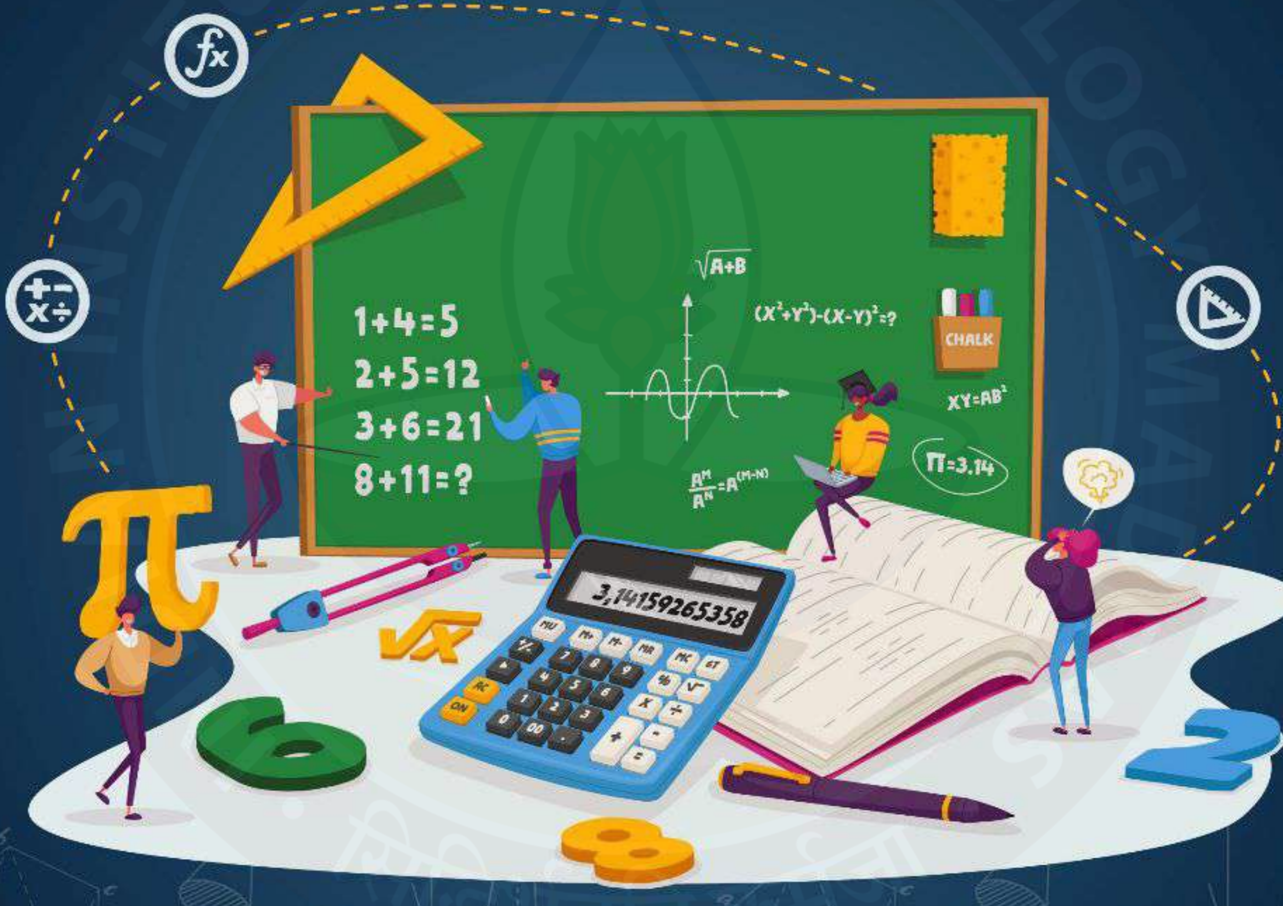
IIT MADRAS **BS DEGREE**



MATHEMATICS

MATHEMATICS FOR DATA SCIENCE-1

Sets and Functions

Created by
Venkatesan Shesha Srikanth
Vicky Kumar Sharma
Shubham Maurya

About the author

Shubham Maurya

Mr. Shubham Maurya is an instructor in IITM BS degree program of IIT Madras. He have completed his graduation from Banaras Hindu University and received BHU Medal for securing first position, and Master of Science in mathematics from prestigious IIT Madras. He have a deep understanding of mathematical concept, theories, and applications, and have delivered his mathematical ideas in a clear and accessible manner to a wide range of audiences of IITM BS degree students.

In this book, he is try to explain his mathematical idea in a such a manner that every student can easily understand the concepts with more examples so that reader get benefited from his unique understanding and inside vision.

Contents

Chapter 1

1 Set theory

1.1 Natural numbers and integers

• Numbers keep a count of objects.


• '7' stones or '7' pencils. Here, 7 represents the count.

1.1.1 Natural numbers

• The set of natural numbers is denoted by $\mathbb{N}$.
• $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$.
• The set of natural numbers includes 0.

1.1.2 Integers

• The set of integers is denoted by $\mathbb{Z}$.
• $\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$.

1.1.3 Arithmetic operations (+,-,×,÷,modulo)

• **Addition(+):** The addition operation gives the sum of two numbers. It involves combining two or more numbers into a single number.
Example $5 + 2 = 7$

• **Substraction(-):** The subtraction operation gives the difference between two numbers.
Example 9 - 4 = 5

• **Multiplication(×):** Multiplication is repeated addition.
Example $3 \times 4 = 12$
Here, $3 \times 4 = 3 + 3 + 3 + 3 = 12$.

• **Division(÷):** Division is repeated subtraction.

Example $18 \div 3 = 6$

• **Modulo:** Modulo operator or Remainder operator gives the remainder when one number is divided by another. It is denoted as "mod".

Example $10 \bmod 3 = 1$
Here, 1 is the remainder we get when we divide 10 by 3 so $10 \bmod 3 = 1$.

1.1.4 Factors

• a is a factor of b if $b \bmod a = 0$.
• b is a multiple of a.

Examples

(i) 2 is a factor of 6.

(ii) 5 is a factor of 10.

1.2 Rational numbers

• The numbers of the form $\frac{p}{q}$, where p, q are integers are called rational numbers.
• The set of rational numbers are denoted by $\mathbb{Q}$.
• The representation of rational numbers may not be unique! For example $\frac{1}{2} = \frac{2}{4} = \frac{10}{20}$.
• The reduced form of any rational number $\frac{p}{q}$ is when p, q have no factors in common.
• Rational numbers extend Natural numbers.

1.2.1 Greatest common divisor

The greatest common divisor (gcd) of two non-zero integers p and q is the greatest positive integer k such that k is a divisor of both p and q.

Examples

(i) gcd(9,12) = 3

(ii) gcd(15,45) = 15

(iii) gcd(0,q) = q, here 0 is multiple of every integer.

(iv) gcd(1,q) = 1, here 1 has no factors other than itself.

1.3 Real numbers

1.3.1 Irrational numbers

The numbers that cannot be written in the form of $\frac{p}{q}$, where p, q are integers are called irrational numbers. Simply, the numbers that are not rationals are called irrationals.

• $\sqrt{2}, \sqrt{3}, \pi$ are some examples of irrational numbers.

Real numbers: All the rational numbers including the irrational numbers are called real numbers.
• Real numbers extend rational numbers.

1.4 Sets

Definition 1.4-1 A *set* is a collection of well defined items.

Examples

Finite sets:

(i) The set of natural numbers less than 10 = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(ii) The set of all months in a year = {January, February, March, April, May, June, July, August, September, October, November, December}.

(iii) The set of all days of the week = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}.

Infinite sets:

(i) The set of all even natural numbers = $\{0, 2, 4, 6, 8, 10, \ldots\}$.

(ii) The set of integers ($\mathbb{Z}$).

• Items in a *set* are called *elements*.
• Order is not important in a set.
• Duplicates in a set does not matter.

Definition 1.4-2 The *Cardinality of a set* S is the number of elements in the set S.

Examples

(i) $S = \{1, 2, 5, 7, 9, 300\}$. The cardinality of the set S is 6.

(ii) A = {Srikanth, Keerthana, Balloon, Cell phone, π}. The cardinality of the set A is 5.

1.4.1 Subsets

Definition 1.4.1-1 A set X is a *Subset* of another set Y if every element in X is also an element in Y. It is denoted by $X \subseteq Y$.

Examples

(i) $X = \{1, 2, 5, 7, 9, 300\}$ and $Y = \{0, 1, 2, 5, 7, 9, 40, 170, 300\}$. Here, $X \subseteq Y$.

(ii) $X = \{4, 16, 32, 64\}$ and $Y = \{64, 16, 32, 4\}$. Here, $X \subseteq Y$.

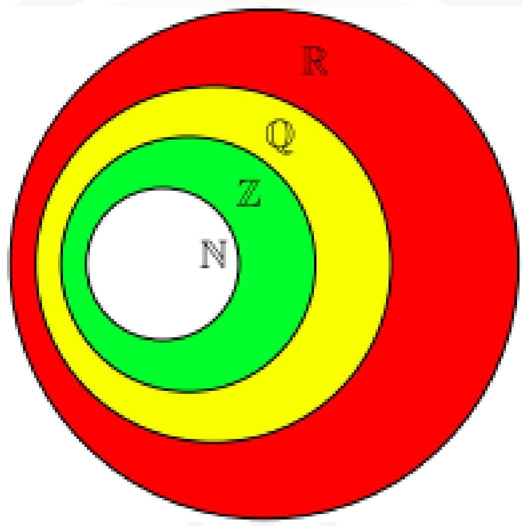
(iii) $\mathbb{N} \subseteq \mathbb{Z}, \mathbb{Z} \subseteq \mathbb{Q}, \mathbb{Q} \subseteq \mathbb{R}$.

Definition 1.4.1-2 A set X is a *proper subset* of another set Y if $X \subseteq Y$ but $X \neq Y$. It is denoted by $X \subset Y$.

(i) $X = \{1, 2, 5, 7, 9, 300\}$ and $Y = \{0, 1, 2, 5, 7, 9, 40, 170, 300\}$. Here, $X \subset Y$.

(ii) $X = \{4, 16, 32, 64\}$ and $Y = \{64, 16, 32, 4\}$. Here, $X = Y$ which implies that X is not a proper subset of Y.

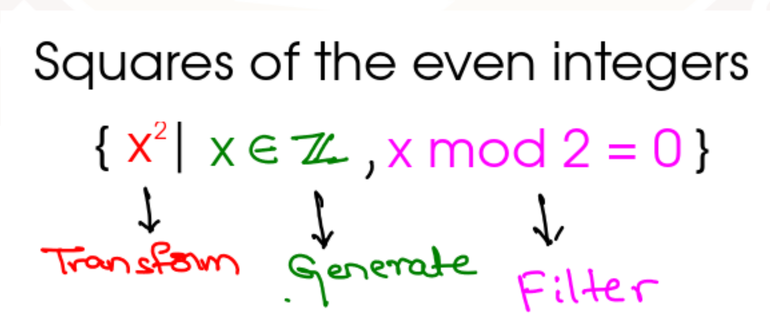
(iii) $\mathbb{N} \subset \mathbb{Z}, \mathbb{Z} \subset \mathbb{Q}, \mathbb{Q} \subset \mathbb{R}$.



1.4.2 Set comprehension

Definition 1.4.2-1 *Set comprehension* is a construction of a subset from the existing sets like $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ etc., by applying some filters on every element in the existing set. It is build with three main components. They are *generator, filter, transformer*

Example



1.5 Relations

1.5.1 Cartesian product

Definition 1.5.1-1 *Cartesian Product* of two non empty sets X and Y is defined as the set of all possible ordered pairs (x, y) such that $x \in X$ and $y \in Y$. The cartesian product is denoted as "$X \times Y$".

Example

$A = \{a, b\}$ and $B = \{1, 2, 3\}$.
The cartesian product $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

1.5.2 Binary relation

Definition 1.5.2-1 A *relation* between two sets (X and Y) is a collection of ordered pairs containing one element from each set. In other words, a *relation* R is a subset of cartesian product of X and Y ($R \subseteq X \times Y$).

Example

Suppose $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Now,
$R_1 = \{(a, 1), (b, 2), (b, 3)\}$.
$R_2 = \{(a, 2), (b, 1)\}$.
$R_3 = \{(b, 3)\}$.
$R_4 = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
Here, R_1, R_2, R_3 and R_4 are relations from set A to set B.

NOTE Order of an element is important in a relation i.e., $(a, 1) \neq (1, a)$.

1.5.3 Properties of relation

(1) Reflexive relation:- Let R be a binary relation on a set S. Then, R is said to be reflexive if and only if for all $x \in S, (x, x) \in R$.

Example

$S = \{1, 2, 3, 4\}$. Now,
$R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.
$R_2 = \{(1, 1), (2, 3), (2, 2), (4, 4), (3, 4)\}$.
$R_3 = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 4), (4, 1)\}$.
Here, R_1 and R_3 are reflexive relations but R_2 is not a reflexive relation because (3,3) $\notin R_2$.
Also, R_1 is called as identity relation.

(2) Symmetric relation:- Let R be a binary relation on a set S. Then R is said to be symmetric if and only if for every $(x, y) \in R \implies (y, x) \in R$,where $x, y \in S$.

Example

$S = \{1, 2, 3, 4\}$. Now,

$R_1 = \{(1,2),(2,1),(3,4),(4,3),(2,2)\}$.
$R_2 = \{(1,2),(2,3),(2,1),(3,2),(4,3),(3,4)\}$.
$R_3 = \{(2,1),(1,2),(3,4),(2,4),(4,2)\}$.
Here, R_1 and R_2 are symmetric relations but R_3 is not a symmetric relation because (3,4) $\in R_3$ but $(4,3) \notin R_3$.

(3) Transitive relation:- Let R be a binary relation on a set S. Then R is said to be transitive if and only if for every pair of elements, (x,y) and $(y,z) \in R \implies (x,z) \in R$, where $x,y,z \in S$.

Example

$S = \{1,2,3,4\}$. Now,
$R_1 = \{(1,2),(2,3),(1,3)\}$.
$R_2 = \{(1,4)\}$.
$R_3 = \{(1,3),(1,4)\}$.
$R_4 = \{(2,4),(1,2),(1,4),(4,1),(1,1),(2,1)\}$.
$R_5 = \{(1,2),(2,1),(1,1),(2,3)\}$ Here, R_1, R_2, R_3 and R_4 are transitive relations but R_5 is not a transitive relation because (1,2) , (2,3) $\in R_5$ but (1,3) $\notin R_5$.

(4) Equivalence relation:- If R is Reflexive, Symmetric, and Transitive, then R is an equivalence relation.

Example

$S = \{1,2,3,4\}$. Now,
$R_1 = \{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2),(4,1),(1,4)\}$.
Here, R_1 is an equivalence relation.

Problem:

Match the following relations with its properties

Relation defined on a set S = { 1,2,3}	Property
$R_1 = \{(1,3),(3,2),(1,1),(2,3),(3,1)\}$	(1) Reflexive relation
$R_2 = \{(1,1),(2,3),(3,2),(2,2),(3,3)\}$	(2) Symmetric relation
$R3 = \{(3,1),(1,1),(2,1),(2,3),(3,3),(2,2)\}$	(3) Transitive relation
$R4 = \{(3,2),(3,1),(2,1)\}$	(4) Equivalence relation

1.6 Functions

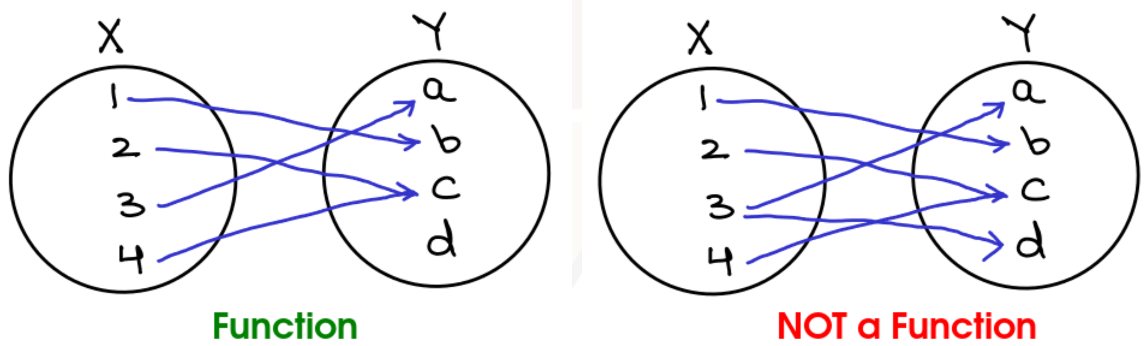
Definition 1.6-1 A *function* is a relation in which each element in set X is mapped/paired with exactly one element in set Y.

- In a function, there should not be two pairs with the same first element.
- In a function, a particular input is given to get a particular output. So, A function

$f : X \longrightarrow Y$ denotes that f is a function from set X to set Y, where X is the domain and Y is the co-domain.

- **Domain of** f= Set of all input values (set X).
  Co-domain of f= Set of all possible output values (set Y).
  Range of f= $\{y \mid y \in Y, y = f(x) \text{ for some } x \in X\}$.

Example



1.6.1 Types of functions

(1) Injective function: If each element in set X is mapped to a distinct element in set Y, then the function is called an injective function or one-to-one function.

(2) Surjective function: If the range is equal to the co-domain, then the function is called a surjective function or onto function.

(3) Bijective function: If the function is both injective and surjective, then it is called a bijective function.

1.6.2 Finding domain and range of a function

(1) If the co-domain is $\mathbb{R}$, then find the domain of the function $f(x) = \sqrt{x}$.
Solution: $\sqrt{x}$ is well-defined only if $x \geq 0$. So, the domain of f is $[0, \infty)$.

(2) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $f(x) = x^2$. Find the range of f and check whether the function f is surjective or not.
Solution: We know that set of all possible output values are positive real numbers including zero. Therefore, the range of the given function is $[0, \infty)$.

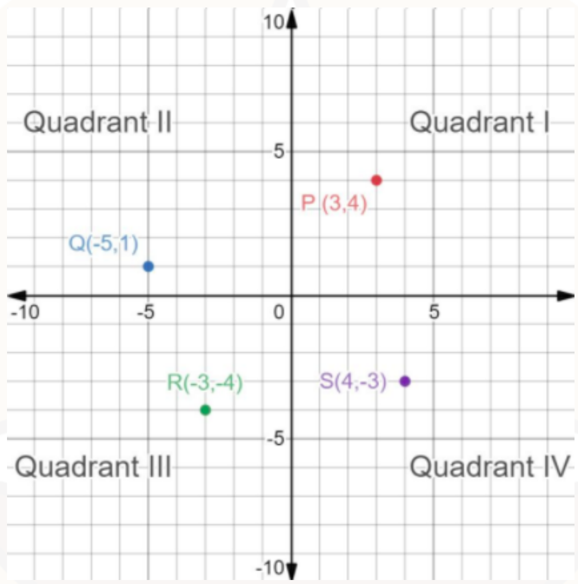
Now, the co-domain is $\mathbb{R}$ and range is $[0, \infty)$. Here co-domain is not equal to the range so the function is not surjective.

Chapter 2

2 Straight lines

2.1 Rectangular coordinate system

A **Cartesian coordinate system** is a coordinate system which specifies each point in a plane by a set of numerical coordinates. These coordinates are the signed distances to the point from two fixed perpendicular oriented lines, measured in the same unit of length. Each reference line is called an **axis** (plural axes) of the system, and the point where two axes meet is called **origin**, whose coordinate is (0, 0).



The coordinate axes split the coordinate plane into four quadrants and two axes.

Quadrant	Abscissa (X-axis)	Ordinate (Y-axis)
I	+ve	+ve
II	-ve	+ve
III	-ve	-ve
IV	+ve	-ve

2.2 Distance between any two points

The **distance** between any two points (x_1,y_1) and (x_2,y_2) in the Cartesian plane(XY plane) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Solved Examples**

(1) Find the distance between the two points (2,4) and (-4,12).

Solution:
We know that, the distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Here, $x_1=2, y_1=4, x_2=-4, y_2=12$.
By using the formula, the distance between the two points (2,4) and (-4,12) will be

$$\sqrt{((-4)-2)^2+(12-4)^2}=\sqrt{(-6)^2+(8)^2}=\sqrt{36+64}=\sqrt{100}=\mathbf{10}$$

(2) If the distance between the two points (-3,y) and (1,4) is 5 units, then find the possible values of y.

Solution:
We know that, the distance between any two points (x_1,y_1) and (x_2,y_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Here, $x_1=-3, y_1=y, x_2=1, y_2=4$.
By using the formula, the distance between the two points (-3,y) and (1,4) will be

$$\sqrt{(1-(-3))^2+(4-y)^2}=\sqrt{(4)^2+(4-y)^2}$$

But, it is given that the distance is 5 units.
Therefore, $\sqrt{(4)^2+(4-y)^2}=5 \implies (4-y)^2=9 \implies 4-y=\pm 3 \implies y=\mathbf{1,7}$

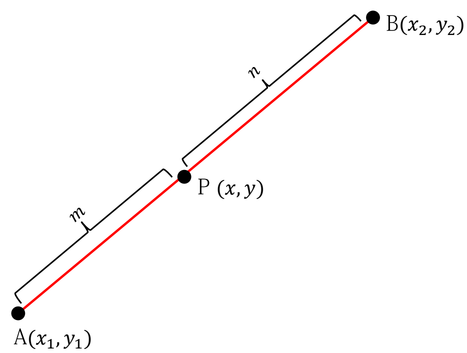
Hence, the possible values of y are 1 and 7.

2.3 Section formula:

If a point P (x, y) **internally** cuts the line segment AB, which connects two points A (x_1,y_1) and B (x_2,y_2) in a ratio $m:n$, then the value of x and y will be

$$x=\frac{mx_2+nx_1}{m+n}$$

$$y=\frac{my_2+ny_1}{m+n}$$

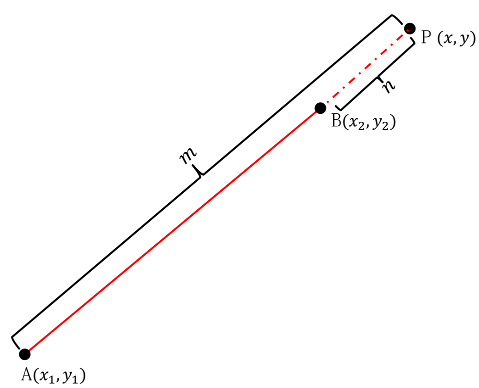


Internal section in the ratio $m : n$

If a point P (x, y) **externally** cuts the line segment AB, which connects two points A (x_1,y_1) and B (x_2,y_2) in a ratio $m : n$, then the value of x and y will be

$$x = \frac{mx_2 - nx_1}{m - n}$$

$$y = \frac{my_2 - ny_1}{m - n}$$



External section in the ratio $m : n$

- **Solved Examples**

(1) Find the point that divides the line segment $P(2, 5)$ and $Q(8, 8)$ internally in the ratio 1:2.

Solution:

Let $S(x, y)$ be the point that divides the line segment PQ in the ratio 1:2.
By using the section formula, the $X-$ coordinate of the point S will be

$$x = \frac{mx_2 + nx_1}{m+n} = \frac{1(8) + 2(2)}{1+2} = \frac{12}{3} = 4$$

Similarly, the $Y-$ coordinate of the point S will be

$$y = \frac{my_2 + ny_1}{m+n} = \frac{1(8) + 2(5)}{1+2} = \frac{18}{3} = 6$$

Hence, the point **(4,6)** divides the given line segment PQ internally in the ratio 1:2.

(2) Find the coordinates of the midpoint of points P(4, -2) and Q(0, 2).

Solution:
We know that, the mid point divides two points in the ratio 1:1.

Therefore, use the section formula by taking $m = 1$ and $n = 1$. The $X-$ coordinate of the mid point of points (4,-2) and (0,2) will be

$$x = \frac{1(0) + 1(4)}{1+1} = \frac{4}{2} = 2$$

Similarly, the $Y-$ coordinate of the mid point of points (4,-2) and (0,2) will be

$$y = \frac{1(2) + 1(-2)}{1+1} = \frac{0}{2} = 0$$

Hence, the mid point of the points P(4, -2) and Q(0,2) is **(2,0)**

2.4 Area of a triangle:

The **area of a triangle** (Δ) formed by three points, (x_1,y_1) , (x_2,y_2) and (x_3,y_3) in XY plane is

$$\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

We use modulus because area of any region is always positive.

• **Solved Examples**

(1) What is the area of the triangle formed by the points, $P(0, 10), Q(-20, -30)$ and $R(10, 30)$.

Solution:

The area of the triangle formed by the points (0,10), (-20,-30) and (10,30) will be

$$\begin{aligned}\Delta &= \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)| \\ &= \frac{1}{2}|0((-30)-30)+(-20)(30-10)+10(10-(-30))| \\ &= \frac{1}{2}|0-20(20)+10(40)| \\ &= \frac{1}{2}|-400+400| \\ &= 0\end{aligned}$$

Hence, the area of the triangle formed by the points $P(0,10), Q(-20,-30)$ and $R(10,30)$ is **0**.

Note: If the area of the triangle formed by three points P, Q, and R is zero, then the three points P, Q and R are colinear.

(2) What is the area of the triangle formed by the midpoints of line segments PQ, QR, and RP where the coordinates of P, Q, and R are (0,0), (4,0), and (4,3) respectively?

Solution:
By using the section formula,
the mid points of the line segments PQ, QR, and RP are (2,0), (4,1.5) and (2,1.5) respectively.
Now, the area of the triangle formed by the points (2,0), (4,1.5) and (2,1.5) will be

$$\begin{aligned}\Delta &= \frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)| \\ &= \frac{1}{2}|2(1.5-1.5)+4(1.5-0)+2(0-1.5)| \\ &= \frac{1}{2}|2(0)+4(1.5)+2(-1.5)| \\ &= \frac{1}{2}|0+6-3| \\ &= 1.5\end{aligned}$$

Hence, the area formed by the midpoints of the line segments PQ, QR, and RP is **1.5 sq.units**

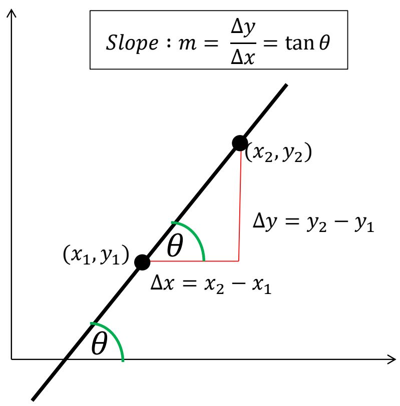
2.5 Straight lines

2.5.1 Slope of a straight line

Slope of a straight line (denoted by m) describes both direction and steepness of a line. Numerically, slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1} = \tan\theta,$$

where (x_1, y_1) and (x_2, y_2) are two points on the line and θ is the inclination of the line with respect to the positive X-axis.



- **Solved Example**

(1) What is the slope of the line passing through the origin and the point (-3,5)?

Solution:
The coordinates of the origin is (0,0). So, the slope of the line passing through the points (0,0) and (-3,5) is $\dfrac{5-0}{-3-0} = \frac{\mathbf{-5}}{\mathbf{3}}$

- **Characterization of parallel lines via slope:**

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively. The two non-vertical lines l_1 and l_2 are parallel if and only if their slopes are equal, i.e. $m_1 = m_2$.

- **Characterization of perpendicular lines via slope:**

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 with inclinations α and β respectively. The two non-vertical lines l_1 and l_2 are perpendicular if and only if the product of their slopes is -1, i.e. $m_1 m_2 = -1$.

2.5.2 Equation of a straight line

In general the equation of the straight lines is often given in the *slope-intercept form*:

$$y = mx + c$$

where:
m is the slope or gradient of the line.
c is the y-intercept of the line.
Point-slope form of straight line
The equation of a straight line having slope m and passing through a point (x_1, y_1) is

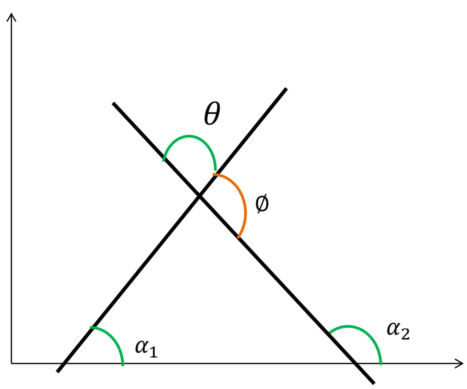
$$(y - y_1) = m(x - x_1)$$

Two-point form of straight line
The equation of a line connecting two points (x_1, y_1) and (x_2, y_2) is

$$(y - y_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

2.5.3 Angle between two lines

Let l_1 and l_2 be two lines with slopes m_1 and m_2 and inclinations α_1 and α_2 respectively (let us assume, $\alpha_2 > \alpha_1$).



If l_1 and l_2 intersect each other forming adjacent angles ϕ and θ, then

$$tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$tan\phi = \frac{m_1 - m_2}{1 + m_1 m_2}$$

2.6 Different forms of equations of a straight line

Forms of equation of straight line	Representation
General form	$ax + by + c = 0$
Slope-point form	$(y - y_0) = m(x - x_0)$
Slope-intercept form (y-intercept)	$y = mx + c$
Slope-intercept form (x-intercept)	$y = m(x - d)$
Intercept form	$\frac{x}{a} + \frac{y}{b} = 1$
Two-point form	$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$

2.7 Condition for parallel and perpendicular lines

Given two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $b_1, b_2 \neq 0$,

- The lines are parallel to each other, if $a_1 \times b_2 = a_2 \times b_1$.
- The lines are perpendicular to each other, if $a_1 \times a_2 = -b_1 \times b_2$.

2.8 Distance of a line from a given point

The distance(d) of a straight line $ax + by + c = 0$ from a point (x_1, y_1) is

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

2.9 Distance between two parallel lines

The distance(D) between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is

$$D = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

lc1-c2l / root(1+m^2)

2.10 Sum Squared Error (SSE)

SSE is the sum of the squares of the deviations of the predicted linear model from the actual data set. Numerically, if we are given a set of n points (x_i, y_i), $i = 1, 2, 3, ..., n$ and we have a line of fit $y = mx + c$, then the SSE will be calculated as

$$SSE = \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Solved examples

(1) A line fit $y = 2x + 2$ is given for the data as shown in the below table. Compute the sum squared error(SSE).

x	1	2	4	9
y	5	6	9	18

Solution
Here 4 points are given and $m = 2, c = 2$. So,

$$
\begin{aligned}
\text{SSE} &= \sum_{i=1}^{4}(y_i - 2x_i - 2)^2 \\
&= (y_1 - 2x_1 - 2)^2 + (y_2 - 2x_2 - 2)^2 + (y_3 - 2x_3 - 2)^2 + (y_4 - 2x_4 - 2)^2 \\
&= (5 - 2(1) - 2)^2 + (6 - 2(2) - 2)^2 + (9 - 2(4) - 2)^2 + (18 - 2(9) - 2)^2 \\
&= (5 - 2 - 2)^2 + (6 - 4 - 2)^2 + (9 - 8 - 2)^2 + (18 - 18 - 2)^2 \\
&= (1)^2 + (0)^2 + (-1)^2 + (-2)^2 \\
&= 1 + 0 + 1 + 4 = \mathbf{6}
\end{aligned}
$$

Chapter 3

3 Quadratic Function

A quadratic function is described as

$$f(x) = ax^2 + bx + c \text{ where } a \neq 0$$

- The curve representing any quadratic function is always a parabola. A simple example of parabola is shown in Figure 1.

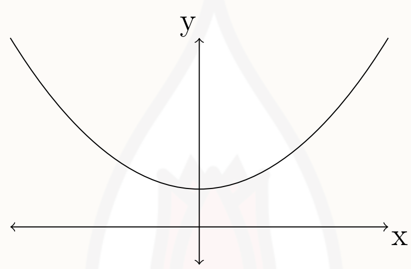


Figure 1 : A parabola

3.1 Important observations

3.1.1 Axis of symmetry

All parabolas have an axis of symmetry. That is, if the graph paper containing the graph of parabola is folded along the axis of symmetry the portion of parabola on either sides will exactly match each other.

- The equation of axis of symmetry of a parabola : $x = \dfrac{-b}{2a}$.

3.1.2 Vertex of a parabola

The point at which the axis of symmetry intersects the parabola is called the vertex.

- The x-coordinate of the vertex of a parabola is $\dfrac{-b}{2a}$.

- The y-coordinate of the vertex of a parabola is $f\left(\dfrac{-b}{2a}\right)$.

3.1.3 Types of parabola

A parabola will

- open towards positive y-axis and has minimum value, if $a > 0$. This is called **Upward parabola**.[Figure 2 : (I)]
- open towards negative y-axis and has maximum value if $a < 0$. This is called **Downward parabola**.[Figure 2 : (II)]

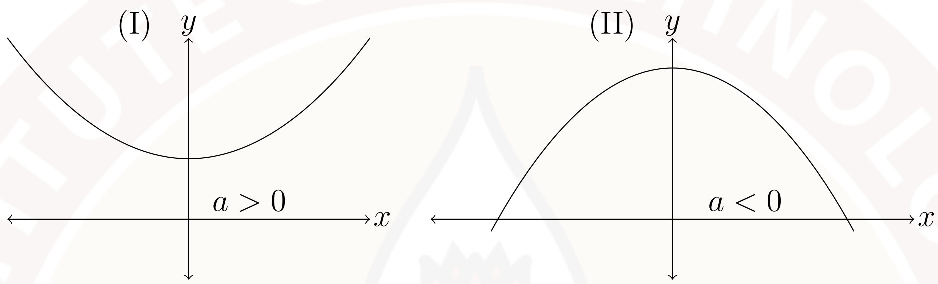


Figure 2 : Two parabolas for (I) $a > 0$ and (II) $a < 0$

3.2 Slope of a quadratic function

For the quadratic function described as $f(x) = ax^2 + bx + c$ where $a \neq 0$, the slope of f at any given point $(x, f(x))$ is $(2ax + b)$.

3.3 Quadratic equation

- If a quadratic function is set equal to a value, then the result is a quadratic equation.
- If $ax^2 + bx + c = 0$, with $a \neq 0$, and a, b, c are integers, then the quadratic equation is said to be in **standard form**.
- The solutions to a quadratic equation are called **roots** of the equation.
- One method for finding the roots of a quadratic equation $f(x) = ax^2 + bx + c = 0$ where $a \neq 0$ and a, b, c are integers, is to find **Zeros** of the quadratic function $f(x)$.

Note: Zeros of a quadratic function $f(x)$ are the x-intercepts of the curve represented by the function $f(x)$ and these are the solutions of the equation $f(x) = 0$.

general formula : (y-h)=a(x-k)

3.3.1 Methods of solving a quadratic equation

(1) **Solve by factoring**

If the quadratic polynomial can be factored, the **Zero Product Property** may be used. This property states that when the product of two factors equals zero, then at least one of the factors is zero.

Steps to solve quadratic equations by factoring

- Write the equation in standard form (equal to 0).
- Factor the polynomial.
- Use the Zero Product Property to set each factor equal to zero.
- Solve each resulting linear equation

Solved examples

(i) Solve the quadratic equation $x^2 + 2x = 24$ by factoring.
Solution:
step-1: $x^2 + 2x - 24 = 0$
step-2:
$x^2 + 6x - 4x - 24 = 0 \implies x(x+6) - 4(x+6) = 0 \implies (x-4)(x+6) = 0$
step-3: $x - 4 = 0$ or $x + 6 = 0$
step-4: $x = 4$ or $x = -6$.

(ii) Solve the quadratic equation $4x^2 + 9x = 9$ by factoring.
Solution:
step-1: $4x^2 + 9x - 9 = 0$
step-2:
$4x^2 + 12x - 3x - 9 = 0 \implies 4x(x+3) - 3(x+3) = 0 \implies (4x-3)(x+3) = 0$
step-3: $4x - 3 = 0$ or $x + 3 = 0$
step-4: $x = \frac{3}{4}$ or $x = -3$.

(2) **Solve by completing the square**

Steps to solve quadratic equations by completing the square

- Transform the equation so that a perfect square is on one side and a constant is on the other side of the equation.
- Take square root on each side. REMEMBER that finding the square root of a constant yields positive and negative values.
- Solve each resulting equation. (If you are finding the square root of a negative number, then there is no real solution)

Solved examples

(i) Solve the quadratic equation $x^2 + 2x = 24$ by completing the square.
Solution:
step-1: $x^2 + 2x + 1 - 1 = 24$
step-2: $(x+1)^2 = 25 \implies x + 1 = \pm\sqrt{25}$
step-3: $x + 1 = +5$ or $x + 1 = -5$
step-4: $x = 4$ or $x = -6$.

(ii) Solve the quadratic equation $4x^2 + 9x = 9$ by completing the square.
Solution:
step-1: $4x^2 + 2.2.\frac{9}{4}x + \frac{81}{16} - \frac{81}{16} = 9$
step-2: $(2x + \frac{9}{4})^2 = \frac{225}{16} \implies 2x + \frac{9}{4} = \pm\sqrt{\frac{225}{16}}$
step-3: $2x + \frac{9}{4} = \frac{15}{4}$ or $2x + \frac{9}{4} = \frac{-15}{4}$
step-4: $x = \frac{3}{4}$ or $x = -3$.

(3) **Solve by quadratic formula**

The roots of a quadratic equation $ax^2 + bx + c = 0$ can be found directly by using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The above formula is called as quadratic formula and $b^2 - 4ac$ is called the discriminant.

Number of real roots depending on the value of the discriminant

Value of the discriminant	Number of real roots
$b^2 - 4ac > 0$	two real roots
$b^2 - 4ac = 0$	one real root
$b^2 - 4ac < 0$	No real roots

Solved examples

(i) Solve the quadratic equation $x^2 + 2x - 24 = 0$ by using the quadratic formula.
Solution:
Here, $a = 1, b = 2$, and $c = -24$. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-24)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{100}}{2} \\ &= \frac{-2 + 10}{2} \text{ or } \frac{-2 - 10}{2} \\ &= 4 \text{ or } -6 \end{aligned}$$

(ii) Solve the quadratic equation $4x^2 + 9x - 9 = 0$ by using the quadratic formula.
Solution:
Here, $a = 4, b = 9$, and $c = -9$. Using the quadratic formula, we get

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-9 \pm \sqrt{9^2 - 4(4)(-9)}}{2(4)} \\
&= \frac{-9 \pm \sqrt{225}}{8} \\
&= \frac{-9 + 15}{8} \text{ or } \frac{-9 - 15}{8} \\
&= \frac{3}{4} \text{ or } -3
\end{aligned}$$

Chapter 4

4 Polynomial function

The word polynomial is derived from the word polynomen, which means poly (many), nomen (name). In this chapter we will be dealing with single variable polynomial.

4.1 Definition of polynomial function:

What is a Polynomial?
A Layman's Perspective: A polynomial is a mathematical expression that consists of numerous mathematical terms added together.
Definition: (A mathematician's Perspective) A polynomial is an algebraic expression in which the only arithmetic is addition, subtraction, multiplication and "natural" exponents of the variables.

- A polynomial function $f(x)$ of degree n is described as

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \dots a_0 x^0, \text{ where } a_n \neq 0 \text{ and } n \in \mathbb{N}$$

  - The above expression can be treated as function from $\mathbb{R} \to \mathbb{R}$.
  - The domain of $f(x)$ is $\mathbb{R}$ and the range depends on the function $f(x)$.

Example: Identification of a polynomial:

4.2 Classification of polynomials:

4.2.1 Based on the number of variables:

Type of polynomial	Example
Polynomial in one variable	$5x^3 + x^2 + x$
Polynomial in two variable	$5x^3y + x^2 + xy$
Polynomial in more than two variable	$5x^3yz + x^2zy + xy$

4.2.2 Based on the degree of polynomial:

The degree of the polynomial

- The exponent on the variable in a term is called the degree of that variable in that term.
  Example: Consider a polynomial in two variables $x^3 + x^5y^4$, the degree of the variable x in the term x^5y^4 is 5.

- The degree of that term is the sum of the degrees of the variables in that term.
  Example: Consider a polynomial in two variables $x^3 + x^5y^4$, the degree of the term x^5y^4 is 9.

- The degree of the polynomial is the highest degree of any one of the terms with non-zero coefficients.
  Example: Consider $f(x) = x^3 + x^{10}$, the degree of the polynomial $f(x)$ is 10.

- The degree of zero polynomial is undefined.

- The various name of the polynomial based on the degree of the polynomial is shown below:

Degree	Name	Example
0	Constant polynomial	$5, 6, e, \pi$
1	Linear polynomial	$2x + 4, -3x$
2	Quadratic polynomial	$2x^2 + 4, -3x^2$
3	Cubic polynomial	$2x^3 + 4, -3x^3$
4	Quartic polynomial	$2x^4 + 4, -3x^4$

4.2.3 Based on the number of terms:

Name	Explanation	Example
Monomial	Polynomial with one term	$x^{10}, 6x^9, e, 5 + 4$
Binomial	Polynomial with two term	$x^{10} + 10, 6x^9 - x^2, x^2 + 3x + x$
Trinomial	Polynomial with three term	$x^{10} + x^2 + 10, 6x^9 - x^2 + 2, x^2 + x + e$

4.3 Operations on polynomial function:

4.3.1 Addition of polynomial:

When two polynomials are added, the like terms in the two polynomials are combined. We use the term "like terms" to refer to terms that have the same variable and exponent. For instance, two terms are similar only if they share the same variable.

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k) x^k,$$

Here $m \vee n$ denotes whichever is maximum.

Solved example:

Q1. The sum of the two cubic polynomials $p(x) = \sum_{k=0}^{3} a_k x^k$ and $q(x) = \sum_{j=0}^{3} b_j x^j$, using the addition algorithm is

Solution:

We know that $p(x) + q(x) = \sum_{k=0}^{m \vee n} (a_k + b_k)\, x^k$, $m \vee n$ denotes whichever is maximum, so, $m \vee n = 3$. On expanding we get,

$\sum_{k=0}^{3} (a_k + b_k)\, x^k = (a_3 + b_3)\, x^3 + (a_2 + b_2)\, x^2 + (a_1 + b_1)\, x^1 + (a_0 + b_0)\, x^0$

Q2. The sum of the two cubic polynomials $p(x) = x^3 + 3x^2 + 5x - 10$ and $q(x) = 3x^3 + 5x^2 - 6x - 20$ is

Solution:

Using addition algorithm to find $p(x) + q(x)$

$\sum_{k=0}^{3}(a_k \ + \ b_k) \ x^k = (a_3 \ + \ b_3) \ x^3 \ + \ (a_2 \ + \ b_2) \ x^2 \ + \ (a_1 \ + \ b_1) \ x^1 \ + \ (a_0 \ + \ b_0) \ x^0$

$\implies (1+3) \ x^3 \ + \ (3+5) \ x^2 \ + \ (5+(-6)) \ x^1 \ + \ (-10+(-20)) \ x^0 \implies 4x^3+8x^2-x-30$

4.3.2 Subtraction of polynomial:

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x) - q(x) = \sum_{k=0}^{m \vee n} (a_k - b_k)x^k,$$

Here $m \vee n$ denotes whichever is maximum.

Like in solved examples of addition of polynomial, a similar subtraction operation can be done here too.

Solved examples:

Q1. Subtract $q(x) = -3x^2 + 2x - 2$ from $p(x) = 1x^3 + 2x^2 + 8x$

$$\begin{array}{c} 1x^3 + 2x^2 + 8x + 0 \\ -(0x^3 - 3x^2 + 2x - 2) \\ \hline \end{array}$$

$$p(x) - q(x) = x^3 + (2+3)x^2 + (8-2)x + 2 = x^3 + 5x^2 + 6x + 2$$

4.3.3 Multiplication of polynomial:

$$\text{Let } p(x) = \sum_{k=0}^{n} a_k x^k \text{ and } q(x) = \sum_{j=0}^{m} b_j x^j. \text{ Then } p(x)q(x) = \sum_{k=0}^{m+n} \sum_{j=0}^{k} (a_j - b_{k-j})x^k,$$

Solved example:

Q. Multiply the polynomials $p(x) = x^2 + x + 1$ and $q(x) = x^2 + 2x + 1$

Solution:

The product of $p(x)$ and $q(x)$ can be done using multiplication algorithm.

For different values of k, the corresponding a_k and b_k are shown the below table

k	a_k	b_k
0	1	1
1	1	2
2	1	1

And for different values of k, the corresponding coefficients and calculations are shown the below table

k	Coefficient	Calculations
0	a_0b_0	1
1	$a_1b_0 + a_0b_1$	$1 + 2 = 3$
2	$a_0b_2 + a_1b_1 + a_2b_0$	$1 + 2 + 1 = 4$
3	$a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0$	$0 + 1 + 2 + 0 = 3$
4	$a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0$	$0 + 0 + 1 + 0 + 0 = 1$

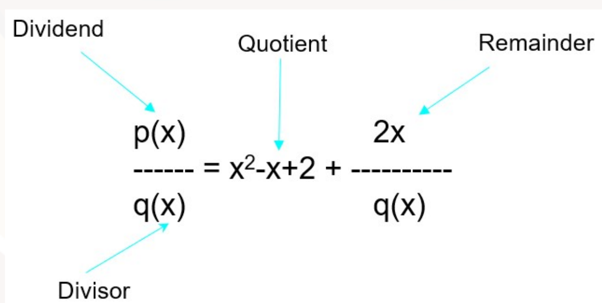
The resultant polynomial is: $p(x)q(x) = x^4 + 3x^3 + 4x^2 + 3x + 1$

4.3.4 Division of polynomial:

The division of a polynomial by a monomial, binomial, or another polynomial using various methods is known as polynomial division.
For example divide $p(x) = x^4 + 2x^2 + 3x + 2$ by $q(x) = x^2 + x + 1$.

$$\begin{array}{r|rrrrr}
 & & & x^2 & -x & +2 \\ \hline
x^2+x+1 & x^4 & & +2x^2 & +3x & +2 \\
 & -x^4 & -x^3 & -x^2 & & \\ \cline{2-4}
 & & -x^3 & +x^2 & +3x & \\
 & & x^3 & +x^2 & +x & \\ \cline{3-5}
 & & & 2x^2 & +4x & +2 \\
 & & & -2x^2 & -2x & -2 \\ \cline{4-6}
 & & & & 2x &
\end{array}$$

Some terminologies:



One of the method to perform polynomial division is Division algorithm which is discussed below.

Division Algorithm:

Step 1. Arrange the terms in descending order of the degree and add the missing exponents with 0 as coefficient.

Step 2. Divide the first term of the dividend by the first term of the divisor and get the monomial.

Step 3. Multiply the monomial with divisor and subtract the result from the dividend.

Step 4.Check if the resultant polynomial has degree less than divisor. If true, write the remainder else go to Step 2.

Example:

Q1. If $p(x) = x^5 - 4x^4 + 3x^3 + 3x^2 - 4$ and $q(x) = (x-2)^2$, then $\frac{p(x)}{q(x)}$ is

a $x^3 + x + 1$

b $x^4 - x^2 - 1$

c $\boldsymbol{x^3 - x - 1}$

d $x^4 - x^2 + 1$

Solution:

The quotient when polynomial $p(x)$ is divided by another polynomial $q(x) = (x-2)^2 = x^2 - 4x + 4$ can be obtained by division algorithm.

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x^3 \quad\quad\quad - x - 1 \\
\hline
x^2 - 4x + 4 & x^5 - 4x^4 + 3x^3 + 3x^2 \quad\quad - 4 \\
 & \underline{-x^5 + 4x^4 - 4x^3} \\
 & \quad\quad\quad\quad -x^3 + 3x^2 \\
 & \quad\quad\quad\quad \underline{x^3 - 4x^2 + 4x} \\
 & \quad\quad\quad\quad\quad\quad -x^2 + 4x - 4 \\
 & \quad\quad\quad\quad\quad\quad \underline{x^2 - 4x + 4} \\
 & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}
$$

Clearly, the answer is $\boldsymbol{x^3 - x - 1}$

Q2. What should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$?

a $4x$

b $\boldsymbol{4x - 3}$

c $6x - 3$

d $2x - 3$

Solution:

Using 4 step division algorithm, we find the remainder when $P(x)$ is divided by $2x^2 + x - 1$. If we subtract the obtained remainder from $P(x)$ then the resultant polynomial will be divisible by $2x^2 + x - 1$.

Now,

$$\begin{array}{r}
3x^2 \quad + x + 1 \\
2x^2 + x - 1 \overline{\big) \; 6x^4 + 5x^3 \qquad\quad + 4x - 4} \\
\underline{-\,6x^4 - 3x^3 + 3x^2 \qquad\qquad} \\
2x^3 + 3x^2 + 4x \qquad \\
\underline{-\,2x^3 \;\; - x^2 \;\; + x} \qquad \\
2x^2 + 5x - 4 \\
\underline{-\,2x^2 \;\; - x + 1} \\
4x - 3
\end{array}$$

Therefore, when $P(x)$ is divided by $2x^2 + x - 1$, we get $4x - 3$ as the remainder.
Hence, $\boldsymbol{4x - 3}$ should be subtracted from the polynomial $P(x) = 6x^4 + 5x^3 + 4x - 4$ to make it divisible by $2x^2 + x - 1$.

Q3. By dividing a polynomial $p(x)$ with another polynomial $q(x)$ we get $h(x)$ as the quotient and $r(x)$ as the remainder.

(a) The maximum degree of $r(x)$ can be,

- ◯ $deg\ p(x)$
- ◯ $deg\ (p(x)) - 1$
- ◯ $deg\ q(x)$
- ◯ $\boldsymbol{deg\ (q(x)) - 1}$

(b) If $deg\ p(x) < deg\ q(x)$, then choose the set of correct answers:

- ◯ $\boldsymbol{h(x) = 0}$
- ◯ $deg\ h(x) = deg\ q(x)$
- ◯ $deg\ r(x) = deg\ q(x)$
- ◯ $\boldsymbol{deg\ r(x) = deg\ p(x)}$

Solution:

(a) The degree of the remainder $r(x)$ should be strictly less than the degree of the polynomial $q(x)$. So the maximum degree of $r(x)$ is $deg\ (q(x)) - 1$.

(b) If $deg\ p(x) < deg\ q(x)$, then quotient will be zero polynomial, hence $deg\ h(x) = 0$. The remainder will be $p(x)$ itself. So $deg\ r(x) = deg\ p(x)$.

Q4. If the polynomials $x^3 + ax^2 + 5x + 7$ and $x^3 + 2x^2 + 3x + 2a$ leave the same remainder when divided by $(x - 2)$, then the value of a is

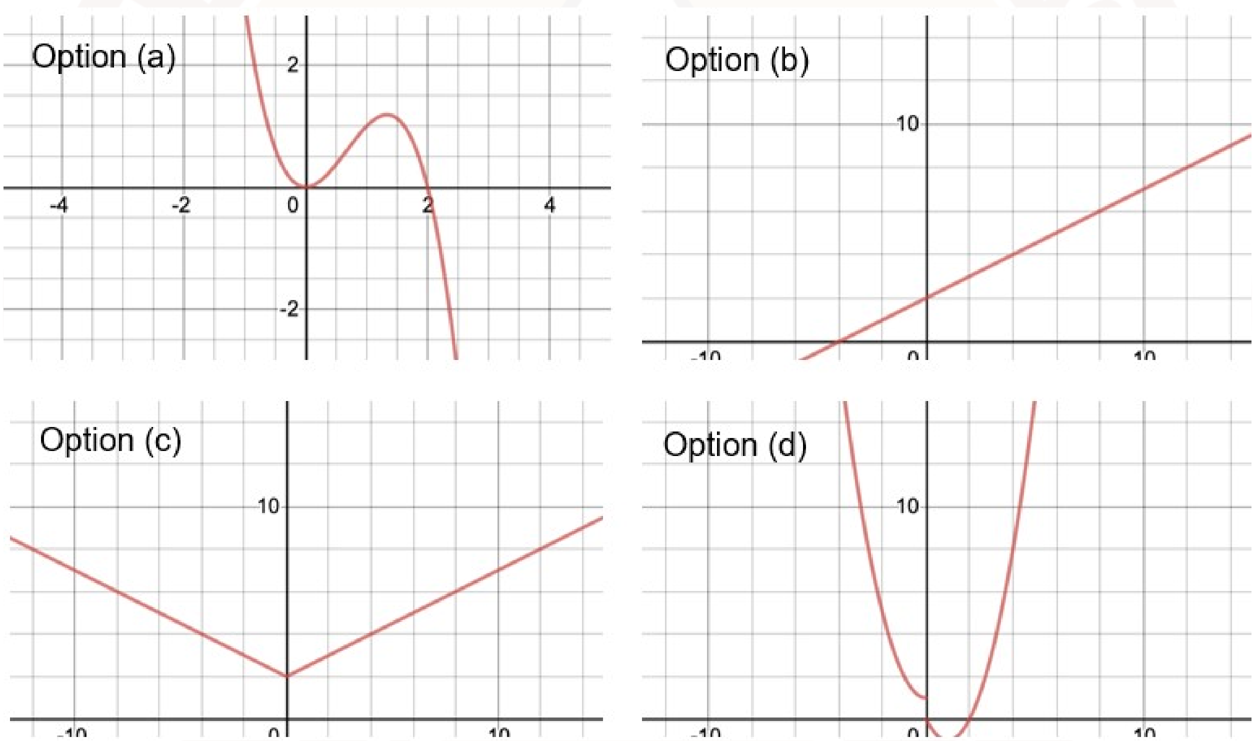
Solution: Given that both the polynomials leave same remainder when divided by $(x - 2)$.
By substituting $x = 2$ both the polynomial should have same value.
By substituting $x = 2$ in $x^3 + ax^2 + 5x + 7$, we get $8 + 4a + 10 + 7 = 4a + 25$.
By substituting $x = 2$ in $x^3 + 2x^2 + 3x + 2a$, we get $8 + 8 + 6 + 2a = 2a + 22$.

So we have,

$$
\begin{aligned}
4a + 25 &= 2a + 22 \\
2a &= -3 \\
a &= -\frac{3}{2}
\end{aligned}
$$

4.4 Characteristics of polynomial function:

Polynomial functions of second degree or higher have graphs that do not have sharp corners. That is, the graphs are smooth curves.
Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous.
Sample question: Which of the graphs given below, represent polynomial functions?



Clearly, the graph represented in option (a) and option (b) are smooth and continuous. Thus it represents polynomial a function. Also, note that graph in option (c) have sharp corner at $x = 0$ and and option (d) is not continuous thus it doesn't represents polynomial function.

4.5 Zeroes of polynomial function:

Recall: If f is a polynomial function, the values of x for which $f(x) = 0$ are called zeros of f.

- If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.
- Also, any value $x = a$ that is a zero of a polynomial function yields a factor of the polynomial, of the form $(x - a)$.
- Given the equation of a polynomial function, we can use this method to find $x-$intercepts because at the $x-$intercepts we find the input values whose output value is zero.
- For general polynomials, this can be a challenging prospect. However quadratic functions can be solved using the quadratic formula.
- The corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember. And formulas do not exist for general higher-degree polynomials.

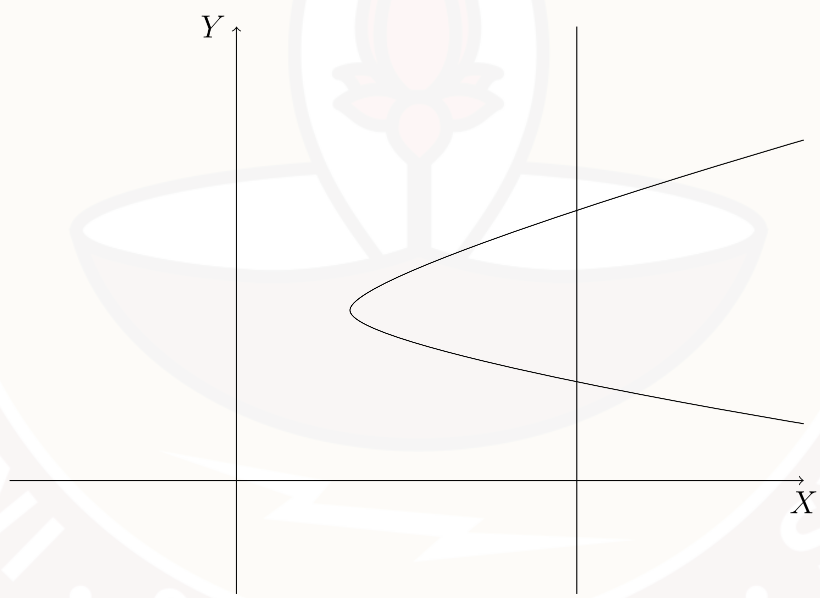
Chapter 5

5 Exponential function:

We have already seen different type of functions previous sections. In this section we are going to see exponential function. Before to proceed with the exponential function there are two test related to the function, one is **vertical line test** (this test is to check a relation is a function or not via. looking to the graph of the relation) and another one is **horizontal line test** (this test is to check given function is one-one or not).

5.1 Vertical Line Test:

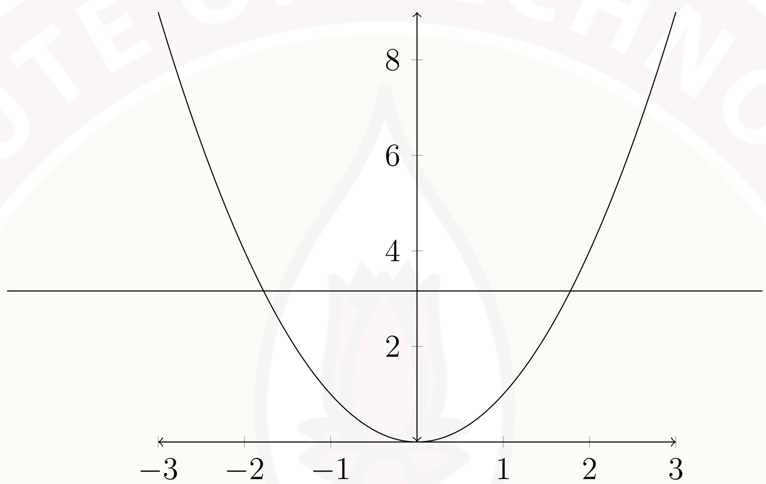
Suppose the graph of a relation is given and we want to check whether given relation is a function or not, for that just draw a vertical line (i.e. a line which is parallel to the $Y-$ axis) which is intersecting to the curve. If the vertical line intersects to the curve more than one point, then the relation fails to vertical line test and given relation does not representation a function.



5.2 Horizontal Line Test:

Suppose the graph of a function is given and we want to check whether given function is a one-one or not, for that just draw a horizontal line (i.e. a line which is parallel to the $X-$ axis) which is intersecting to the curve. If the horizontal line intersects to the curve more than one point, then the function fails to horizontal line test and given function is not a one-one function.

If any horizontal line intersects the graph of a function f in at most one point, then f in one -to- one.



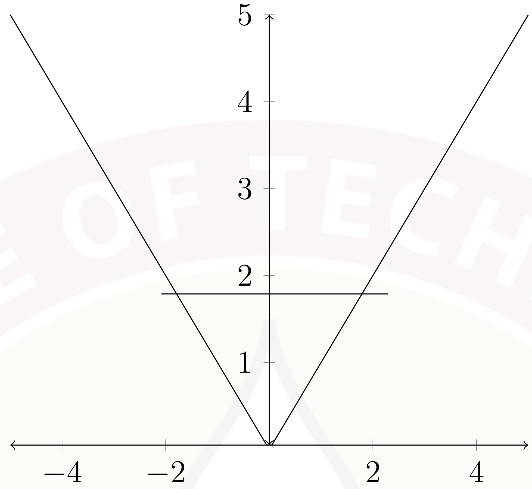
- **Increasing function:** Let $f : A \to B$ be a function and $x_1, x_2 \in A$ such that $x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$, then f is an increasing function.

- **Decreasing function:** Let $f : A \to B$ be a function and $x_1, x_2 \in A$ such that $x_1 \leq x_2 \implies f(x_1) \geq f(x_2)$, then f is an decreasing function.

- **One-one function:** Let $f : A \to B$ be a function and $x_1, x_2 \in A$ such that $f(x_1) = f(x_2) \implies x_1 = x_2$ or $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$, then f is one-one.

- **Theorem:** If f is an increasing or decreasing function, then the f is one-one.

Example: Check whether the given function $f(x) = |x|$ one-one or not.
Lets draw the graph of the function and check with the horizontal line test:



$f(x) = |x|$

We can see that horizontal line intersects the graph at two points. Hence the function $f(x) = |x|$ is not one-one.

5.3 Exponential function:

Definition: An exponential function in standard form is given by $f(x) = a^x$, where $a > 0, a \neq 1$.

5.4 Laws of exponents:

For $s, t \in \mathbb{R}$ and $a, b > 0$,

i) $a^s \times a^t = a^{s+t}$

ii) $(a^s)^t = a^{st}$

iii) $(ab)^s = a^s b^s$

Example: Solve the following the expression

$$2^3 \times (4^5)^3$$

$$= 2^3 \times 4^{5\times 3}$$

$$= 2^3 \times 4^{15}$$

$$= 2^3 \times (2^2)^{15}$$

$$= 2^3 \times (2)^{2\times 15}$$

$$= 2^3 \times (2)^{30}$$

$$= 2^{3+30}$$

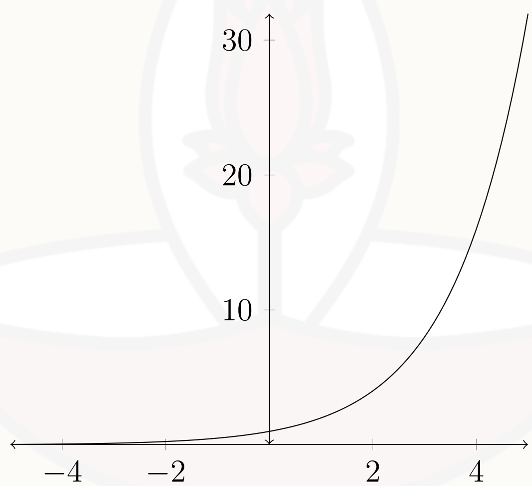
$$= 2^{33}$$

Hence, $2^3 \times (4^5)^3 = 2^{33}$

5.5 Graphing the exponential function $f(x) = 2^x$

To graph of the exponential function $f(x) = 2^x$, let's find the following things:

- Domain of the function $f(x)$ is $\mathbb{R}$.
- Range of the function $f(x)$ is $(0, \infty)$.
- $y-$ intercept is the point $(0, 1)$.
- $x-$ intercept is Nil i.e. graph of the function does not intersect the $X-$axis.
- End behavior of the function $x \to \infty, 2^x \to \infty$ and $x \to -\infty, 2^x \to 0$.
- There is no root of the function as $x-$ intercept is Nil.
- Function $f(x) = 2^x$ is an increasing function as,
  let $x_1 \neq x_2$ such that $x_1 < x_2 \implies 2^{x_1} < 2^{x_2}$

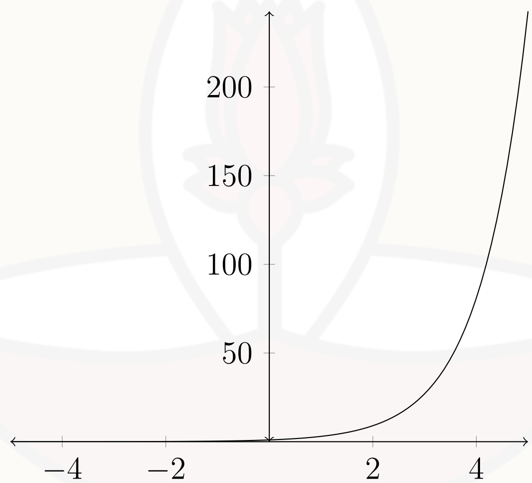


$f(x) = 2^x$

5.6 Graph of the function $f(x) = a^x$, when $a > 1$

- Domain of the function $f(x)$ is $\mathbb{R}$.
- Range of the function $f(x)$ is $(0, \infty)$.
- $y-$ intercept is the point $(0, 1)$.
- $x-$ intercept is Nil i.e. graph of the function does not intersect the $X-$axis.
- End behavior of the function $x \to \infty, a^x \to \infty$ and $x \to -\infty, a^x \to 0$.
- There is no root of the function as $x-$ intercept is Nil.
- Function $f(x) = a^x$ is an increasing function as,
  let $x_1 \neq x_2$ such that $x_1 < x_2 \implies a^{x_1} < a^{x_2}$ and so $f(x) = a^x$ is one-to-one function.

For example we can see the graph of the function $f(x) = 3^x$

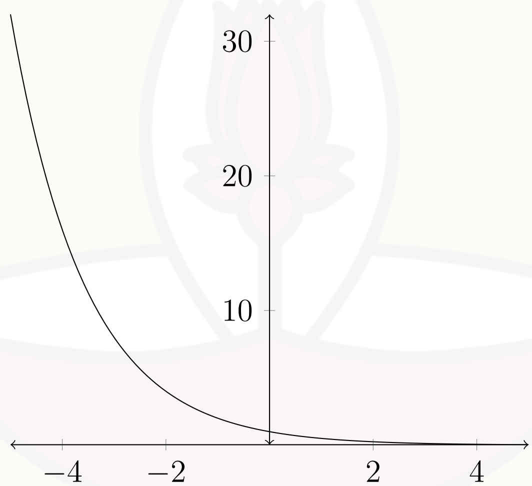


$f(x) = 3^x$

5.7 Graph of the function $f(x) = a^x$, when $0 < a < 1$

- Domain of the function $f(x)$ is $\mathbb{R}$.
- Range of the function $f(x)$ is $(0, \infty)$.
- $y-$ intercept is the point $(0, 1)$.
- $x-$ intercept is Nil i.e. graph of the function does not intersect the $X-$axis.
- End behavior of the function $x \to \infty, a^x \to 0$ and $x \to -\infty, a^x \to \infty$.
- There is no root of the function as $x-$ intercept is Nil.
- Function $f(x) = a^x$ is an decreasing function as,
  let $x_1 \neq x_2$ such that $x_1 < x_2 \implies a^{x_1} > a^{x_2}$ and so $f(x) = a^x$ is one-to-one function.

For example we can see the graph of the function $f(x) = (\frac{1}{2})^x$

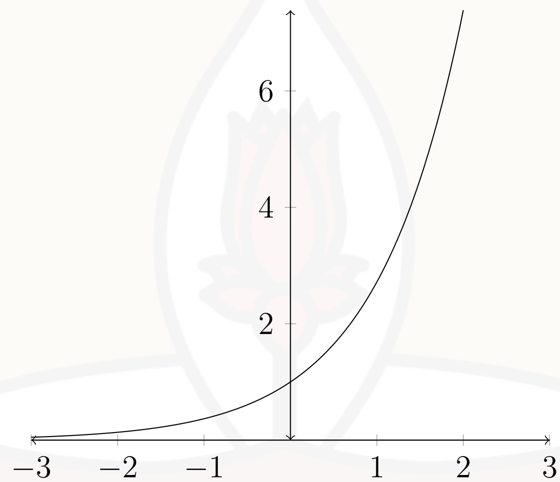


$f(x) = (\frac{1}{2})^x$

5.8 Natural exponential:

The natural exponential function is defined as $f(x) = e^x$.
Let's see some properties:

- Value of e is greater than 1 i.e. $e > 1$
- Domain of the function is $\mathbb{R}$.
- Range of the function is $(0, \infty)$.
- e is the slope of the tangent line to the $f(x) = e^x$ at the point $(1, e)$.
- The area under the $f(x) = e^x$ from $(-\infty, 1)$ is e.



$f(x) = e^x$

Example:
Let $R(t)$ be the percent of people who respond to affiliate links under YouTube descriptions and purchase the product in t minutes, given by $R(t) = 50 - 100e^{-0.2t}$.

(a) What is the percentage of people responding after 10 minutes?

(b) What is the highest percent expected?

(c) How long before $R(t)$ exceeds 30%

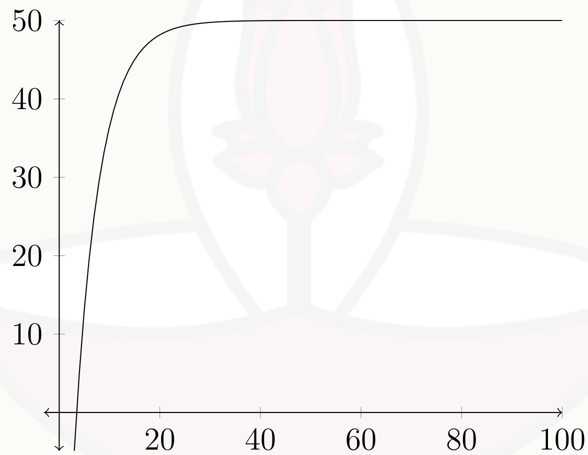
Lets solve the problem:

(a) The percentage of people responding after 10 minutes is given by $R(10)$

$$\begin{aligned} R(10) &= 50 - 100e^{-0.2(10)} \\ &= 50 - 100e^{-2} \\ &\approx 36.46 \end{aligned}$$

That means approx 36.46% people purchased the product after 10 minutes

(b) The highest percent expected is 50% and it can be seen from the graph given below



As

$$t \to \infty \implies e^{-0.2t} \to 0$$

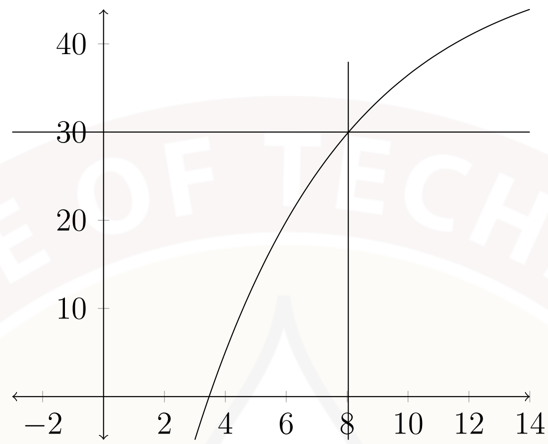
Therefore,

$$R(t) = 50 - 100e^{-0.2t} \to 50$$

(c) Let

$$\begin{aligned} R(t) &= 30 \\ \implies 50 - 100e^{-0.2t} &= 30 \\ \implies 100e^{-0.2t} &= 20 \\ \implies \frac{1}{5} &= e^{-0.2t} \end{aligned}$$

When we solve the above equations using logarithmic (which we will see it ahead), we will get $t \approx 8$.
We can see it via the graph also:



5.9 Composition of function:

The composition of the function f and g is denoted by $f \circ g$ is defined by
$(f \circ g)(x) = f(g(x))$.
The domain of the composite function $f \circ g$ is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.
For example:
Let $f(x) = 3x - 4$ and $g(x) = x^2$, then find the $f \circ g$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2) \\ &= 3x^2 - 4\end{aligned}$$

Determination of the domain for the composition function:

$$(f \circ g)(x) = f(g(x))$$

Two rules must be followed while finding the domain of composition function

$$(f \circ g)(x) = f(g(x))$$

- If $x \notin Domain(g)$, then this implies $x \notin Domain(fog)$.
- $\{x \mid g(x) \notin Domain(f)\}$ must not be include in the domain of $(f \circ g)$.

Example:
A shop is offering two discounts on a product with an MRP of ₹14000 . The first discount is a 15% discount on the MRP, and the second discount is a flat ₹3000 discount. If both discounts are applied, what is the final cost of the product?
Lets solve it:
Let x be the price of the product and discounts can be think as functions,
The first discount is a 15% discount on the MRP so price of the product after applying the first discount can be written as $f(x) = 0.85x$.
And the second discount is a flat ₹3000 so price of the product after applying the second discount can be written as $g(x) = x - 3000$.
Let $h(x)$ be the price of the product applying the fist discount and then the second discount, this we will get after substituting $f(x)$ into $g(x)$ such that :

$$\begin{aligned} h(x) &= g(f(x)) \\ \text{so } h(x) &= f(x) - 3000 \\ \implies h(x) &= 0.85x - 3000 \\ \text{given } x &= 14000 \\ \implies h(14000) &= 0.85(14000) - 3000 \\ \implies h(14000) &= 11900 - 3000 \\ \implies h(14000) &= 8900 \end{aligned}$$

The final cost of the product is ₹8900

Example:
Let $f(x) = \frac{2}{x-1}$ and $g(x) = \frac{3}{x}$, then find the $(f \circ g)(x)$ and domain of $(f \circ g)(x)$.
$(f \circ g)(x) = f(g(x)) = f(\frac{3}{x}) = \frac{2}{\frac{3}{x}-1} = \frac{2x}{3-x}$
Since,the function g is not defined at $x = 0$ and the function f is not defined at $x = 1$, so

- As $0 \notin domain(g)$, so the $0 \notin domain(f \circ g)$.
- As $x = 3$ $g(3) = 1$, so $x = 3 \notin domain(f \circ g)$

So the $domain(f \circ g)$ is $\mathbb{R} \setminus \{0, 3\}$
Example: Let $f(x) = 3x - 4$ and $g(x) = x^2$, find $(g \circ f)(x)$ and $(f \circ g)(x)$.
To find $(g \circ f)(x)$, we first apply $f(x)$ to x and then apply $g(x)$ to the result. This can be written as $g(f(x))$. Therefore, we get:

$$\begin{aligned}
&(g \circ f)(x) = g(f(x)) \\
\implies\ & g(f(x)) = g(3x - 4) \\
\implies\ & g(f(x)) = (3x - 4)^2 \\
\implies\ & g(f(x)) = 9x^2 - 24x + 16
\end{aligned}$$

Therefore, $(g \circ f)(x) = 9x^2 - 24x + 16$.
To find $(f \circ g)(x)$, we first apply $g(x)$ to x and then apply $f(x)$ to the result. This can be written as $f(g(x))$. Therefore, we get:

$$\begin{aligned}
&(f \circ g)(x) = f(g(x)) \\
\implies\ & f(g(x)) = f(x^2) \\
\implies\ & f(g(x)) = 3x^2 - 4
\end{aligned}$$

Therefore, $(f \circ g)(x) = 3x^2 - 4$.

5.10 Inverse Function:

If the function f is one-to-one then f^{-1} exist and the inverse of a function f, f^{-1} is the function such that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, \ \forall\, x \in Domain(f) = Range(f^{-1})$$

and

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \ \forall\, x \in Range(f) = Domian(f^{-1})$$

Remark: If f is be one- to - one function $\implies$ f^{-1} exists for f.
Warning: $f^{-1} \neq \frac{1}{f}$
Remark: If $(a, f(a))$ is on the graph of f, then $(f(a), a)$ is on the graph of f^{-1}. Hence the graph of the inverse function is symmetric about the line $y = x$
Example:
Let $g(x) = 4x$ and $h(x) = \frac{x}{4}$, check whether the function g is the inverse function of h or not.
First, we have to check that the function h is one-to-one or not.
let $x_1, x_2 \in \mathbb{R}$ such that $h(x_1) = h(x_2) \implies \frac{x_1}{4} = \frac{x_2}{4} \implies x_1 = x_4$
Hence, function h is one-to-one function.
Lets find the composition of the function g and h,

$$(g \circ h)(x) = g(h(x)) = g\left(\frac{x}{4}\right) = 4 \times \frac{x}{4} = x$$

and

$$(h \circ g)(x) = h(g(x)) = h(4x) = \frac{4x}{4} = x$$

Hence, the function g is the inverse function of h.

Example: Given that $g(x) = x^3$ and $g^{-1}(x) = x^{1/3}$, verify that $g^{-1}(g(x)) = g(g^{-1}(x)) = x$.
We have $g(x) = x^3$ and $g^{-1}(x) = x^{1/3}$.
Lets verify that $g^{-1}(g(x)) = x$,
substitute $g(x)$ in place of x into $g^{-1}(x)$ we get,

$$\begin{aligned} g^{-1}(g(x)) &= g^{-1}(x^3) \\ &= (x^3)^{1/3} \\ \implies g^{-1}(g(x)) &= x \end{aligned}$$

Therefore, $g^{-1}(g(x)) = x$.
Now, lets verify that $g(g^{-1}(x)) = x$, substitute $g^{-1}(x)$ in place of x into $g(x)$ we get,

$$\begin{aligned} g(g^{-1}(x)) &= g(x^{1/3}) \\ &= (x^{1/3})^3 \\ \implies g(g^{-1}(x)) &= x \end{aligned}$$

Therefore, $g(g^{-1}(x)) = x$.

Example: Check whether $f(x) = \frac{x-5}{2x+3}$ is the inverse of $g(x) = \frac{3x+5}{1-2x}$ or not.
To verify whether f is the inverse of g, we need to check whether $f \circ g = f(g(x)) = x$ and $g \circ f = g(f(x)) = x$ for all x in their domains.
Lets find $f \circ g$:

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f\left(\frac{3x+5}{1-2x}\right) \\
&= \frac{\left(\frac{3x+5}{1-2x}\right) - 5}{2\left(\frac{3x+5}{1-2x}\right) + 3} \\
&= \frac{3x+5-5(1-2x)}{10+3} \\
&= \frac{13x}{13} \\
\implies f \circ g &= f(g(x)) = x
\end{aligned}$$

Now, lets find $g \circ f$:

$$\begin{aligned}
g \circ f &= g(f(x)) \\
&= \frac{3f(x)+5}{1-2f(x)} \\
&= \frac{3(\frac{x-5}{2x+3})+5}{1-2(\frac{x-5}{2x+3})} \\
&= \frac{3(x-5)+5(2x+3)}{2x+3-2x+10} \\
&= \frac{13x}{13} \\
\implies g \circ f &= g(f(x)) = x
\end{aligned}$$

Hence,

$$f \circ g = f(g(x)) = x = g(f(x)) = g \circ f$$

Thus $f(x)$ is the inverse of $g(x)$

Chapter 6

6 Logarithmic function:

As we know that exponential function $f(x) = a^x$ $(a > 0, a \neq 1)$ is one-to-one function so it exponential function is invertible. Once, if the function is invertible then we can talk about the inverse function of the given function and so the logarithmic function is the inverse of function of the exponential function.
The logarithmic function (to the base a) in the standard form is $\log_a x$ and is defined to be the inverse of $f(x) = a^x$

$$y = \log_a x \iff x = a^y$$

Recall the domain and range of the inverse function,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, \ \forall \ x \in Domain(f) = Range(f^{-1})$$

and

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \ \forall \ x \in Range(f) = Domian(f^{-1})$$

Hence, as $Domain(a^x)$ of is $\mathbb{R}$ so the $Range(\log_a x)$ is the $\mathbb{R}$. And as $Range(a^x)$ of is the interval $(0, \infty)$ so the $Domain(\log_a x)$ is the interval $(0, \infty)$.

1. Example: Find the domain of the function $f(x) = \log_4(1 - x)$.

   Since input values of the logarithmic function is only positive real values so

   $$1 - x > 0 \implies x < 1$$

   Hence the domain of the function $f(x) = \log_4(1 - x)$ is the interval $(-\infty, 1)$

2. Example: Find the domain of the function $f(x) = \log_3(\frac{1+x}{1-x}), x \neq 1$.

   Since input values of the logarithmic function is only positive real values so

   $$\frac{1+x}{1-x} > 0$$

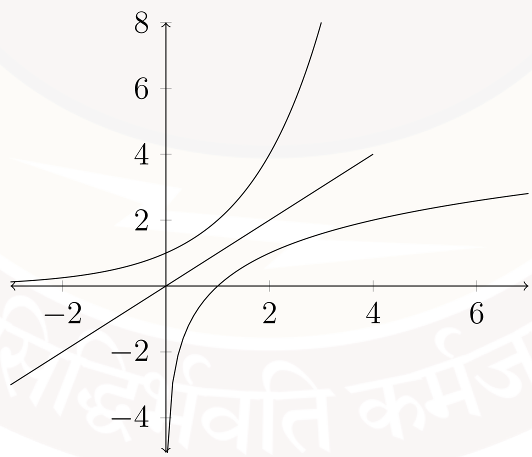
   Observe that the given fraction should positive when
   either $1 + x > 0 \implies x > -1$ *i.e.* $x \in (-1, \infty)$ and
   $1 - x > 0 \implies x < 1$ *i.e.* $x \in (-\infty, 1)$ and so $x \in (-1, \infty) \cap (-\infty, 1)$ i.e. $x \in (-1, 1)$
   or $1 + x < 0 \implies x < -1$ *i.e.* $x \in (-\infty, -1)$ and $1 - x < 0 \implies x > 1$ *i.e.* $x \in (1, \infty)$
   so $x \in (-\infty, -1) \cap (1, \infty)$ i.e. there is no such x exist. $x \in (-1, 1)$

   Hence the domain of the function $f(x) = \log_3(\frac{1+x}{1-x})$ is the interval $(-1, 1)$

6.1 Graphing the logarithmic function $f(x) = \log_2 x$

Note that graph of the inverse function symmetric about the graph of the given function with respect to the line $y = x$ Hence to draw the graph of the logarithmic function $f(x) = \log_2 x$, first draw the graph of the exponential function 2^x, then the graph of the curve which is symmetric about the line $y = x$ and find the following things via observing the exponential function 2^x:

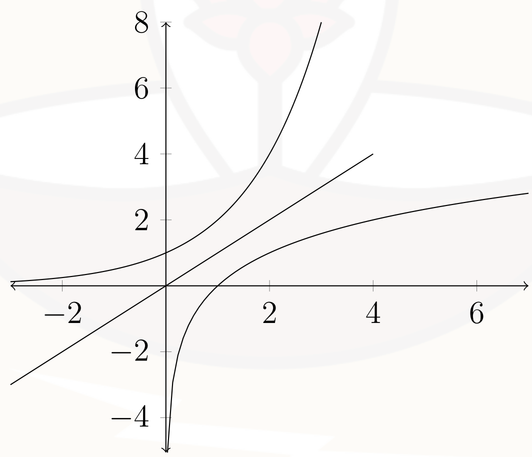
- Since the range of the function 2^x is the interval $(0, \infty)$ therefore domain of the function $\log_2 x$ is the interval $(0, \infty)$.
- Since the domain of the function 2^x is $\mathbb{R}$ therefore range of the function $\log_2 x$ is $\mathbb{R}$.
- Since $y-$ intercept of the function 2^x is the point $(0, 1)$ therefore the function $\log_2 x$ has x-intercept which is the point (1, 0).
- Since there is no $x-$ intercept of the function 2^x similarly there is no $y-$ intercept for the function $\log_2 x$.
- End behavior of the function $x \to \infty,\ 2^x \to \infty$ and $x \to -\infty,\ 2^x \to 0$ similarly end behavior of the function $x \to \infty,\ \log_2 x \to \infty$ and $x \to 0^+,\ \log_2 x \to -\infty$
- Since there is no root of the function 2^x as there is no $x-$ intercept but the function $\log_2 x$ has x- intercept i.e there is one root which is at $x = 1$
- As function 2^x is an increasing function similarly $\log_2 x$ is also a increasing function. And so the function $\log_2 x$ is one- one.



$2^x \quad Vs \quad \log_2 x$

6.2 Graph of the function $f(x) = \log_a x$, when $a > 1$

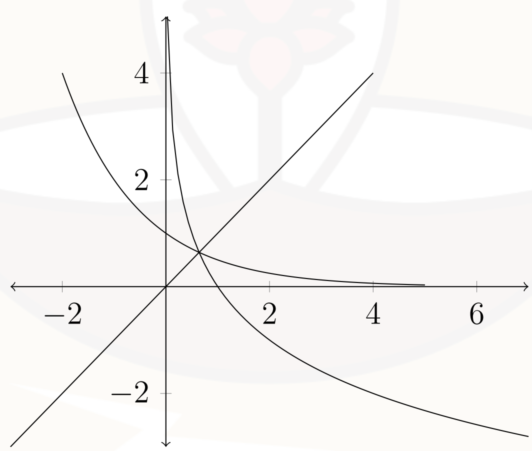
- Since the range of the function a^x is the interval $(0, \infty)$ therefore domain of the function $\log_a x$ is the interval $(0, \infty)$.
- Since the domain of the function a^x is $\mathbb{R}$ therefore range of the function $\log_a x$ is $\mathbb{R}$.
- Since $y-$ intercept of the function a^x is the point $(0, 1)$ therefore the function $\log_a x$ has x-intercept which is the point (1, 0).
- Since there is no $x-$ intercept of the function a^x similarly there is no $y-$ intercept for the function $\log_a x$.
- End behavior of the function $x \to \infty,\ a^x \to \infty$ and $x \to -\infty,\ a^x \to 0$ similarly end behavior of the function $x \to \infty,\ \log_a x \to \infty$ and $x \to 0^+,\ \log_a x \to -\infty$
- Since there is no root of the function a^x as there is no $x-$ intercept but the function $\log_a x$ has x- intercept i.e there is one root which is at $x = 1$
- As function a^x is an increasing function similarly $\log_a x$ is also a increasing function. And so the function $\log_a x$ is one- one.



$a^x \;\; Vs \;\; \log_a x$

6.3 Graph of the function $f(x) = \log_a x$, when $0 < a < 1$

- Since the range of the function a^x is the interval $(0, \infty)$ therefore domain of the function $\log_a x$ is the interval $(0, \infty)$.
- Since the domain of the function a^x is $\mathbb{R}$ therefore range of the function $\log_a x$ is $\mathbb{R}$.
- Since $y-$ intercept of the function a^x is the point $(0, 1)$ therefore the function $\log_a x$ has x-intercept which is the point (1, 0).
- Since there is no $x-$ intercept of the function a^x similarly there is no $y-$ intercept for the function $\log_a x$.
- End behavior of the function $x \to \infty,\ a^x \to 0$ and $x \to -\infty,\ a^x \to \infty$ similarly end behavior of the function $x \to \infty,\ \log_a x \to -\infty$ and $x \to 0^+,\ \log_a x \to \infty$
- Since there is no root of the function a^x as there is no $x-$ intercept but the function $\log_a x$ has x- intercept i.e there is one root which is at $x = 1$
- As function a^x is a decreasing function similarly $\log_a x$ is also a decreasing function. And so the function $\log_a x$ is one- one.

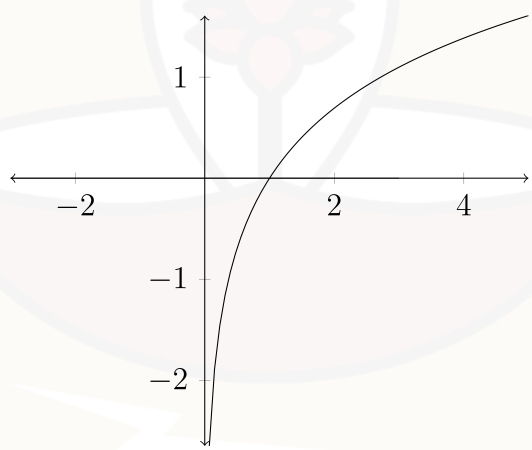


$a^x \ \ Vs \ \ \log_a x$

6.4 Natural logarithmic:

The natural logarithmic is logarithmic function which have base as e i.e
$f(x) = \log_e x = \ln x$.
Let's see some properties:

- Value of e is greater than 1 i.e. $e > 1$
- Domain of the function is $(0, \infty)$.
- Range of the function is $\mathbb{R}$.
- $x-$ intercept is the point $(1, 0)$.
- $y-$ intercept is Nil i.e. graph of the function does not intersect the $X-$axis.
- End behavior of the function $x \to \infty, \ln x \to \infty$ and $x \to 0, \ln x \to -\infty$.
- $\ln x$ has only one root at $x = 1$ as it has $x-$ intercept.
- Function $f(x) = \ln x$ is an increasing function and so $f(x) = \ln x$ is one-to-one function.

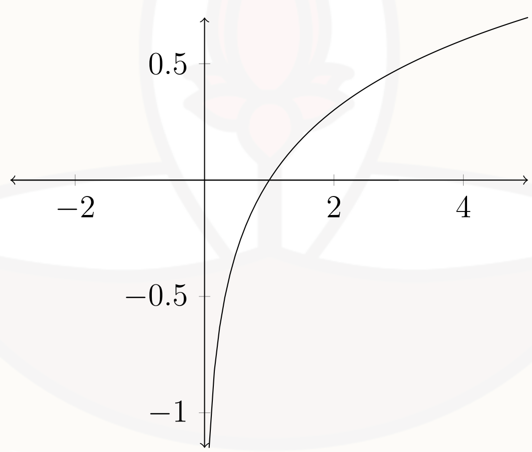


$f(x) = \ln x$

6.5 Common logarithmic:

The common logarithmic is logarithmic function which have base as 10 i.e
$f(x) = \log_{10} x = \log x$.
Let's see some properties:

- Domain of the function is $(0, \infty)$.
- Range of the function is $\mathbb{R}$.
- $x-$ intercept is the point $(1, 0)$.
- $y-$ intercept is Nil i.e. graph of the function does not intersect the $X-$axis.
- End behavior of the function $x \to \infty, \log x \to \infty$ and $x \to 0, \log x \to -\infty$.
- $\log x$ has only one root at $x = 1$ as it has $x-$ intercept.
- Function $f(x) = \log x$ is an increasing function and so $f(x) = \log x$ is one-to-one function.



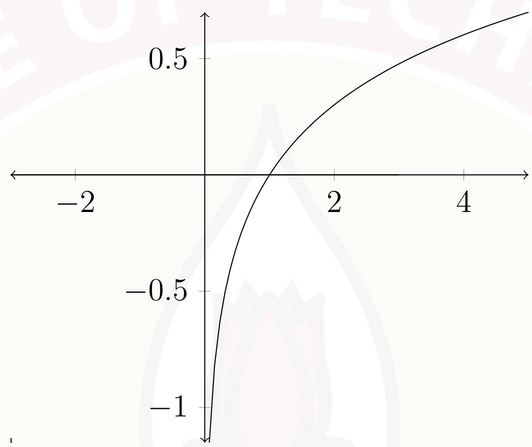
$f(x) = \log x$

Example: Draw the graph of $-\log(x+1)$.

To draw the graph the of $-\log(x+1)$, we can start by considering the graph of $\log(x)$ and then making some transformations.

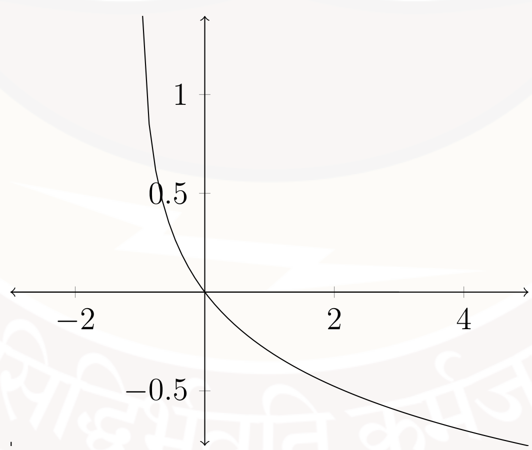
Observe that, graph of $\log(x)$ has a vertical asymptote at $x = 0$ and passes through the point $(1, 0)$. As we are interested in the negative of the logarithm, we need to reflect the graph across the x-axis.And then, we can shift the graph to the left by 1 unit to obtain the graph of $-\log(x+1)$.

The graph of $\log(x)$:



$\log x$

To obtain the graph of $-\log(x+1)$, we reflect this graph across the x-axis and shift it left by 1 unit:



$-\log(x+1)$

As we can see from the graph, $-\log(x+1)$ is undefined for $x \leq -1$, and as x approaches -1 from the right, the function goes to ∞. The function $-\log(x+1)$ is decreasing on its domain, and its range is $(0, \infty)$.

6.6 Laws of Logarithmic function:

As we have seen some laws of exponential function similarly the logarithmic function have some laws which can proved easily using the exponential laws.
Here is some properties: Assume $a > 1$ or $0 < a < 1$ and $M,\ N > 0$, and $r \in \mathbb{R}$

- $a^{\log_a M} = M$
- $\log_a a^M = M$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a MN = \log_a M + \log_a N$
- $\log_a \frac{M}{N} = \log_a M - \log_a N$
- $\log_a \frac{1}{N} = -\log_a N$
- $\log_a M^r = r \log_a M$

Theorem: Let $a > 1$ or $0 < a < 1$ and $M, N > 0$ then $M = N$ if and only if $\log_a M = \log_a N$.
Proof: Let $M = N$, taking logarithmic with base a both sides, we get,

$$\log_a M = \log_a N$$

.
conversely,
Let $\log_a M = \log_a N$, taking exponential with base a both sides. we get,

$$a^{\log_a M} = a^{\log_a N}$$

$$\implies M = N$$

Theorem: Let $a, b > 1$ or $0 < a, b < 1$. Then for $x > 0$,

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Proof: Let

$$M = \log_a x \implies a^M = x$$
$$N = \log_b x \implies b^N = x$$
$$R = \log_b a \implies b^R = a$$

Form the above three equations

$$x = (b^M)^R = b^{RM}$$

Taking log with base b both sides,

$$\log_b x = log_b b^{RM} = RM$$
$$\iff \log_b x = (log_b a)(\log_a x)$$
$$\iff \log_a x = \frac{\log_b x}{\log_b a}$$

Example: Solve the equation $e^{-x^2} = \frac{(e^x)^2}{e^3}$
Solution: We have $e^{-x^2} = \frac{(e^x)^2}{e^3}$, we can write,

$$e^{-x^2} = e^{2x-3}$$

Taking the natural logarithm of both sides, we get,

$$\ln(e^{-x^2}) = \ln(e^{2x-3})$$

Using the properties of logarithms, we get,

$$-x^2 = 2x - 3$$

After rearranging terms, we get,

$$x^2 + 2x - 3 = 0$$

After factoring the quadratic equation, we get,

$$(x + 3)(x - 1) = 0$$

Therefore, we have $x = -3$ and $x = 1$.

Example: Solve the exponential equation $9^x - 2 \cdot 3^{x+1} - 27 = 0$
Solution:
To solve the equation, lets simplify it, let $y = 3^x$, then we get,

$$\begin{aligned} 9^x - 2 \cdot 3^{x+1} - 27 &= 0 \\ \implies (3^2)^x - 2 \cdot 3 \cdot 3^x - 27 &= 0 \\ \implies 3^{2x} - 6 \cdot 3^x - 3^3 &= 0 \\ \implies y^2 - 6y - 27 &= 0 \end{aligned}$$

Now, lets solve the quadratic equation

$$\begin{aligned} y^2 - 6y - 27 &= 0 \\ \implies (y-9)(y+3) &= 0 \end{aligned}$$

So, we have, $y = 9, y = -3 \implies 3^x = 9$ and $3^x = -3$
(Exponential value is never negative)
Hence, $3^x = 9 \implies x = 2$

Example: Solve the equation $5^{x-2} = 3^{3x+2}$
Solution:
To solve the equation, after taking logarithmic both sides with base 10, we get,

$$
\begin{aligned}
\log_{10}(5^{x-2}) &= \log_{10}(3^{3x+2}) \\
\implies (x-2)\log_{10}(5) &= (3x+2)\log_{10}(3) \\
\implies x\log_{10}(5) - 2\log_{10}(5) &= 3x\log_{10}(3) + 2\log_{10}(3) \\
\implies x\log_{10}(5) - 3x\log_{10}(3) &= 2\log_{10}(5) + 2\log_{10}(3) \\
\implies x(\log_{10}(5) - 3\log_{10}(3)) &= 2\log_{10}(5) + 2\log_{10}(3) \\
\implies x &= \frac{2(\log_{10}(5) + \log_{10}(3))}{\log_{10}(5) - 3\log_{10}(3)} \\
\implies x &= \frac{2\log_{10}(15)}{\log_{10}(5/27)} \\
\implies x &= \frac{2\log_{10}(15)}{\log_{10}(5) - \log_{10}(27)} \\
\implies x &= \frac{2\log_{10}(15)}{\log_{10}(\frac{5}{27})}
\end{aligned}
$$

Therefore, the solution to the equation is:

$$x = \frac{2\log_{10}(15)}{\log_{10}(\frac{5}{27})}$$

Example: Solve the equation $\log_8(x+1) + \log_8(x-1) = 1$.
Solution: Using the product rule of logarithms, we get:

$$\begin{aligned} \log_8((x+1)(x-1)) &= 1 \\ \implies \log_8(x^2-1) &= 1 \\ \implies x^2 - 1 &= 8 \\ \implies x^2 &= 9 \end{aligned}$$

Hence we get, $x = 3$ or $x = -3$
However, we need to check if both solutions satisfy the original equation. We can see that $x = -3$ does not work since $\log_8(-2)$ and $\log_8(-4)$ are not defined in the real numbers. Therefore, the only solution is $x = 3$.

Example: Solve the following equation $\ln(x^2) = (\ln(x))^2$

Solution: We have $\ln(x^2) = (\ln(x))^2$
Using the properties of logarithms, we get,

$$2\ln(x) = (\ln(x))^2$$

Let $\ln(x) = t$
After substituting, we get,

$$t^2 - 2t = 0$$

$$\implies t = 0 \text{ or } t = 2$$

Hence $\ln(x) = 0 \implies x = 1$
Or $\ln(x) = 2 \implies x = e^2$

Example: Find the value of x which satisfying the equation

$$\log_3 x + \log_4 x = 4$$

Solution: We have

$$\log_3 x + \log_4 x = 4$$

After using base change formula, we get,

$$\frac{\ln(x)}{\ln 3} + \frac{\ln(x)}{\ln(4)} = 4$$

$$\implies \ln x = \frac{4 \ln 3 \ln 4}{\ln 12}$$

$$\implies x = e^{\frac{4\ln(3)\ln(4)}{\ln(12)}}$$

7 Exercise:

1. If $\log_{\sqrt{2}}(x+4) - \log_2(\frac{1}{2}x+2) = 1$ then x is
   (a) -3
   (b) 1
   (c) -4
   (d) 5

Use the following information for the questions 2 and 3.
Let N_0 be the number of atoms of a radioactive material at the initial stage i.e., at time $t = 0$, and $N(t)$ be the number of atoms of the same radioactive material at a given time t, which is given by the equation $N(t) = N_0 e^{-\lambda t}$, where λ is the decay constant.

2. If at time t_1, the number of atoms reduces to the half of N_0 and at the time t_2 the number of atoms reduces to the one fourth of N_0, then which one of the following equations is correct?
   - ○ $e^{\frac{t_1}{t_2}} = 2$
   - ○ $e^{\frac{t_2}{t_1}} = 2$
   - ○ $e^{\lambda(t_2-t_1)} = 2$
   - ○ $e^{\lambda(t_1-t_2)} = 2$

3. If $N_{\frac{1}{\lambda}}$ is the number of atoms at the time $t = \frac{1}{\lambda}$, then what will be the ratio of N_0 to $N_{\frac{1}{\lambda}}$?
   - ○ $1 : e$
   - ○ $e : 1$
   - ○ $1 : e^{-\lambda}$
   - ○ $1 : e^{\lambda}$

4. Selvi deposits ₹P in a bank A which provides an interest rate of 10% per year. After 10 years, she withdraws the whole amount from bank A and deposits it in another bank B for n years which provides an interest rate of 12.5% per year. $M_A(x)$ represents the amount in Selvi's account after x years of depositing in bank A. $M_B(y)$ represents the amount in Selvi's account after y years of depositing in bank B. If the interests are compounded yearly, then choose the set of correct options.
   - ○ $M_A(x)$ is an one-one function of x, for $x \in (0, 10)$.
   - ○ $M_B(y)$ is an one-one function of y.
   - ○ $M_A(12) = P \times 1.1^{12}$
   - ○ $M_A(12) = 0$

- ◯ $M_A(x)$ is a strictly increasing function of x, for $x \in (0, 10)$.
- ◯ $M_B(y)$ is a decreasing function of y.
- ◯ $M_B(n) = (P \times 1.1^{10}) \times (1.125)^n$
- ◯ $M_B(n) = (P \times 1.1^n) \times (1.125)^{10}$

Use the following information for questions 5-9.
Given two real valued functions $f(x) = \frac{5x+9}{2x}$, $g(y) = \sqrt{y^2 - 9}$. If $h(x) = f(g(x))$, then answer the following questions.

5. If the domain of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $\mathbb{R} \setminus (-n, n)$ respectively, then find the value of $m + n$.

6. If the range of $f(x)$ and $g(x)$ are $(-\infty, m) \cup (m, \infty)$ and $[n, \infty)$ respectively, then find the value of $2(m + n)$.

7. If the domain of $h(x)$ is $(-\infty, -3) \cup (m, \infty)$, then find the value of m.

8. If the domain of $f^{-1}(x)$ is $(-\infty, m) \cup (m, \infty)$, then find the value of $2m$.

9. If $f^{-1}(5) = 9/m$, then find the value of m.

10. Choose the set of correct options.
    (a) $\log_5 2$ is a rational number
    (b) If $0 < b < 1$ and $0 < x < 1$ then $\log_b x < 0$
    (c) If $\log_{\mathbf{3}}(\log_{\mathbf{5}} x) = \mathbf{1}$ then $x = 125$
    (d) If $0 < b < 1, 0 < x < 1$ and $x > b$ then $\log_b x > 1$
    (e) If $0 < b < 1$ and $0 < x < y$ then $\log_{\mathbf{b}} x > \log_{\mathbf{b}}\ y$

11. If $b > 0$ and $4 \log_x b + 9 \log_{b^5 x} b = 1$, then the possible value(s) of x is(are)
    (a) b^{10}
    (b) b^9
    (c) b^{-2}
    (d) b^5
    (e) b^4

12. Suppose that two types of insects are found in a pond. Their growth in number after t seconds is given by the equations $f(t) = 5^{3t-2}$ and $h(t) = 3^{2t+1} (t \neq 0)$. For what value of t will both insects be of same number in the pond?
    (a) $\frac{\ln 3 + 2 \ln 5}{3 \ln 5 - 2 \ln 3}$
    (b) $\frac{\ln 75}{\ln \frac{125}{9}}$

(c) $\log_{\frac{125}{9}} 75$

(d) $\frac{\ln 5+2\ln 3}{3\ln 3-2\ln 5}$

13. Find the number of values of x satisfying the equation $(5x)^{\log_{(5x)^{\frac{1}{5}}}(6x^3-36x^2+66x-35)^{\frac{1}{5}}} = 1$.