HW 1 Yoon Tae Park (yp2201@nyu.edu)

1. (a) True, since A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$

Also, $P(B) = P(B \cap A) + P(B \cap A^c)$

$= P(B) \cdot P(A) + P(B \cap A^c)$

$\Rightarrow P(B)(1 - P(A)) = P(B \cap A^c)$

$\Rightarrow P(B) \cdot P(A^c) = P(B \cap A^c)$

$\boxed{\therefore A^c \text{ and } B \text{ are independent}}$

(b) False, Suppose you flip a fair coin twice.

Let A: the event that head occurs for first coin flip, (1H)

B: the event that tail occurs for first coin flip, or head occurs for second coin flip (1T or 2H)

C: the event that head occurs for second coin flip (2H)

First, $P(A \mid B, C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)} = \dfrac{(1H \text{ and } 2H)}{(2H)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$P(A \mid C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{(1H \text{ and } 2H)}{(2H)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

So, A and B are conditionally independent given C.

Next, $P(A \mid B, C^c) = \dfrac{P(A \cap B \cap C^c)}{P(B \cap C^c)} = \dfrac{(1H \text{ and } (1T \text{ or } 2H) \text{ and } (\neq 2H))}{((1T \text{ or } 2H) \text{ and } (\neq 2H))} = \dfrac{\phi}{(1T \text{ and } 2T)}$

($\because$ $A \cap B = 1H$ and $2H$, but cannot add $(\neq 2H)$) $= 0$

$P(A \mid C^c) = \dfrac{P(A \cap C^c)}{P(C^c)} = \dfrac{1H \text{ and } (\neq 2H)}{(2T)} = \dfrac{1H \text{ and } 2T}{2T} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

Since $P(A \mid B, C^c) \neq P(A \mid C^c)$,

A and B are not conditionally independent given C^c.

c) True, Any Events (i.e. A and B) in a partiton are disjoint.

So, $P(A \cap B) = 0$.

If A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$.

Since $P(A) \neq 0$, $P(B) \neq 0$ but $P(A \cap B) = 0$ so equation is incorrect.

Therefore, events in a partition cannot be independent.

d) True. Since $P(A|B) = 1$, $\dfrac{P(A \cap B)}{P(B)} = 1$, $P(A \cap B) = P(B)$

$$P(B^c|A^c) = \frac{P(B^c \cap A^c)}{P(A^c)} = \frac{P((A \cup B)^c)}{P(A^c)} \quad (\because \text{by de morgan's law})$$

$$= \frac{1 - P(A \cup B)}{1 - P(A)} = \frac{1 - (P(A) + P(B) - P(A \cap B))}{1 - P(A)}$$

$$= \frac{1 - (P(A) + P(B) - P(B))}{1 - P(A)} = \frac{1 - P(A)}{1 - P(A)} = 1$$

e) True. $P(B|A \cup B) = P(B \cap A | A \cup B) + P(B \cap A^c | A \cup B)$

$$= \frac{P((B \cap A) \cap (A \cup B))}{P(A \cup B)} + \frac{P((B \cap A^c) \cap (A \cup B))}{P(A \cup B)}$$

(since, $P(B/A) = P(B|(A \cup B) \cap A)$, change formula as below)

$$= \frac{P((B \cap A) \cap (A \cup B))}{P((A \cup B) \cap A)} \cdot \frac{P((A \cup B) \cap A)}{P(A \cup B)} + \frac{P((B \cap A^c) \cap (A \cup B))}{P((A \cup B) \cap A^c)} \cdot \frac{P((A \cup B) \cap A^c)}{P(A \cup B)}$$

$$= P(B|(A \cup B) \cap A) \cdot P(A|A \cup B) + P(B|(A \cup B) \cap A^c) \cdot P(A^c|A \cup B)$$

(Since $(A \cup B) \cap A = A$, $(A \cup B) \cap A^c = B \cap A^c$)

$$= P(B|A) \cdot P(A|A \cup B) + P(B|B \cap A^c) \cdot P(A^c|A \cup B)$$

(also, $P(B|B \cap A^c) = \dfrac{P(B \cap B \cap A^c)}{P(B \cap A^c)} = 1$)

$$= P(B|A) \cdot P(A|A \cup B) + P(A^c|A \cup B)$$

$$\geq P(B|A) \cdot P(A|A \cup B) + P(B|A) \cdot P(A^c|A \cup B) \quad (\because P(B|A) \leq 1)$$

$$= P(B|A)(P(A|A \cup B) + P(A^c|A \cup B)) = P(B|A)$$

2. (a) -1) For F_A to be a valid σ-algebra, we need to prove below

i) $\Omega_A \in F_A$, set of $A \cap F$ should be partition of every A.

and contains total A.

$\therefore \quad A = \Omega_A \in F_A$.

ii) If $B \in F_A$, then $B^c \in F_A$,

If sample space is A, then $B^c = A - B$.

If $B \in F_A$, there is some set $S \in F$ such that $B = A \cap S$

Since F is a σ-algebra, $S^c \in F$.

$A = (A \cap S) + (A \cap S^c) = B + (A \cap S^c)$

$\Rightarrow \quad A - B = A \cap S^c$

$\therefore \quad B^c = A - B = A \cap S^c \in F_A$

iii) if $B_1, B_2 \in F_A$, then $B_1 \cup B_2 \in F_A$. Also, if $B_1, B_2, \ldots \in F_A$, then $\bigcup_{i=1}^{\infty} A_i \in F_A$

If $B_1, B_2 \in F_A$, then there exist $S_1, S_2 \in F$ such that $B_1 = A \cap S_1$, $B_2 = A \cap S_2$

Since F is a σ-algebra, $S_1 \cup S_2 \in F$, so $A \cap (S_1 \cup S_2)$ is in F_A

$\therefore \quad A \cap (S_1 \cup S_2) = (A \cap S_1) \cup (A \cap S_2) = B_1 \cup B_2 \in F_A$

By the same argument for the finite case,

if $B_1, B_2, \ldots \in F_A$, then $\bigcup_{i=1}^{\infty} A_i \in F_A$.

-2) Let $B = A \cap S$, then $P_A(S \cap A) = P_A(B) = \dfrac{P(B)}{P(A)}$

i) $P_A(B) \geq 0$ for any event $B \in F_A$. ($\because P(B) \geq 0$, $P(A) > 0$)

ii) $P_A(\text{sample space}) = 1$, $\quad P_A(A) = \dfrac{P(A)}{P(A)} = 1$

iii) If $B_1, B_2, \cdots B_n \in F_A$ are disjoint, then $P(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} P(B_i)$

(i.e $P(B_1 \cup B_2) = P(B_1) + P(B_2) - \underbrace{P(B_1 \cap B_2)}_{0} = P(B_1) + P(B_2)$)

$$P_A(\bigcup_{i=1}^{n} B_i) = \frac{P(\bigcup_{i=1}^{n} B_i)}{P(A)} = \frac{\sum_{i=1}^{n} P(B_i)}{P(A)} = \sum_{i=1}^{n} P_A(B_i).$$

By the same argument for the finite case,

for a countably infinite disjoint sets $B_1, B_2, \ldots \in F_A$,

$P(\lim_{n\to\infty} \bigcup_{i=1}^{n} B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i)$.

2-(b) Yes, P should satisfy the conditions below

i) $P(B) \geq 0$ for any event $B \in F$.

(∵ numerator and denominator are both non-negative)

ii) P(sample space) $= 1$, $P(\Omega) = \dfrac{\#\text{ of data points with value in } \Omega}{N} = 1$

iii) If $S_1, S_2, \ldots, S_n \in F$ are disjoint, then $P(\bigcup_{i=1}^{n} S_i) = \sum_{i=1}^{n} P(S_i)$

$$P\left(\bigcup_{i=1}^{n} S_i\right) = \frac{\#\text{ of data points with value } \bigcup_{i=1}^{n} S_i}{N}$$

$$= \frac{\#\text{ of data points with value in } S_1 + \#\text{ of} \ldots S_2 + \cdots + \#\text{ of} \ldots S_n}{N}$$

$$= \sum_{i=1}^{n} P(S_i)$$

By the same argument for the finite case,

for a countably infinite disjoint sets $S_1, S_2, \ldots \in F$,

$$P\left(\lim_{n \to \infty} \bigcup_{i=1}^{n} S_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} P(S_i)$$

3. (a) Yes, since employees have not been in contact with each other, it is reasonable to assume that Test i only depends on particular employee i.

(b) Define I_i ($1 \leq i \leq 10$) = each employee i being ill

T_i ($1 \leq T \leq 10$) = each employee i test is positive

The event that at least one test is positive is $\bigcup_{i=1}^{10} T_i$.

$$P\left(\bigcup_{i=1}^{10} T_i\right) = 1 - P\left(\left(\bigcup_{i=1}^{10} T_i\right)^c\right) = 1 - P\left(\bigcap_{i=1}^{10} T_i^c\right) \text{ (by DeMorgan's laws)}$$

~~Since~~ $$P\left(\bigcap_{i=1}^{10} T_i^c\right) = \prod_{i=1}^{10} P(T_i^c) \quad (\because T_i \text{ is independent})$$

$$= \prod_{i=1}^{10} \left(P(I_i) P(T_i^c | I_i) + P(I_i^c) P(T_i^c | I_i^c) \right)$$

$\left(\begin{array}{l} P(I_i) = 0.01 \\ P(I_i^c) = 0.99 \end{array}, \begin{array}{l} P(T_i | I_i) = 0.98, \\ P(T_i^c | I_i) = 0.02, \end{array} \begin{array}{l} P(T_i | I_i^c) = 0.05 \\ P(T_i^c | I_i^c) = 0.95 \end{array} \right)$

$$= \left((0.01)(0.02) + (0.99)(0.95)\right)^{10}$$

$$\fallingdotseq 0.543$$

$$\boxed{\therefore P\left(\bigcup_{i=1}^{10} T_i\right) \fallingdotseq 1 - 0.543 \fallingdotseq 0.457}$$

(c) $$P\left(\bigcap_{i=1}^{10} I_i^c \,\middle|\, \bigcup_{j=1}^{10} T_j\right) = \frac{P\left(\bigcap_{i=1}^{10} I_i^c, \bigcup_{j=1}^{10} T_j\right)}{P\left(\bigcup_{j=1}^{10} T_j\right)}$$

$P\left(\bigcup_{j=1}^{10} T_j\right) = 0.457$ (3-b))

$$P\left(\bigcap_{i=1}^{10} I_i^c, \bigcup_{j=1}^{10} T_j\right) = P\left(\bigcap_{i=1}^{10} I_i^c\right) \cdot P\left(\bigcup_{j=1}^{10} T_j \,\middle|\, \bigcap_{k=1}^{10} I_k^c\right)$$

$$= \prod_{i=1}^{10} P(I_i^c) \times \left(1 - P\left(\bigcap_{j=1}^{10} T_j^c \,\middle|\, \bigcap_{k=1}^{10} I_k^c\right)\right)$$

Assumption) T_j^c is conditionally independent of T_i^c, $j \neq i$, conditioned on $\bigcap_k I_k^c$

since T_j^c doesn't provide information about T_i^c.

$$P\left(\bigcap_{j=1}^{10} T_j^c \,\middle|\, \bigcap_{k=1}^{10} I_k^c\right) = \prod_{j=1}^{10} P\left(T_j^c \,\middle|\, \bigcap_{k=1}^{10} I_k^c\right)$$

Assumption2) T_i is conditionally independent of I_j^c $(i \neq j)$, conditioned on I_i

since T_i only depends on I_i.

$$\prod_{j=1}^{10} P\left(T_j^c \,\middle|\, \bigcap_{k=1}^{10} I_k^c\right) = \prod_{j=1}^{10} P(T_j^c \mid I_j^c)$$

$$\therefore P\left(\bigcap_{i=1}^{10} I_i^c, \bigcup_{j=1}^{10} T_j\right) = \frac{\prod_{i=1}^{10} P(I_i^c) \times \left(1 - \prod_{j=1}^{10} P(T_j^c \mid I_j^c)\right)}{P\left(\bigcup_{j=1}^{10} T_j\right)}$$

$$= \frac{(0.99)^{10} \times (1 - (0.95)^{10})}{0.457} \doteq \boxed{0.793}$$

4. (a) Total count : $2^5 = 32$.

$x=1)$

i) H:1	ii) H:2	iii) H:3
HTTTT	HTHTT	HTHTH
THTTT	HTTHT	
TTHTT	HTTTH	
TTTHT	THTHT	
TTTTH	THTTH	
	TTHTH	
5	6	1

$12/32 \doteqdot 0.375$

$x=2)$

i) H:2	ii) H:3	iii) H:4
HHTTT	HHTHT	HHTHH
THHTT	HHTTH	
TTHHT	THHTH	
TTTHH	HTHHT	
	HTTHH	
4	THTHH	1
	6	

$11/32 = 0.344$

$x=3)$

i) H:3	ii) H:4
HHHTT	HHHTH
THHHT	HTHHH
TTHHH	
3	2

$5/32 \doteqdot 0.156$

$x=4)$

i) H:4
HHHHT
THHHH
2

$2/32 \doteqdot 0.063$

$x=5)$ HHHHH

1

$1/32 = 0.031$

$x=0)$ TTTTT

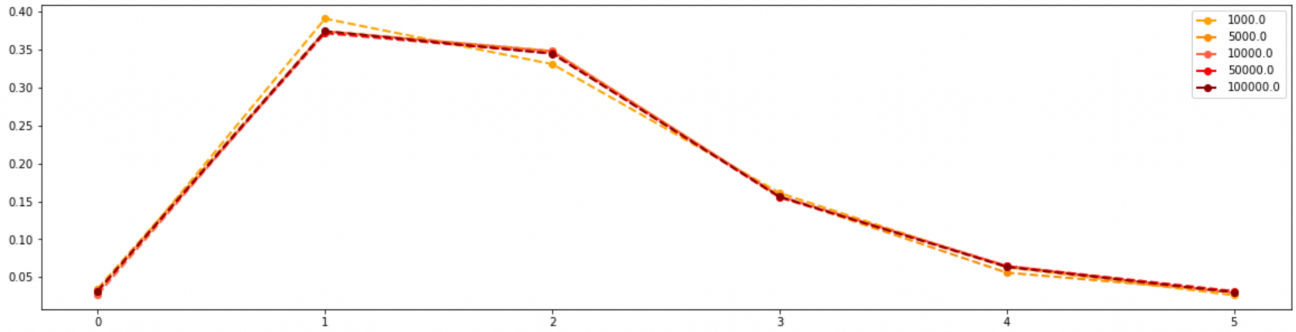
1

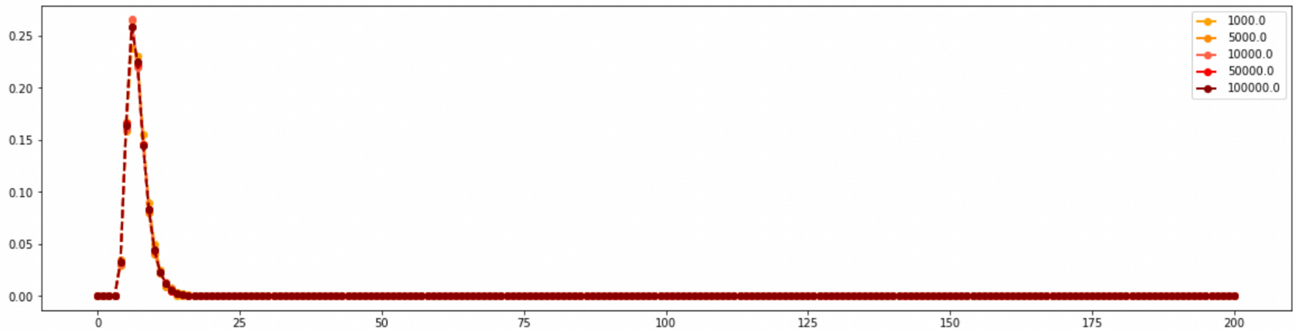
In [15]:
```
# 4-b) probability is similar to 4-a)
print(p_longest_streak(5, 100000)[1:])
```

```
[0.37544, 0.34286, 0.15678, 0.06302, 0.03089]
```

In [3]:
```
# 4-b) code
import numpy as np
import matplotlib.pyplot as plt
plt.close("all")
np.random.seed(2017)

def p_longest_streak(n, tries):
    # initialize list
    result = [0] * (n+1)

    # iterate # of tries
    for _ in range(tries):
        # initialize continue_count and maxCount
        continue_cnt = 0
        maxCount = 0

        # iterate flip by n times
        for i in range(n):
            # create random coin
            coin = np.random.random()

            # if coin >= 0.5, then add to continuous count
            if coin >= 0.5:
                continue_cnt += 1

            # if not, cnt is not continuous so cnt = 0
            else:
                continue_cnt = 0

            # put into maxCount before next iteration
            maxCount = max(maxCount, continue_cnt)

        # add values into result list
        result[maxCount] += 1

        # divide by total probability to get probability
        result_prob = [result[i] / tries for i in range(len(result))]

    # return list of probability
    return result_prob

n_tries = [1e3,5e3,1e4,5e4,1e5]
n_vals = [5,200]

color_array = ['orange','darkorange','tomato','red', 'darkred', 'tomato', 'purple', 'grey', 'deepskyblue',
               'maroon','darkgray','darkorange', 'steelblue', 'forestgreen', 'silver']
for ind_n in range(len(n_vals)):
    n = n_vals[ind_n]
    plt.figure(figsize=(20,5))
    for ind_tries in range(len(n_tries)):
        tries = n_tries[ind_tries]
        print ("tries: " + str(tries))
        p_longest_tries = p_longest_streak(n, int(tries))
        plt.plot(range(n+1),p_longest_tries, marker='o',markersize=6,linestyle="dashed",lw=2,
                 color=color_array[ind_tries],
                 markeredgecolor= color_array[ind_tries],label=str(tries))
    plt.legend()
```

```
tries: 1000.0
tries: 5000.0
tries: 10000.0
tries: 50000.0
tries: 100000.0
tries: 1000.0
tries: 5000.0
tries: 10000.0
tries: 50000.0
tries: 100000.0
```





In [4]:
```
# 4-c) about 0.32
# The program may not be generating truly random sequences,
# because it may follow the same and repeated algorithm to generate random numbers.
# (starting from the same seed)
# So the numbers it produces may not be truly random.

print ("The probability that the longest streak of ones in a Bernoulli iid sequence of length 200 has length 8 or mo
print (np.sum(p_longest_streak(200, 100000)[8:])) # Compute the probability and print it here
```

```
The probability that the longest streak of ones in a Bernoulli iid sequence of length 200 has length 8 or more is
0.32003000000000004
```