HALTING PROBLEM AND INCOMPLETENESS THEOREM

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1. Introduction and the statement of the result

Theorem 1. *The following statements are equivalent, that is, we can prove each clause from the other clauses.*

1. *No sound, recursively axiomatizable arithmetic which contains Robinson Arithmetic Q can prove its own consistency. (Gödel's second incompleteness theorem)*
2. *Halting problem of Turing machine is undecidable.*
3. *There is no sound, recursively axiomatizable and complete arithmetic which contains Robinson Arithmetic Q. (Gödel's first incompleteness theorem)*

Proof. We prove $1 \Rightarrow 2, 2 \Rightarrow 3, 3 \Rightarrow 1$.

$(3 \Rightarrow 1)$

Let T be a sound, recursively axiomatizable arithmetic which contains Robinson Arithmetic Q. By 3, T cannot prove its own consistency Con_T. On the other hand $T \nvdash \neg Con_T$ since T is sound and $\neg Con_T$ is false in the standard model. Therefore T is not complete.

The case $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are proved in Section 2 and 3 respectively. $\square$

2. Incompleteness theorem to undecidability of halting problem

In this section, prove $1 \Rightarrow 2$, that is, second incompleteness theorem implies undecidability of halting problem.

We prove the contraposition. Assume that there is a decision procedure A of halting problem. Let T be a enough strong, sound and recursively axiomatizable theory of arithmetic. For example, we can take $T \equiv PA$. Halting problem can be formalized by the arithmetical formula $H(x)$. Then we can formalize the statement that A solves halting problem by the formula

$$\phi_A := \forall x.\{Comp(\ulcorner A \urcorner, x) = \text{true} \Leftrightarrow H(x)\}. \tag{1}$$

Let $T' := T + \phi_A$. By assumption of A, ϕ_A is true. Hence T' is sound. In particular T' is consistent.

Now, consider the program e which enumerates the theorems of T' and halt if it finds $0 = 1$. Halting problem of e and consistency of T' are equivalent, and its proof requires only a weak fragment of arithmetic. Since T' is consistent, e never terminates. Since A solves halting problem, $A(e) = \text{false}$. We assume that T contains Robinson arithmetic Q. Then,

$$T \vdash A(e) = \text{false}. \tag{2}$$

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By axiom ϕ_A, we have

$$T' \vdash \neg H(e). \tag{3}$$

If we take enough strong theory as T, from (3) we can derive

$$T' \vdash Con_{T'}. \tag{4}$$

However, T' is sound, recursively axiomatized theory of arithmetic which contains Q. This contradicts 1.

□

3. Undecidability of halting problem to incompleteness theorem

In this section, we prove $2 \Rightarrow 3$. The proof is folklore [1].

Again, we prove the contraposition. Assume that T is sound, recursively axiomatizable and complete theory of arithmetic which contains Robinson Arithmetic Q. As Section 2, let $H(x)$ be an arithmetical formula stating halting problem. Since T is complete, for given code e of a program, either $H(e)$ or $\neg H(e)$ is a theorem of T. Further, if T proves $H(e)$ then by soundness, $H(e)$ is true in the standard model, that is, e will halt, and if T proves $\neg H(e)$ then by soundness, $H(e)$ is false in the standard model, that is, e will not halt.

Consider a procedure A which searches $H(e)$ or $\neg H(e)$ from theorems of T, and if it finds $H(e)$ then returns true, and if it finds $\neg H(e)$, then returns false. Since either $H(e)$ or $\neg H(e)$ is a theorem of T, A will terminate. Further the answer of A is the correct answer to halting problem, since T is sound.

This contradicts clause 2.

□

References

[1] Halting Problem (Wikipedia). .

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