Analysis in $\mathbb{R}^n$
IMECC - Unicamp

1st list of excercices
March 13^{th}, 2024

Name: Jaider Torres **RA: 241343**

1) Let $\|\cdot\|_{\mathbb{R}^n}$, $\mathbb{R}^n$, $\|\cdot\|_S$ and $\mathbb{R}^n$, $\|\cdot\|_M$ be the euclidean norm, the sum norm and the maximum norm of $\mathbb{R}^n$, respectively. We want to prove that

$$\|x\|_\infty \leq \|x\|_{\mathbb{R}^n} \leq \|x\|_S \leq n\|x\|_\infty,$$

for each $x \in \mathbb{R}^n$.

For the first inequality, let $l = \text{máx}\{|x_i|\}_{i\in[n]}$, where $[n] := \mathbb{N} \cap [1, n]$. Note that if $l \geq 1$, we have that

$$\begin{aligned} l &\leq l^2 \\ &\leq \sum_{i\in[n]\setminus j} |x_i|^2 + l^2 \\ &= \sum_{i\in[n]} |x_i|^2. \end{aligned}$$

Thus, $l \leq \|x\|_{\mathbb{R}^n}$.

Now, if $l \in (0, 1)$, we have that

$$l^2 \leq \sum_{i\in[n]\setminus j} |x_i|^2 + l^2.$$

Thus, $\|x\|_M \leq \|x\|_{\mathbb{R}}$.

Let's prove now that $\|x\|_{\mathbb{R}^n} \leq \|x\|_S$. For this, we remark that

$$\sum_{i\in[n-1]} |x_i|^2 + |x_n| \leq \left(\sum_{i\in[n-1]} |x_i|^2\right)^2 + 2\left(\sum_{i\in[n-1]} |x_i|\right)|x_n| + |x_n|^2.$$

Doing this inductively on the left factor of the right part of the previous inequality, we get that

$$\sum_{i\in[n]} |x_i|^2 \leq \left(\sum_{i\in[n]} |x_i|\right)^2,$$

showing $\|x\|_{\mathbb{R}^n} \leq \|x\|_S$.

Last, to prove that $\|x\|_S \leq n\|x\|_\infty$, we just note that

$$\sum_{i\in[n]} |x_i| \leq \sum_{i\in[n]} \text{máx}\{|x_i|\} = n\,\text{máx}\{|x_i|\}.$$

Hence, we have the inequality of norms.

2) a. To prove that $\|x\|_p$ defines a norm, we have to check that

i) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, for each $x, y \in \mathbb{R}^n$,

ii) $\|\alpha x\|_p = |\alpha|\|x\|_p$, for each $x \in \mathbb{R}^n$, and

iii) $\|x\|_p > 0$ for $0 \neq x \in \mathbb{R}^n$ and $\|x\|_p = 0$ if and only if $x = 0 \in \mathbb{R}^n$.

For the second item, note that

$$\begin{aligned}\|\alpha x\|_p &= \left(\sum_{i\in[n]} |\alpha x_i|^p\right)^{\frac{1}{p}} \\ &= \left(\sum_{i\in[n]} |\alpha|^p|x_i|^p\right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \sum_{i\in[n]} |x_i|^p\right)^{\frac{1}{p}} \\ &= |\alpha| \left(\sum_{i\in[n]} |x_i|^p\right)^{\frac{1}{p}}.\end{aligned}$$

For the third item, it's straightforward that $\|x\| = 0$ if and only if $x = 0 \in \mathbb{R}^n$. Given $x \in \mathbb{R}^n$ nonzero, we have that

$$\left(\sum_{i\in[n]} |x_i|^p\right)^{\frac{1}{p}} > 0,$$

since $\{|x_i| > 0\}_{i\in J}$, where $J \subseteq [n]$ is a subcollection of index of $[n]$, is nonempty.

Now, for the firs item, note that if $x, y \in \mathbb{R}^n$ are such that $\|x\|_p + \|y\|_p = 0$, then there is nothing to prove as item **iii)** is satisfied. Since item **ii)** holds, let $x, y \in \mathbb{R}^n$ such that $\|x\|_p + \|y\|_p = 1$ and let $t = \|y\|_p$, in such a way that $\|x\|_p = 1 - t$. If $t = 0, 1$, the result would follows. Thus, let's consider $t \in (0, 1)$. Taking $z = \frac{x}{(1-t)}$ and $w = \frac{y}{t}$, we have that

$$\|x + y\|_p = \|(1 - t)z + tw\|_p \leq 1$$

since $\|z\|_p = \|w\|_p = 1$ and $t \in [0, 1]$. But since $x \mapsto |x|^p$ is a convex function for $p \geq 1$, denoting $z_i = \frac{x_i}{(1-t)}$ and $w_i = \frac{y_i}{t}$ (where $x_i, y_i \in \mathbb{R}$ are the i−th entry of x, y, respectively), we have that $|(1 - t)z_i + tw_i|^p \leq (1 - t)|z_i|^p + t|w_i|^p$. Then,

$$\sum_{i\in[n]} |(1 - t)z_i + tw_i|^p \leq \sum_{i\in[n]} (1 - t)|z_i|^p + t|w_i|^p,$$

implying

$$\|(1 - t)z + tw\|_p^p \leq 1 = \|x\|_p + \|y\|_p.$$

Hence, $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ for $p \in [1, \infty)$.

Now, prove that $\|\cdot\|_\infty$ is a norm is very straightforward since it clearly satisfies the previous properties, i.e., given $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have that $\text{máx}\{|x_i + y_i|\} =$

$\left|x_j + y_h\right| \leq \left|x_j\right| + \left|y_j\right|$, $|\alpha x_i + \alpha y_i| = |\alpha||x_i + y_i|$ and clearly $\text{máx}\{|x_i|\} \geq 0$ whenever $x \neq 0 \in \mathbb{R}^n$ and the equality happens when $x = 0$.

b. Now, if $p \in (0,1)$, taking $n = 2$, $\{e_i\}_{i=1,2}$ the canonical basis of R^2 and $p = \frac{1}{2}$, we have that $\|e_1 + e_2\|_{\frac{1}{2}} = (1^{1/2} + 1^{1/2})^2 \geq (1^{\frac{1}{2}})^2 + (1^{\frac{1}{2}})^2 = \|e_1\|_p + \|e_2\|_p$. As i) is not satisfied in this case, we note that this doesn't define a norm on $\mathbb{R}^n$.

We'll show here an attempt using properties of concavity of real valued functions and a classical approach:

(·) (Attempt) In **a.** we used that the map $x \mapsto x^p$ on $\mathbb{R}^+$ was convex for $p \geq 1$. We'll try to recreate the proof using that $x \mapsto x^p$ is concave for $p \in (0,1)$. Let $x, y \in \mathbb{R}^n$ such that $x_i, y_i \geq 0$ for every $i \in [n]$ and $\|x\|_p + \|y\|_p = q < 1$. Let $t = \|y\|_p$ so that $\|x\|_p = q - t$. Write $z = \frac{x}{q-t}$ and $w = \frac{y}{t}$. Note that

$$\|x + y\|_p = \|(q - t)z + tw\|,$$

since $\|z\|_p = \|w\|_p = 1$ and $t \in (0, c) \subset (0, 1)$.

Denote $z_i = \frac{x_i}{q-t}$ and $w_i = \frac{y_i}{t}$. Using the fact that $x \mapsto x^p$ is concave, we have

$$|(q - t)z_i + tw_i|^p \geq (q - t)|x|^p + t|w|^p,$$

implying that

$$\sum_{i\in[n]} |(q - t)z_i t w_i|^p \geq \sum_{i\in[n]} (q - t)|z_i|^p + t|w_i|^p,$$

i.e.,

$$\|(q - t)z + tw\|_p^p \geq (q - t)\|z\|_p^p + t\|w\|_p^p = \|x\|_p^p + \|y\|_p^p.$$

As $\frac{1}{p} > 1$,

$$\|x + y\|_p = \|(q - t)z + tw\|_p \geq \|x\|_p + \|y\|_p.$$

Couldn't figure it out.

(··) (Zhao. C, Cheung. W. **HÖLDER'S REVERSE INEQUALITY AND ITS APPLICATIONS**, PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE, 2016).

Firs, we'll remember two results that will be useful (Kreyszig. E, **Introductory functional analysis with applications**, John Wiley & Sons Inc ,1978).

- (**Hölder Inequality**) Given $x = (\zeta_i) \in l^p$ and $y = (\xi_j) \in l^p$, $p > 1$, the equality

$$\sum_{j\in\mathbb{N}} |\zeta_j \xi_j| \leq \left(\sum_{i\in\mathbb{N}} |\zeta_i|^p\right)^{\frac{1}{p}} \left(\sum_{k\in\mathbb{N}} |\xi_k|^q\right)^{\frac{1}{q}}$$

  holds.

- (**Minkowski Inequality**) Given $x = (\zeta_i) \in l^p$ and $y = (\xi_j) \in l^p$, where $p > 1$, the equality

$$\left(\sum_{k\in\mathbb{N}} |\zeta_k + \xi_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i\in\mathbb{N}} |\zeta_i|^p\right)^{\frac{1}{p}} + \left(\sum_{n\in\mathbb{N}} |\xi_n|^p\right)^{\frac{1}{p}}$$

  holds.

With this results, **a.**, i) can be shown as follows:

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i\in[n]} |x_i+y_i||x_i+y_i|^{p-1} \\
&\overset{*_1}{\leq} \sum_{i\in[n]} |x_i||x_i+y_i|^{p-q} + \sum_{j\in[n]} \left|y_j\right|\left|x_j+y_j\right|^{p-q} \\
&\overset{*_2}{\leq} \left(\sum_{i\in[n]} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i\in[n]} |x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i\in[n]} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i\in[n]} |x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\sum_{i\in[n]} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i\in[n]} |x_i+y_i|^p\right)^{\frac{p-1}{p}} + \left(\sum_{i\in[n]} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i\in[n]} |x_i+y_i|^p\right)^{\frac{p-1}{p}} \\
&= \|x\|_p\|x+y\|_p^{p-q} + \|y\|_p\|x+y\|_p^{p-1} \\
&= \left(\|x\|_p + \|y\|_p\right)\|x+y\|_p^{p-1},
\end{aligned}$$

where q is such that $\frac{1}{p}+\frac{1}{q}=1$.

Note that Hölder inequality can be expressed in terms of integrals:

- Let $f(x)$ and $g(x)$ be positive continuous functions on $[a,b]$. If $p>1$ and $1=\frac{1}{p}+\frac{1}{q}$ for some q, then

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \geq \int_a^b f(x)g(x)dx.$$

Define for $h \neq 1$

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}.$$

Lemma If $a,b\in\mathbb{R}^+$ and $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$, then

$$S\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \geq \frac{a}{p}+\frac{b}{q}.$$

Now, we can state Hölder's inverse inequality.

Theorem (Hölder's inverse inequality) Let $\frac{1}{p}+\frac{1}{q}=1$, with $p>1$. If $f(x)$ and $g(x)$ are non-negative continuous functions and $f^{\frac{1}{p}(x)}g^{\frac{1}{q}(x)}$ is integrable on $[a,b]$, then

$$\left(\int_a^b f^p(x)dx\right)\left(\int_a^b g^q(x)\right) \leq \int_a^b S\left(\frac{Y}{X}\frac{f^p(x)}{g^q(x)}\right)\cdot f(x)g(x)dx,$$

where

$$X=\int_a^b f^p(x)dx, \qquad Y=\int_a^b g^q(x)dx.$$

If $f(x)$ and $g(x)$ reduces to positive real sequences $\{(a_i)_{i\in\mathbb{N}}\}_{i\in[n]}$ and $\{(b_i)_{i\in\mathbb{N}}\}_{i\in[n]}$, respectively, then the previous theorem implies that

$$\left(\sum_{i\in[n]} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i\in[n]} b_i^q\right)^{\frac{1}{q}} \leq \sum_{i\in[n]} S\left(\frac{Y' a_i^p}{X' b_i^q}\right) a_i b_i,$$

where $X' = \sum_{i\in[n]} a_i^p$ and $Y' = \sum_{i\in[n]} b_i^p$.

So, letting $x, y \in \mathbb{R}^n$ nonzero vector with positive entries, the inequality in $*_1$ above turns to be an equality and using the Hölder's inverse inequality, the inequality in $*_2$ changes to $\geq$, showing that $\|x+y\|_p^p \geq \left(\|x\|_p + \|y\|_p\right) \|x+y\|_p^{p-1}$

c. In **a.** we have proved i) using the fact that the mapping $x \mapsto |x|^p$ is convex for $p > 1$. As the case $p = 1$ was done in **1.**, let's consider $p > 1$. Let $q = \frac{1}{p}$. As $p \geq 1$, $q \in (0,1)$. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $x \mapsto x^q$. We note that $f'(x) = qx^{q-1} > 0$ and $f''(x) = q(q-1)x^{q-2} < 0$ for every $x \in \mathbb{R}^+$. Thus, f is strictly increasing and is concave, respectively. So,

$$\begin{aligned} f\left(\sum_{i\in[n]} |x_i|^p\right) &\leq f\left(\sum_{i\in[n]} \text{máx}\{|x_i|^p\}\right) \\ &= f\left(n \cdot |x_{j^*}|^p\right) \\ &= n^{\frac{1}{p}} |x_{j^*}|. \end{aligned}$$

This implies that $\|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty$.

Now, let $a = \|x\|_\infty$. As $a^p \leq \sum_{i\in[n]\setminus j} |x_i|^p + a^p$ we have that $a \leq \left(\sum_{i\in[n]\setminus j} |x_j|^p + a^p\right)^{\frac{1}{p}}$. Thus $\|x\|_\infty \leq \|x\|_p$.

Hence, $\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty$.

As $\alpha = 1, \beta = n^{\frac{1}{p}}$ are such that $\|x\|_\infty \leq 1 \cdot \|x\|_p$ and $\|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty$, we have furthermore that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are equivalents.

3. Let $f : \mathbb{C} \to \mathbb{C}$ defined by $z \mapsto ze^{i\theta}$, where $\theta \in [0, 2\pi)$. We firs note that $\frac{df(z)}{dz} = e^{i\theta} \neq 0$ and $|f(z)| = |z| \cdot |e^{i\theta}| = |z|$. Thus, f holomorphic and preserves modules. Now, writing $f = u(x+y) + v(x,y)$, where $u(x,y) = x\cos\theta - y\sin\theta$ and $v(x,y) = x\sin\theta + y\cos\theta$, we have that, looking at f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, the jacobian of f is

$$\begin{aligned} |\text{Jac}(f)| &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \begin{vmatrix} u_x & -v_x \\ v_x & u_x \end{vmatrix} \\ &= u_x^2 + v_x^2 \\ &= \cos^2\theta + \sin^2\theta \\ &= |f'|^2 \\ &\neq 0, \end{aligned}$$

where the jacobian matrix of f have this form for the Cauchy-Riemann equations. Thus, since every transformation of $\mathbb{R}^2$ such that it's associated matrix have this form and have positive determinant is conformal, i.e., preserves angles and orientation, we deduce that f is conformal, and as it preserves angles and orientation.

This remarks, i.e., f being conformal and modulus preserving, prove that f is in fact an isometry of $\mathbb{C}$ given by rotation in θ degrees. This implies that rotation in $\mathbb{R}^2$ are isometries.

4. As suggested, let's prove that if h is an isometry of $\mathbb{C}$, then necessarily it's the identity map.

Let $h : \mathbb{C} \to \mathbb{C}$ an isometry of $\mathbb{C}$ such that have $0, 1, i$ as fixed points. Thus, $|h(z) - h(w)| = |z - w|$ for every $z, w \in \mathbb{C}$. Note that in $w = 0, 1, i$ we have $|h(z)| = |z|$, $|h(z) - 1| = |z - 1|$ and $|h(z) - i| = |z - i|$. Using that $u\overline{u} = |u|^2$ for every $u \in \mathbb{C}$, we have that $(h(z)-1)(\overline{h(z)}-1) = (z-1)(\overline{z}-1)$ and $(h(z) - i)(\overline{h(z)} + i) = (z-1)(\overline{z}+i)$. Using the fact that $|h(z)| = |z|$, we have

$$\begin{aligned}(h(z) - 1)(\overline{h(z)} - 1) &= (z-1)(\overline{z}-1)\\ |h(z)|^2 - h(z) - \overline{h(z)} + 1 &= |z|^2 - z - \overline{z} + 1\\ h(z) + \overline{h(z)} &= z + \overline{z}\end{aligned}$$

and

$$\begin{aligned}(h(z) - i)(\overline{h(z)} + i) &= (z-i)(\overline{z}+i)\\ |h(z)|^2 + i(h(z) - \overline{h(z)}) + 1 &= |z|^2 i(z - \overline{z}) + 1\\ h(z) - \overline{h(z)} &= z - \overline{z}.\end{aligned}$$

This implies $h(z)$ and z have the same real and imaginary part. Hence, $h(z) = z$, i.e., $h = id_{\mathbb{C}}$ and the proof is complete.

Let $I = \{f : \mathbb{C} \to \mathbb{C} : f \text{ is isometry}\}$. We want to prove that the elements of I have the form $h(z) = \alpha z + \beta$ or $j(z) = \alpha\overline{z} + \beta$, $\alpha, \beta \in \mathbb{C}$, where $|\alpha| = 1$.

Thus, let $f \in I$ an unknown isometry, and let $\alpha = f(1) - f(0)$ and $\beta = f(0)$. Consider the function

$$k(z) = \frac{f(z) - \beta}{\alpha} = \frac{f(z) - f(0)}{f(1) - f(0)}.$$

We note that

$$\begin{aligned}|\alpha| &= |f(1) - f(0)|\\ &= |1 - 0|\\ &= 1\end{aligned}$$

and

$$\begin{aligned}|k(z)-k(w)| &= \left|\frac{f(z)-\beta-f(w)+\beta}{\alpha}\right| \\ &= \left|\frac{f(z)-f(w)}{\alpha}\right| \\ &= |f(z)-f(w)| \\ &= |z-w|.\end{aligned}$$

So, $k \in I$. Moreover, $k(0)=0$ and $k(1)=1$ by definition of k. Now, for $i \in \mathbb{C}$ we have

$$\begin{aligned}|k(i)| &= |k(i)-k(0)| \\ &= |i-0| \\ &= 1,\end{aligned}$$

and thus

$$\begin{aligned}|k(i)-k(1)| &= \left|\frac{f(i)-\beta-f(1)+\beta}{\alpha}\right| \\ &= |f(i)-f(1)| \\ &= |i-1| \\ &= \sqrt{2}.\end{aligned}$$

Hence, $k(i)$ must be either i or $-i$ since this are the only two complex numbers satisfying that their distance to 0 and 1 are 1 and $\sqrt{2}$, respectively, i.e., $k(i)=\overline{\pm i.}$ If $k(i)=i$, using the fact shown before we deduce $k=\mathrm{id}_{\mathbb{C}}$. If $k(i)=-i$, then $\overline{k(z)}=i$ and thus $\overline{k(z)}=z$, i.e., $k(z)?\bar{z}$.

As $f(z)=\alpha k(z)+\beta$ and k sends z to either z or $\bar{z}$, we have complete the proof.

This describes the isometries in I as combinations of translations, rotations and reflections.

5. Again, as suggested, we're going to prove first that any isometry $K \in \mathrm{Mor}_{\mathrm{Set}}(\mathbb{R}^n, \mathbb{R}^n)$ such that $K(0)=0 \in \mathbb{R}^n$ is orthogonal, i.e., $\langle K(v), K(w)\rangle = \langle v, w\rangle$ for every $v, w \in \mathbb{R}^n$.

We note that

$$\|K(v)\| = \|K(0)+K(v)\| = \|0+v\| = \|v\|.$$

Thus, K preserves norms. Now, as K is an isometry, for every $x, y \in \mathbb{R}^n$ we have that $\|x-y\| = \|K(x)-K(y)\|$. Thus, $\|K(x)-K(y)\|^2 = \|x-y\|^2$. Operating the left side of the equality we get

$$\begin{aligned}\|K(x)-K(y)\|^2 &= \sqrt{\langle K(x)-K(y), K(x)-K(y)\rangle}^2 \\ &= \langle K(x)-K(y), K(x)-K(y)\rangle \\ &= \langle K(x), K(x)\rangle - 2\langle K(x), K(y)\rangle + \langle K(y), K(y)\rangle \\ &= \|K(x)\|^2 - 2\langle K(x), K(y)\rangle.\end{aligned}$$

By operating in the same way in the right side of the equality, we obtain

$$-2\langle K(x), K(y)\rangle = -2\langle x, y\rangle,$$

i.e,

$$\langle K(x), K(y)\rangle = \langle x, y\rangle.$$

Hence, K preserves the inner product.

Now, let's prove that if $K \in \mathrm{Mor}_{\mathrm{Set}}(\mathbb{R}^n, \mathbb{R}^n)$ is an isometry such that $K(0) = 0 \in \mathbb{R}^n$ and $\{e_i\}_{i\in[n]}$ is the canonical basis of $\mathbb{R}^n$, then $\{K(e_i)\}_{i\in[n]}$ is an orthogonal basis of $\mathbb{R}^n$.

As $\{e_i\}_{i\in[n]}$ is orthonormal, $\{K(e_i)\}_{i\in[n]}$ is an orthonormal set of vector of $\mathbb{R}^n$ since K preserves inner product for the previous result, i.e., $\langle K(e_i), K(e_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij}$ and $\|K(e_i)\| = \|e_i\|$, for each $i, j \in [n]$.

We remark that given $V \in \mathrm{Vec}_k$, where $\dim_k V = n < \infty$, with inner product and $\mathcal{A} = \{v_i\}_{i\in[n]}$ a orthogonal subset of form by nonzero vectors, then for any $v \in \mathrm{Span}\{v_i\}_{i\in[n]}$,

$$v = \sum_{i\in[n]} \frac{\langle v, v_i\rangle}{\|v_i\|^2} v_i.$$

This fact comes from realizing that, writing $v = \sum_{i\in[n]} \alpha_i v_i$, $\langle v, v_j\rangle = \alpha_j\langle v_j, v_j\rangle$, i.e., $\alpha_j = \frac{\langle v, v_j\rangle}{\|v_j\|^2}$.

So, in our case, for every $x \in \mathbb{R}^n$ we can write

$$K(x) = \sum_{i\in[n]} \langle K(x), K(e_i)\rangle K(e_i)$$

($\|K(e_i)\| = 1$). But as $\langle K(x), K(e_i)\rangle = \langle x, e_i\rangle = x_i$, we have that

$$K(x) = \sum_{i\in[n]} x_i K(e_i),$$

showing that $\{K(e_i)\}_{i\in[n]}$ is an orthonormal basis of $\mathbb{R}^n$.

Moreover, given $\alpha \in \mathbb{R}$, we have $K(\alpha x+y) = \alpha K(x)+K(y)$ (after a simple computation), showing that K is a linear transformation.

With this in mind, let's consider $T : \mathbb{R}^n \to \mathbb{R}^n$ any isometry. We want to prove that $T(x) = H(x) + x_0$, where $H \in \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{R}}}(R^n, R^n)$ is orthogonal and $x_0 \in \mathbb{R}^n$.

Thus, let $T : \mathbb{R}^n \to \mathbb{R}^n$ any isometry of $\mathbb{R}^n$. Analogously to the case in **4.**, for any $i \in [n]$, consider $\alpha = T(e_i) - T(0)$ and $\beta = T(0)$. We note that

$$\|\alpha\| = \|T(e_i) - T(0)\| = \|1 - 0\| = 1.$$

Let

$$L(x) = T(x) - \beta.$$

Then $\|L(x)\| = \|T(x) - T(0)\| = \|x\|$, and

$$\|L(x) - L(y)\| = \|T(x) - T(y)\| = \|x - y\|.$$

Thus, L is an isometry of $\mathbb{R}^n$. Furthermore, note that

$$L(0) = H(0) - \beta = 0.$$

Thus, the previous results L preserves inner product of $\mathbb{R}^n$, i.e., $\langle L(x), L(y)\rangle = \langle x, y\rangle$ for every $x, y \in \mathbb{R}^n$, and also it's a linear transformation that preserves orthonormality of the canonical basis of $\mathbb{R}^n$.

Hence, $T(x) = L(x) + \beta$, as expected.