Homework 0 Yoon Tae Park (yp2201@nyu.edu)

1. (a) True.

If $S_1, \dots, S_n$ is a partition of Ω, then

$\Omega = \cup_i S_i$ and $S_i \cap S_j = \phi$ for $i \neq j$

Then, $(S_i \cap A) \cap (S_j \cap A) = (S_i \cap S_j) \cap A = \phi \cap A = \phi$ for $i \neq j$

and $\cup_i (S_i \cap A) = (\cup_i S_i) \cap A = \Omega \cap A = A$.

$\boxed{\therefore\ S_i \cap A \text{ for every } i \text{ is a partation of } A.}$

(b) False, if $\Omega = \{1,2,3\}$, $A = \{1,2\}$, $B = \{2,3\}$,

Then $A^c \cup B^c = \{3\} \cup \{1\} = \{1,3\}$,

but $(A \cup B)^c = \{1,2,3\}^c = \phi$.

$\boxed{\therefore\ A^c \cup B^c \neq (A \cup B)^c}$.

(c) False, if $A = \{1,2,3,4\}$, $B = \{2,3,5,6\}$, $C = \{3,4,6,7\}$,

Then $(A \cup B) \cap C = \{1,2,3,4,5,6\} \cap \{3,4,6,7\} = \{3,4,6\}$

but $A \cup (B \cap C) = \{1,2,3,4\} \cup \{3,6\} = \{1,2,3,4,6\}$

$\boxed{\therefore\ (A \cup B) \cap C \neq A \cup (B \cap C)}$

2. (a)

$$
\begin{aligned}
S_n &= r^m + \cancel{r^{m+1}} + \cdots + r^n \\
-\ \ r \cdot S_n &= \cancel{r^{m+1}} + \cdots + \cancel{r^n} + r^{n+1}
\end{aligned}
$$

$$(1-r) S_n = r^m - r^{n+1}$$

Since $r \neq 1$, $\boxed{S_n = \dfrac{r^m - r^{n+1}}{1-r}}$

(b) Based on expression (a), $S_n = \dfrac{r^m - r^{n+1}}{1-r}$ $(r \neq 1)$

$$\sum_{i=m}^{\infty} r^i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r^m - r^{n+1}}{1-r}$$

Since m is a positive integer, r^{n+1} should remain bounded

~~$\mathscr{r}$ ~~

Also, if $r=1$, $\sum_{i=m}^{\infty} r^i = \infty$, so it will not converge

$\boxed{\therefore \ |r| < 1}$

(c) Base step: For $n=2$ $\sum_{i=1}^{2} i = 1+2=3=\dfrac{2\cdot(2+1)}{2}$ (o)

Induction step:

Assume for any fixed $n \geq 2$ that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$

$$
\begin{aligned}
\text{Then } \sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \quad \text{(o)}
\end{aligned}
$$

3. (a) Since $\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)-f(x)}{(x+h)-(x)}$, this means an average rate of change which has interval h (in axis x). If h moves close to 0, two points gets closer to a certain point and rate of change is instantaneous.

(b) $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0}\frac{2xh+h^2}{h}$$
$$= \lim_{h\to 0} 2x+h$$
$$\boxed{= 2x}$$

(c) $L_y(y) = ay+b = f(y)$, $b = f(y) - f'(y)y$

$L_y'(y) = a = f'(y)$

$\therefore L_y(x) = f'(y)x + f(y) - f'(y)y$

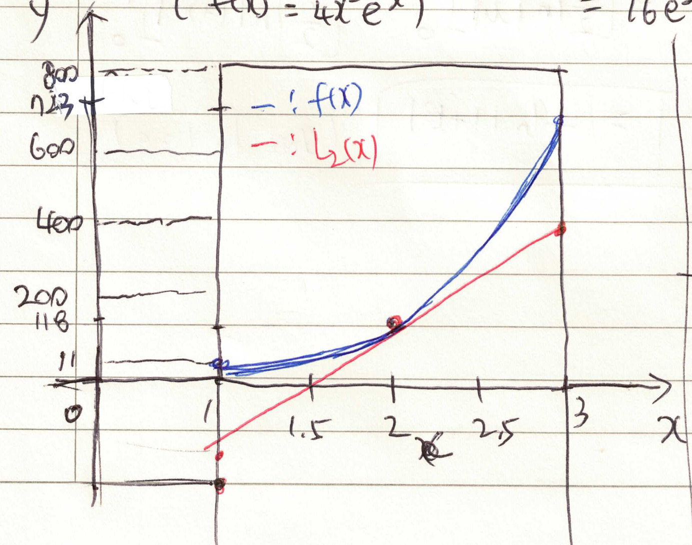
$= f'(y)(x-y) + f(y)$

(d) By the product rule, $f'(x) = 8xe^x + 4x^2e^x = (4x^2+8x)e^x$

$L_2(x) = f'(2)(x-2)+f(2) = 32e^2(x-2)+16e^2$

$(f(x) = 4x^2e^x)$ $= 16e^2(2x-3)$

*plot)



$f(0) = 4e \fallingdotseq 11$

$f(2) = 16e^2 \fallingdotseq 118$

$f(3) = 36e^3 \fallingdotseq 723$

$L_2(0) = -48e^2 = -355$

$L_2(2) = 16e^2 = 118$

$L_2(3) = 48e^2 = 355$

4. (a) On the upper right quadrant must equal to $y=(x-1)^2$

since it goes through $(0,1)$, $(\frac{1}{2},\frac{1}{4})$, and $(1,0)$

By symmetry, the area below this curve, above the horizontal axis between 0 and 1, and the vertical axis between 0 and 1 is equal to one-fourth of the area of the shape.

Therefore the area equals to,

$$4\times\int_0^1 (x-1)^2\,dx = 4\int_0^1 (x^2-2x+1)\,dx$$

$$=4\times\left[\frac{1}{3}x^3-x^2+x\right]_0^1$$

$$=4\times\left(\frac{1}{3}-0\right)=\boxed{\frac{4}{3}}$$

(b) Let $u=1+x^2$, then $du=2x\,dx$, $x\,dx=\frac{1}{2}\,du$

$$f(t)=\int_0^t \frac{x\,dx}{u}=\frac{1}{2}\int_0^t \frac{1}{u}\,du$$

)

$$=\left[\frac{1}{2}\ln|u|\right]_0^t=\left[\frac{1}{2}\ln|1+x^2|\right]_0^t$$

$$\boxed{=\frac{1}{2}\ln|1+t^2|}$$ (since $\ln 1=0$)