Analysis in $\mathbb{R}^n$
IMECC

4th list of exercises
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Notation: If τ_X is a topology on X, then $\overline{\tau}_X$ denotes the family of closed sets in the topology τ_X. Again, $[n] =: \mathbb{N} \cap [0,1]$.

1) Let $f : \mathbb{R} \to (0,\infty)$ satisfying the conditions. Let $x, y \in f(\mathbb{R}) \subset (0,\infty)$. Como $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $q \in \mathbb{Q}$ such that $x < q < y$. Then $(0,q) \cup (q,\infty)$ is a nontrivial split of $f(\mathbb{R})$ that induces a nontrivial split on $\mathbb{R}$, contradicting the continuity of f. The only way such a function exist is that $f \equiv e$. This a continuous function that sends every real number to a transcendental number and $f(e) = e$. Thus, $f(\pi) = f(\sqrt{2}^{\sqrt{2}}) = f(e^{\sqrt{2}}) = e$.

2) a) We want to prove there exist $A, B \in \tau_{\mathbb{R}^n}$ disjoints such that $F \subset A$ and $G \subset B$. Note that $f(x) = 0$ for every $x \in F$ and $f(x) = 1$ for every $x \in G$. Thus, $Ran(f) = [0,1]$. If we prove f is continuous, every opens on $[0,1]$ of the forms $[0,a)$, $(a+\epsilon, 1]$ would separate F and G, for $a \in (0,1)$ and $\varepsilon \in \mathbb{R}^+$.

Let $C \subset \mathbb{R}^n$. Note that $g : \mathbb{R}^n \to \mathbb{R}$ defined by $x \mapsto dist(x,C)$ is continuous since

$$\begin{aligned} |g(x) - g(y)| dist(x,C) + dist(y,C) \\ &= \Big| \|x - \hat{a}\| - \|y - \hat{b}\| \Big| \\ &\leq \|x - y\|, \end{aligned}$$

for every $x, y \in \mathbb{R}^n$. To see this, note that

$$d(x,z) \leq d(x,y) + d(y,z)$$

for every $z \in \mathbb{R}^n$. Thus,

$$dist(x,C) \leq d(x,y) + dist(y,C),$$

leading us to

$$dist(x,C) - dist(y,C) \leq d(y,z).$$

Inverting the roles of x and y we get the desired conclusion.

This implies f is in fact uniformly continuous (taking $\varepsilon = \delta$).

Now, as $F, G \in \overline{\tau}_{\mathbb{R}^n}$, then $\partial F \subset F$ and $\partial G \subset G$ and thus $\partial F \cap \partial G = \emptyset$. This implies $dist(x,F) + dist(xG) \neq 0$ for every $x \in \mathbb{R}^n$. Given that $f_1(x) = dist(x,F)$ and $f_2(x) = dist(x,G)$ are continuous, then $f(x) = \frac{f_1(x)}{f_1(x)+f_2(x)}$ is a continuous function since it's the product of continuous functions where the denominator $f_1(x) + f_2(x)$ have no poles.

So, going back to the original question, let $\varepsilon \in \mathbb{R}^+$ and $a \in [0,1]$ such that $(a-\varepsilon, a+\varepsilon) \subset (0,1)$, and consider the opens subsets $[0,a), (a+\varepsilon 1] \in \tau_{[0,1]}$. As f is continuous, $A = f^{-1}([0,a))$ and $B = f^{-1}((a+\varepsilon, 1])$ are opens such that $F \subset A$, $G \subset B$ and $A \cap B = \emptyset$. Hence, f separates topologically F and G.

b) Let $F, G \subset \mathbb{R}^n$ such that $dist(F,G) > 0$. Let's prove f is uniformly continuous.

Note that

$$|f(x) - f(y)| = \left| \frac{dist(x,F)}{dist(x,F) + dist(x,G)} - \frac{dist(y,F)}{dist(y,F) + dist(y,G)} \right|.$$

Let $a_1 = dist(x,F)$, $b_1 = dist(x,G)$, $a_2 = dist(y,G)$ and $b_2 dist(y,G)$. From the right hand of the previous equality we get

$$\begin{aligned} \left| \frac{a_1}{a_1+b_1} - \frac{a_2}{a_2+b_2} \right| &= \left| \frac{a_1a_2 + a_1b_2 - a_2a_1 - a_2b_1}{(a_1+b_1)(a_2+b_2)} \right| \\ &= \left| \frac{a_1b_2 - a_2b_1 + a_1b_1 - a_1b_1}{(a_1+b_1)(a_2+b_2)} \right| \\ &= \left| \frac{a_1(b_2 - b_1) + b_1(a_1 - a_2)}{(a_1+b_1)(a_2+b_2)} \right| \\ &\leq \frac{a_1|b_2 - b_1| + b_1|a_1 - a_2|}{(a_1+b_1)(a_2+b_2)} \\ &= \frac{a_1|dist(y,G) - dist(x,G)| + b_1|dist(x,F) - dist(y,F)|}{(a_1+b_1)(a_2+b_2)} \end{aligned}$$

From the previous item we get

$$\begin{aligned}\frac{a_1\|x-y\|+b_1\|x-y\|}{(a_1+b_1)(a_2+b_2)} &\leq \frac{\|x-y\|}{a_2+b_2}\\ &\leq \frac{\|x-y\|}{dist(F,G)}.\end{aligned}$$

Hence, f is uniformly continuous.

c) Suppose f is uniformly continuous. We want to prove $dist(F,G)>0$.

   By contradiction, suppose $dist(F,G)=0$. So, there exist $a\in G\backslash F$ such that $\text{lím}_{X\to a} dis(x,F)=0$. Let $(x_n)_{n\in\mathbb{N}}\subset F$ such that $x_n\to x$ as $n\to\infty$. Let $\varepsilon\in(0,1)$. As f is continuous, there exist $\delta\in\mathbb{R}^+$ such that if $\|x-a\|<\delta$ then $|f(x)-f(a)|<\varepsilon$. Also, as $x_n\to x$, there exist $N\in\mathbb{N}$ such that $\|x_n-x\|<\delta$ whenever $n\geq N$. However, we get that

$$\begin{aligned}|f(x_n)-f(a)| &\leq |f(x_n)-1|\\ &= \left|\frac{dist(x_n,F)}{dist(x_n,F)+dist(x_n,G)}-1\right|\\ &= |-1|\geq\varepsilon\end{aligned}$$

   This contradicts the continuity of f. Hence F,G are geometrically disjoint, i.e., $dist(F,G)>0$.

d) Note that, as $B[0,r]$ and $\partial B[0,R]$ are centered at the origin, for every $x\in\mathbb{R}^n$, $dist(x,B[0,r])=\|x\|-r$ and $dist(x,\partial B[0,R])=\|x\|-R$. Thus, the Urysohn function for $B[0,r]$ and $\partial B[0,R]$, $f:\mathbb{R}^n\to[0,1]$, is given by

$$\begin{aligned}x\mapsto &\frac{dist(x,\partial B[0,R])}{dist(x,\partial B[0,R])+dist(x,B[0,r])}\\ &=\frac{\|x\|-R}{2\|x\|-(R+r)}.\end{aligned}$$

   Note that $f(x)=0$ if $x\in\partial B[0,R]$ and $f(x)=1$ if $x\in B[0,r]$

3) Before we proceed with the solution, we'll show that every $\emptyset\neq U\subset\mathbb{R}^n$ open and connected, U is path-connected.

   Attempt: As $U\subset\mathbb{R}^n$ is open, for every $x\in U$ there exist $\varepsilon\in\mathbb{R}^+$ such that $B_\varepsilon(x)\subset U$. Let $x,y\in U$ be any. Since U is open, we can choose a collection $\{x_i\}_{i\in[m]}\subset U$ such that there is a collection $\{\varepsilon_i\}_{i\in[m]}\subset\mathbb{R}^+$ satisfying $B_{\varepsilon_i}(x_i)\subset U$, $x\in B_{\varepsilon_1}(x_1)$, $y\in B_{\varepsilon_m}(x_m)$ and $B_{\varepsilon_i}(x_i)\cap B_{\varepsilon_{i+1}}(x_{i+1})\neq\emptyset$ for every $i\in[m]$. As each $B_{\varepsilon_i}(x_i)$ is convex, for every $x_i',y_i'\in B_{\varepsilon_i}(x_i)$ we have that $tx_i'+(1-t)y_i'\in B_{\varepsilon_i}(x_i)$ for every $t\in[0,1]$. Now, let $\hat{x}_i\in B_{\varepsilon_i}(x_i)\cap B_{\varepsilon_{i+1}}(x_{i+1})$. Let $\gamma_i:[0,1]\to B_{\varepsilon_i}(x_i)$ defined by $\gamma_i(t)=t\hat{x}_i+(1-t)\hat{x}_{i-1}$ for $i\in[m]\backslash\{1,m\}$, and $\gamma_1:[0,1]\to B_{\varepsilon_1}(x_1)$ defined by $\gamma_1(t)=t\hat{x}_1+(1-t)x$ and $\gamma_m:[0,1]\to B_{\varepsilon_m}(x_m)$ defined by $\gamma_1(t)=ty+(1-t)\hat{x}_{m-1}$. Then $\gamma:[0,1]\to U$ defined by

$$\gamma(t)=\begin{cases}\gamma_1(mt) & \text{for } 0\leq t\leq\frac{1}{m}\\ \gamma_2(m(m-1)t-(m+1)) & \text{for } \frac{1}{m}\leq t\leq\frac{1}{m-1}\\ \vdots & \\ \gamma_m(h(m,t)) & \text{for } \frac{1}{m-(m-2)}\leq t\leq 1,\end{cases}$$

   where $h(m,t)$ is such that $h(m,\frac{1}{2})=0$ and $h(m,1)=1$. Then γ is a piece-wise continuous map from joining x and y in U. As $x,y\in U$ are arbitrary, we conclude U is path-connected.

   Now, if U is closed and int(U) is connected, then U is path-connected since int(U) is path-connected by the same argument.

   - Let's prove $GL_n^{\pm}(\mathbb{R})$ is path-connected. For that, remember that we defined previously the linear function

$$\begin{aligned}f_{\text{det}} &: \mathbb{R}^{n^2}\to\mathbb{R}\\ X=(x_{11},\ldots,x_{nn}) &\mapsto f_{\text{det}}(X)=\sum_{\sigma\in S_n} sgn(\sigma)\prod_{i\in[n]} x_{i\sigma(i)}.\end{aligned}$$

     Then $GL_n^{\pm}=f_{\text{det}}^{-1}((-\infty,0))\cup f_{\text{det}}^{-1}((0,\infty))$, i.e., $GL_n^-=f_{\text{det}}^{-1}((-\infty,0))$ and $GL_n^+=f_{\text{det}}^{-1}((0,\infty))$. As each one of the are open since f_{det} is continuous, we have that GL_n^+ and GL_n^- are path-connected.

- Let $Diag_n(\mathbb{R})$ be the set of all diagonalizable matrices of order n with real entries. Given Any $A \in Diag_n(\mathbb{R})$, there exist a Diagonal matrix

$$D_A = \begin{pmatrix} d_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & d_n \end{pmatrix} := [d_1, \cdots, d_n]$$

  and a invertible matrix $P_A \in GL_n(\mathbb{R})$ such that $A = P^{-1}DP$. Define $\gamma : [0,1] \to Diag_n(\mathbb{R})$ by $\gamma(t) = P_A^{-1}[t + (1-t)d_1, \cdots, t + (1-t)d_n]P$. Then $\gamma(0) = P_A^{-1}AP_A = A$ and $\gamma(1) = Id_n$, and $\gamma(t) \in Diag_n(\mathbb{R})$ for every $t \in [0,1]$. Thus, given any other matrix $B \in Diag_n(\mathbb{R})$, we compose the path to the identity and the reverse path of the other to join both of the matrix with a path. This proves $Diag_n(\mathbb{R})$ is path-connected.

- Finally, let's show that $SL_n(\mathbb{R})$ is path-connected. Define $f : GL_n^+(\mathbb{R})$ by $A \mapsto \frac{1}{\det A}A$. Note that f is continuous since, letting $A = (a_{ij})_{i,j\in[n]} \in SL_n(\mathbb{R})$, the coordinate functions of f are $\frac{a_{ij}}{\det(A)}$, for $i, j \in [n]$, and $f_{\det}(A) \neq 0$ for every $A \in GL_n^+(\mathbb{R})$, implying this each one of them are continuous. Hence, given that $GL_n^+(\mathbb{R})$ is path-connected and path-connected is a topological invariant over continuous functions, we deduce $SL_n(\mathbb{R})$ is path-connected.

4) a) Let $\gamma : [0,1] \to C$ defined by $t \mapsto t\hat{x} + (1-t)\hat{y}$. Note that

$$\begin{aligned} \|\hat{x} - x + t(y - \hat{x})\| &= \|(1-t)\hat{x} + t\hat{y} - x\| \\ &\geq \|x - \hat{x}\|, \end{aligned}$$

for every $t \in [0,1]$ since $\gamma(t) \in C$.

Considering the squares of the previous expressions, we get

$$\begin{aligned} \langle a, a\rangle &\leq \langle a + tb, a + tb\rangle \\ &= \langle a, a\rangle + 2t\langle a, b\rangle + t^2\langle b, b\rangle, \end{aligned}$$

and thus $t\left(2\langle a, b\rangle + t\|b\|^2\right) \geq 0$ for every $t \in [0,1]$, where $a = \hat{x} - x$ and $b = y - \hat{x}$. Note that we took a in this form since $\langle a, b\rangle = \langle \hat{x} - x, y - \hat{x}\rangle$ and thus proving $\langle a, b\rangle \geq 0$ would be enough to prove our result. Suppose, by contradiction, that $\langle a, b\rangle < 0$. Then, for $0 \leq t < -\frac{2\langle a,b\rangle}{\|b\|^2}$, we have that $t(2\langle a, b\rangle + t\|b\|^2) < 0$, contradicting that $t\left(2\langle a, b\rangle + t\|b\|^2\right) \geq 0$ for every $t \in [0,1]$. Thus, we must have $\langle a, b\rangle \geq 0$, i.e., $\langle \hat{x} - x, y - \hat{x}\rangle = -\langle x - \hat{x}, y - \hat{x}\rangle \geq 0$. Hence, for every $y \in C$, $\langle x - \hat{x}, y - \hat{x}\rangle \leq 0$.

b) As C is closed, for every $x \in \mathbb{R}^n$ there is $\hat{x} \in C$ such that $\|x - \hat{x}\| = dist(x, C)$. Let $x, y \in \mathbb{R}$. From a) we know that $\langle x - \hat{x}, \hat{y} - \hat{x}\rangle \leq 0$ and $\langle y - \hat{y}, \hat{x} - \hat{y}\rangle \leq 0$, since $\hat{x}, \hat{y} \in C$. Then

$$\begin{aligned} \langle x - \hat{x}, \hat{y} - \hat{x}\rangle + \langle y - \hat{y}, \hat{x} - \hat{y}\rangle &= \langle x - \hat{x}, \hat{y} - \hat{x}\rangle + \langle \hat{y} - y, \hat{y} - \hat{x}\rangle \\ &= \langle (x - y) + (\hat{y} - \hat{x}), \hat{y} - \hat{x}\rangle \\ &\langle x - y, \hat{y} - \hat{x}\rangle + \|\hat{y} - \hat{x}\|^2 \leq 0. \end{aligned}$$

Thus we get

$$\begin{aligned} \|\hat{y} - \hat{x}\|^2 &\leq \langle y - x, \hat{y} - \hat{x}\rangle \\ &\overset{\text{Cauchy-Schwarz}}{\leq} \|x - y\|\|\hat{y} - \hat{x}\|. \end{aligned}$$

Hence $\|f(x) - f(y)\| = \|\hat{x} - \hat{y}\| \leq \|x - y\|$, as desired.

5) Note that $f : (0,1] \to \mathbb{R}$ defined by $x \mapsto \cos\left(\frac{1}{x}\right)$ is continuous since it's the composition of two continuous functions on $(0,1]$. Thus, the graph of f, $\mathfrak{A}$, is path-connected since for any two elements $a, b \in \mathfrak{A}$ there exits $t_0, t_1 \in (0,1]$ such that $f([t_0, t_1])$ is a continuous path joining a and b. We'll use this to prove $\mathfrak{A}$ is connected (i.e., a particular case of path-connected implies connected). Suppose, by contradiction, that $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$, with $\mathfrak{A}_1, \mathfrak{A}_2 \subsetneq \mathfrak{A}$ opens such that $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \emptyset$. Let $a \in \mathfrak{A}_1$ and $b \in \mathfrak{A}_2$ any. Since $\mathfrak{A}$ is path-connected, there exist a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = a$ and $\gamma(1) = b$, i.e., a continuous path joining a and b in $\mathfrak{A}$. By the continuity of γ we get that $[0,1] = \gamma^{-1}(X_1) \cup \gamma^{-1}(X_2)$, where $\gamma^{-1}(X_1) \cap \gamma^{-1}(X_2) = \emptyset$. This contradicts that the interval $[0,1]$ connected. Hence $\mathfrak{A}$ is connected.

Now, denoting $l = \{(0, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$ the set of accumulation points of $\mathfrak{A}$ and, given that $\mathfrak{A}$ is connected, we must have that $L = \overline{\mathfrak{A}} = \mathfrak{A} \cup l$ is connected. Explicitly, suppose $L = A \cup B$, for $A, B \in \tau_{\mathbb{R}^2}$ disjoint and nonempty. As $\mathfrak{A}$ is connected, from $\mathfrak{A} = (A \cap \mathfrak{A}) \cup (B \cap \mathfrak{A})$ we must have $\mathfrak{A} = A \cap \mathfrak{A}$ or $\mathfrak{A} = A \cap \mathfrak{B}$. Assuming $\mathfrak{A} = A \cap \mathfrak{A}$, B should be and open containing an accumulation point of $\mathfrak{A}$ and so $B \cap \mathfrak{A} \neq \emptyset$, which is a contradiction.

Let's prove L is not path-connected. Suppose $\gamma : [0,1] \to L$ is a continuous map joining such that $\gamma(0) = (0,0) \in l$ and $\gamma(1) = p \in \mathfrak{A}$. As γ is continuous, it is continuous in $(0,1)$, and there exist $\delta \in \mathbb{R}^+$ $t \in B(0,\delta)$ implies $\gamma(t) \in B(\gamma(0), \frac{1}{2})$. As $\cos(2k\pi) = 1$ and $\cos((2k+1)\pi) = -1$ for every $k \in \mathbb{Z}$, we have $(x, \cos(1/x)) = (x, 1)$ if $x = \frac{1}{2k\pi}$ and $(x, \cos(1/x)) = (x, -1)$ if $x = \frac{1}{\pi(2k+1)}$, for every $k \in \mathbb{Z}$. This values of x approaches arbitrarily to 0 when k tends to infinity, and so there are values of this form in $B(0,\delta)$. Let $t', t'' \in B(0,\delta)$ such that $p(t') = (b,1)$ y $p(t'') = (c,-1)$, with $b, c \in B(0,\delta)$. As $\|p(t')\| > \frac{1}{2}$ y $\|p(t'')\| > \frac{1}{2}$, we get a contradiction since γ is continuous. Hence, L is not path-connected.