

Homework 0

Solutions

1. (Sets)

- (a) True. For $i \neq j$ we have $(S_i \cap A) \cap (S_j \cap A) = \emptyset$ because if it is not empty, then there exists an element that is both in S_i and S_j , and this would contradict $S_i \cap S_j = \emptyset$. We still need to establish that $\cup_i S_i \cap A = A$. To prove this we show that the sets contain each other. $\cup_i S_i \cap A \subseteq A$ because every element in $S_i \cap A$ is in A . Now assume that there is an element in A that is not in $\cup_i S_i \cap A$. Then the element would not belong to any S_i , but this would imply that it does not belong to Ω , which is impossible because $A \subseteq \Omega$. This implies $A \subseteq \cup_i S_i \cap A$ and concludes the proof.
- (b) False. Let the universe $\Omega := \{1, 2\}$, $A := \{1\}$, and $B := \{2\}$. Then $A^c \cup B^c = \{2\} \cup \{1\} = \Omega$, but $(A \cup B)^c = \Omega^c = \emptyset$.
- (c) False. Let A and B be nonempty, $A = B$, and $A \cap C = \emptyset$. Then $(A \cup B) \cap C = A \cap C = \emptyset$ but $A \cup (B \cap C) = A \cup \emptyset = A$.

2. (Series)

(a)

$$\sum_{i=m}^n r^i = \frac{1}{1-r} \sum_{i=m}^n (r^i - r^{i+1}) \quad (1)$$

$$= \frac{1}{1-r} \left(\sum_{i=m}^n r^i - \sum_{j=m+1}^{n+1} r^j \right) \quad (2)$$

$$= \frac{r^m - r^{n+1}}{1-r}. \quad (3)$$

(b) We need r^{n+1} to remain bounded, so $|r| < 1$.

(c) *Base step:*

For $n = 2$

$$\sum_{i=1}^n i = 3 = \frac{n(n+1)}{2}. \quad (4)$$

Induction step:

Assume

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}, \quad (5)$$

then

$$\sum_{i=1}^n i = \frac{n(n-1)}{2} + n \quad (6)$$

$$= \frac{n^2 + n}{2} \quad (7)$$

$$= \frac{n(n+1)}{2}. \quad (8)$$

3. (Derivatives)

- (a) The change in the function over an interval of length h is given by $f(x+h) - f(x)$. Dividing by the length of the interval yields the rate of change. Taking the limit as $h \rightarrow 0$ makes it instantaneous.

(b)

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (9)$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \quad (10)$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \quad (11)$$

$$= \lim_{h \rightarrow 0} 2x + h \quad (12)$$

$$= 2x. \quad (13)$$

- (c) To ensure the derivative is the same we need $a := f'(y)$, since

$$L'_y(y) := a. \quad (14)$$

To ensure that the value is the same, we need $b := f(y) - f'(y)y$ because

$$L_y(y) := f'(y)y + b \quad (15)$$

$$= f(y). \quad (16)$$

This yields

$$L_y(x) := f'(y)(x - y) + f(y). \quad (17)$$

- (d) By the chain rule, we have

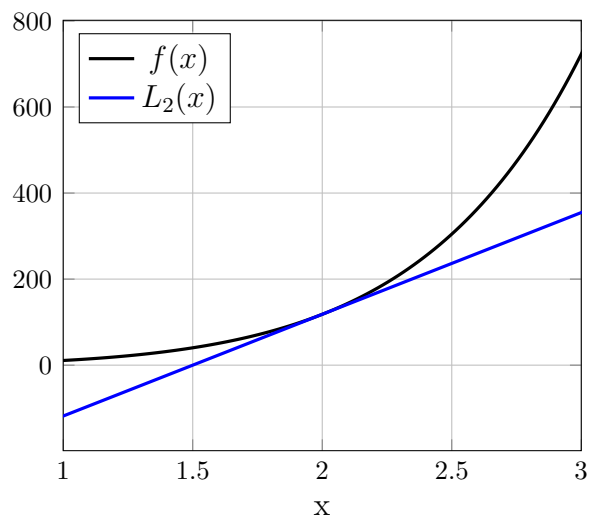
$$f'(x) = (4x^2 + 8x)e^x, \quad (18)$$

so

$$L_2(x) = 32e^2(x - 2) + 16e^2 \quad (19)$$

$$= 16e^2(2x - 3). \quad (20)$$

Here is the plot



4. (Integrals) We will use integrals to compute probabilities and expectations related to continuous quantities.

- (a) The quadratic on the upper right quadrant must equal $(x-1)^2$ to go through $(0,1)$, $(1/2,1/4)$, and $(1,0)$. By symmetry, the area below this curve and above the horizontal axis between 0 and 1 is equal to one fourth of the area of the shape. As a result, we have that the area equals,

$$4 \int_0^1 (x-1)^2 dx = 4 \int_0^1 x^2 - 2x + 1 dx \quad (21)$$

$$= 4 \left(\frac{1}{3} - 1 + 1 \right) \quad (22)$$

$$= \frac{4}{3}. \quad (23)$$

- (b) Let $u = x^2$, then

$$f(t) := \int_0^t \frac{x}{1+x^2} dx \quad (24)$$

$$= \frac{1}{2} \int_0^{t^2} \frac{1}{1+u} du \quad (25)$$

$$= \frac{\ln(1+t^2)}{2}. \quad (26)$$