Homework 4

Solutions

1. (Halloween)

   (a) Mike grabs $\tilde{m}$ chocolate bars, where $\tilde{m}$ is a random variable with values between 0 and 3. Since all three values have the same probability, $p_{\tilde{m}}(m) = \frac{1}{3}$, for $m \in \{0, 1, 2\}$ and zero otherwise. Laura grabs $\tilde{l}$ chocolate bars, where $\tilde{l}$ is a random variable with values between 0 and $3 - \tilde{m}$. Since they have the same probability, the conditional pmf equals

$$p_{\tilde{l}|\tilde{m}}(l|m) = \begin{cases} \frac{1}{3-m} & \text{if } 0 \leq l \leq 2 - m, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

   By the chain rule, the joint pmf is:

$$p_{\tilde{l},\tilde{m}}(l, m) = \frac{1}{3(3-m)} \text{ for } m = 0, .., 2 \text{ and } l = 0, .., 2 - m. \tag{2}$$

   Note that the joint pmf is defined on triangular points in the (l, m) plane.

   (b) To find the marginal, we sum over m. For every l the sum has a different number of terms:

$$p_{\tilde{l}}(0) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{2} + \frac{1}{1}\right) = \frac{11}{18}, \tag{3}$$

$$p_{\tilde{l}}(1) = \frac{1}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{18}, \tag{4}$$

$$p_{\tilde{l}}(2) = \frac{1}{3}\frac{1}{3} = \frac{1}{9}. \tag{5}$$

   (c) When $\tilde{l} = 1$, $\tilde{m}$ must be either 0 or 1. By the definition of conditional pmf we have:

$$p_{\tilde{m}|\tilde{l}}(0|1) = \frac{p_{\tilde{l},\tilde{m}}(1, 0)}{p_{\tilde{l}}(1)} \tag{6}$$

$$= \frac{\frac{1}{3}\frac{1}{3}}{\frac{5}{18}} = \frac{2}{5}, \tag{7}$$

$$p_{\tilde{m}|\tilde{l}}(1|1) = \frac{p_{\tilde{l},\tilde{m}}(1, 1)}{p_{\tilde{l}}(0)} \tag{8}$$

$$= \frac{\frac{1}{3}\frac{1}{2}}{\frac{5}{18}} = \frac{3}{5}. \tag{9}$$

2. (Interview)

(a) We have

$$
\begin{aligned}
p_{\tilde{i}_1,\tilde{i}_2}(1,1) &= \mathrm{P}(\tilde{e}_1\tilde{q}=1,\tilde{e}_2\tilde{q}=1) && (10)\\
&= \mathrm{P}(\tilde{e}_1=1,\tilde{e}_2=1,\tilde{q}=1)+\mathrm{P}(\tilde{e}_1=-1,\tilde{e}_2=-1,\tilde{q}=-1) && (11)\\
&= \mathrm{P}(\tilde{e}_1=1)\mathrm{P}(\tilde{e}_2=1)\mathrm{P}(\tilde{q}=1)+\mathrm{P}(\tilde{e}_1=-1)\mathrm{P}(\tilde{e}_2=-1)\mathrm{P}(\tilde{q}=-1) && (12)\\
&= 0.19. && (13)
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
p_{\tilde{i}_1}(1) &= \mathrm{P}(\tilde{e}_1\tilde{q}=1) && (14)\\
&= \mathrm{P}(\tilde{e}_1=1,\tilde{q}=1)+\mathrm{P}(\tilde{e}_1=-1,\tilde{q}=-1) && (15)\\
&= \mathrm{P}(\tilde{e}_1=1)\mathrm{P}(\tilde{q}=1)+\mathrm{P}(\tilde{e}_1=-1)\mathrm{P}(\tilde{q}=-1) && (16)\\
&= 0.35. && (17)\\
p_{\tilde{i}_2}(1) &= 0.35. && (18)
\end{aligned}
$$

Since $0.19 \neq 0.35^2$, the random variables are not independent.

(c) For any x_1, x_2 and q in $\{-1,1\}$, we have

$$
\begin{aligned}
p_{\tilde{i}_1,\tilde{i}_2\,|\,\tilde{q}}(x_1,x_2\,|\,q) &= \mathrm{P}(\tilde{e}_1\tilde{q}=x_1,\tilde{e}_2\tilde{q}=x_2\,|\,\tilde{q}=q) && (19)\\
&= \mathrm{P}(\tilde{e}_1=x_1/q,\tilde{e}_2=x_2/q\,|\,\tilde{q}=q) && (20)\\
&= \mathrm{P}(\tilde{e}_1=x_1/q)\mathrm{P}(\tilde{e}_2=x_2/q) \quad \text{by independence} && (21)\\
&= \mathrm{P}(\tilde{e}_1=x_1/q\,|\,\tilde{q}=q)\mathrm{P}(\tilde{e}_2=x_2/q\,|\,\tilde{q}=q) \quad \text{by independence} && (22)\\
&= \mathrm{P}(\tilde{e}_1\tilde{q}=x_1\,|\,\tilde{q}=q)\mathrm{P}(\tilde{e}_2\tilde{q}=x_2\,|\,\tilde{q}=q) && (23)\\
&= p_{\tilde{i}_1\,|\,\tilde{q}}(x_1\,|\,q)p_{\tilde{i}_2\,|\,\tilde{q}}(x_2\,|\,q), && (24)
\end{aligned}
$$

so $\tilde{i}_1$ and $\tilde{i}_2$ are conditionally independent given $\tilde{q}$.

3. (Markov chain) The transition matrix of the Markov chain is

$$
T := \begin{bmatrix} 0 & 0.1 & 0 \\ 1 & 0 & 1 \\ 0 & 0.9 & 0 \end{bmatrix}. \tag{25}
$$

T has three eigenvectors

$$
q_1 := \begin{bmatrix} 0.0741 \\ 0.741 \\ 0.667 \end{bmatrix}, \qquad q_2 := \begin{bmatrix} 0.0741 \\ -0.741 \\ 0.667 \end{bmatrix}, \qquad q_3 := \begin{bmatrix} 0.707 \\ 0 \\ -0.707 \end{bmatrix}. \tag{26}
$$

The corresponding eigenvalues are $\lambda_1 := 1$, $\lambda_2 := -1$ and $\lambda_3 := 0$. The initial state vector for $i = 0$ is

$$
p_{\tilde{x}[1]} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{27}
$$

We use the eigenvector matrix

$$Q := \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \tag{28}$$

to compute

$$Q^{-1} p_{\tilde{x}[1]} = \begin{bmatrix} 0.675 \\ 0.675 \\ 1.27 \end{bmatrix}. \tag{29}$$

So

$$p_{\tilde{x}[1]} = 0.675\, q_1 + 0.675\, q_2 + 1.27\, q_3. \tag{30}$$

We consequently have

$$\begin{align} p_{\tilde{x}[i]} &= T^{\,i-1} p_{\tilde{x}[1]} \tag{31} \\ &= T^{\,i} \left(0.675\, q_1 + 0.675\, q_2 + 1.27\, q_3\right) \tag{32} \\ &= 0.675\, \lambda_1^{i-1} q_1 + 0.675\, \lambda_2^{i-1} q_2 + 1.27\, \lambda_3^{i-1} q_3 \tag{33} \\ &= 0.675 \left(q_1 + (-1)^{i-1}\, q_2\right) = \begin{cases} \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix} & \text{if } i \text{ is odd,} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & \text{if } i \text{ is even.} \end{cases} \tag{34} \end{align}$$

For even i the chain is always at B. For odd i it is either at A or C with probability 0.5.

4. (Precipitation data)

   https://github.com/cfgranda/prob_stats_for_data_science/blob/main/modeling_multivariate_discrete_data/precipitation_EXERCISE-solution.ipynb