# 1002. HW3 Yoon Tae Park (yp2201@nyu.edu)

1. (a) From the assumptions we know that the density is zero outside of the wall, equal to a constant $C_b$ in the behind of the painting, and to another constant $C_r$ in the rest of the wall.

Since $P(\text{spider is behind painting}) = 2 \times p(\underbrace{\text{spider is on the rest of the wall}}_{p(\text{rest})})$,
(I will redefine as $p(\text{behind})$, $p(\text{rest})$)

and $p(\text{behind}) + p(\text{rest}) = 1$ So, $p(\text{behind}) = \frac{2}{3}$, $p(\text{rest}) = \frac{1}{3}$

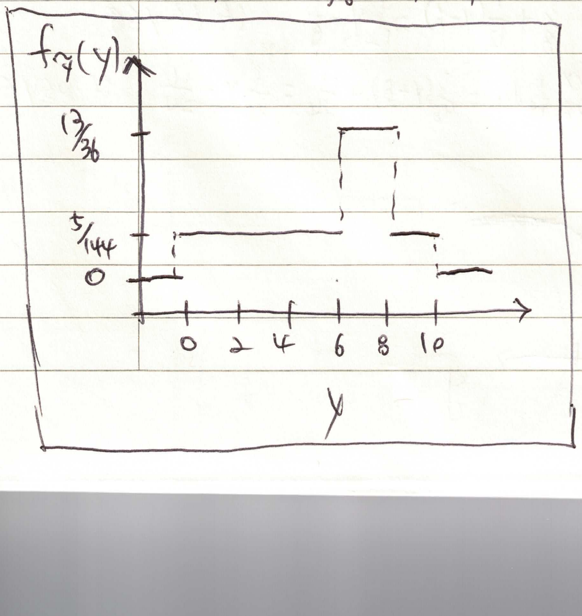
$$p(\text{behind}) = \int_6^8 \int_4^6 C_b \, dx\,dy = 4C_b = \frac{2}{3}, \quad \boxed{\therefore C_b = \frac{1}{6}}$$

Similarly, $p(\text{rest}) = \int_0^6 \int_0^{10} C_r \, dx\,dy + \int_6^8 \int_0^4 C_r \, dx\,dy + \int_6^8 \int_6^{10} C_r \, dx\,dy + \int_8^{10} \int_0^{10} C_r \, dx\,dy = 96 C_r = \frac{1}{3}$, $\boxed{\therefore C_r = \frac{1}{288}}$

Therefore,
$$\boxed{f_{\tilde{x},\tilde{y}}(x,y) = \begin{cases} \frac{1}{6} & \text{if } 4 \le x \le 6,\ 6 \le y \le 8 \\ \frac{1}{288} & \text{if } 0 \le y \le 6 \text{ or } 8 \le y \le 10 \text{ or } 0 \le x \le 4,\ 6 \le y \le 8 \text{ or } 6 \le x \le 10,\ 6 \le y \le 8 \\ 0 & \text{otherwise} \end{cases}}$$
(assuming $x$ as width, and $y$ as height)

(b) Pdf of the height at which the spider is located is given by,

$$f_{\tilde{y}}(y) = \int_0^{10} f_{\tilde{x},\tilde{y}}(x,y)\,dx = \begin{cases} \int_0^{10} \frac{1}{288} dx = \frac{10}{288} = \frac{5}{144} & \text{if } 0 \le y \le 6,\ 8 \le y \le 10 \\ \int_0^4 \frac{1}{288} dx + \int_4^6 \frac{1}{6} dx + \int_6^{10} \frac{1}{288} dx = \frac{4}{288} + \frac{2}{6} + \frac{4}{288} = \frac{13}{36} & \text{if } 6 \le y \le 8 \\ 0 & \text{otherwise} \end{cases}$$



1, (c) Conditioned that the spider is located where I can see,

joint pdf is a constant $c$ such that

$$p(\text{rest}) = \int_0^6\int_0^{10} c\,dx\,dy + \int_6^8\int_0^4 c\,dx\,dy + \int_6^8\int_6^{10} c\,dx\,dy + \int_8^{10}\int_0^{10} c\,dx\,dy = 96c = 1, \therefore \boxed{c = \frac{1}{96}}$$

Let's assume condition 'you can see it' as $B$.

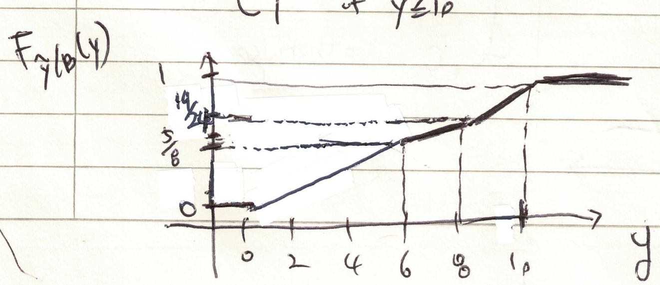
Then conditional cdf of the height is,

$$F_{\tilde{y}/B}(y) = \int_{-\infty}^{y} f_{\tilde{y}/B}(u)\,du$$

$$f_{\tilde{x},\tilde{y}/B}(x,y) = \begin{cases} \frac{1}{96} & \text{if } 0\le y\le 6 \text{ or } 8\le y\le 10 \text{ or } \\ & 0\le x\le 4,\ 6\le y\le 8 \text{ or } 6\le x\le 10,\ 6\le y\le 8 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\tilde{y}/B}(y) = \int_0^{10} f_{\tilde{x},\tilde{y}/B}(x,y)\,dx = \begin{cases} \int_0^{10}\frac{1}{96}dx = \frac{5}{48} & \text{if } 0\le y\le 6,\ 8\le y\le 10 \\ \int_0^4\frac{1}{96}dx + \int_6^{10}\frac{1}{96}dx = \frac{1}{12} & \text{if } 6\le y\le 8 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore\ = \begin{cases} 0 & \text{if } y\le 0 \\ \int_0^y \frac{5}{48}du = \frac{5}{48}y & \text{if } 0\le y\le 6 \\ \int_0^6\frac{5}{48}du + \int_6^y\frac{1}{12}du = \frac{5}{8}+\frac{1}{12}(y-6) = \frac{1}{12}y+\frac{1}{8} & \text{if } 6\le y\le 8 \\ \int_0^6\frac{5}{48}du+\int_6^8\frac{1}{12}du+\int_8^y\frac{5}{48}du = \frac{5}{48}(y-8)+\frac{19}{24} = \frac{5}{48}y-\frac{1}{24} & \text{if } 8\le y\le 10 \\ 1 & \text{if } y\ge 10 \end{cases}$$



2.(a) We know that density outside of the ponds is zero. Also, by putting a constant $c_L$ as a density in the large pond, and constant $c_S$ as a density in the small pond, and knowing that $p(\text{large pond}) = \frac{1}{4}$, $p(\text{small pond}) = \frac{3}{4}$, we can get

$$p(\text{large pond}) = \int_0^{10}\int_0^{10} c_L \, dx\,dy = 100c_L = \frac{1}{4}, \quad \boxed{\therefore c_L = \frac{1}{400}}$$

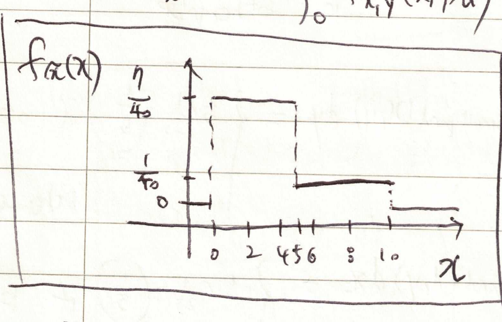
(assuming $x$ as width and $y$ as height)

$$p(\text{small pond}) = \int_{12}^{17}\int_0^{5} c_S \, dx\,dy = 25c_S = \frac{3}{4} \quad \boxed{\therefore c_S = \frac{3}{100}}$$

$$\therefore f_{X,Y}(x,y) = \begin{cases} \frac{1}{400} & \text{if } 0 \leq x \leq 10,\ 0 \leq y \leq 10 \\ \frac{3}{100} & \text{if } 0 \leq x \leq 5,\ 12 \leq y \leq 17 \\ 0 & \text{otherwise} \end{cases}$$

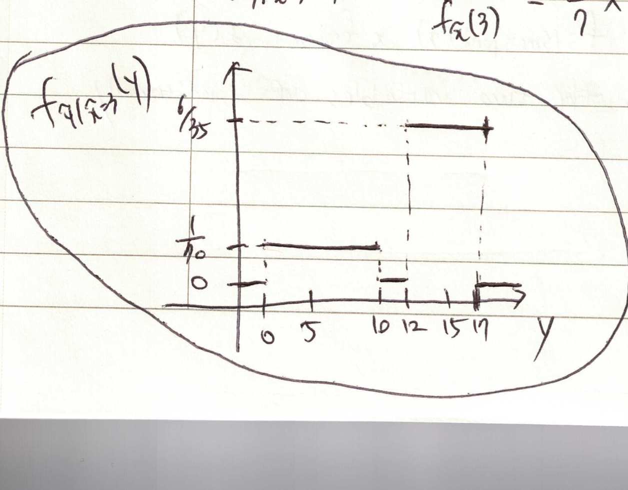
(b) Marginal pdf of horizontal ($x$) position of the frog is

$$f_X(x) = \int_0^{17} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^{10}\frac{1}{400}dy + \int_{12}^{17}\frac{3}{100}dy = \frac{7}{40} & \text{if } 0 \leq x \leq 5 \\ \int_0^{10}\frac{1}{400}dy = \frac{1}{40} & \text{if } 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$




(c) $$f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{40}{7} \times f(3,y) = \frac{40}{7} \times \begin{cases} \frac{1}{400} & \text{if } 0 \leq y \leq 10 \\ \frac{3}{100} & \text{if } 12 \leq y \leq 17 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{70} & \text{if } 0 \leq y \leq 10 \\ \frac{6}{35} & \text{if } 12 \leq y \leq 17 \\ 0 & \text{otherwise} \end{cases}$$



2-(d) No, the vertical position of the frog is not independent from the horizontal position of the frog.

Giving counter example, $\left(\exists f_{\tilde{y}}(y) \neq f_{\tilde{y}|\tilde{x}}(y,x)\right)$

$$f_{\tilde{y}}(15) = \int_0^5 f_{\tilde{x},\tilde{y}}(x,y)\,dx = 5\times\frac{3}{100} = \frac{15}{100} = \frac{3}{20} \quad (\text{from 2-a})$$

$\neq$

$$f_{\tilde{y}|\tilde{x}=3}(15) = \frac{6}{35} \quad (\text{from 2-c})$$

(e) Conditioned on the frog in the small pond, joint pdf is a constant $c$ such that,

$$p(\text{small pond}) = \int_{12}^{17}\int_0^5 c\,dx\,dy = 25c = 1, \quad \therefore c = \frac{1}{25}$$

Joint pdf : $f_{\tilde{x},\tilde{y}|\text{small pond}}(x,y) = \begin{cases} \frac{1}{25} & \text{if } 0 \leq x \leq 5,\ 12 \leq y \leq 17 \\ 0 & \text{otherwise} \end{cases}$

$$f_{\tilde{x}|\text{small pond}}(x) = \int_{12}^{17} f_{\tilde{x},\tilde{y}|\text{small pond}}(x,y)\,dy = \begin{cases} 5\times\frac{1}{25} = \frac{1}{5} & \text{if } 0 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{\tilde{y}|\text{small pond}}(y) = \int_0^5 f_{\tilde{x},\tilde{y}|\text{small pond}}(x,y)\,dx = \begin{cases} 5\times\frac{1}{25} = \frac{1}{5} & \text{if } 12 \leq y \leq 17 \\ 0 & \text{otherwise} \end{cases}$$

Since $f_{\tilde{x},\tilde{y}|\text{small pond}}(x,y) = f_{\tilde{x}|\text{small pond}}(x) \times f_{\tilde{y}|\text{small pond}}(y)$ for any values of $x$ and $y$, the two variables are conditionally independent.

3. (a) For fixed $x$, $f_{\tilde{S}_1|\tilde{X}}(s|x)=0$ if $x<s-0.25$ or $x>s+0.25$

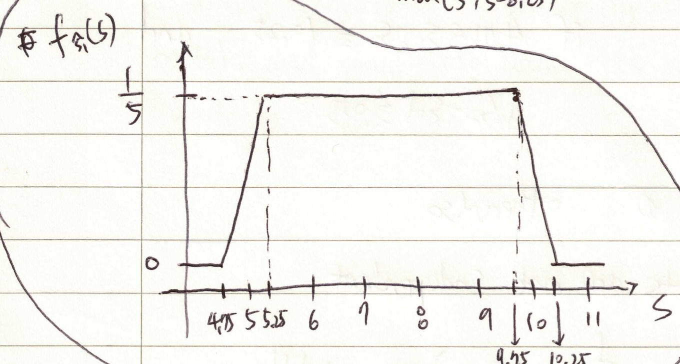
($\therefore$ $x-0.25<s<x+0.25$)

↳ $x$ should be between $s-0.25 \sim s+0.25$)

$$f_{\tilde{S}_1}(s) = \int_5^{10} \underbrace{f_{\tilde{S}_1|\tilde{X}}(s|x)}_{\Downarrow} \times \underbrace{f_{\tilde{X}}(x)}_{\Downarrow}\, dx$$

( $0.5\times c_1=1$, $c_1=2$ ; $5\times c_2=1$, $c_2=\frac{1}{5}$ )

$$= \int_{\max(5,\, s-0.25)}^{\min(10,\, s+0.25)} \frac{2}{5}\, dx = \begin{cases} 0 & \text{if } s<4.75 \\ \frac{2}{5}(s+0.25-5)=\frac{2}{5}(s-4.75) & \text{if } 4.75\le s\le 5.25 \\ \frac{2}{5}(s+0.25-s+0.25)=\frac{1}{5} & \text{if } 5.25\le s\le 9.75 \\ \frac{2}{5}(10-s+0.25)=\frac{2}{5}(10.25-s) & \text{if } 9.75\le s\le 10.25 \\ 0 & \text{if } s>10.25 \end{cases}$$




(b) For fixed $s_1$, $f_{\tilde{S}_1|\tilde{X}}(s_1|x)=0$ if $x<s_1-0.25$ or $x>s_1+0.25$

Also, for fixed $s_2$, $f_{\tilde{S}_2|\tilde{X}}(s_2|x)=0$ if $x<s_2-0.25$ or $x>s_2+0.25$

$$\underline{f_{\tilde{S}_1,\tilde{S}_2}(7,7.1)} = \int_5^{10} \underbrace{f_{\tilde{X}}(x)}_{\frac{1}{5}} \cdot \underbrace{f_{\tilde{S}_1|\tilde{X}}(7|x)}_{2} \cdot \underbrace{f_{\tilde{S}_2|\tilde{X}}(7.1|x)}_{2}\, dx$$

(using conditional independence)

$$= \int_{\max(5,\,6.75,\,6.85)}^{\min(10,\,7.25,\,7.35)} \frac{4}{5}\, dx = \int_{6.85}^{7.25} \frac{4}{5}\, dx = \frac{8}{25}$$

$$\therefore f_{\tilde{X}|\tilde{S}_1,\tilde{S}_2}(x|7,7.1) = \frac{f_{\tilde{X},\tilde{S}_1,\tilde{S}_2}(x,7,7.1)}{f_{\tilde{S}_1,\tilde{S}_2}(7,7.1)} = \frac{f_{\tilde{X}}(x)\times f_{\tilde{S}_1|\tilde{X}}(7|x)\cdot f_{\tilde{S}_2|\tilde{X}}(7.1|x)}{f_{\tilde{S}_1,\tilde{S}_2}(7,7.1)}$$

$$= \begin{cases} \frac{4/5}{8/25} = \frac{5}{2} & \text{if } 6.85\le x\le 7.25 \\ 0 & \text{otherwise} \end{cases}$$

(c) For fixed $s_1$, $f_{\tilde{s}_1|\tilde{x}}(s_1|x) = 0$ if $x < s_1 - 0.25$ or $x > s_1 + 0.25$

Also, for fixed $s_2$, $f_{\tilde{s}_2|\tilde{x}}(s_2|x) = 0$ if $x < s_2 - 0.25$ or $x > s_2 + 0.25$

$$f_{\tilde{s}_1,\tilde{s}_2}(s_1, s_2) = \int_5^{10} f_{\tilde{x}}(x) \cdot f_{\tilde{s}_1|\tilde{x}}(s_1|x) \cdot f_{\tilde{s}_2|\tilde{x}}(s_2|x)\, dx$$

$$= \int_{\max(5,\, s_1-0.25,\, s_2-0.25)}^{\min(10,\, s_1+0.25,\, s_2+0.25)} \frac{4}{5}\, dx$$

$$= \begin{cases} \frac{4}{5}\left(\min(10, s_1+0.25, s_2+0.25) - \max(5, s_1-0.25, s_2-0.25)\right) & \text{if } 4.75 \leq s_1, s_2 \leq 10.25 \text{, and } |s_2 - s_1| \leq 0.5 \\ 0 & \text{otherwise} \end{cases}$$

(f) No, two measurements are not independent.

Giving counter example, $f_{\tilde{s}_1,\tilde{s}_2}(5, 10) = 0$ while

$f_{\tilde{s}_1}(5) \neq 0$ and $f_{\tilde{s}_2}(10) \neq 0$

It is intuitively correct, since $\tilde{s}_1$ provides information about the location of $\tilde{x}$, which in turn provides information about $\tilde{s}_2$.

4. Create independent samples from a uniform distribution in [0,1]. By using rejection sampling method, count if sample is in triangle and reject if sample is out of triangle. By repeating this, we can simulate joint pdf $f_{\tilde{x}}(x)$ that is uniformly distributed in the shaded region (triangle).

This makes sense because we are actually doing conditional probability which is getting $(\tilde{u}_1, \tilde{u}_2)$ given the shaded region.

Samples included in the shaded region would be normalized by the shaded region, and since total density is constant, density in the triangle would be also constant.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

cnt = 0
u1,u2 = [],[]

while cnt < 1000:
    x, y = np.random.rand(2)

    if y <= 2*x and y <= 2*(1-x):
        u1.append(x)
        u2.append(y)
        cnt += 1
```

In [2]:
```
plt.scatter(u1, u2)
plt.show()
```