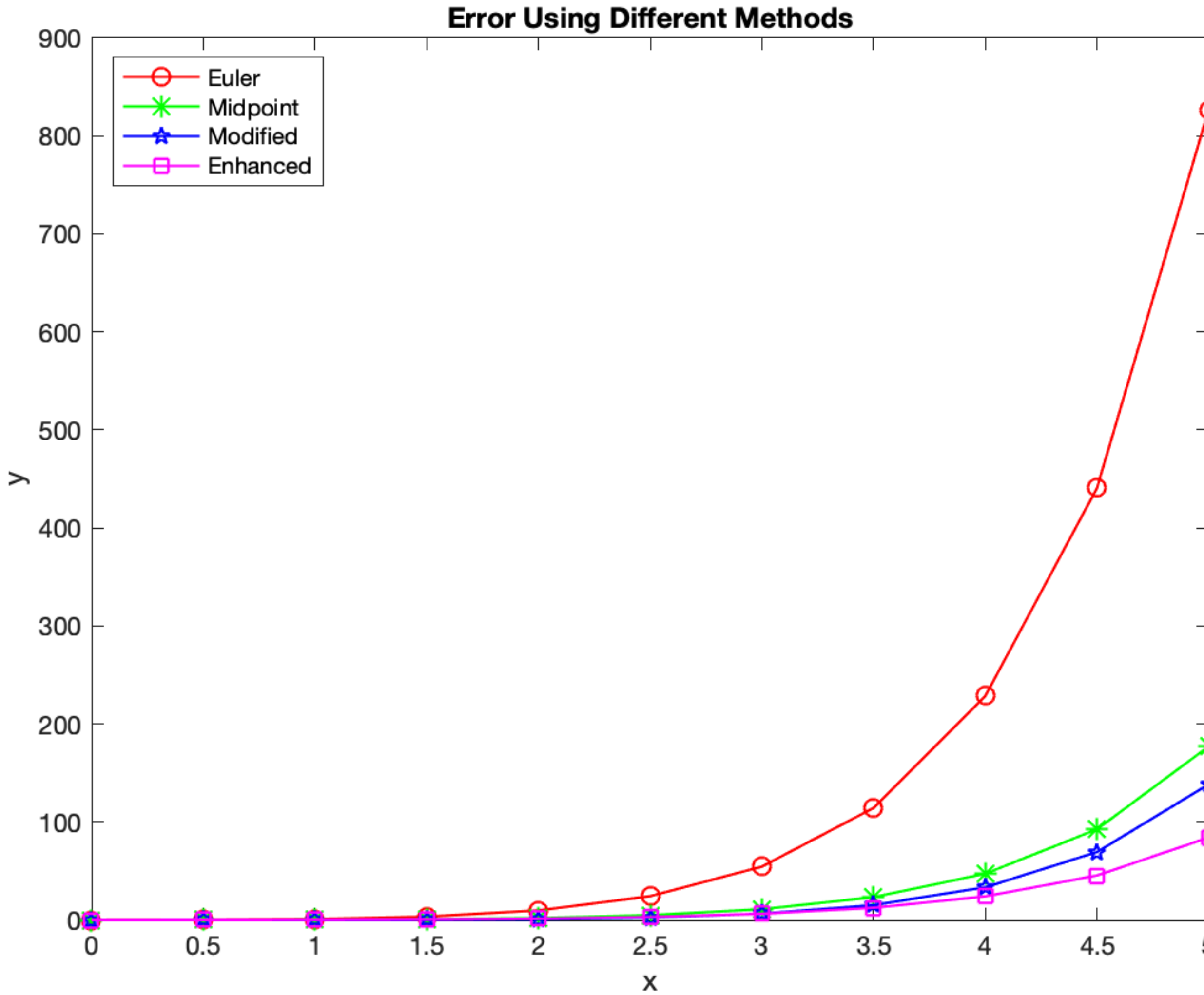

Figure 8: Graphical Comparison Among the Errors with Different Methods

Although the Euler method is one of the fundamentals in this class, it was interesting to have the chance to dive deeper into the graphical explanations of these different methods and how they have evolved better and better. I wonder if there will ever be even more enhancements to these equations so that the error rate is nearly negligible for all timesteps.