HW3 _ DSGA1002 (yp2201@nyu.edu)

1. (a) Since $f_{\tilde{t}}(t) = \begin{cases} \lambda \exp(-\lambda t), & \text{if } t \geq 0, \\ 0, & \text{otherwise}, \end{cases}$

$P(\tilde{t} > t_{1/2})$ can be shown as $\int$ of pdf.

$$P(\tilde{t} > t_{1/2}) = \int_{t_{1/2}}^{\infty} \lambda \exp(-\lambda x)\, dx$$

$$= \cancel{\lambda} \times \frac{1}{-\cancel{\lambda}} \times \left[\exp(-\lambda x)\right]_{t_{1/2}}^{\infty}$$

$$= -(\underbrace{\exp(-\lambda \cdot \infty)}_{= 0} - \exp(-\lambda t_{1/2}))$$

$$= \exp(-\lambda t_{1/2}) = \frac{1}{2}$$

Since $\exp(-\lambda t_{1/2}) = \frac{1}{2}$

$\hookrightarrow -\lambda \cdot t_{1/2} = -\ln 2$

$$\boxed{t_{1/2} = \frac{\ln 2}{\lambda}}$$

It is a reasonable definition. Since all particles follow the same exponential distribution, we may think result as an experiment of whether a particle survives after $t_{1/2}$. By this intuitive definiton of p(event), half of the particles will survive since p(event) $= \frac{1}{2}$

b) $p(t_{1/2} < \tilde{t} < t) = \int_{t_{1/2}}^{t} \lambda \exp(-\lambda x)\, dx$

$= -\left(\left[\exp(-\lambda x)\right]_{t_{1/2}}^{t} \right)$

$= -\exp(-\lambda t) + \exp(-\lambda t_{1/2})$

Since $p(\tilde{t} > t_{1/2}) = \exp(-\lambda \cdot t_{1/2}) = \frac{1}{2}$,

$= -\exp(-\lambda t) + \frac{1}{2}$

$-\exp(-\lambda t) + \frac{1}{2} = \frac{1}{4}$, so $\exp(-\lambda t) = \frac{1}{4}$, $-\lambda t = -\ln 4$

since $t_{1/2} = \frac{\ln 2}{\lambda}$, $\boxed{t = \frac{2\ln 2}{\lambda} = 2 \cdot t_{1/2}}$ $t = \frac{2\ln 2}{\lambda}$

This means that it takes twice the time of half particles to be decayed. It is consitent with the intuitive meaning of half life, since after 1/2 particles are decayed, 1/4 (half of the remaining ones) should decay after the half time.

(c) $p(\tilde{t} > k \cdot t_{1/2}) = \int_{k t_{1/2}}^{\infty} \lambda \exp(-\lambda x)\, dx = -\left(\left[\exp(-\lambda x)\right]_{k t_{1/2}}^{\infty} \right)$

$= -(\underbrace{\exp(-\lambda\infty)}_{=0} - \exp(-\lambda k t_{1/2})) = \exp(-\lambda k t_{1/2})$

Since $t_{1/2} = \frac{\ln 2}{\lambda}$, $\exp(-\lambda k t_{1/2}) = \exp(-\lambda \cdot k \cdot \frac{\ln 2}{\lambda})$

$= \exp(-k \ln 2) = \exp(-\ln 2^k)$

$= \boxed{\frac{1}{2^k}}$

This means that if we wait for k · half times, then there will be remaining particles of k times halved. This is consistent with the intuitive meaning of half life.

2. (a) Let's define $\tilde{\gamma}$: time the particle takes to decay

$\tilde{r}$: reading from the device

$r = 1, 2, 3, \ldots$

pmf of $\tilde{r}$ $= P(\tilde{r}=r) = P(r-1 \leq \tilde{\gamma} < r) = \int_{r-1}^{r} \lambda \exp(-\lambda x)\, dx$

$= -(\exp(-\lambda r) - \exp(-\lambda(r-1)))$

$= e^{-\lambda(r-1)} - e^{-\lambda r} = \boxed{e^{-\lambda(r-1)}(1-e^{-\lambda})}$

$\therefore$ pmf of $\tilde{r} = [\tilde{\gamma}]$ is a geometric of parameter $(1-e^{-\lambda})$

(b) Let's define error $\tilde{x}$: $[\tilde{\gamma}] - \tilde{\gamma}$, $0 \leq \tilde{x} \leq 1$

CDF : $F_{\tilde{x}}(x) = P(\tilde{x} \leq x) = P([\tilde{\gamma}] - \tilde{\gamma} \leq x)$

(Since $[\tilde{\gamma}] - \tilde{\gamma} \leq x$, $[\tilde{\gamma}] - x \leq \tilde{\gamma}$

also, by round-up definition of this gaussian symbol, $\tilde{\gamma} \leq [\tilde{\gamma}]$)

Therefore, $[\tilde{\gamma}] - x \leq \tilde{\gamma} \leq [\tilde{\gamma}]$

$= P([\tilde{\gamma}] - x \leq \tilde{\gamma} \leq [\tilde{\gamma}])$ (This can be formed as union of disjoint events)

$= P\left(\bigcup_{i=1}^{\infty} \{i - x \leq \tilde{\gamma} \leq i\}\right) = \sum_{i=1}^{\infty} P(i-x \leq \tilde{\gamma} \leq i)$

$= \sum_{i=1}^{\infty} \int_{i-x}^{i} \lambda \cdot e^{-\lambda x}\, dx = \sum_{i=1}^{\infty} e^{-\lambda(i-x)} - e^{-\lambda i}$

$= (e^{\lambda x} - 1) \sum_{i=1}^{\infty} e^{-\lambda i} = (e^{\lambda x} - 1) \cdot \dfrac{e^{-\lambda}(1 - \cancel{e^{-\lambda\infty}})}{1 - e^{-\lambda}} \dfrac{(\times e^{\lambda})}{(\times e^{\lambda})}$

$\boxed{= \dfrac{e^{\lambda x} - 1}{e^{\lambda} - 1}}$

PDF : Differentiating CDF, $\left(\dfrac{e^{\lambda x} - 1}{e^{\lambda} - 1}\right)' = \dfrac{1}{e^{\lambda} - 1} \cdot \lambda \cdot e^{\lambda x} = \dfrac{\lambda \cdot e^{\lambda x}}{e^{\lambda} - 1}$

$$\boxed{\therefore f_{\tilde{x}}(x) = \begin{cases} \dfrac{\lambda \cdot e^{\lambda x}}{e^{\lambda} - 1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}}$$

3. (a) Since $0 \leq x \leq w$, $w \geq \max(x_1, x_2, \cdots x_n) \boxed{= 1.5}$

(observed values : 1.25 , 0.4, 1.5, 1, 1.2)

(b) $L(w) = \prod_i^n f_w(x_i) = \prod_i^n \left(\frac{2}{w^2} \cdot x_i\right)$

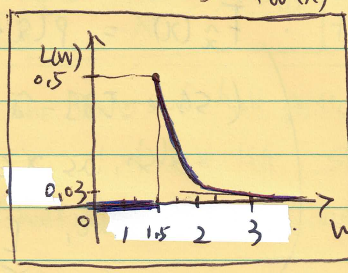
(Since $n=5$, $x_1 = 1.25$, $x_2 = 0.4$, $x_3 = 1.5$, $x_4 = 1$, $x_5 = 1.2$)

$= \frac{2^5}{w^{10}} \cdot \prod_i^5 x_i = \frac{2^5}{w^{10}} (1.25 \times 0.4 \times 1.5 \times 1 \times 1.2) = \boxed{\frac{28.8}{w^{10}}}$

If $w \geq 1.5$, this formula follows $L(w) = \frac{28.8}{w^{10}}$

If not, (i.e $w=1$) this formula $= 0$ as $f_w(x) \begin{cases} \frac{2x}{w^2} & \text{for } 0 \leq x \leq w \\ 0 & \leftarrow \text{otherwise,} \end{cases}$

and $f_w(\underbrace{\max(x_1, x_2 \ldots x_n)}_{= 1.5}) = 0$



(c) The maximum likelihood estimate of $w = 1.5$.

Likelihood is decreasing over $[1.5, \infty]$.

(d) No. ML estimate is correct only when we observe a sample $= w$,

and this probability is zero because random variable has a continuous cdf.

(e) Since $w=2$, cdf $F_w(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{w^2} & 0 \leq x \leq w \quad ((x^2)' = 2x) \\ 1 & x \geq w \end{cases}$

cdf) $y = \frac{x^2}{4}$, inverse cdf) $x = \frac{y^2}{4}$, $y = 2\sqrt{x}$

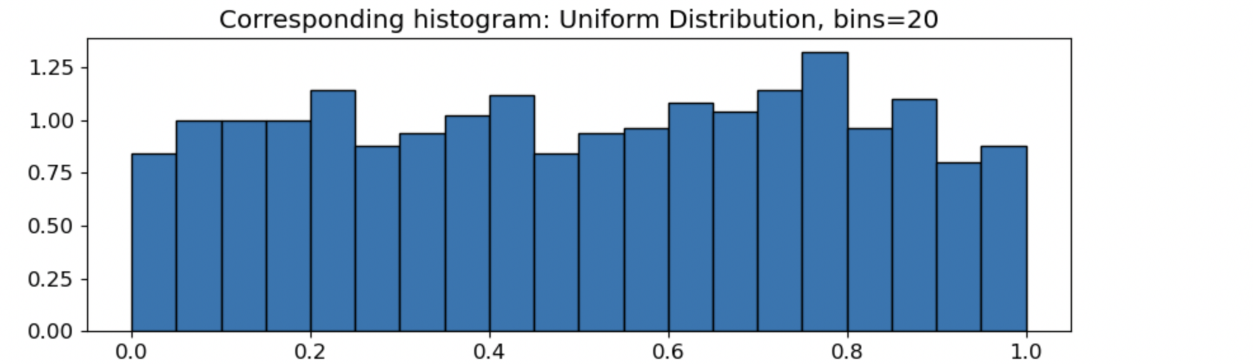
$\uparrow$

0.64

$\boxed{\therefore \text{sample is } 2\sqrt{0.64} = 1.6}$

Q4-a) Corresponding histogram shows a Uniform Distribution

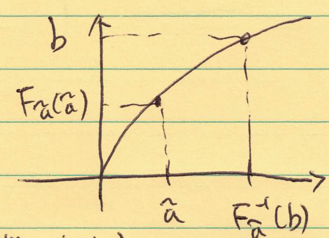
```python
# 4-a)
import matplotlib.pyplot as plt
import numpy as np

plt.rcParams['figure.figsize'] = (10, 3)
plt.rcParams['font.size'] = 12

data = np.load('samples.npy')
param = 1.0
x = data
y = 1 - np.exp(-param * x)

plt.title('Corresponding histogram: Uniform Distribution, bins=20')
plt.hist(y, bins=20, edgecolor = 'black', density=True)

plt.show()
```



(b) $\tilde{a}$: random variable with an invertible cdf $F_{\tilde{a}}$.

Distribution of $F_{\tilde{a}}(\tilde{a})$?

Let's define $\tilde{b} = F_{\tilde{a}}(\tilde{a})$,



$\boxed{F_{\tilde{b}}(b)} = P(\tilde{b} \le b) = P(F_{\tilde{a}}(\tilde{a}) \le b)$

Since F is increasing as value increases, (monotonic)

it is true that $\tilde{a} \le F_{\tilde{a}}^{-1}(b)$.

$$= P(\tilde{a} \le F_{\tilde{a}}^{-1}(b)) = F_{\tilde{a}}(F_{\tilde{a}}^{-1}(b)) = \boxed{b.}$$

$\therefore$ cdf $\tilde{b}$: $F_{\tilde{b}}(b) = b$ (constant value), uniform distribution.