Homework 5

Solutions

1. (Short questions)

(a) True. In the notes we showed that

$$\mathrm{E}\left(\tilde{a}^2\right) - \mathrm{E}^2(\tilde{a}) = \mathrm{E}\left((\tilde{a} - \mathrm{E}(\tilde{a}))^2\right). \tag{1}$$

For discrete random variables this equals the sum of a nonnegative quantity multiplied by a nonnegative quantity:

$$\mathrm{E}\left((\tilde{a} - \mathrm{E}(\tilde{a}))^2\right) = \sum_{a \in R_{\tilde{a}}} (a - \mathrm{E}(\tilde{a}))^2 p_{\tilde{a}}(a), \tag{2}$$

so the result must be nonnegative, which implies $\mathrm{E}\left(\tilde{a}^2\right) - \mathrm{E}^2(\tilde{a}) \geq 0$. For continuous random variables the argument is exactly the same with integrals instead of sums.

(A more straightforward solution): Let $\tilde{a}$ take values $\{-1, 0, 1\}$ with equal probability. $E(\tilde{a}) = 0$ but $E(\tilde{a}^2) > 0$. That is $E^2(\tilde{a})$ can be less than $E(\tilde{a}^2)$

(b) True. Let m denote the median of $\tilde{a}$, we have

$$\begin{aligned} F_{\tilde{a}+b}(m+b) &= \mathrm{P}\left(\tilde{a} + b \leq m + b\right) &(3)\\ &= \mathrm{P}\left(\tilde{a} \leq m\right) &(4)\\ &= \frac{1}{2} &(5)\end{aligned}$$

so $m + b$ is the median of $\tilde{a} + b$.

(c) True. Since $\tilde{a}$ and $\tilde{b}$ have the same distribution $\mathrm{E}(\tilde{a}) = \mathrm{E}(\tilde{b})$ (since the expectation operator only depends on the pmf or pdf). By independence $\mathrm{E}(\tilde{a}\tilde{b}) = \mathrm{E}(\tilde{a})\mathrm{E}(\tilde{b}) = \mathrm{E}^2(\tilde{a})$.

(d) The probability of the event occurring is $1/n$ so

$$\begin{aligned} p_{I_i}(0) &= 1 - \frac{1}{n}, &(6)\\ p_{I_i}(1) &= \frac{1}{n}. &(7)\end{aligned}$$

By linearity of expectation

$$\begin{aligned} \mathrm{E}\left(\text{Number of kids that get their own present}\right) &= \mathrm{E}\left(\sum_{i=1}^{n} I_i\right) &(8)\\ &= \sum_{i=1}^{n} \mathrm{E}\left(I_i\right) &(9)\\ &= \sum_{i=1}^{n} p_{I_i}(1) &(10)\\ &= 1. &(11)\end{aligned}$$

On average one kid gets their parents' own present.

2. (Paste and Rice)

   (a) The constraints are $100 \leq \max\{\tilde{x}, \tilde{r}\} \leq 300$. The joint pdf is constant over that region. Figure 1 contains the diagram.

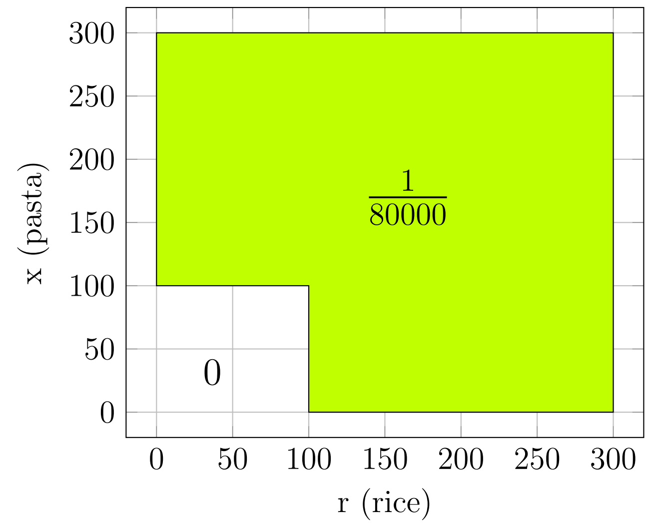

Figure 1: Joint pdf of r and x.

   (b) We compute

$$\begin{aligned} \mathrm{E}(\tilde{r}\tilde{x}) &= \int_{x=100}^{300}\int_{r=0}^{300} \frac{rx}{80000}\,\mathrm{d}x\,\mathrm{d}r + \int_{x=0}^{100}\int_{r=100}^{300} \frac{rx}{80000}\,\mathrm{d}x\,\mathrm{d}r && (12)\\ &= \frac{1}{80000}\left(\frac{x^2}{2}\Big]_{100}^{300}\frac{r^2}{2}\Big]_{0}^{300} + \frac{x^2}{2}\Big]_{0}^{100}\frac{r^2}{2}\Big]_{100}^{300}\right) && (13)\\ &= 25000. && (14) \end{aligned}$$

$$\begin{aligned} \mathrm{E}(\tilde{x}) &= \int_{x=100}^{300}\int_{r=0}^{300} \frac{x}{80000}\,\mathrm{d}x\,\mathrm{d}r + \int_{x=0}^{100}\int_{r=100}^{300} \frac{x}{80000}\,\mathrm{d}x\,\mathrm{d}r && (15)\\ &= \frac{1}{80000}\left(300\frac{x^2}{2}\Big]_{100}^{300} + 200\frac{x^2}{2}\Big]_{0}^{100}\frac{r^2}{2}\Big]_{100}^{300}\right) && (16)\\ &= 162.5. && (17) \end{aligned}$$

   By symmetry $\mathrm{E}(\tilde{r}) = 162.5$. The covariance $\mathrm{Cov}(\tilde{r}, \tilde{x}) = \mathrm{E}(\tilde{r}\tilde{x}) - \mathrm{E}(\tilde{r})\,\mathrm{E}(\tilde{x}) = -1406.25$ so $\tilde{r}$ and $\tilde{x}$ are negatively correlated.

   (c) The variables are correlated, which implies they cannot be independent.

3. (Law of conditional variance)

   (a) Given $\tilde{a} = a$, $\tilde{a}$ is constant, and its value is x, therefore once x is fixed $\mathrm{Var}(\tilde{b}|\tilde{a} = a)$ is a **number**. It represents the variance of the random variable $\tilde{b}$ given the information

$\tilde{a} = a$. In other words, given two random variables $\tilde{a}$ and $\tilde{b}$, look at their joint density, condition $\tilde{a}$ to the value x. This slice gives rise to a new distribution of the random variable $\tilde{b}|(\tilde{a} = a)$, $\mathrm{Var}(\tilde{b}|\tilde{a} = a)$ is precisely the variance of this random variable.

$$\mathrm{Var}(\tilde{b}|\tilde{a} = a) = E((\tilde{b} - E(\tilde{b}|\tilde{a} = a))^2|\tilde{a} = a) \tag{18}$$

(b) Now we don't keep x fixed but rather we treat it as a variable, for any x we assign a value $\mathrm{Var}(\tilde{b}|\tilde{a} = a)$, this is a function from the range of $\tilde{a}$ to real numbers, which we call by h, so $h(x) = \mathrm{Var}(\tilde{b}|\tilde{a} = a)$. Since this function is defined on a probability space we can regard it as a **random variable** by $h(\tilde{a}) = \mathrm{Var}(\tilde{b}|\tilde{a})$.

(c) Using iterated expectations we get:

$$\begin{aligned}
E(\mathrm{Var}(\tilde{b}|\tilde{a})) &= E(E(\tilde{b}^2|\tilde{a})) - E(E(\tilde{b}|\tilde{a})^2) && (19)\\
&= E(\tilde{b}^2) - E(E(\tilde{b}|\tilde{a})^2) && (20)\\
\mathrm{Var}(E(\tilde{b}|\tilde{a})) &= E(E(\tilde{b}|\tilde{a})^2) - E(E(\tilde{b}|\tilde{a}))^2 && (21)\\
&= E(E(\tilde{b}|\tilde{a})^2) - E(\tilde{b})^2 && (22)\\
\implies E(\mathrm{Var}(\tilde{b}|\tilde{a})) + \mathrm{Var}(E(\tilde{b}|\tilde{a})) &= E(\tilde{b}^2) - E(\tilde{b})^2 = \mathrm{Var}(\tilde{b}) && (23)
\end{aligned}$$

Average of the variance of $\tilde{b}$ given $\tilde{a}$, plus the variance of the average of $\tilde{b}$ given $\tilde{a}$.

(d) Let $\tilde{t}$ be the time at which a runner gets injured. And let $\tilde{a}$ be the random variable that describes the age group: $\tilde{a} = 1$ be the group of runners below 30, and $\tilde{a} = 2$ be the group above 30. So that $\tilde{t}|\{\tilde{a} = 1\} \sim exp(1)$ and $\tilde{t}|\{\tilde{a} = 2\} \sim exp(2)$. We are also given that $P(\tilde{a} = 2) = 0.2$. Also note that if $\tilde{a} \sim exp(\lambda)$ then $E(\tilde{a}^2) = Var(\tilde{a}) + E(\tilde{a})^2 = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$.

$$\begin{aligned}
E(\tilde{t}) &= E(E(\tilde{t}|\tilde{a})) = E(\tilde{t}|\tilde{a} = 1)P(\tilde{a} = 1) + E(\tilde{t}|\tilde{a} = 2)P(\tilde{a} = 2) && (24)\\
&= 1 \cdot 0.8 + \frac{1}{2} \cdot 0.2 = 0.9 && (25)\\
E(\tilde{t}^2) &= E(E(\tilde{t}^2|\tilde{a})) = E(\tilde{t}^2|\tilde{a} = 1)P(\tilde{a} = 1) + E(\tilde{t}^2|\tilde{a} = 2)P(\tilde{a} = 2) && (26)\\
&= 2/1^2 \cdot 0.8 + 2/2^2 \cdot 0.2 = 1.7 && (27)\\
\implies \mathrm{Var}(\tilde{t}) &= 1.7 - (0.9)^2 = 0.89 && (28)
\end{aligned}$$

4. (Water salinity and temperature)

(a) Please refer to Fig 2. The linear estimate has a negative slope, implying that the salinity decreases with temperature.

(b) Please refer to Fig 2. According to the conditional means, the salinity decreases with temperature and then increases again.

(c) The reliability of the conditional mean estimator depends on the number of data points that fell into the bin. In Fig 2 the conditional mean estimates to the left and more reliable that the one on the right, as each bin have more data points.

(d) I have no idea!

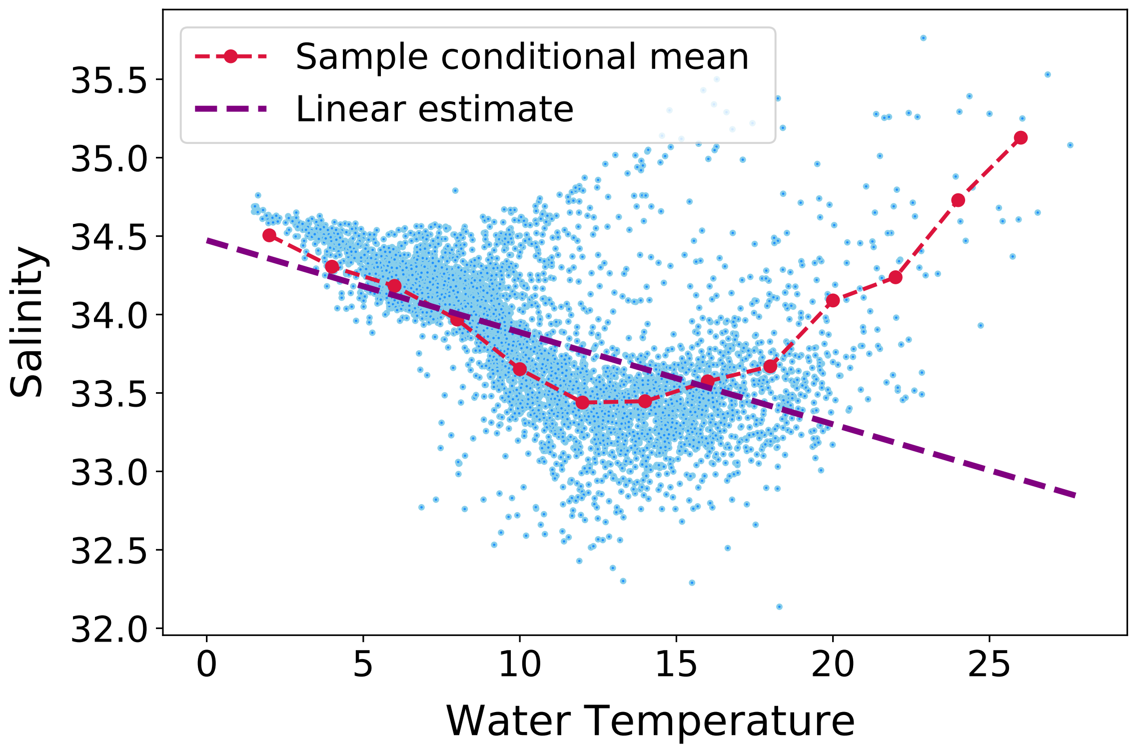
Sample conditional mean
Linear estimate
35.5
35.0
34.5
34.0
33.5
33.0
32.5
32.0
Salinity
0
5
10
15
20
25
Water Temperature

Figure 2