# Homework 3

Solutions

1. (Half life)

   (a) We have

$$
\begin{aligned}
\mathrm{P}(\tilde{t} > t_{1/2}) &= \int_{t_{1/2}}^{\infty} \lambda \exp(-\lambda x) \, \mathrm{d}x && (1)\\
&= \exp\left(-\lambda t_{1/2}\right). && (2)
\end{aligned}
$$

   Setting equal to $1/2$ yields $t_{1/2} = \frac{\ln 2}{\lambda}$. This is the time that it takes for the particle to decay with probability $1/2$. This seems like a reasonable definition, because if e have a group of particles all following the same distribution, we can consider each as a realization of an experiment where we check whether a particle survives after $t_{1/2}$. By our *intuitive* definition of the probability of an event, about half of them will survive because the probability of the event is $1/2$.

   (b) We have

$$
\begin{aligned}
\mathrm{P}(t_{1/2} < \tilde{t} < t) &= \int_{t_{1/2}}^{t} \lambda \exp(-\lambda x) \, \mathrm{d}x && (3)\\
&= \exp\left(-\lambda t_{1/2}\right) - \exp(-\lambda t) && (4)\\
&= \frac{1}{2} - \exp(-\lambda t). && (5)
\end{aligned}
$$

   Setting equal to $1/4$ yields $t_{1/4} = \frac{\ln 4}{\lambda} = \frac{2\ln 2}{\lambda} = 2t_{1/2}$. Intuitively, this means that after half the particles have decayed, it takes the same time for one quarter of the particles to decay. This is consistent with the intuitive definition of half life because once half of the particles are gone, half of the remaining ones (i.e. one quarter) should decay after the half time.

   (c)

$$
\begin{aligned}
\mathrm{P}(\tilde{t} > k t_{1/2}) &= \int_{k t_{1/2}}^{\infty} \lambda \exp(-\lambda x) \, \mathrm{d}x && (6)\\
&= \exp\left(-\lambda k t_{1/2}\right) && (7)\\
&= \exp\left(-\ln 2^k\right) && (8)\\
&= \frac{1}{2^k}. && (9)
\end{aligned}
$$

   This says that if we wait for $k$ half times, the remaining particles are halved $k$ times, which is consistent with the intuitive definition of half time.

2. (Measurements)

(a) Let $\tilde{d}$ be the time the particle takes to decay. The pmf of the reading $\tilde{r} = \lceil \tilde{d} \rceil$ is a geometric of parameter $1 - e^{-\lambda}$,

$$\mathrm{P}\left(\tilde{r} = r\right) = \mathrm{P}\left(r - 1 \leq \tilde{d} < r\right) = \int_{r-1}^{r} \lambda\, e^{-\lambda x} \mathrm{d}x = e^{-\lambda(r-1)} - e^{-\lambda r} \tag{10}$$

$$= \left(e^{-\lambda}\right)^{r-1} \left(1 - e^{-\lambda}\right) \quad \text{for } r = 1, 2, 3, \ldots \tag{11}$$

(b) Let $\tilde{x}$ be the error, clearly $0 \leq \tilde{x} \leq 1$. Its cdf is

$$F_{\tilde{x}}\left(x\right) = \mathrm{P}\left(\tilde{x} \leq x\right) \tag{12}$$

$$= \mathrm{P}\left(\lceil \tilde{d} \rceil - \tilde{d} \leq x\right) \tag{13}$$

$$= \mathrm{P}\left(\cup_{i=1}^{\infty} \left\{i - x \leq \tilde{d} \leq i\right\}\right) \quad \text{union of disjoint events} \tag{14}$$

$$= \sum_{i=1}^{\infty} \mathrm{P}\left(i - x \leq \tilde{d} \leq i\right) \tag{15}$$

$$= \sum_{i=1}^{\infty} \lambda \int_{i-x}^{i} e^{-\lambda x} \mathrm{d}x \tag{16}$$

$$= \sum_{i=1}^{\infty} e^{-\lambda(i-x)} - e^{-\lambda i} \tag{17}$$

$$= \left(e^{\lambda x} - 1\right) \sum_{i=1}^{\infty} e^{-\lambda i} \tag{18}$$

$$= \frac{e^{-\lambda}\left(e^{\lambda x} - 1\right)}{1 - e^{-\lambda}} = \frac{e^{\lambda x} - 1}{e^{\lambda} - 1}. \tag{19}$$

Differentiating we obtain

$$f_{\tilde{x}}\left(x\right) = \begin{cases} \frac{\lambda\, e^{\lambda x}}{e^{\lambda} - 1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

3. (Triangular pdf)

(a) The possible values of $w$ are $w \geq \max\left(x_1, \ldots, x_n\right) = 1.5$.

(b) The likelihood equals

$$\mathcal{L}\left(w\right) = \prod_{i}^{n} f_w\left(x_i\right) \tag{21}$$

$$= \frac{2^5}{w^{10}} \prod_{i}^{n} x_i \tag{22}$$

$$= \frac{28.8}{w^{10}} \tag{23}$$

if $w \geq \max\left(x_1, \ldots, x_n\right)$ and 0 otherwise because in that case $f_w\left(\max\left(x_1, \ldots, x_n\right)\right) = 0$.

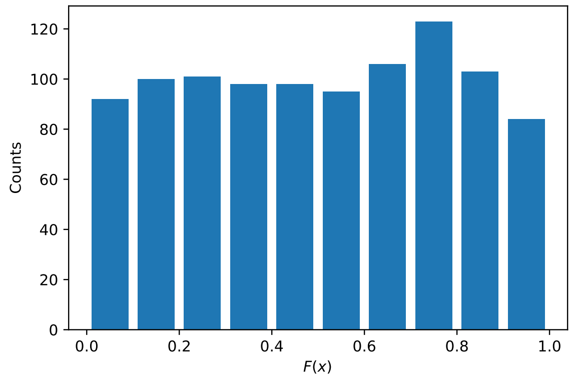
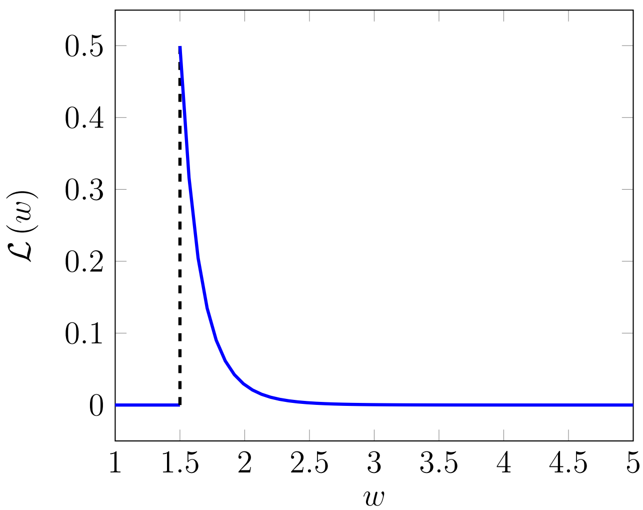



Figure 1: The CDF histogram of data samples




(c) The maximum likelihood estimate is 1.5 because the likelihood is decreasing over $[1.5, \infty)$.

(d) The probability is zero. The ML estimate is only correct if we observe a sample equal to $w$, which is an event with probability zero because the random variable has a continuous cdf.

(e) The cdf equals

$$F_w(x) = \begin{cases} 0, & x \leq 0, \\ \frac{x^2}{w^2}, & \text{for } 0 \leq x \leq w, \\ 1, & \text{for } x \geq w. \end{cases} \tag{24}$$

The inverse of the cdf in the interval of interest is $w\sqrt{y}$. The sample is $2\sqrt{0.64} = 1.6$.

4. (Applying the cdf)

(a) The histogram is shown in Figure 1. It looks more or less uniform between 0 and 1.

(b) Applying the definition of cdf,

$$F_{\tilde{b}}(b) = \mathrm{P}\left(\tilde{b} \leq b\right) \tag{25}$$

$$= \mathrm{P}\left(F_{\tilde{a}}(\tilde{a}) \leq b\right). \tag{26}$$

If $F_{\tilde{a}}$ is invertible, then $a \leq b$ is equivalent to $F_{\tilde{a}}(a) \leq F_{\tilde{a}}(b)$. Let us prove it by showing that each statement implies the other one.

(1) If $a \leq b$ then $F_{\tilde{a}}(a) \leq F_{\tilde{a}}(b)$ because cdfs are non decreasing.

(2) If $F_{\tilde{a}}(a) \leq F_{\tilde{a}}(b)$ then either $F_{\tilde{a}}(a) = F_{\tilde{a}}(b)$ which implies $a = b$ because $F_{\tilde{a}}$ is invertible or $F_{\tilde{a}}(a) < F_{\tilde{a}}(b)$ which implies $a \leq b$ because cdfs are non increasing.

As a result, $F_{\tilde{a}}(\tilde{a}) \leq b$ implies $\tilde{a} \leq F_{\tilde{a}}^{-1}(b)$, so

$$F_{\tilde{b}}(b) = \mathrm{P}\left(\tilde{a} \leq F_{\tilde{a}}^{-1}(b)\right) \tag{27}$$

$$= F_{\tilde{a}}\left(F_{\tilde{a}}^{-1}(b)\right) \tag{28}$$

$$= b, \quad 0 \leq b \leq 1. \tag{29}$$

It turns out that $\tilde{b}$ is a uniform random variable between 0 and 1.