

Optimal control of Hybrid Quantum Systems

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Abstract

In the field of quantum information and computing, a promising design is that of quantum hybrid systems in which weakly interacting atomic spins are coupled with a cavity of photons which provide effective means of transmission of information. Together they provide a physical basis for which quantum information can be stored and processed. The computational power required for exact simulations of such a system far exceeds that of any computer today, so it is in this project a method that uses the low excitation regime approximation is used to accurately simulate such a system and significantly reduced computational requirement. It is also studied ways in which manipulation of the time varying coupling strength between the spins and the cavity can be used to optimise the fidelity that information is stored.

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1 Summary

In this project the time evolution of a low excitation state in a coupled spin and cavity system is calculated, and its overlap is produced with this initial state in order to study how information in the system is lost and retained. The spin basis vectors and its associated Hamiltonian are first expressed in terms of a new basis system known as the Krylov subspace, where a small number of these basis states can well approximate the information contained in the exact case of the spin vectors. The validity of this approximation is investigated, and it is explored how increasing the number of these Krylov basis states produces results that are expected in the exact case.

With this approximation, the spin system is coupled to a cavity system with a coupling strength whose time dependence can be given. Different functional forms are investigated, as well as the magnitude of this coupling in an attempt to optimise the system to better retain information.

Finally, it is suggested to find a more formal formulation of the method to evaluate the quality with which information is stored, and the implications this might have for possible implementations of machine learning for finding an optimum form for the coupling constant.

2 Introductory Chapter

2.1 Quantum Computing

Over the last 60 to 70 years, there have been steady improvements to the performance of our computers, often driven by breakthroughs in physics which result in better and more innovative engineering solutions. However recent developments in the field of quantum information and computing promises a breakaway from our traditional ideas of computing, and performance capabilities far greater than any traditional computing that exists today. The most notable example of which being Google's own quantum computer, which using 53 qubits, the quantum equivalent of a classical cubit, performed an operation in 200 s, which would have taken today's classical supercomputer around 10000 years [1].

This is an extraordinary achievement, which if scalable would have a tremendous impact on our lives in areas such as our ability to maintain secure communications and modern cryptography methods [2][3]. Already it has been demonstrated the uses of a quantum systems to simulate other quantum systems with the use of superconducting circuits [4][5].

However, the very small number of qubits used is because building a stable quantum computing has proved very difficult since the systems are very sensitive to the environment. Any small amount of interference results in decoherence, ruining the systems ability to store or process information, which is why today a major problem is designing a system in which gates and processing of information can occur quickly, and be stored for a long enough time.

2.2 Quantum Hybrid Systems

A major contender for future designs in the field of quantum computers is that of hybrid quantum systems [2][6][7]. These systems take advantage of the fact that atoms and spins are well decoupled from the environment and exhibit long coherence times, whereas

superconducting circuits or resonators are more strongly coupled to electromagnetic fields but have shorter coherence times [2]. Combined, these two systems interact with some interaction Hamiltonian H_{int} with high fidelity, and the system is able to be controlled very precisely. Already today there have been many demonstrations of this technology, which have also used atoms and solid state elements coupled to some superconducting element, and also the ability to produce this in the form of a circuit board, making the idea of mass production of such element a near possibility [2][6]. The importance and the activity of this research area therefore cannot be understated.

However to study these systems is very difficult, because these systems involve a resonator coupled to a spin system with perhaps hundreds of thousands of spins. The spin system alone would require a matrix of size 2^N , where N would be number of spins in the system. For $N > 100$ on a traditional computer, even any simple calculations or simulations becomes impossible. Given the how difficult and expensive it is to build such systems, having a way to simulate possible designs that is feasible with existing computers would provide a tremendous asset in the developments of the next generation of hybrid quantum systems [1].

2.3 The Aim of the Project

It is therefore in this project that it is demonstrated that these 2^N dimensional matrices in the low excitement regime can be expressed very accurately in terms of a different basis regime, for which there is a linear relationship between N and the size of the matrix which makes higher N calculations significantly more feasible [8]. This new system is evaluated, in particular in terms in comparison to the exact solution, and also in terms of changing the number of new basis states and the how many are required for a good approximation to the exact case. Furthermore, for the case of a resonator coupled to a spin system, different functional forms of the coupling constant $g(t)$ are investigated in terms of optimising the use of the spin system as form of quantum storage.

3 Method Discussion

3.1 The Krylov Subspace

In order to simulate this system, the spin system was first by itself expressed in terms of this new basis, the result of which was compared to an exact simulation. It was also investigated to what degree a small number of Krylov states could be used, but still retain agreement with the exact case. Next the so called Fock space was introduced, or the cavity in which there are photons. This meant introducing the interaction term H_{int} in the system. At this point, an Rk4 integrator was implemented since the errors are of fourth order, better compared to the former diagonalisation method. It was with this new integrator that different functional forms of the coupling constant $g(t)$ could be introduced and compared.

The new basis system introduced is the Krylov subspace. If it is given that the spin-space being dealt with is a low energy state, which in other words only a very small number of spins point up, then the Hilbert space can be expressed in terms of the Krylov subspace, where a much smaller number of basis which can capture the information of a full Hilbert space [8].

To first study the spin system, the Hamiltonian for the spins H_s , needed to be expressed in terms of this new Krylov basis. The following derivation for the Krylov basis states and the associated Hamiltonian is given in full (Himadri. Optimal control of spin ensemble in a cavity) [8], but given here briefly for completeness.

The spin component of the Hamiltonian for a system with N spins is given as

$$H_s = \frac{1}{2} \sum_{i=1}^N \omega_i \sigma_i^z, \quad (1)$$

where ω_i are the spin ensemble transition frequencies and σ_i^z are the spin half Pauli operators. Now consider the state $|\psi\rangle$. The Krylov subspace is spanned by the following vectors:

$$\{|\psi\rangle, H|\psi\rangle, H^1|\psi\rangle, H^2|\psi\rangle, \dots, H^{k-1}|\psi\rangle\}, \quad (2)$$

where the Krylov vectors $|\phi_k\rangle$ are given by making the states in (2) orthogonal. The matrix elements can therefore be expressed as the following

$$|\tilde{\phi}_{n+1}\rangle = H|\phi_n\rangle - \alpha_n|\phi_n\rangle - \beta_n|\phi_{n-1}\rangle, \quad (3)$$

where

$$\alpha_n = \langle \phi_n | H | \phi_n \rangle; \quad \beta_n^2 = \langle \tilde{\phi}_n | \tilde{\phi}_n \rangle. \quad (4)$$

From this definition, the tridiagonal Hamiltonian expressed in the Krylov subspace is given by:

$$H_s = \begin{pmatrix} \alpha_1 & \beta_2 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & \beta_3 & \alpha_3 & \beta_4 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \beta_4 & \alpha_4 & \beta_5 & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_5 & \alpha_5 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \beta_{k-1} & \alpha_{k-1} & \beta_k \\ 0 & 0 & 0 & 0 & 0 & \cdot & 0 & \beta_k & \alpha_k \end{pmatrix}. \quad (5)$$

Given this exact Hamiltonian in the desired basis, the low energy regime can now be considered, with the goal of using a small number of Krylov basis in order to approximate an effective Hamiltonian that is computationally less cumbersome than the exact case. Consider the case where the initial state consists of just a single excitation, which is given by the following:

$$|e_k\rangle = |0^{\otimes p}\rangle \otimes |1\rangle_k \otimes |0^{\otimes N-1-p}\rangle, \quad (6)$$

where the k^{th} spin is excited. Acting the Hamiltonian as given in (5) on this state gives the following

$$\begin{aligned} H_s |e_k\rangle &= \frac{1}{2} \sum_i^N \omega_i \sigma_z |e_k\rangle = \frac{\omega_k}{2} |e_k\rangle - \frac{1}{2} \sum_{i \neq k}^N \omega_i |e_k\rangle \\ &= \left(\omega_k - \sum_i^N \frac{\omega_i}{2} \right) |e_k\rangle = \left(\omega_k - \frac{N}{2} \Omega \right) |e_k\rangle \end{aligned}, \quad (7)$$

where it has been shown that the state (6) is an eigenstate of the Hamiltonian and where the summation $\sum_i^N \frac{\omega_i}{2}$ is replaced by a mean value Ω . The case is then considered when

there is not just an excitation in the k^{th} spin, but when there is an equal superposition of single excitation states given by the following state:

$$|\phi_1\rangle = \frac{1}{\sqrt{N}} \sum_k^N |e_k\rangle, \quad (8)$$

which is not an eigenstate of the Hamiltonian H_s . This Hamiltonian acting on this state gives the following:

$$H_s |\phi_1\rangle = \frac{1}{\sqrt{N}} \sum_k^N \left(\omega_k - \frac{N}{2} \Omega \right) |e_k\rangle, \quad (9)$$

which by using (3) and (4) can be used to derived the next basis:

$$|\phi_2\rangle = \frac{1}{\sigma\sqrt{N}} \sum_k^N (\omega_k - \Omega) |e_k\rangle, \quad (10)$$

where σ is the standard deviation in the frequencies ω_i . Given a basis can be derived from the previous basis, it is now possible to build up the system of basis from this method. However consider now the following basis:

$$|\psi(x)\rangle = \frac{1}{\sqrt{N}} \sum_k^N (\omega_k - \Omega)^x |e_k\rangle. \quad (11)$$

The reason for this choice becomes clear when the moments of a Gaussian distribution are considered as given as the following:

$$E(x) = \frac{1}{N} \sum_k^N (\omega_k - \Omega)^x = \begin{cases} \sigma^x (x-1)!! & \forall \text{ even } x \\ 0 & \forall \text{ odd } x. \end{cases}, \quad (12)$$

where the spin frequencies ω_i are taken to have a Gaussian Distribution with mean Ω and standard deviation σ . Now consider the spin Hamiltonian H_s acting on the state $|\psi(x)\rangle$:

$$H_s |\psi(x)\rangle = |\psi(x+1)\rangle - \frac{N-2}{2} \Omega |\psi(x)\rangle, \quad (13)$$

where the above equation gives the relation $\langle \psi(x) | H_s | \psi(x) \rangle = -\eta E(2x)$ where $\eta = \frac{N-2}{2} \Omega$ and $\langle \psi(x+1) | H_s | \psi(x) \rangle = -E(2x+2)$ since $E(2x+1) = 0$. With these information, the basis states can be derived, not included here, and given as the following:

$$\begin{aligned} |\phi_0\rangle &= |0^{\otimes N}\rangle \\ |\phi_1\rangle &= \frac{1}{\sqrt{N}} \sum_k^N |e_k\rangle = |\psi(0)\rangle \\ |\phi_2\rangle &= \frac{1}{\sqrt{2}\sigma^2} (|\psi(2)\rangle - \sigma^2 |\psi(0)\rangle) \\ |\phi_3\rangle &= \frac{1}{\sqrt{6}\sigma^3} (|\psi(3)\rangle - 3\sigma^2 |\psi(1)\rangle) \\ |\phi_4\rangle &= \frac{1}{\sqrt{24}\sigma^4} (|\psi(4)\rangle - 6\sigma^2 |\psi(2)\rangle + 3\sigma^4 |\psi(0)\rangle) \end{aligned}, \quad (14)$$

which results in the following effective Hamiltonian H_s^{ef} :

$$H_s^{ef} = \begin{pmatrix} -\eta - \Omega & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & -\eta & \sqrt{2}\sigma & 0 & 0 & \cdot & 0 & 0 \\ 0 & \sqrt{2}\sigma & -\eta & \sqrt{3}\sigma & 0 & \cdot & 0 & 0 \\ 0 & 0 & \sqrt{3}\sigma & -\eta & \sqrt{4}\sigma & \cdot & 0 & 0 \\ 0 & 0 & 0 & \sqrt{4}\sigma & -\eta & \cdot & 0 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & -\eta & \sqrt{k}\sigma \\ 0 & 0 & 0 & 0 & 0 & \cdot & \sqrt{k}\sigma & -\eta \end{pmatrix}. \quad (15)$$

From (15), it can be seen that the Hamiltonian depends on the frequency distribution with mean Ω , standard deviation σ , and the number of basis states k which suitably chosen as will be discussed. With this Hamiltonian, the behaviour of the spin system can be approximated in this new basis.

However to introduce a cavity and coupling between this spin system and the cavity, the total Hamiltonian H in the new coupled Hilbert space can be written as the following:

$$H = H_s^{ef} \otimes I + \omega_c I \otimes \hat{a}_c^\dagger \hat{a}_c + g(t) (|\phi_0\rangle\langle\phi_1| \otimes \hat{a}_c^\dagger + |\phi_1\rangle\langle\phi_0| \otimes \hat{a}_c), \quad (16)$$

where ω_c is the resonant frequency of the cavity, and \hat{a}_c^\dagger and \hat{a}_c are the annihilation and creation operators in the photon space. The first term in (16) is simply the Hamiltonian derived in (15), and the second term is Hamiltonian for the cavity spin system, which is in the so called Fock space.

The second line is the interaction term, with strength $g(t)$. The first term in the bracket $|\phi_0\rangle\langle\phi_1| \otimes \hat{a}_c^\dagger$ represents the case in which a photon is annihilated, which results in an increase in excitation of the spin state, and like wise for the second term in the case when a decrease in spin in the spin state results in the creation of photon in the cavity space.

3.2 Calculating the Overlap State $\chi^2(t)$

Given now the full expression for the Hamiltonian, it was then proposed in order to study the ability of the system initially just to store information, which in terms of the initial state $|\psi(t=0)\rangle$, would simply be the overlap between the time evolved state using the Hamiltonian H would be

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \chi^2(t). \quad (17)$$

Clearly, (17) would start at a value of $\chi^2(0) = 1$, and then decrease as information is lost in the system, a result which is discussed in more detail in the next section.

In order to investigate this behaviour, firstly a diagonalisation method was implemented as the following:

Consider the state of the spin system which is given as $|\psi(t)\rangle$, which is expressed in the Krylov basis. The time evolution of the system is given by the time evolution operator \hat{U} acting on the initial state $|\psi(0)\rangle$:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \quad (18)$$

At this point consider the matrix M which contains the eigenvectors of the effective Hamiltonian and the conjugate operator M^\dagger . The diagonalised effective Hamiltonian can be written as:

$$H_D = M^\dagger H M, \quad (19)$$

therefore by applying M^\dagger once to the expression for $|\psi(t)\rangle$,

$$M^\dagger |\psi(t)\rangle = e^{-iM^\dagger H t} M M^\dagger |\psi(0)\rangle, \quad (20)$$

which can be simplified to

$$M^\dagger |\psi(t)\rangle = e^{-iH_d t} M^\dagger |\psi(0)\rangle. \quad (21)$$

Finally if the M matrix operates once again on the left hand side, an expression for $|\psi(t)\rangle$ is left.

$$|\psi(t)\rangle = e^{-iH_D t} |\psi(0)\rangle. \quad (22)$$

The quantity that is wished to be found is given by

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \chi(t), \quad (23)$$

which is a measure of the overlap for the initial state and the evolved state.

Given this method, different k values could be chosen and the result plotted. Its worth noting that for the coupled system, this would result in a square matrix with dimension of $k + 2$. This was because if the initial state contained only one excitation, in either the spin state or the photon space, the dimension of the Fock space at most had to be 2, which corresponded to 0 or 1 photons in the cavity.

While the diagonalisation method was working, the RK4 integrating method was implemented as a replacement. This was because it has a local truncation error of $\mathcal{O}(5)$, which meant when choosing the time interval density, for a similar level of accuracy the RK4 method needed a smaller time interval density which was computationally advantageous.

The final part of this project was to experiment with different functional forms of $g(t)$ in order optimise the systems ability to retain information. As will be discussed, this problem is very difficult, given that there is not an obvious functional form for $g(t)$ and its parameters, neither any obvious physical starting point. However a number of examples were performed and compared to the base case when $g(t) = 1$.

4 Results

For the results obtained, three different stages are presented here. The first stage is the behaviour of the spin system by itself with no interaction terms, with the effect of different k values investigated. Next is the introduction of the cavity system and a non-zero constant interaction term g . And finally, different functional forms and parameters are considered for $g(t)$, and the effect it has on the ability of the spin-cavity system to retain information.

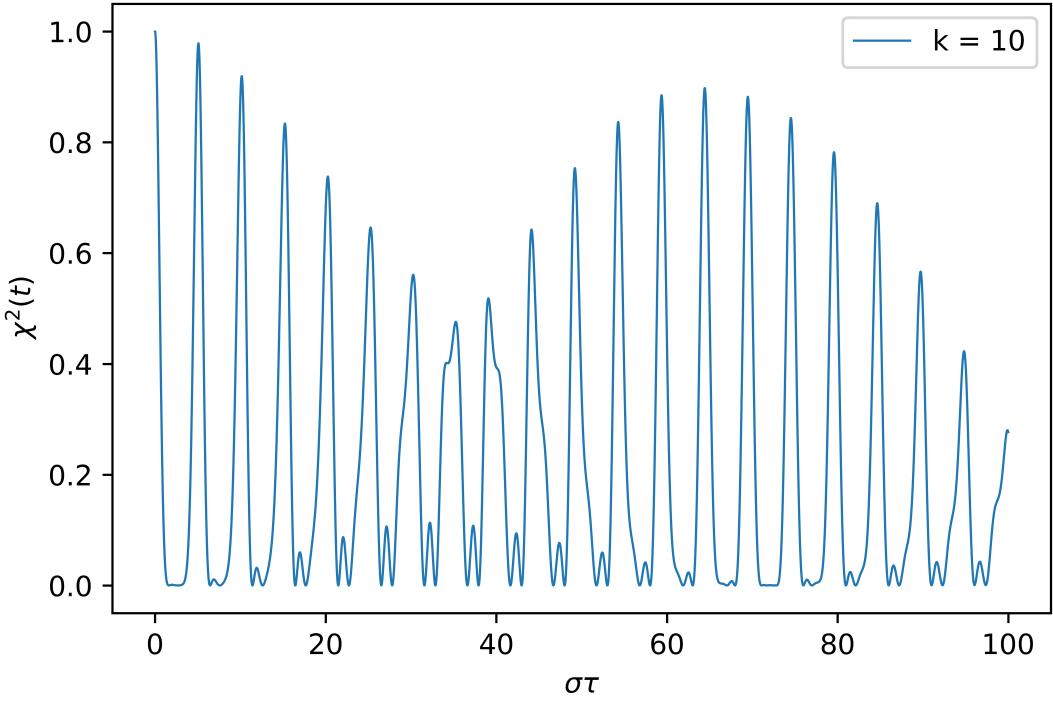


Figure 1: The evolution of the first excited spin state in the Krylov subspace for $k = 10$.

4.1 The Spin System

Initially, just the spin system is considered with the Hamiltonian H_s^{ef} as derived in (15), with k number of Krylov basis. The result is plotted and is shown in Fig. 1, where the measure of the overlap between the initial state $\psi(0)$ and time evolved state $\psi(t)$ which is denoted as $\chi^2(t)$ is plotted the scaled time $\sigma\tau$ for $k = 10$. The use of the scaled time $\sigma\tau$ arises from (15), where a factor of σ is taken outside, however it is just a scaling factor that does not affect the analysis of the results.

As discussed in the method, both a diagonalisation method and a RK4 method was implemented, but for producing these results the RK4 method was used because it was faster, but both produced identical results.

Shown in Fig. 2, is the same simulation but for larger k values. The most striking difference is that the system behaviours do retain the sudden drop initially, but the revival is clearly less oscillatory, and appears to be a decreasing function where each peak is considered, in other words the values of these sudden peaks will not rise again. The second important thing to note is that as k increases, the period of these revivals denoted here as τ_r also increases.

4.2 The Spin-Cavity System

Fig. 3 shows the $\chi^2(t)$ evolution of the spin system that is now coupled to a cavity system of two dimensions with coupling strength $g(t) = 1$. This is shown in orange and the non-coupled state is shown in blue.

The coupled state exhibits the same behaviour of a sudden drop very close to $\sigma\tau = 0$,

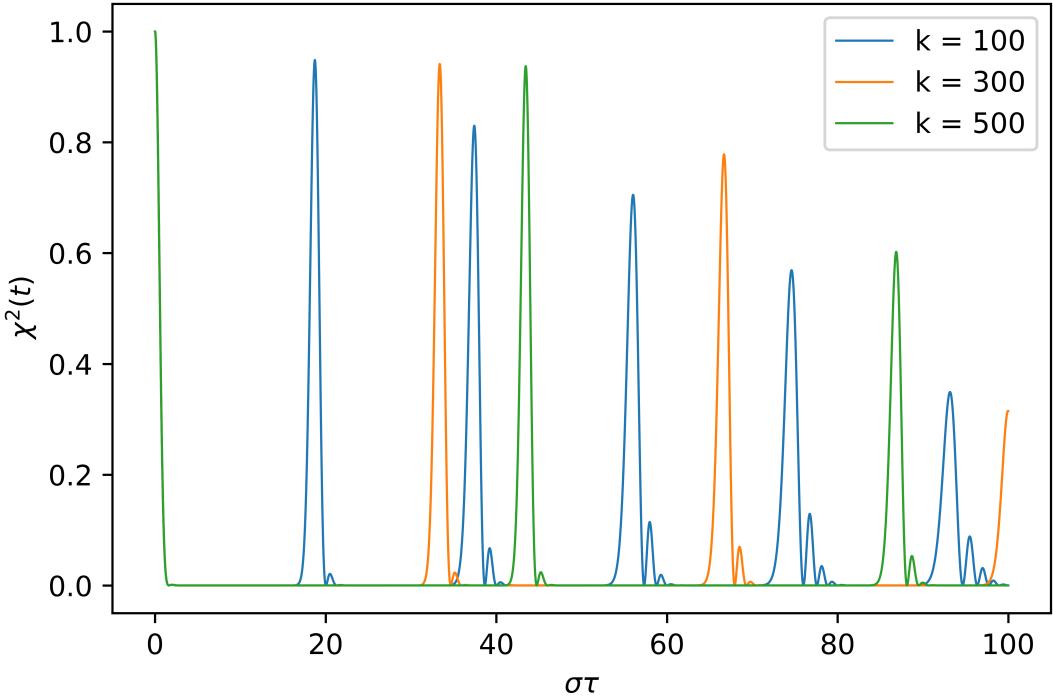


Figure 2: A comparison of how the simulation behaved with increasing k value. Note in particular the lack of such oscillatory behaviour for these large k values.

and then evenly spaced peaks whose height decreases with time. However the important difference to note is that for a comparable time, the height of the peak of the coupled system is always higher, which shows a stronger retention of the initial state, reasons for which will be discussed in the results discussion.

4.3 Different Functional Forms for $g(t)$

Plotted in Fig. 4 are different functional forms of $g(t)$, where $S(t)$ represents a square wave of amplitude and frequency 1, and similarly $T(t)$ represents a saw tooth wave with the same properties. It can be seen that there is similar behaviour amongst the time varying coupling functions with peak like behaviour occurring initially at similar times. However, compared to the case $g(t) = 1$, they all seem to perform worse in the sense that the size of the peaks are always smaller.

Given that the simple functional forms did seem to be an improvement on the case of time invariant constant compiling, effect of changing the coupling strength was investigated as shown in Fig. 5. Its not immediately clear what this relationship is, and it appears changing the strength of the interaction leads to very complex changes. However it can be said that for larger interaction strengths, a much faster oscillating behaviour emerges at a reduced peak height.

However, increasing this coupling strength very quickly results in the behaviour shown in Fig. 6. The result is very rapidly oscillating system, whose peaks decrease very slowly in time. For coupling strengths $g > 5$, very similar behaviour is observed with slower

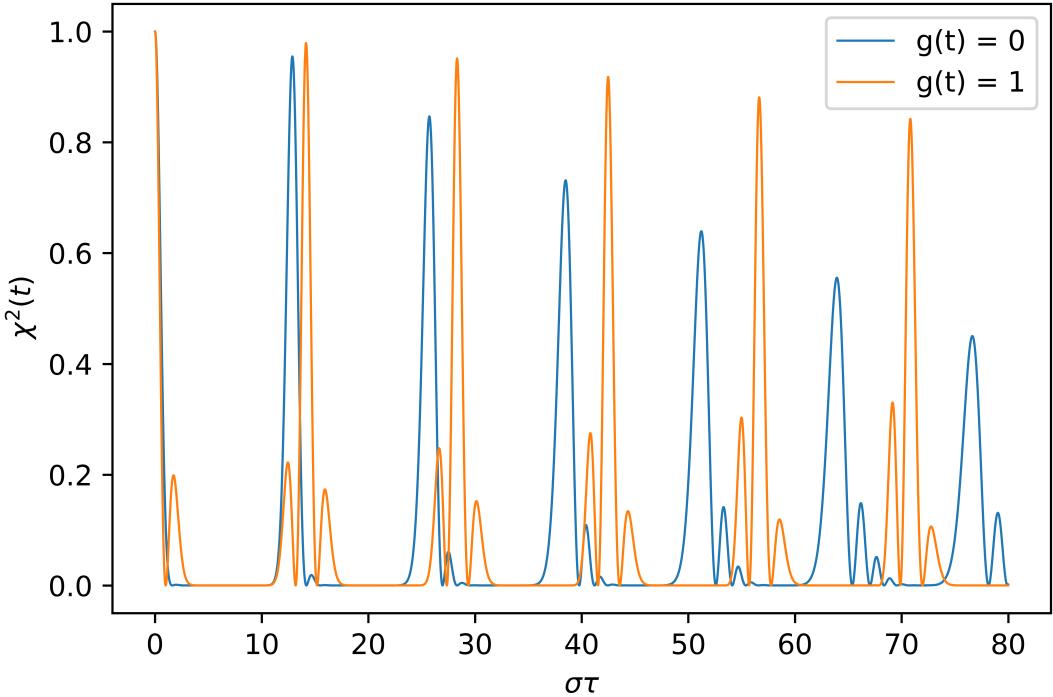


Figure 3: The spin system coupled with the cavity system is shown in orange, and a non-coupled system shown in blue.

decrease in peak height.

One final aspect of the different g values was investigated. Initially, the height of peaks were analysed, since the closer these values were to 1, the higher fidelity that this quantum information could be retained with. Now it is investigated the time it takes for this peak to arise, which is plotted in Fig. 7 and is denoted t_τ . It is observed that the larger the g value, the shorter it takes for this revival, a result that agrees with the observation that larger g values oscillate with a greater frequency.

It is also observed that there appears to be a cut off value of $g \approx 100$ where for any $g > 100$ the revival time does not decrease which indicates that there is some upper limit to the strength coupling constant beyond which there is no advantage to trying increasing this constant.

5 Discussion of Results

5.1 The Spin System

In order to evaluate the results shown in Fig. 1, which shows the time evolution of $\chi^2(t)$ for the spin system alone, it is worth reviewing the physical conditions which are being simulated. It was stated that by implementing a number of Krylov basis k , a simulation of a much larger system of N spins where $N \gg k$ could be simulated. In this case of large N , physically it would be expected that this initial state $|\psi(0)\rangle$ would be very quickly be lost as information is lost into the system.

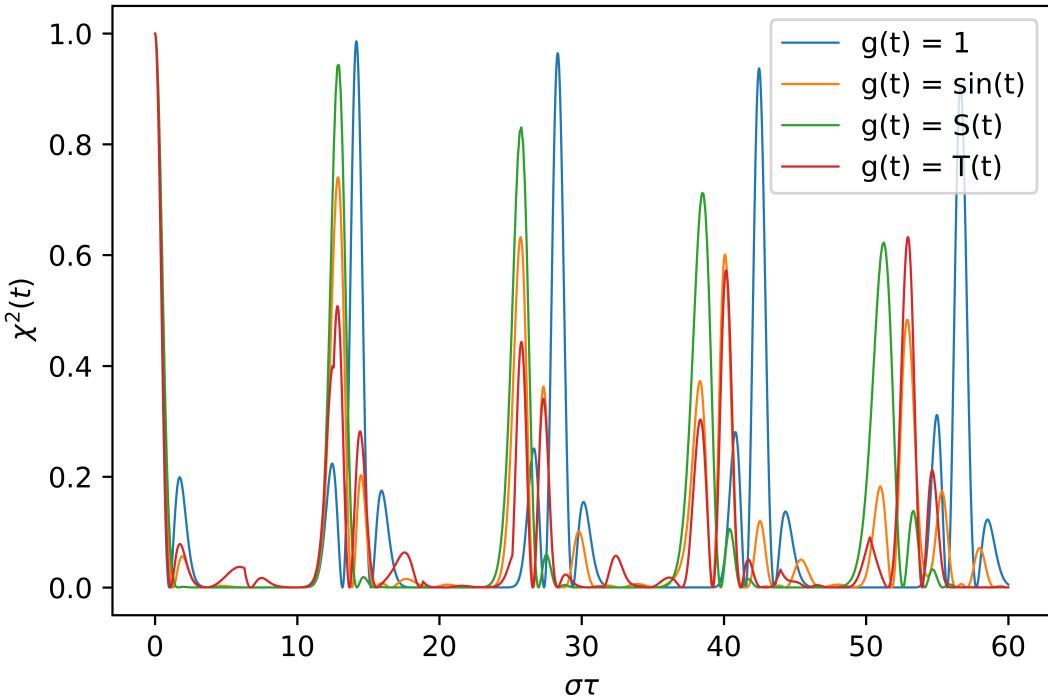


Figure 4: Different functional forms are plotted, where $S(t)$ represents a square wave and $T(t)$ represents a saw tooth wave.

The system consists of a large N and given the huge number of allowed configurations with no mechanism for the information to flow back into the system, and it can be said that once this information is lost and it is incredibly improbable to see the state return. This physical argument is clearly at odds to the resurgences as shown in Fig. 1. The initial drop is expected, but this oscillating function is nonphysical and non-representative of the physical case. This simulation was for $k = 10$, and now it is explored the effect of simulations with larger k values.

Fig. 2 shows the same simulation for much larger k values, and as discussed larger k values result in less frequent peaks, and it takes longer for a revival to occur. In comparison to Fig. 1, it is an improvement in terms of what is expected physically and with larger values of k the behaviour seems to approach that of what is expected in the exact case, which is evidence to suggest that even larger k values will provide suitably accurate models and these other peaks that occur which are not deemed physical would be suitably reduced.

It is also important to note that while it appeared that for $k > 1000$ for the length of time used, simulations would take on the order of $10h$, running the simulation exactly would be impossible computationally on any kind of computer today. Therefore this is evidence that with proper optimisation and dedicated hardware it is an accurate simulation such that the results could be used to inform design could be feasible.

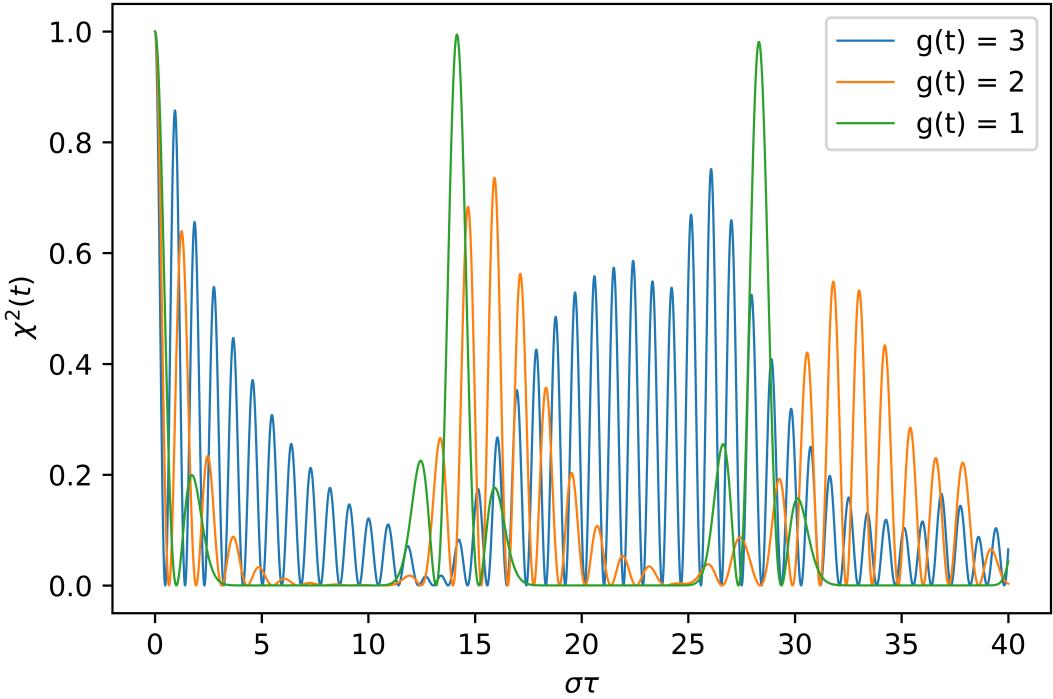


Figure 5: Different coupling strengths plotted and compared.

5.2 The Spin-Cavity System

With the introduction of the cavity system, so introduces a mechanism for information to flow back into the system, since information from the spin system and result formation of a photon which is coupled to the lowest state of the spin system resulting in information coming back as a spin activation.

The most simple case when $g(t) = 1$ is shown in Fig. 3, which as discussed shows that in the coupled case information is lost more slowly compared to the un-coupled case, which suggests similarly as the theory does that by correctly choosing the functional form of $g(t)$, the systems ability to retain the initial state can be optimised.

However as will be shown, given the large size of the system there is little in terms of physical arguments that can be made in terms of what might constitute an effective functional form. It is suggested here that given the oscillating nature of the system and the need to keep reviving the system, a period function is a likely candidate. Arguments to the strength of $g(t)$ are discussed in a later section but it was concluded that an optimum $g(t)$ was best found using some form of machine learning method or similar optimisation technique given the complexity of the problem.

Ideas and the requirements for this implementation are discussed in the conclusion, however some investigation was performed into the behaviour of the system for different functional forms of $g(t)$.

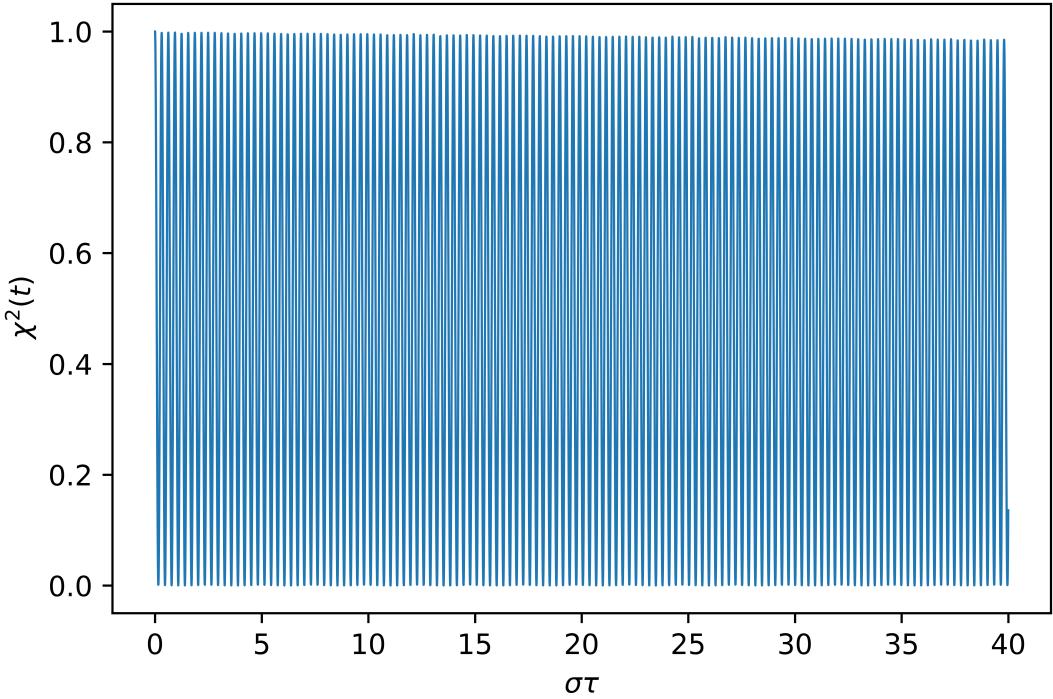


Figure 6: A coupling form of $g(t) = 5$ is plotted, resulting in a very different behaviour to that observed in Fig. 5.

5.3 Different Functional Forms for $g(t)$

Fig. 4 shows how different functional forms of $g(t)$ compare, where each periodic function has the same amplitude and frequency. While it can be argued that the best performing function is the square wave shown in green as $S(t)$ since the green peaks are the highest, the case of $g(t) = 1$ clearly is still the best form.

Therefore these functional forms did contribute greatly to the understanding of what functional form of $g(t)$ would best. What was tested was varying the frequency of these periodic functions, but it was found that there were better results when the frequency of functional forms was very small. In other words for the time interval being considered, $g(t) \approx 1$ since a large period means that for example the $\sin(\omega t) \approx 1$, affirming the result that $g(t) = 1$ yields the best results.

5.4 Strength of Time Invariant $g(t)$

Given the case $g(t) = 1$ worked best, the effect of increasing this coupling constant is investigated. The result is shown in Fig. 5. However the relationships between these three different plots is not immediately clear. It can be said that it appears larger strengths results in a deviation from single peak behaviour, and results in a higher frequency revival pattern, whose distribution is quite complex. It initially appeared that effect on increasing $g(t)$ seemed to be complex too be useful, however an interesting result occurs for any $g > 4$.

Fig. 6 represents the evolution of $\chi^2(t)$ for $g(t) = 5$. Its clear that the behaviour has become much more steady, and that $\chi^2(t)$ very rapidly oscillates between 0 and a value

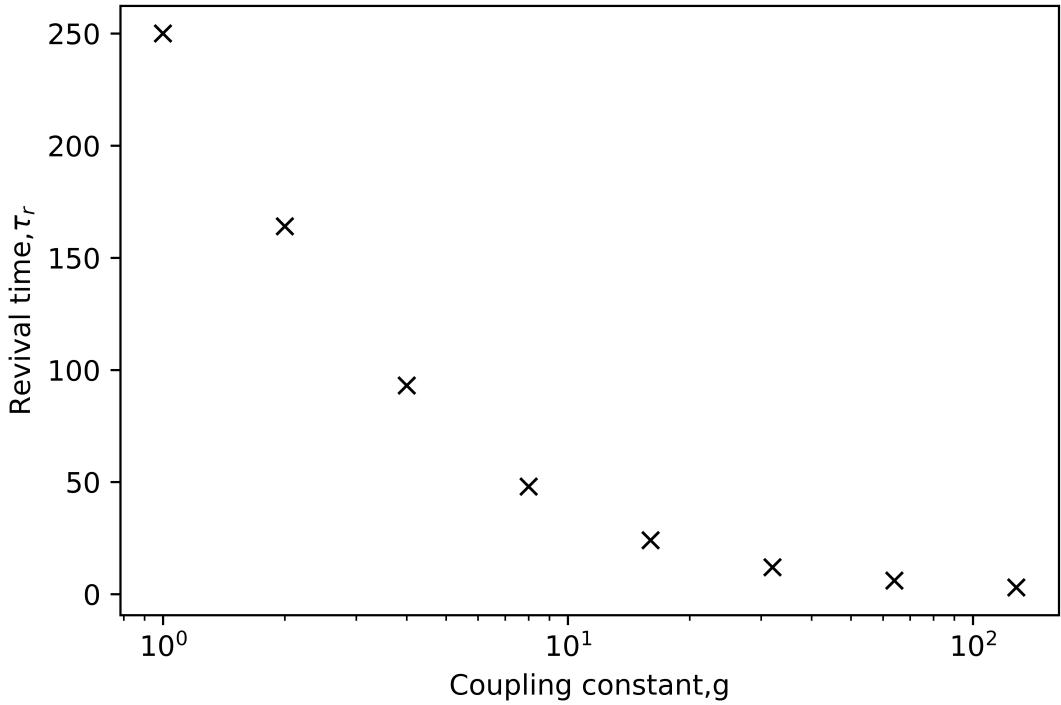


Figure 7: The amount of time it takes for the first peak to revive for different g values.

very close to 1. This value does very slowly decrease in time, and it retains the initial state the best for any case tested so far, and for $g > 5$, a very similar distribution is found, with the only difference being a higher frequency and slower decrease in peak height.

A physical explanation for this could be that when the spin system is very strongly coupled to the cavity, information at very high frequency enters and leaves the system and conserves the initial state, and indeed looking at (16), the total Hamiltonian contains three terms. The first two contain information about the time evolution of each individual state, and the last term with pre-factor $g(t)$ is responsible for the interaction between the two systems. For large g values, it is this final term that becomes most important and is large in comparison to the independent system terms. This results in a final Hamiltonian H that is interaction dominant resulting in this highly oscillating behaviour shown in Fig. 6. How physically viable such large $g(t)$ values are is an important design consideration that could be a limiting factor in trying to implement the parameters presented here.

Finally, one final aspect in which the different $g(t)$ functions can be judged is the notion of the revival time t_τ . Producing a method in which $g(t)$ functions can be holistically evaluated is very broad problem, and the extent to which the results of such a method would be made redundant due to unforeseen design limitations is certainly a case to investigate more simple means of evaluation at this early stage such as peak height, and in this case the time it takes for this peak to occur.

The obvious limitation to this test is of course this is the time taken for a revival to occur, but nothing is said about height of this peak. The result is shown in Fig. 7. Despite all these limitations there a smooth relationship is observed between increasing $g(t)$ and revival time τ_r . Clearly there is a plateau as discussed before beyond $g = 100$ there

is no change.

Again, physically there is not obvious interpretation, and perhaps it can be only said that there is some other mechanism or limiting factor that means this effect does not occur indefinitely. This in itself is another argument for a more empirical machine learning method for evaluating $g(t)$ functions, however this investigation does perhaps show sound boundary conditions on the parameters of $g(t)$ that an end ideal result produced using a machine learning method would be yield would lie within.

6 Conclusion

6.1 The Aims of the Project

The aim of this project was to implement a method in order to simulate accurately a large system of spins coupled to a cavity, and to use the time dependence of the coupling strength to optimise the systems ability to retain its initial state so that information can be stored and measured at a later stage.

In terms of implementing the Krylov subspace, this was done successfully and it has been shown that with a sufficiently large number of Krylov basis states, the simulation behaves as it would in the exact case in this low energy regime. It was also demonstrated that the simulation required significantly less computational power than the exact case. However it is also true that even for $k = 500$, as plotted in Fig. 2 there remained some artifacts such as the reviving peaks which are a reflection of the fact the simulation is an approximation. To find a suitably larger k value that is computationally feasible but also accurate is a problem that with more time, careful optimisation and computational power would be investigated.

It is also worth stating here that the simulation only allowed the existence of one excitation of the system, and there was only coupling between these states. Modifying the states and the Hamiltonian would be a simple extension to allow higher energy states but would become more computationally intensive.

In terms of optimising the system using the coupling constant $g(t)$, some ground was made in terms of studying the system in terms of the strength of this interaction, as well measuring the revival time τ_r for these different strengths. The simple functional forms as shown in Fig. 4 did not prove to be any better than the time invariant case, which suggests a better functional form is more complex than any of the cases tested.

6.2 Further Work

It is for this reason it is suggested that further work would compromise largely of two parts. Firstly, in this project, different functional forms were evaluated in terms of the heights of the peaks produced, and the time taken for these peaks to appear. However, this is a very simplistic analysis and in reality it is suggested there are many qualities which would make up an ideal functional form such as how regular the function is, for how long it retains for example $\chi^2(t) = 0.9$ and the aforementioned revival time.

Developing a method that can evaluate the different functional forms on several different qualities would be important in being able to rigorously compare results. A starting point that was thought of but not implemented would be half-life measures. In other words measure for each $g(t)$, for how long does each function retain a peak that is of

height $\chi^2(t) > 0.5$, along with some other value that is measure of how uniform the frequency is.

This method is emphasised since the second piece of further work suggested here is implementing a machine learning method in order to optimise the functional form of $g(t)$. Given the complexity of the system, it is hard to physically reason what the optimum form of $g(t)$ might be. However using a machine learning method would avoid this.

However it is required in implementing a machine learning method a way in which the program can evaluate the quality with which the initial state is stored in the system.

The fundamental aims of this project were fulfilled, with an implementation of the Krylov space, an introduction of the cavity system, and testing of different $g(t)$ functions in order to optimise the system. In order to move forwards it is suggested that a method of evaluation is formulated and implemented with a machine learning method in order to find an optimum functional form of $g(t)$.

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