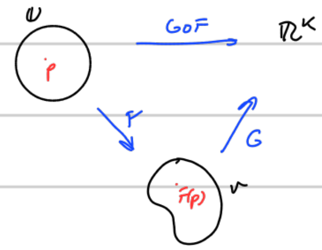
AULA 09 - 08/04

REGRA DA CADEIA



Sejam $U \subseteq \mathbb{R}^n$ e $V \subseteq \mathbb{R}^m$ abertos, $F: U \to \mathbb{R}^m$ dif. em $p \in U$ t.q. $F(U) \subseteq V$ e $G: V \to \mathbb{R}^k$ dif. em $F(p)$.

TEO. Nas hip. acima, $G \circ F$ é dif. em $p \in U$ e $D(G\circ F)(p) = DG(F(p)) \cdot DF(p)$ → A der. da comp. é a comp. das der.

Diagramaticamente:

$$\begin{tikzcd} U \arrow[rr, "G\circ F"] \arrow[dr, "F"'] & & \mathbb{R}^k \\ & V \arrow[ur, "G"'] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathbb{R}^n \arrow[rr, "D(G\circ F)(p)"] \arrow[dr, "DF(p)"'] & & \mathbb{R}^k \\ & \mathbb{R}^m \arrow[ur, "DG(q)"'] & \end{tikzcd}$$

(FUNTOR COVARIANTE)

DEM. Denotemos, $q = F(p)$, $L = DF(p)$, $M = DG(q)$, $H = G\circ F$.

Queremos mostrar: $H(p+v) = H(p) + (M\circ L)\cdot v + o(\|v\|)$, $v \to 0$

Considere os restos:

$r_H(v) = H(p+v) - H(p) - (M\circ L)\cdot v$, queremos $r_H(v) = o(\|v\|)$, $v\to 0$

$r_F(v) = F(p+v) - F(p) - L\cdot v$, sabemos $r_F(v) = o(\|v\|)$, $v \to 0$

$r_G(w) = G(q+w) - G(q) - M\cdot w$, sabemos $r_G(w) = o(\|w\|)$, $w \to 0$

Podemos reescrever r_H em função de r_F, r_G:

$$r_H(v) = G(F(p+v)) - G(F(p)) - M(Lv) \quad\quad Lv = F(p+v) - F(p) - r_F(v)$$

$$= G(F(p+v)) - G(q) - M(\underbrace{F(p+v) - q}_{w:=}) + M\cdot r_F(v)$$

$$= G(q+w) - G(q) - M\cdot w + M\cdot r_F(v) = r_G(w) + M\cdot r_F(v)$$

Portanto,

$$\frac{r_H(v)}{\|v\|} = \frac{r_G(w)}{\|w\|}\cdot\frac{\|w\|}{\|v\|} + M\cdot\left(\frac{r_F(v)}{\|v\|}\right)$$

⚠ Temos dois casos: $w=0$ e $w \neq 0$
OK!

Note: $\|v\| \to 0 \Rightarrow M\cdot\left(\frac{r_F(v)}{\|v\|}\right) \to 0$ (pois $r_F(v) = o(\|v\|)$ e M é cont.)

$\Rightarrow \|w\| \to 0$ (pois F é cont.) $\Rightarrow \frac{r_G(w)}{\|w\|} \to 0$ (pois $r_G(w) = o(\|w\|)$)

Resta analisar $\frac{\|w\|}{\|v\|}$

Mas,

$$\frac{\|w\|}{\|v\|} = \frac{\|F(p+v) - F(p)\|}{\|v\|} = \frac{\|Lv - r_F(v)\|}{\|v\|} \leq \frac{\|Lv\|}{\|v\|} + \frac{\|r_F(v)\|}{\|v\|} \leq \|L\| + \frac{\|r_F(v)\|}{\|v\|}$$

→ Conseq., $\frac{\|w\|}{\|v\|}$ é lim. quando $v\to 0$

Assim,

$$0 \leq \frac{\|r_H(v)\|}{\|v\|} \leq \underbrace{\frac{\|r_G(w)\|}{\|w\|}}_{\to 0}\underbrace{\left(\|L\| + \underbrace{\frac{\|r_F(v)\|}{\|v\|}}_{\to 0}\right)}_{\text{lim.}} + \underbrace{\left\|M\cdot\left(\frac{r_F(v)}{\|v\|}\right)\right\|}_{\to 0}$$

□

Ex. $[n=k=1]$

$$\begin{tikzcd} I \subseteq \mathbb{R} \arrow[rr, "f\circ c"] \arrow[dr, "c"'] & & \mathbb{R} \\ & U \subseteq \mathbb{R}^m \arrow[ur, "f"'] & \end{tikzcd}$$

$$D(f\circ c)(t)\cdot 1 = \big(Df(c(t))\cdot Dc(t)\big)\cdot 1 = Df(c(t))\cdot\big(Dc(t)\cdot 1\big)$$

$$(f\circ c)'(t) \qquad\qquad df(c(t))\cdot c'(t) = \frac{\partial f}{\partial x_1}(c(t))\cdot c_1'(t) + \dots + \frac{\partial f}{\partial x_n}(c(t))\cdot c_n'(t)$$

Um exemplo particular (deste exemplo) é $c(t) = p + tv$ e $u(t) = f\circ c(t)$, já sabíamos $u'(t) = df(p+tv)\cdot v$.

Cor. Nas hipóteses do TEO., matricialmente temos $J(G\circ F)(p) = JG(F(p))\cdot JF(p)$.

Com coord. $x_1,\dots,x_n \in \mathbb{R}^n$ e $y_1,\dots,y_m \in \mathbb{R}^m$, obtemos

$$\frac{\partial(G_i\circ F)}{\partial x_j}(p) = \sum_{l=1}^{m} \frac{\partial G_i}{\partial x_l}(F(p))\cdot\frac{\partial F_l}{\partial x_j}(p)$$

Cor. $F\in C^j(U,\mathbb{R}^m)$, $G\in C^j(V;\mathbb{R}^k) \Rightarrow G\circ F \in C^j(U;\mathbb{R}^k)$ → P/ demonstrar esse resultado, não precisamos de toda a força da REGRA DA CADEIA, precisamos apenas do caso particular dentro do ex. acima

Por abuso, pensando em termos de variáveis:

$$\begin{cases} y_1 = F_1(x_1,\dots,x_n) \\ \vdots \\ y_m = F_m(x_1,\dots,x_n) \end{cases} \quad \text{e} \quad \begin{cases} z_1 = G_1(y_1,\dots,y_m) \\ \vdots \\ z_k = G_k(y_1,\dots,y_m) \end{cases}$$

temos MATRICIALMENTE: $\dfrac{\partial(z_1,\dots,z_k)}{\partial(x_1,\dots,x_n)} = \dfrac{\partial(z_1,\dots,z_k)}{\partial(y_1,\dots,y_m)}\,\dfrac{\partial(y_1,\dots,y_m)}{\partial(x_1,\dots,x_n)}$

EM COORD.:

Ex. [Eq. onda dim. 1] $\dfrac{\partial^2 f}{\partial t^2} = c^2\dfrac{\partial^2 f}{\partial x^2}$, $c\in\mathbb{R}$ fixo $\left(f\in C^2(\mathbb{R}^2);\ f = f(t,x)\right)$

P/ encontrar uma sol. geral, considere a mudança de variáveis (D'ALAMBERT):

$$\begin{cases} u = x + ct \\ v = x - ct \end{cases} \iff \begin{cases} x = \frac{1}{2}(u+v) \\ t = \frac{1}{2c}(u-v) \end{cases} ; \quad g(u,v) = f(t,x) = f\left(\tfrac{1}{2}(u+v), \tfrac{1}{2}(u-v)\right)$$

Temos $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial t}\dfrac{\partial t}{\partial u} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} = \dfrac{1}{2c}\dfrac{\partial f}{\partial t} + \dfrac{1}{2}\dfrac{\partial f}{\partial x}$

f satisfaz a eq. onda

$$\frac{\partial^2 g}{\partial v\,\partial u} = \frac{1}{2c}\left(\frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial v} + \frac{\partial^2 f}{\partial x\,\partial t}\frac{\partial x}{\partial v}\right) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t\,\partial x}\frac{\partial t}{\partial v} + \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v}\right) \overset{(\dots)}{=} 0$$