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Q1~4. Bayesian setting, prior: $p(w)$ on $w \in \mathbb{R}^d$

Input space $\mathcal{X} = \mathbb{R}^d$, Outcome space $Y_\pm = \{-1, 1\}$

$D = ((x^{(1)}, y^{(1)}), \cdots, (x^{(n)}, y^{(n)}))$

1. By bayes rule,

Posterior density $p(w|D) = \dfrac{p(D|w) \cdot p(w)}{p(D)}$

Considering both sides as functions of w, for fixed D, then both sides are densities on w and we can write

Posterior: $p(w|D)$ = likelihood: $p(D|w)$ · prior: $p(w)$ / $p(D)$

$NLL_D(w) = -\log p(D|w), \Rightarrow p(D|w) = \exp(-NLL_D(w))$

$$\therefore p(w|D) = \frac{\exp(-NLL_D(w)) \cdot p(w)}{p(D)}$$

2. No, it is not a conjugate prior.

Prior) $p(w) : w \sim N(0, \Sigma)$, Gaussian

$$= |2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2} w^T \Sigma^{-1} w\right)$$

Likelihood) $p(D|w)$

$$= \sum_{i=1}^{n} \left[(1+y_i) \log(1+\exp(-x^T w)) + (1-y_i) \log(1+\exp(-x^T w)) \right]$$

We need to show that posterior is also an Gaussian family.

$$p(w|D) \simeq p(D|w) \cdot p(w)$$

$$\simeq \left(|2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2} w^T \Sigma^{-1} w\right) \right) \times$$

$$\sum_{i=1}^{n} \left[(1+y_i) \log(1+\exp(-x^T w)) + (1-y_i) \log(1+\exp(-x^T w)) \right]$$

However, this doesn't result in Gaussian family.
(can't revise or fix to show as a Gaussian)

$\therefore$ It is not a conjugate prior

3. Let's assume that there exists a covariance matrix Σ, such that $\hat{w}_{MAP} = \text{argmin}\,[-\log(P(w|D))] = \frac{1}{n}\sum_{i=1}^{n}\log(1+\exp(-y_i w^T x_i)) + \lambda\|w\|^2$ ERM

$$-\log(P(w|D)) = -\log\left(\frac{\exp(-NLL_D(w)\cdot P(w))}{P(D)}\right)$$

$$= NLL_D(w) - \log(P(w)) + \log(P(D))$$

From HW4, $NLL = n\cdot\hat{R}_n$, and $\log(P(D))$ is a constant ($\hat{R}_n$: Empirical risk)

$\therefore -\log(P(w)) = -\frac{1}{2}\log(|2\pi\Sigma|) + \frac{1}{2}w^T\Sigma^{-1}w$ should correspond to the regularization term.

$$\Rightarrow \frac{1}{2}w^T\Sigma^{-1}w = n\lambda\|w\|^2 = n\lambda w^T w$$

$$\Rightarrow \Sigma^{-1} = 2n\lambda$$

$$\Rightarrow \Sigma = \frac{1}{2n\lambda}I \quad \left(w \sim N\left(0, \frac{1}{2n\lambda}I\right)\right)$$

$\therefore$ There exist a covariance matrix Σ, and its value is $\Sigma = \frac{1}{2n\lambda}I$.

4. In Q3, we figured out that there exists a covariance matrix Σ, and its value is $\Sigma = \frac{1}{2n\lambda} I$ $\left(w \sim N(0, \frac{1}{2n\lambda} I) \right)$

Now prior is defined as $w \sim N(0, I)$, $\Sigma = I$.

As ERM = MAP for $\Sigma = \frac{1}{2n\lambda} I$,

$$\frac{1}{2n\lambda} I = I \Rightarrow \frac{1}{2n\lambda} = 1$$

$$\therefore \lambda = \frac{1}{2n}$$

5. $L(\theta_1, \theta_2) = p(D_r, D_c \mid \theta_1, \theta_2)$

$= p(D_c \mid \theta_1) \cdot p(D_r \mid \theta_1, \theta_2)$

$= \theta_1^{C_h} (1-\theta_1)^{C_t} \cdot (\theta_1\theta_2)^{n_h} \cdot (1-\theta_1\theta_2)^{n_t}$

$\log L(\theta_1, \theta_2) = C_h \log\theta_1 + C_t \log(1-\theta_1) + \underbrace{n_h \log(\theta_1\theta_2)}_{n_h(\log\theta_1 + \log\theta_2)} + n_t \log(1-\theta_1\theta_2)$

* $\frac{\partial}{\partial\theta_1} \log L(\theta_1, \theta_2) = \frac{C_h}{\theta_1} - \frac{C_t}{(1-\theta_1)} + \frac{n_h}{\theta_1} + \frac{n_t}{1-\theta_1\theta_2} \cdot (-\theta_2) = 0$

$\Rightarrow C_h - \frac{C_t\theta_1}{(1-\theta_1)} + n_h - \frac{\theta_1\theta_2 n_t}{1-\theta_1\theta_2} = 0$

$\Rightarrow C_h - \frac{C_t\theta_1}{(1-\theta_1)} = 0 \quad \left(n_h = \frac{\theta_1\theta_2 n_t}{1-\theta_1\theta_2}\right)$

$\Rightarrow (1-\theta_1) C_h - C_t\theta_1 = 0$

$\Rightarrow C_h - \theta_1 C_h - \theta_1 C_t = 0$

$\therefore C_h = (C_h + C_t)\theta_1 \quad \therefore \boxed{\theta_1 = \frac{C_h}{C_h + C_t}}$

* $\frac{\partial}{\partial\theta_2} \log L(\theta_1, \theta_2) = \frac{n_h}{\theta_2} + \frac{n_t}{1-\theta_1\theta_2}(-\theta_1) = 0$

$\Rightarrow n_h - \frac{\theta_1\theta_2 n_t}{1-\theta_1\theta_2} = 0$

$\Rightarrow (1-\theta_1\theta_2) n_h - \theta_1\theta_2 n_t = 0$

$\Rightarrow n_h - \theta_1\theta_2 n_h - \theta_1\theta_2 n_t = 0$

$\Rightarrow \theta_2 = \frac{n_h}{n_h + n_t} \times \frac{1}{\theta_1}$

$\Rightarrow \theta_2 = \frac{n_h}{n_h + n_t} \times \frac{C_h + C_t}{C_h}$

$\boxed{\theta_2 = \frac{n_h}{n_h + n_t} \times \frac{C_h + C_t}{C_h}}$

$n_h = \frac{\theta_1\theta_2 n_t}{1-\theta_1\theta_2}$

6. $p(\theta_1) = \text{Beta}(h, t)$

Then, $L(\theta_1, \theta_2) = \underbrace{p(\theta_1)} \cdot \underbrace{p(D_r, D_c \mid \theta_1, \theta_2)}$

$$= \underbrace{\theta_1^{(h-1)} \cdot (1-\theta_1)^{(t-1)}} \cdot \underbrace{\theta_1^{C_h} \cdot (1-\theta_1)^{C_t} \cdot (\theta_1\theta_2)^{n_h} \cdot (1-\theta_1\theta_2)^{n_t}}$$

$$\log L(\theta_1, \theta_2) = (h-1)\log\theta_1 + (t-1)\log(1-\theta_1) + C_h \log\theta_1 + C_t \log(1-\theta_1) + n_h \log(\theta_1\theta_2) + n_t \log(1-\theta_1\theta_2)$$

$$= (h-1+C_h)\log\theta_1 + (t-1+C_t)\log(1-\theta_1) + n_h \log(\theta_1\theta_2) + n_t \log(1-\theta_1\theta_2)$$

$$* \frac{\partial}{\partial\theta_1} \log L(\theta_1, \theta_2) = \frac{(h-1+C_h)}{\theta_1} - \frac{(t-1+C_t)}{1-\theta_1} + \frac{n_h}{\theta_1} - \frac{n_t\theta_2}{(1-\theta_1\theta_2)} = 0$$

$$\Rightarrow (h-1+C_h) - \frac{\theta_1(t-1+C_t)}{(1-\theta_1)} + n_h - \frac{n_t\theta_1\theta_2}{(1-\theta_1\theta_2)} = 0$$

$$\Rightarrow (h-1+C_h) - \frac{\theta_1(t-1+C_t)}{(1-\theta_1)} = 0$$

$$\Rightarrow (1-\theta_1)(h-1+C_h) - \theta_1(t-1+C_t) = 0$$

$$\Rightarrow h-1+C_h = \theta_1(h+t-2+C_h+C_t)$$

$$\therefore \theta_1 = \frac{h-1+C_h}{h+t-2+C_h+C_t}$$

$$* \frac{\partial}{\partial\theta_2} \log L(\theta_1, \theta_2) = \frac{n_h}{\theta_2} - \frac{n_t\theta_1}{(1-\theta_1\theta_2)} = 0$$

$$\Rightarrow (1-\theta_1\theta_2)n_h = n_t \cdot \theta_1\theta_2 \quad , \quad n_h = \frac{n_t\theta_1\theta_2}{1-\theta_1\theta_2}$$

$$\Rightarrow n_h = (n_h+n_t)\theta_1\theta_2$$

$$\Rightarrow \theta_2 = \frac{n_h}{n_h+n_t} \times \frac{1}{\theta_1} = \frac{n_h}{n_h+n_t} \times \frac{(h+t-2+C_h+C_t)}{(h-1+C_h)}$$

$$\therefore \theta_2 = \frac{n_h}{(n_h+n_t)} \times \frac{(h+t-2+C_h+C_t)}{(h-1+C_h)}$$

7. $J(w) = \underbrace{\lambda \|w\|^2}_{①} + \underbrace{\frac{1}{n}\sum_{i=1}^{n} \max_{y \in Y} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]}_{②}$

① $\lambda \|w\|^2$: norm ⇒ convex

② $\frac{1}{n}\sum_{i=1}^{n} \max_{y \in Y} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$

$\max_{y \in Y} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$ is maximum,

and based on notes on Convex Optimization,

it is convex.

So, the point-wise sum

$\frac{1}{n}\sum_{i=1}^{n} \max_{y \in Y} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$

is also convex.

∴ As summation of convex functions is still convex,

$J(w) = \lambda \|w\|^2 + \frac{1}{n}\sum_{i=1}^{n} \max_{y \in Y} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$

is convex.

8. $\hat{y}_i = \arg\max_{y \in \mathcal{Y}} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$

$J(w) = \lambda \|w\|^2 + \frac{1}{n} \sum_{i=1}^{n} \max_{y \in \mathcal{Y}} [\Delta(y_i, y) + \langle w, \psi(x_i, y) - \psi(x_i, y_i) \rangle]$

Subgradient of $J(w)$:

$$2\lambda w + \frac{1}{n} \sum_{i=1}^{n} (\psi(x_i, y) - \psi(x_i, y_i))$$

9. Based on Q8, expression is:

$$2\lambda w + (\psi(x_i, \hat{y}_i) - \psi(x_i, y_i))$$

10. Based on Q8, expression is:

$$2\lambda w + \frac{1}{m} \sum_{j=i}^{i+m-1} (\psi(x_j, \hat{y}_j) - \psi(x_j, y_j))$$

[Optional]

$$\mathcal{Y} = \{-1, 1\}, \quad \Delta(y, \hat{y}) = \mathbb{1}(y \neq \hat{y})$$

$$\begin{cases} h(x, 1) = g(x)/2 \\ h(x, -1) = -g(x)/2 \end{cases}$$

$$\ell(h, (x, y)) = \max_{y' \in \mathcal{Y}} [\Delta(y, y') + h(x, y') - h(x, y)] = \max\{0, 1 - yg(x)\}$$

:) $y = y'$, $\ell(h, (x, y)) = 0$

:(($y \neq y'$

if $y = 1$:

$$\ell(h, (x, y)) = \max_{y' \in \mathcal{Y}} [1 + h(x, -1) - h(x, 1)]$$

$$= \max_{y' \in \mathcal{Y}} [1 - g(x)/2 - g(x)/2]$$

$$= \max_{y' \in \mathcal{Y}} [1 - g(x)] = \max_{y' \in \mathcal{Y}} \{0, 1 - yg(x)\}$$

if $y = -1$:

$$\ell(h, (x, y)) = \max_{y' \in \mathcal{Y}} [1 + h(x, 1) - h(x, -1)]$$

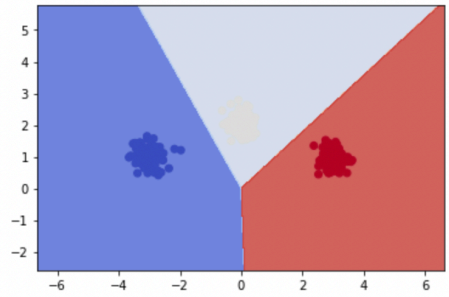
$$= \max_{y' \in \mathcal{Y}} [1 + g(x)] = \max_{y' \in \mathcal{Y}} \{0, 1 - yg(x)\}$$

```python
# Q11
from sklearn.base import BaseEstimator, ClassifierMixin, clone

class OneVsAllClassifier(BaseEstimator, ClassifierMixin):
    """
    One-vs-all classifier
    We assume that the classes will be the integers 0,..,(n_classes-1).
    We assume that the estimator provided to the class, after fitting, has a "decision_function" that
    returns the score for the positive class.
    """
    def __init__(self, estimator, n_classes):
        """
        Constructed with the number of classes and an estimator (e.g. an
        SVM estimator from sklearn)
        @param estimator : binary base classifier used
        @param n_classes : number of classes
        """
        self.n_classes = n_classes
        self.estimators = [clone(estimator) for _ in range(n_classes)]
        self.fitted = False

    def fit(self, X, y=None):
        """
        This should fit one classifier for each class.
        self.estimators[i] should be fit on class i vs rest
        @param X: array-like, shape = [n_samples,n_features], input data
        @param y: array-like, shape = [n_samples,] class labels
        @return returns self
        """
        #Your code goes here
        # for each selected class, convert selected y labels into 1 and -1 for others
        # then, fit X with y_values
        for i in range(self.n_classes):
            y_converted = np.where(y==i,1,-1)
            self.estimators[i].fit(X, y_converted)

        self.fitted = True
        return self

    def decision_function(self, X):
        """
        Returns the score of each input for each class. Assumes
        that the given estimator also implements the decision_function method (which sklearn SVMs do),
        and that fit has been called.
        @param X : array-like, shape = [n_samples, n_features] input data
        @return array-like, shape = [n_samples, n_classes]
        """
        if not self.fitted:
            raise RuntimeError("You must train classifer before predicting data.")

        if not hasattr(self.estimators[0], "decision_function"):
            raise AttributeError(
                "Base estimator doesn't have a decision_function attribute.")

        #Replace the following return statement with your code

        #initialize score as all zeros
        score=np.zeros([self.n_classes,X.shape[0]])

        # for each rows, update score by using decision_function method
        for i in range(self.n_classes):
            score[i]=self.estimators[i].decision_function(X)

        # as we are returning shape = [n_samples, n_classes], retun transposed score
        return score.T

    def predict(self, X):
        """
        Predict the class with the highest score.
        @param X: array-like, shape = [n_samples,n_features] input data
        @returns array-like, shape = [n_samples,] the predicted classes for each input
        """
        #Replace the following return statement with your code

        # calling score results
        score = self.decision_function(X)

        # initialize y_pred as zeros
        y_pred = np.zeros([score.shape[0]])

        # now iterate for each row and update class that has the max score
        for i in range(len(y_pred)):
            y_pred[i] = np.where(score[i] == max(score[i]))[0][0]

        return y_pred
```

```
#Q12 Here we test the OneVsAllClassifier
from sklearn import svm
svm_estimator = svm.LinearSVC(loss='hinge', fit_intercept=False, C=200)
clf_onevsall = OneVsAllClassifier(svm_estimator, n_classes=3)
clf_onevsall.fit(X,y)

for i in range(3) :
    print("Coeffs %d"%i)
    print(clf_onevsall.estimators[i].coef_) #Will fail if you haven't implemented fit yet

# create a mesh to plot in
h = .02  # step size in the mesh
x_min, x_max = min(X[:,0])-3,max(X[:,0])+3
y_min, y_max = min(X[:,1])-3,max(X[:,1])+3
xx, yy = np.meshgrid(np.arange(x_min, x_max, h),
                     np.arange(y_min, y_max, h))
mesh_input = np.c_[xx.ravel(), yy.ravel()]

Z = clf_onevsall.predict(mesh_input)
Z = Z.reshape(xx.shape)
plt.contourf(xx, yy, Z, cmap=plt.cm.coolwarm, alpha=0.8)
# Plot also the training points
plt.scatter(X[:, 0], X[:, 1], c=y, cmap=plt.cm.coolwarm)


from sklearn import metrics
metrics.confusion_matrix(y, clf_onevsall.predict(X))
```

```
/Users/yoontaepark/opt/anaconda3/lib/python3.8/site-packages/sklearn/svm/_base.py:985: ConvergenceWarning: Liblinea
r failed to converge, increase the number of iterations.
  warnings.warn("Liblinear failed to converge, increase "
```

```
Coeffs 0
[[-1.05852865 -0.90296547]]
Coeffs 1
[[ 0.27026037 -0.13034023]]
Coeffs 2
[[ 0.89164616 -0.82601495]]

array([[100,   0,   0],
       [  0, 100,   0],
       [  0,   0, 100]])
```



Multiclass SVM

```python
# Q13
def zeroOne(y,a) :
    '''
    Computes the zero-one loss.
    @param y: output class
    @param a: predicted class
    @return 1 if different, 0 if same
    '''
    return int(y != a)

def featureMap(X,y,num_classes) :
    '''
    Computes the class-sensitive features.
    @param X: array-like, shape = [n_samples,n_inFeatures] or [n_inFeatures,], input features for input data
    @param y: a target class (in range 0,..,num_classes-1)
    @return array-like, shape = [n_samples,n_outFeatures], the class sensitive features for class y
    '''
    #The following line handles X being a 1d-array or a 2d-array
    num_samples, num_inFeatures = (1,X.shape[0]) if len(X.shape) == 1 else (X.shape[0],X.shape[1])
    #your code goes here, and replaces following return

    # defining n_outFeatures as class * features, and initialize feature map
    n_outFeatures = num_classes * num_inFeatures
    feature_map = np.zeros([num_samples, n_outFeatures])

    # reshaping for 1d-array case
    if num_samples == 1:
        X = X.reshape((1,-1))

    # update feature map, so for selected target class columns, update x values
    for i, X_i in enumerate(X):
        feature_map[i, (y*num_inFeatures) : (y*num_inFeatures)+num_inFeatures] = X_i

    return feature_map
```

```python
# Q14
def sgd(X, y, num_outFeatures, subgd, eta = 0.1, T = 10000):
    '''
    Runs subgradient descent, and outputs resulting parameter vector.
    @param X: array-like, shape = [n_samples,n_features], input training data
    @param y: array-like, shape = [n_samples,], class labels
    @param num_outFeatures: number of class-sensitive features
    @param subgd: function taking x,y,w and giving subgradient of objective
    @param eta: learning rate for SGD
    @param T: maximum number of iterations
    @return: vector of weights
    '''
    num_samples = X.shape[0]
    #your code goes here and replaces following return statement

    # initialize w vector
    w_vect = np.zeros(num_outFeatures)

    # iterate T times. For each iteration, shuffle index and update w_vect by using subgd param
    for _ in range(T):
        idx = np.random.randint(num_samples)
        w_vect = w_vect - eta*subgd(X[idx],y[idx],w_vect)

    return w_vect
```

```python
# Q15
class MulticlassSVM(BaseEstimator, ClassifierMixin):
    '''
    Implements a Multiclass SVM estimator.
    '''
    def __init__(self, num_outFeatures, lam=1.0, num_classes=3, Delta=zeroOne, Psi=featureMap):
        '''
        Creates a MulticlassSVM estimator.
        @param num_outFeatures: number of class-sensitive features produced by Psi
        @param lam: l2 regularization parameter
        @param num_classes: number of classes (assumed numbered 0,..,num_classes-1)
        @param Delta: class-sensitive loss function taking two arguments (i.e., target margin)
        @param Psi: class-sensitive feature map taking two arguments
        '''
        self.num_outFeatures = num_outFeatures
        self.lam = lam
        self.num_classes = num_classes
        self.Delta = Delta
        self.Psi = lambda X,y : Psi(X,y,num_classes)
        self.fitted = False
    def subgradient(self,x,y,w):
        '''
        Computes the subgradient at a given data point x,y
        @param x: sample input
        @param y: sample class
        @param w: parameter vector
        @return returns subgradient vector at given x,y,w
        '''
        #Your code goes here and replaces the following return statement

        # initialze y_hat as 0, and get the result based on y_hat
        y_hat = 0
        res_tmp = self.Delta(y, y_hat) + np.dot(w, (self.Psi(x,y_hat)-self.Psi(x,y)).T)

        # iterate for given classes, and for each classes, get the result
        # compare and update if result for given class is larger
        for y_i in range(self.num_classes):
            res_y_i = self.Delta(y, y_i) + np.dot(w, (self.Psi(x,y_i)-self.Psi(x,y)).T)
            if res_tmp < res_y_i:
                res_tmp = res_y_i
                y_hat = y_i

        # retrun subgradient vector at given x,y,w
        return (2*self.lam*w + self.Psi(x,y_hat) - self.Psi(x,y))
    def fit(self,X,y,eta=0.1,T=10000):
        '''
        Fits multiclass SVM
        @param X: array-like, shape = [num_samples,num_inFeatures], input data
        @param y: array-like, shape = [num_samples,], input classes
        @param eta: learning rate for SGD
        @param T: maximum number of iterations
        @return returns self
        '''
        self.coef_ = sgd(X,y,self.num_outFeatures,self.subgradient,eta,T)
        self.fitted = True
        return self

    def decision_function(self, X):
        '''
        Returns the score on each input for each class. Assumes
        that fit has been called.
        @param X : array-like, shape = [n_samples, n_inFeatures]
        @return array-like, shape = [n_samples, n_classes] giving scores for each sample,class pairing
        '''
        if not self.fitted:
            raise RuntimeError("You must train classifer before predicting data.")

        #Your code goes here and replaces following return statement
        # initialize score
        score = np.zeros((X.shape[0], self.num_classes))

        # update score
        for i in range(X.shape[0]):
            for j in range(self.num_classes): # j values are y classes
                score[i][j] = np.dot(self.coef_, self.Psi(X[i], j).T)

        # return score
        return score
```

```python
def predict(self, X):
    '''
    Predict the class with the highest score.
    @param X: array-like, shape = [n_samples, n_inFeatures], input data to predict
    @return array-like, shape = [n_samples,], class labels predicted for each data point
    '''

    #Your code goes here and replaces following return statement
    # recall score by using pre-defined decision function
    score = self.decision_function(X)

    y_pred = np.zeros(X.shape[0])

    for i in range(X.shape[0]):
        y_pred[i] = np.where(score[i] == max(score[i]))[0][0]

    return y_pred
```

```
#Q16 the following code tests the MulticlassSVM and sgd
#will fail if MulticlassSVM is not implemented yet
est = MulticlassSVM(6,lam=1)
est.fit(X,y,eta=0.1)
print("w:")
print(est.coef_)
Z = est.predict(mesh_input)
Z = Z.reshape(xx.shape)
plt.contourf(xx, yy, Z, cmap=plt.cm.coolwarm, alpha=0.8)
# Plot also the training points
plt.scatter(X[:, 0], X[:, 1], c=y, cmap=plt.cm.coolwarm)

from sklearn import metrics
metrics.confusion_matrix(y, est.predict(X))
```

```
w:
[[-0.32248241 -0.02890479 -0.00091071  0.1444795   0.32339312 -0.1155747 ]]

array([[100,   0,   0],
       [  0, 100,   0],
       [  0,   0, 100]])
```