Forecasting Time Series Homework 7

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Recall that $\{x_t\}$ is a martingale if $E[x_{n+h} | x_n, x_{n-1}, \cdots] = x_n$ for all n and for all lead times $h > 0$. Actually, to establish that $\{x_t\}$ is a martingale, one simply needs to prove the above formula for $h = 1$ since it can be shown that if it holds for $h = 1$ it must hold for all $h > 0$.

1) Suppose $x_t = x_{t-1} + \varepsilon_t$ where $\varepsilon_t = e_t + \beta e_{t-1} e_{t-2}$, $\beta \neq 0$, and $\{e_t\}$ is strict white noise.

a) What is the best linear predictor of x_{n+1} based on $x_n, x_{n-1}, \cdots$? Justify your answer.

Ans) 1-a) $x_t = x_{t-1} + \varepsilon_t$,

where $\varepsilon_t = e_t + \beta e_{t-1} e_{t-2}$, $\beta \neq 0$, $\{e_t\}$ is strict white noise.

→ Best linear predictor of x_{n+1} based on $x_n, x_{n-1}, \cdots$

The best linear forecast $f_{n,1}^{lin}$ is defined by two characteristics,

(1) $f_{n,1}^{lin}$ is a linear combination of $x_n, x_{n-1}, \cdots$

⇒ As we are forecasting by linear prediction, and are using currently available information, we meet this condition.

(2) The forecast error $x_{n+1} - f_{n,1}^{lin}$ is uncorrelated with all linear combinations of $x_n, x_{n-1}, \cdots$.

⇒ We can write $x_{n+1} = x_n + \varepsilon_{n+1}$, and say that ε_{n+1} is uncorrelated with all linear combination of $x_n, x_{n-1}, \cdots$ where ε_t is uncorrelated zero mean white noise

∴ $f_{n,1}^{lin} = x_n$, and is the best possible linear forecast.

b) What is the best possible predictor of x_{n+1} based on x_n , x_{n-1} , $\cdots$? Justify your answer.

1-b) Best possible predictor of x_{n+1} based on $x_n, x_{n-1}, \ldots$?

The optimal forecast, in terms of mean squared error, is the conditional expectation, $E[x_{n+h} \mid x_n, x_{n-1}, \cdots]$.

As we have AR(1) model $x_{t+1} = x_t + \varepsilon_{t+1}$ $(x_t = x_{t-1} + \varepsilon_t)$, conditional mean can be written as:

$$E[x_{n+1} \mid x_n, x_{n-1}, \cdots] = x_n + E[\varepsilon_{n+1} \mid x_n, \cdots]$$

As $\varepsilon_t = e_t + \beta e_{t-1} e_{t-2}$, where $\{e_t\}$ is strict white noise, then $\{\varepsilon_t\}$ is zero mean white noise but can be predicted by the non-linear formula $\varepsilon_t = e_t + \beta e_{t-1} e_{t-2}$.

Then, the conditional expectation for ε_{n+1} is not zero and therefore the best possible predictor of x_{n+1} is

$$E[x_{n+1} \mid x_n, x_{n-1}, \cdots] = x_n + \underbrace{E[\varepsilon_{n+1} \mid x_n, \cdots]}_{\neq 0}$$

As $\varepsilon_t = e_t + \beta e_{t-1} e_{t-2}$,

$\varepsilon_{t+1} = e_{t+1} + \beta e_t e_{t-1}$.

Since $\{e_t\}$ are strict white noise, $\{e_t\}$ are strictly independent and e_{t+1} is not forecastable.

So, we may replace e_{t+1} by its mean zero to obtain the best forecast $f_{n,1} = \beta e_t e_{t-1}$

$$E[x_{n+1} \mid x_n, x_{n-1}, \cdots] = x_n + \underbrace{E[\varepsilon_{n+1} \mid x_n, \cdots]}_{\neq 0}$$

$$= x_n + \beta e_t e_{t-1}$$

c) Compare your answers to a) and b) to decide whether $\{x_t\}$ is a martingale. (Keep in mind the discussion at the top of this handout).

1-c) $\{x_t\}$ is not a martingale.

$\{x_t\}$ is a martingale if $E[x_{n+1} / x_n, x_{n-1}, \cdots] = x_n$

However, as shown in 1-b), the best possible predictor of x_{n+1} : $E[x_{n+1} / x_n, x_{n-1}, \cdots] \neq x_n$

Therefore, $\{x_t\}$ is not a martingale.

2) Suppose $x_t = \alpha x_{t-1} + e_t$ where $\{e_t\}$ is strict white noise.

a) If $|\alpha| < 1$, is $\{x_t\}$ a martingale? Justify your answer.

$x_t = \alpha x_{t-1} + e_t$ where $\{e_t\}$ is strict white noise.

We know that $x_t = E[x_t \mid x_{t-1}, x_{t-2}, \cdots]$.

Combining two equations above gives

$$E[x_t \mid x_{t-1}, x_{t-2}, \cdots] = \alpha x_{t-1} + \underbrace{E[\varepsilon_t \mid x_{t-1}, x_{t-2}, \cdots]}_{=0 \text{ (strict white noise)}}$$

(To become a martingale, we have to satisfy $E[x_{n+h} \mid x_n, x_{n-1}, \cdots] = x_n$ for all h)

If $|\alpha| < 1$,

$E[x_t \mid x_{t-1}, x_{t-2}, \cdots] \neq x_{t-1}$

$\Rightarrow$ $\{x_t\}$ is not a martingale.

b) If $\alpha = 1$, is $\{x_t\}$ a martingale? Justify your answer.

If $\alpha = 1$, using the same reasoning above,

$E[x_t \mid x_{t-1}, x_{t-2}, \cdots] = x_{t-1}$

$\Rightarrow$ $\{x_t\}$ is a martingale.

3) Suppose $\{\varepsilon_t\}$ are martingale differences. Suppose we "integrate" $\{\varepsilon_t\}$ to obtain a series $\{y_t\}$. Specifically, define $y_1 = \varepsilon_1$, $y_2 = \varepsilon_1 + \varepsilon_2$, etcetera.

a) Show that $y_t = y_{t-1} + \varepsilon_t$.

3-a) $y_1 = \varepsilon_1$, $y_2 = \varepsilon_1 + \varepsilon_2$

$$y_t = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{t-1} + \varepsilon_t$$

$$- \quad y_{t-1} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{t-1}$$

$$y_t - y_{t-1} = \varepsilon_t$$

$$\therefore \; y_t = y_{t-1} + \varepsilon_t$$

b) Use the results of a) to show that $\{y_t\}$ is a martingale. (Thus, integrating a martingale difference yields a martingale.)

3-b)

Recall that $\{y_t\}$ is a martingale if $E[y_{n+h} \mid y_n, y_{n-1} \cdots] = y_n$ for all n and for all lead times $h > 0$,

Also, from 3-a) $y_t = y_{t-1} + \varepsilon_t$,

this implies that $y_{t+1} = y_t + \varepsilon_{t+1}$.

The best possible forecast of y_{t+1} given the available information

$$E[y_{t+1} \mid y_t, y_{t-1} \cdots] = y_t + E[\varepsilon_{t+1} \mid y_t, y_{t-1} \cdots]$$

As $\{\varepsilon_t\}$ are martingale differences,

$$E[\varepsilon_{t+1} \mid \varepsilon_t, \varepsilon_{t-1}, \cdots] = 0$$

$$E[\varepsilon_{t+1} \mid y_t, y_{t-1}, \cdots] = 0,$$

$$E[y_{t+1} \mid y_t, y_{t-1}, \cdots] = y_t + 0 = y_t.$$

$\therefore$ $\{y_t\}$ is a martingale