Forecasting Time Series Homework 3

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1) Consider the $MA(1)$ model $x_t = \varepsilon_t + \beta\varepsilon_{t-1}$, where $\{\varepsilon_t\}$ is zero mean white noise.

A) Use the formula $\rho_1 = Corr(x_t, x_{t-1}) = \beta \frac{var\ \varepsilon_t}{var\ x_t}$ to show that $\rho_1 = \frac{\beta}{1+\beta^2}$.

Ans)

$$\rho_1 = corr(x_t, x_{t-1}) = \beta \frac{var(\varepsilon_t)}{var(x_t)}$$

For MA(1) model,

$$var(x_t) = var(\varepsilon_t + \beta\varepsilon_{t-1})$$

$$= var(\varepsilon_t) + \beta^2 \underbrace{var(\varepsilon_{t-1})}_{= var(\varepsilon_t)} + \underbrace{2\beta cov(\varepsilon_t, \varepsilon_{t-1})}_{=0}$$

since all ε's are uncorrelated

$$= (1+\beta^2)\, var(\varepsilon_t)$$

By rewriting the equation, we obtain

$$\rho_1 = corr(x_t, x_{t-1})$$

$$= \beta \frac{var(\varepsilon_t)}{var(x_t)}$$

$$= \beta \frac{\cancel{var(\varepsilon_t)}}{(1+\beta^2)\cancel{var(\varepsilon_t)}}$$

$$= \frac{\beta}{1+\beta^2}$$

B) Using the result from Part A), determine the maximum possible value for ρ_1 for the $MA(1)$ model. At what value of β is this maximum attained?

Ans) For MA(1) model to converge, the β needs to be between -1 and 1.
For ρ_1 to be the maximum using the equation in Part A, the β needs to be 1 and it makes the ρ_1 be $\frac{1}{1+1^2} = \frac{1}{2}$. The mathematical proof for this can be taking a derivative of the equation, $\frac{\beta}{1+\beta^2}$, with respect to β and setting it equal to 0. Calculations are below.

To find the maximum,

Set $\frac{d}{d\beta}\left(\frac{\beta}{1+\beta^2}\right) = \frac{(1)(1+\beta^2) - \beta(2\beta)}{(1+\beta^2)^2} = 0$

$1+\beta^2 - 2\beta^2 = 0$

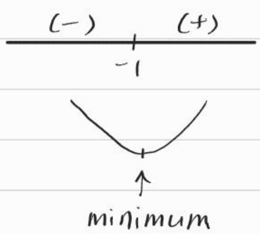
$1-\beta^2 = 0$

$(1+\beta)(1-\beta) = 0$

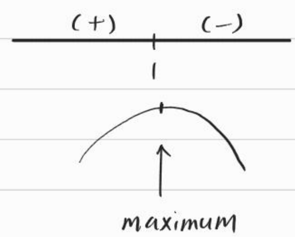
$\beta = -1$ or 1

Using the first derivative test,

At $\beta = -1$,



At $\beta = 1$,



Therefore, the local maximum is at $\beta = 1$.

At $\beta = 1$, $\rho_1 = \frac{\beta}{1+\beta^2} = \frac{1}{2}$

To check the end behavior of the function, use the Limit Theorem,

$\lim_{\beta\to-\infty} \frac{\beta}{1+\beta^2} = 0$ and $\lim_{\beta\to\infty} \frac{\beta}{1+\beta^2} = 0$

$\Rightarrow$ This shows that $\rho_1 = \frac{1}{2}$ is the maximum at $\beta = 1$.

C) Based on your answer to Part B), is the $MA(1)$ model capable of producing very smooth (i.e., very highly autocorrelated) realizations?

Ans) No, based on part B, we expect the maximum value of the autocorrelation coefficient to be ½. This value is not close to 1, meaning that MA(1) model is not capable of producing very smooth, or highly autocorrelated, realizations.

D) What is the correlation between x_t and x_{t-1} if $\{x_t\}$ is the first difference of a white noise series?

Ans) The white noise is already stationary and if we take another difference of stationary series, we are over-differencing it resulting in negative autocorrelation. We get yt = et - e(t-1), which is an MA(1) series with negative autocorrelation coefficient of -0.5. (β = -1, so the correlation between x_t and x_{t-1} is -0.5 based on Part A.)

D) $x_t = \varepsilon_t - \varepsilon_{t-1}$

This can be interpreted as MA(1) model with $B = -1$

$$(x_t = \varepsilon_t + \underbrace{B}_{-1} \varepsilon_{t-1})$$

Lag 1 (auto)correlation : $\rho_1 = \dfrac{B}{1+B^2} = -\dfrac{1}{2}$

2) Granger, p. 56, Problem 2

2. You are told that the monthly change in sales of paperback books in a certain store obeys an MA(2) model of the form

$$x_t = \varepsilon_t + 0.6\varepsilon_{t-1} + 0.3\varepsilon_{t-2}$$

and are given the following recent values for x_t: $x_{20} = 180$, $x_{21} = -120$, $x_{22} = 90$, $x_{23} = 10$.

(a) What forecast would you have made for x_{20} and x_{21} at time $t = 19$ if you had assumed $\varepsilon_{17} = -10$, $\varepsilon_{18} = 30$, $\varepsilon_{19} = 70$?

(b) Using these same values for $\varepsilon_{17}, \varepsilon_{18}$, and ε_{19}, what do you forecast for x_{24}, x_{25}, x_{26}, and x_{27} at time $t = 23$?

2(a) As we are in $t=19$, $\varepsilon_{20} = \varepsilon_{21} = \varepsilon_{22} \cdots = 0$

$$- x_{20} = \underset{\uparrow \atop 0}{\varepsilon_{20}} + 0.6\underset{\uparrow \atop 70}{\varepsilon_{19}} + 0.3\underset{\uparrow \atop 30}{\varepsilon_{18}} = 0 + 0.6\times 70 + 0.3\times 30 = 51$$

$$- x_{21} = \underset{\uparrow \atop 0}{\varepsilon_{21}} + 0.6\underset{\uparrow \atop 0}{\varepsilon_{20}} + 0.3\underset{\uparrow \atop 70}{\varepsilon_{19}} = 0+0+0.3\times 70 = 21$$

$$- x_{22} = \underset{\uparrow \atop 0}{\varepsilon_{22}} + 0.6\underset{\uparrow \atop 0}{\varepsilon_{21}} + 0.3\underset{\uparrow \atop 0}{\varepsilon_{20}} = 0$$

(b) $\boxed{t=23}$ $X_{20} = 180$, $X_{21} = -120$, $X_{22} = 90$, $X_{23} = 10$

As we are in $t=23$, $\varepsilon_{24} = \varepsilon_{25} = \varepsilon_{26} \cdots = 0$

$$\boxed{- x_{24}} = \underset{\uparrow \atop 0}{\varepsilon_{24}} + 0.6\underset{\uparrow \atop -33.884}{\varepsilon_{23}} + 0.3\underset{\uparrow \atop 182.34}{\varepsilon_{22}} = 0 + (-20.3304) + 54.702 = \boxed{34.3716}$$

$$\boxed{- x_{25}} = \underset{\uparrow \atop 0}{\varepsilon_{25}} + 0.6\underset{\uparrow \atop 0}{\varepsilon_{24}} + 0.3\underset{\uparrow \atop -33.884}{\varepsilon_{23}} = \boxed{-10.1652}$$

$$\boxed{- x_{26}} = \underset{\uparrow \atop 0}{\varepsilon_{26}} + 0.6\underset{\uparrow \atop 0}{\varepsilon_{25}} + 0.3\underset{\uparrow \atop 0}{\varepsilon_{24}} = \boxed{0}$$

$$\boxed{- x_{27}} = \underset{\uparrow \atop 0}{\varepsilon_{27}} + 0.6\underset{\uparrow \atop 0}{\varepsilon_{26}} + 0.3\underset{\uparrow \atop 0}{\varepsilon_{25}} = \boxed{0}$$

$$180 = x_{20} = \varepsilon_{20} + 0.6\varepsilon_{19} + 0.3\varepsilon_{18} = \varepsilon_{20} + 0.6\times 70 + 0.3\times 30$$
$$= \varepsilon_{20} + 42 + 9$$
$$\therefore \varepsilon_{20} = 180 - 51 = 129$$

$$-120 = x_{21} = \varepsilon_{21} + 0.6\varepsilon_{20} + 0.3\varepsilon_{19} = \varepsilon_{21} + 0.6\times 129 + 0.3\times 70$$

$$= \varepsilon_{21} + 98.4$$

$$\therefore \varepsilon_{21} = -120 - 98.4 = -218.4$$

$$90 = x_{22} = \varepsilon_{22} + 0.6\,\varepsilon_{21} + 0.3\,\varepsilon_{20}$$

$$= \varepsilon_{22} + 0.6 \times -218.4 + 0.3 \times 129$$

$$= \varepsilon_{22} + (-131.04) + 38.7$$

$$= \varepsilon_{22} - 92.34$$

$$\therefore \varepsilon_{22} = 182.34$$

$$10 = x_{23} = \varepsilon_{23} + 0.6\,\varepsilon_{22} + 0.3\,\varepsilon_{21}$$

$$= \varepsilon_{23} + 0.6 \times 182.34 + 0.3 \times (-218.4)$$

$$= \varepsilon_{23} + 109.404 + (-65.52)$$

$$= \varepsilon_{23} + 43.884$$

$$\therefore \varepsilon_{23} = -33.884$$

3) Granger, p. 56, Problem 3

3. Suppose that you are unsure whether a certain series has been generated by an MA(1) or an MA(2) model. How would the value of the second autocorrelation coefficient ρ_2 help you choose between these alternative models?

Ans) In ACF, if there is a statistical cut-off beyond the lag-1 or the ρ_2 is 0, it can be said that the series has been generated by an MA(1) model. If there is a statistical cut-off beyond the lag-2 or the ρ_2 is not 0, it can be said that the series has been generated by an MA(2) model.

4) Granger, p. 57, Problem 4

4. Prove that the two series x_t and y_t, generated by

$$x_t = \varepsilon_t + 0.8\varepsilon_{t-1}$$

and

$$y_t = \eta_t + 1.25\eta_{t-1},$$

where ε_t, η_t are each zero-mean white noise series and $\operatorname{var}(\varepsilon_t) = 1$, $\operatorname{var}(\eta_t) = 0.64$, have identical variances and autocorrelation sequences ρ_k, $k = 0, 1, 2, \ldots$. Prove that if z_t is generated by

$$z_t = \theta_t + a\theta_{t-1} + b\theta_{t-2},$$

where θ_t is white noise, and if $b \neq 0$, then z_t cannot have the same autocorrelation sequence as x_t and y_t.

Answers for this problem continue on the next page.

Let $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$ to find the generalized formulas for variance and autocorrelation coefficient for MA (1) model.

$$Var(x_t) = Var(\varepsilon_t + \theta\varepsilon_{t-1})$$
$$= Var(\varepsilon_t) + \theta^2 \underbrace{Var(\varepsilon_{t-1})}_{= Var(\varepsilon_t)}$$
$$= (1+\theta^2)\,Var(\varepsilon_t)$$

$$\rho_0 = corr(x_t, x_t) = 1$$
$$\rho_1 = corr(x_t, x_{t+1})$$
$$= \frac{cov(x_t, x_{t+1})}{\sqrt{var(x_t)}\sqrt{var(x_{t+1})}}$$
$$= \frac{cov(\varepsilon_t + \theta\varepsilon_{t-1},\ \varepsilon_{t+1} + \theta\varepsilon_t)}{\sqrt{var(x_t)}\sqrt{var(x_{t+1})}}$$
$$= \frac{\cancel{cov(\varepsilon_t, \varepsilon_{t+1})} + cov(\varepsilon_t, \theta\varepsilon_t) + \cancel{cov(\theta\varepsilon_{t-1}, \varepsilon_{t+1})} + \cancel{cov(\theta\varepsilon_{t-1}, \theta\varepsilon_t)}}{\sqrt{var(x_t)}\sqrt{var(x_{t+1})}}$$
$$= \frac{\theta\ var(\varepsilon_t)}{Var(x_t)}$$

from calculation above

$$= \frac{\theta\ \cancel{var(\varepsilon_t)}}{(1+\theta^2)\ \cancel{Var(\varepsilon_t)}}$$

$$\rho_2 = corr(x_t, x_{t+2})$$
$$= \frac{cov(x_t, x_{t+2})}{\sqrt{var(x_t)}\sqrt{var(x_{t+2})}}$$
$$= \frac{cov(\varepsilon_t + \theta\varepsilon_{t-1},\ \varepsilon_{t+2} + \theta\varepsilon_{t+1})}{\sqrt{var(x_t)}\sqrt{var(x_{t+2})}}$$
$$= \frac{\cancel{cov(\varepsilon_t, \varepsilon_{t+2})} + \cancel{cov(\varepsilon_t, \theta\varepsilon_{t+1})} + \cancel{cov(\theta\varepsilon_{t-1}, \varepsilon_{t+2})} + \cancel{cov(\theta\varepsilon_{t-1}, \theta\varepsilon_{t+1})}}{\sqrt{var(x_t)}\sqrt{var(x_{t+2})}}$$
$$= 0$$

$\rho_3, \rho_4, \rho_5, \cdots = 0$ since all covariances $k > 1$ will be 0 using the same logic.

Apply the formulas above to the given equations.

① $X_t = \varepsilon_t + 0.8\,\varepsilon_{t-1}$

$$Var(X_t) = (1+\theta^2)\ Var(\varepsilon_t)$$
$$= (1+0.8^2)(1)$$
$$= 1.64$$

$$\rho_0 = 1$$
$$\rho_1 = \frac{\theta}{1+\theta^2}$$
$$= \frac{0.8}{1+0.8^2}$$
$$= \frac{20}{41} \quad (\approx 0.4878)$$
$$\rho_2,\ \rho_3,\ \rho_4,\ \cdots = 0$$

② $y_t = \eta_t + 1.25\,\eta_{t-1}$

$$Var(y_t) = (1+\theta^2)\ Var(\eta_t)$$
$$= (1+1.25^2)(0.64)$$
$$= 1.64$$

$$\rho_1 = 1$$
$$\rho_2 = \frac{\theta}{1+\theta^2}$$
$$= \frac{1.25}{1+1.25^2}$$
$$= \frac{20}{41} \quad (\approx 0.4878)$$
$$\rho_2,\ \rho_3,\ \rho_4,\ \cdots = 0$$

$\Rightarrow$ Therefore, X_t and y_t have identical variances and autocorrelation sequences ρ_k.

$Z_t = \theta_t + a\theta_{t-1} + b\theta_{t-2}$: θ is white noise, $b \neq 0$

To prove that Z_t cannot have the same autocorrelation sequence as MA(1) model, we focus on the ρ_2.

$$\rho_2 = corr(Z_t, Z_{t+2})$$

$$= \frac{cov(Z_t, Z_{t+2})}{\sqrt{var(Z_t)}\sqrt{Var(Z_{t+2})}}$$

rewriting numerator

$$= cov(\theta_t + a\theta_{t-1} + b\theta_{t-2},\ \theta_{t+2} + a\theta_{t+1} + b\theta_t)$$

$$= \cancel{cov(\theta_t, \theta_{t+2})} + \cancel{cov(\theta_t, a\theta_{t+1})} + cov(\theta_t, b\theta_t)$$
$$+ \cancel{cov(a\theta_{t-1}, \theta_{t+2})} + \cancel{cov(a\theta_{t-1}, a\theta_{t+1})} + \cancel{cov(a\theta_{t-1}, b\theta_t)}$$
$$+ \cancel{cov(b\theta_{t-2}, \theta_{t+2})} + \cancel{cov(b\theta_{t-2}, a\theta_{t+1})} + \cancel{cov(b\theta_{t-2}, b\theta_t)}$$

$$= cov(\theta_t, b\theta_t)$$

$$= b^2\, var(\theta_t)$$

$$= \frac{b^2\, Var(\theta_t)}{\sqrt{Var(Z_t)}\sqrt{\underbrace{Var(Z_{t+2})}_{= Var(Z_t)}}}$$

$$= \frac{b^2\, var(\theta_t)}{var(Z_t)}$$

rewriting denominator

$$Var(Z_t) = Var(\theta_t + a\theta_{t-1} + b\theta_{t-2})$$

$$= Var(\theta_t) + a^2 \underbrace{Var(\theta_{t-1})}_{= Var(\theta_t)} + b^2 \underbrace{var(\theta_{t-2})}_{= Var(\theta_t)}$$

$$= (1 + a^2 + b^2)\, var(\theta_t)$$

$$= \frac{b^2\, \cancel{Var(\theta_t)}}{(1 + a^2 + b^2)\, \cancel{Var(\theta_t)}}$$

$$= \frac{b^2}{1 + a^2 + b^2}$$

$\neq 0$ since $b \neq 0$ by given

$\Rightarrow$ Therefore, Z_t (MA(2) model) cannot have the same autocorrelation sequence as x_t and y_t (MA(1) model)

5) Granger, p. 62, Problem 2

2. A company economist believes that a rival firm's quarterly advertising expenditures obey the AR(3) model

$$x_t = 0.2x_{t-1} + 0.6x_{t-2} - 0.3x_{t-3} + \varepsilon_t.$$

If the values for the last three quarters are $x_{10} = 180$, $x_{11} = 140$, and $x_{12} = 200$, forecast expenditures for the next three quarters, $t = 13$, 14, and 15.

ANS) $x_{13} = 70$, $x_{14} = 92$, $x_{15} = 0.4$

$$X_t = 0.2X_{t-1} + 0.6X_{t-2} - 0.3X_{t-3} + \varepsilon_t$$

$t = 13$ $\quad X_{13} = 0.2X_{12} + 0.6X_{11} - 0.3X_{10} + \underset{0}{\varepsilon_{13}}$

$= 0.2(200) + 0.6(140) - 0.3(180)$

$= 40 + 84 - 54$

$= 70$

$t = 14$ $\quad X_{14} = 0.2\underset{\hat{X}_{13}}{X_{13}} + 0.6X_{12} - 0.3X_{11} + \underset{0}{\varepsilon_{14}}$

$= 0.2(70) + 0.6(200) - 0.3(140)$

$= 14 + 120 - 42$

$= 92$

$t = 15$ $\quad X_{15} = 0.2\underset{\hat{X}_{14}}{X_{14}} + 0.6X_{13} - 0.3X_{12} + \underset{0}{\varepsilon_{15}}$

$= 0.2(92) + 0.6(70) - 0.3(200)$

$= 18.4 + 42 - 60$

$= 0.4$

6) Granger, p. 63, Problem 3

3. A time series is constructed on a computer by an AR(1) model. You are told that the series has zero mean, that the second autocorrelation is $\rho_2 = 0.64$, and that $x_{50} = 10$. Forecast x_{51} and x_{52}. Why would knowing the value of ρ_3 help forecast x_{51} but not x_{52}?

Ans)

6. - AR(1) model: $x_{t+1} = \beta x_t + \varepsilon_{t+1}$

at time t, - $\underline{\hat{x}_{t+1} = \beta x_t}$ $(\varepsilon_{t+1} = 0)$

$$x_{t+2} = \beta x_{t+1} + \varepsilon_{t+2}$$

$$= \beta(\beta x_t + \varepsilon_{t+1}) + \varepsilon_{t+2}$$

$$= \beta^2 x_t + \beta \underbrace{\varepsilon_{t+1}}_{0} + \underbrace{\varepsilon_{t+2}}_{0}$$

at time t, - $\underline{\hat{x}_{t+2} = \beta^2 x_t}$

- $\mathrm{cov}(x_t, x_{t+2}) = \mathrm{Cov}(x_t, \beta^2 x_t + \beta\varepsilon_{t+1} + \varepsilon_{t+2})$

$$= \mathrm{cov}(x_t, \beta^2 x_t) + \underbrace{\mathrm{cov}(x_t, \beta\varepsilon_{t+1})}_{=0} + \underbrace{\mathrm{cov}(x_t, \varepsilon_{t+2})}_{=0}$$

$$= \beta^2 \mathrm{Var}(x_t)$$

- $\rho^2 = \dfrac{\mathrm{Cov}(x_t, x_{t+2})}{\sqrt{\mathrm{Var}(x_t)}\sqrt{\mathrm{Var}(x_{t+2})}} = \dfrac{\mathrm{Cov}(x_t, x_{t+2})}{\mathrm{Var}(x_t)} = \beta^2 = 0.64$

- As $\beta^2 = 0.64$, $\beta = \pm 0.8$

$\hat{x}_{t+1} = \beta x_t \Leftrightarrow \hat{x}_{51} = \beta x_{50} = \pm 0.8 \times 10 = \pm 8$

$\hat{x}_{t+2} = \beta^2 x_t \Leftrightarrow \hat{x}_{52} = \beta^2 x_{50} = 0.64 \times 10 = 6.4$

Also, If we know $\rho_3 = \beta^3$, we can identify plus or minus sign of value β. In this example, if $\beta^3 > 0$, then $\beta > 0$ and therefore $\hat{x}_{51} = 8$, which helps to forecast x_{51}.

However, $\hat{x}_{52}$ is calculated by β^2, so knowing β^3 doesn't helps to forecast $\hat{x}_{52}$.

7) Granger, p. 63, Problem 4

4. If a series x_t is generated by

$$x_t = 0.8x_{t-1} + \varepsilon_t,$$

where ε_t is zero-mean white noise with variance 25, find the variances of the one-, two-, and three-step forecast errors, i.e., $\mathrm{var}(e_{n,h})$, $h = 1, 2,$ and 3.

7). $x_t = 0.8x_{t-1} + \varepsilon_t$, $E(\varepsilon_t)=0$, $Var(\varepsilon_t)=25$

- One-step forecast: $\hat{x}_{t+1} = 0.8\,x_t$
- Two-step forecast: $\hat{x}_{t+2} = (0.8)^2 x_t$
- Three-step forecast: $\hat{x}_{t+3} = (0.8)^3 x_t$, as $\varepsilon_{t+1} = \varepsilon_{t+2} \cdots = 0$ at time t.

- One-step forecast error

$$\underline{x_{t+1} - \hat{x}_{t+1}} = 0.8x_t + \varepsilon_{t+1} - 0.8x_t = \varepsilon_{t+1}$$

$$\boxed{Var(x_{t+1} - \hat{x}_{t+1}) = Var(\varepsilon_{t+1}) = Var(\varepsilon_t) = 25.}$$

- Two-step forecast error

$$\underline{x_{t+2} - \hat{x}_{t+2}} = 0.8\,x_{t+1} + \varepsilon_{t+1} - (0.8)^2 x_t$$
$$= 0.8(0.8\,x_t + \varepsilon_t) + \varepsilon_{t+1} - (0.8)^2 x_t$$
$$= \cancel{(0.8)^2 x_t} + 0.8\varepsilon_t + \varepsilon_{t+1} - \cancel{(0.8)^2 x_t}$$
$$= 0.8\,\varepsilon_t + \varepsilon_{t+1}$$

$$\boxed{Var(x_{t+2} - \hat{x}_{t+2})} = Var(0.8\,\varepsilon_t + \varepsilon_{t+1})$$
$$= 0.64\,Var(\varepsilon_t) + Var(\varepsilon_{t+1}) + 2\cdot 0.8\,\underbrace{Cov(\varepsilon_t, \varepsilon_{t+1})}_{=0}$$
$$= 0.64 \times 25 + 25$$
$$= 1.64 \times 25 = \boxed{41}$$

- Three-step forecast error

$$\underline{x_{t+3} - \hat{x}_{t+3}} = 0.8\,x_{t+2} + \varepsilon_{t+2} - (0.8)^3 x_t$$
$$= 0.8(0.8\,x_{t+1} + \varepsilon_{t+1}) + \varepsilon_{t+2} - (0.8)^3 x_t$$
$$= 0.8(0.8(0.8x_t + \varepsilon_t) + \varepsilon_{t+1}) + \varepsilon_{t+2} - (0.8)^3 x_t$$
$$= 0.8((0.8)^2 x_t + 0.8\,\varepsilon_t + \varepsilon_{t+1}) + \varepsilon_{t+2} - (0.8)^3 x_t$$
$$= \cancel{(0.8)^3 x_t} + (0.8)^2\varepsilon_t + 0.8\,\varepsilon_{t+1} + \varepsilon_{t+2} - \cancel{(0.8)^3 x_t}$$

$$\boxed{Var(x_{t+3} - \hat{x}_{t+3})} = Var[(0.8)^2\varepsilon_t + 0.8\varepsilon_{t+1} + \varepsilon_{t+2})$$

$(Cov(\varepsilon_t, \varepsilon_{t+1}) = Cov(\varepsilon_{t+1}, \varepsilon_{t+2}) = Cov(\varepsilon_t, \varepsilon_{t+2}) = 0)$

$$= (0.8)^4\,Var(\varepsilon_t) + (0.8)^2\,Var(\varepsilon_{t+1}) + Var(\varepsilon_{t+2})$$
$$= (0.8)^4 \times 25 + (0.8)^2 \times 25 + 25$$
$$= ((0.8)^4 + (0.8)^2 + 1) \times 25 = \boxed{51.24}$$