IIT MADRAS BS DEGREE

STATISTICS

Multiple discrete random variables

By : Nikita Kumari

About the author

Nikita Kumari is a course instructor for the Statistics and Machine Learning Foundation course in the IITM BS degree program. With a Masters degree in Mathematics from the Indian Institute of Technology, Madras, she has taught Statistics to students with diverse backgrounds.

She is an enthusiastic and dedicated instructor who is committed to helping her students develop a strong understanding of Statistics. With her extensive knowledge, enthusiasm, and effective teaching skills, she greatly enhances the quality of this Statistics textbook.

Contents

Chapter 1

1 Joint, Marginal and Conditional of multiple discrete random variables

We have seen previously some usual outcomes and experiments, they are quite complex in nature. There are a lot of different outcomes that is of interest to us and lot of different random variables defined on the same probability space that are of interest to us. In the first two weeks, we will see how to deal with multiple random variables in a proability space.

1.1 Examples: Toss a coin thrice

A fair coin is tossed thrice. We will get three outcomes and naturally, there can be three random variables. Let

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is a head} \\ 0, & \text{if the } i\text{-th toss is a tail} \end{cases}$$

for $i : 1, 2, 3$.

The three random variables are defined on the same probability space.

Observations:

- The three random variables X_1, X_2, X_3 together completely describe the outcome of the experiment.
- X_1, X_2 and X_3 are independent to each other. Events defined using X_1 alone will be independent of events defined using X_2 alone and will be independent of events defined using X_3 alone. Even though all the random variables live in the same probability space, if we define events with each of the random variables separately, the events will be be independent.

1.2 Example: Random 2-digit number

A 2-digit number from 00 to 99 is selected at random. Partial information is available about the number as two random variables. Let X be the digit in units place. Let Y be the remainder obtained when the number is divided by 4.

X can take any value from 0 to 9 uniformly.
$P(X = 0)$ is same as the probability of a two digit number being $00, 10, \ldots, 90$. Since total number of outcomes is 100, therefore, $P(X = 0) = \dfrac{1}{10}$. Similarly, $P(X = 1)$ is same as the

probability of a two digit number being $01, 11, \ldots, 91$. Since total number of outcomes is 100, therefore, $P(X = 1) = \frac{1}{10}$, and so on. Hence,

$$X \in \text{Uniform}\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

We will have the remainder as $0, 1, 2, 3$ for any number divided by 4.

Now, $P(X = 0)$ is same as the probability of two digit number being 00, 04, 08, ..., 96. Total number of favourable outcomes is 25. Therefore, $P(Y = 0) = \frac{25}{100} = 1/4$. Similarly for $Y = 1$, the favourable outcomes are $01, 05, 09, \ldots, 97$, that again count up to 25. So, $P(Y = 1) = 1/4$. Similarly we can try for $Y = 2$ and $Y = 3$. Hence,

$$Y \in \text{Uniform}\{0, 1, 2, 3\}$$

So, we have defined two different random variables on a common probability space.

Now, suppose event $X = 1$ has occurred. It means, the last digit of the number is 1. Then, what about the event $Y = 0$? Can these two events happen simultaneously? The answer is NO. Any number that ends with 1 cannot be a multiple of 4. So, the events $X = 1$ and $Y = 0$ are not independent.

We saw such things can also happen unlike the coin toss example where any event defined using one random variable was independent of any event defined using the another random variable. In the next example, we will see how this can play a role in modeling and other things in practice.

1.3 Example: IPL powerplay over

We will look at the IPL powerplay over data. The experiment is one over of IPL powerplay. Lets define two random variables X and Y, where X represent the number of runs scored in the over and Y represent number of wickets in the over.

Consider the events: $Y = 0, Y = 1, Y = 2$. We know that the event $Y = 0$ will be the most common, 0 wickets falling in an over. One possible model we can think of is if $Y = 0$, we expect X to take larger values than when $Y = 1$ or $Y = 2$. So, we will have to take into account all these relationships while building a model for complex experiments.

So, we saw that there are many experiments where defining multiple random variables, understanding relationships between them, trying to model distributions and between these random variables is of great use. In the next section, we will see the objects that we will focus on when we want to jointly study multiple random variables.

1.4 Two random variables: Joint, marginal, conditional PMFs

We will start with two random variables and eventually will move to multiple random variables. We had defined the PMF for one discrete random variable, but when we have two discrete random variables defined in the same probability space, many PMFs can be defined. We have joint PMF, Marginal PMF and conditional PMF, we will see their properties, structure, conditions and relationships between them.

1.4.1 Two discrete random variables: Joint PMF

Definition: Suppose X and Y are discrete random variables defined in the same probability space. Let the range of X and Y be T_X and T_Y, respectively. The joint PMF of X and Y , denoted f_{XY} , is a function from $T_X \times T_Y$ to $[0, 1]$ defined as

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{and } Y = t_2), t_1 \in T_X, t_2 \in T_Y$$

- Joint pmf can be written in the form of a table or a matrix.
- $P(X = t_1 \text{and } Y = t_2)$ can also be written as $P(X = t_1, Y = t_2)$.

Properties of joint PMF:

- Each entry in the joint PMF table will take values between 0 and 1.
- All the values in the table will sum up to 1.

In the next section, we will look at some of the examples from the joint distribution perspective.

Example 1: Toss a fair coin twice

Suppose a fair coin is tossed twice independently and we define the random variable X_i as

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is a head} \\ 0, & \text{if the } i\text{-th toss is a tail} \end{cases}$$

for $i : 1, 2$

(i) Find $f_{X_1X_2}(0, 0)$.

<u>Solution:</u>

$X_1 = 0, X_2 = 0$ implies that the first toss is a tail and the second toss is a tail. Also, the events $X_1 = 0$ and $X_2 = 0$ are independent.

$$\begin{aligned} f_{X_1X_2}(0, 0) =& P(X_1 = 0, X_2 = 0) \\ =& \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \end{aligned}$$

(ii) Find $f_{X_1X_2}(0,1)$.

Solution:

$X_1 = 0, X_2 = 1$ implies that the first toss is a tail and the second toss is a head. Also, the events $X_1 = 0$ and $X_2 = 1$ are independent.

$$f_{X_1X_2}(0,1) = P(X_1 = 0, X_2 = 1)$$
$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

The joint PMF table of X_1 and X_2 is

X_2 \ X_1	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

Joint PMF of X_1 and X_2

To generalize this, suppose you have two random variables X_1 and X_2. You put all the values taken by X_1 in the first row of the table, and all the values taken by X_2 in the first column of the table. The ij-th position in the matrix will give the probability that X_1 taking the value i and X_2 taking the value j.

Example 2: Random 2-digit number

A 2-digit number from 00 to 99 is selected at random. Let X be the digit in units place. Let Y be the remainder obtained when the number is divided by 4.

(i) Find $f_{XY}(0,0)$.

Solution:

$f_{XY}(0,0)$ is same as $P(X = 0, Y = 0)$.
$X = 0, Y = 0$ implies that the unit place digit is 0 and the number is divisible by 4. The favourable outcomes are
$(0,0), (2,0), (4,0), (6,0)$ and $(8,0)$.

$$f_{XY}(0,0) = P(X = 0, Y = 0)$$
$$= \frac{5}{100} = \frac{1}{20}$$

(ii) Find $f_{XY}(1, 0)$.

Solution:

$f_{XY}(1, 0)$ is same as $P(X = 1, Y = 0)$.
$X = 1, Y = 0$ implies that the unit place digit is 1 and the number is divisible by 4, which is not possible. So, the number of favourable outcomes is zero.

$$f_{XY}(1, 0) = 0$$

(iii) Find $f_{XY}(4, 2)$.

Solution:

$f_{XY}(4, 2)$ is same as $P(X = 4, Y = 2)$.
$X = 4, Y = 2$ implies that the unit place digit is 4 and the number is 2 modulo 4. The favourable outcomes are
$(1, 4), (3, 4), (5, 4), (7, 4)$ and $(9, 4)$.

$$f_{XY}(4, 2) = \frac{1}{20}$$

Observations:

- If you write down all the possibilities, the probability is either 0 or 1/20.

$Y \backslash X$	0	1	2	3	4	5	6	7	8	9
0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0
1	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$
2	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0
3	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$

Joint PMF of X and Y

- Every entry in the joint PMF is between 0 and 1, and if we add all the entries in the table, it will give 1. So, it is a reasonable joint PMF for this problem.

We can notice that things are getting slightly more complex and as we go more and more into complex outcomes, this kind of dependency is very crucial and it will become more and more complicated to write down. We will stop here with the first part of Joint PMF and in the next section, we will look at the marginal PMFs.

1.4.2 Two random variables: Marginal PMF

Definition: Suppose X and Y are jointly distributed discrete random variables with joint PMF f_{XY} . The PMF of the individual random variables X and Y are called as marginal PMFs.

$$f_X(t) = P(X = t) = \sum_{t' \in T_Y} f_{XY}(t, t') \tag{1}$$

$$f_Y(t) = P(Y = t) = \sum_{t' \in T_X} f_{XY}(t', t) \tag{2}$$

where, T_X and T_Y are the ranges of X and Y respectively.

Here, (1) and (2) are marginalization equations. There is unique way of defining the marginal PMF given the joint PMFs.

Proof:

Let $T_Y = \{y_1, y_2, \ldots, y_k\}$.

$$\begin{aligned} P(X = t) =& P(X = t, Y = y_1) \text{or} \ldots \text{or} P(X = t, Y = y_k) \\ =& P(X = t, Y = y_1) + \ldots + P(X = t, Y = y_k) \end{aligned}$$

This process is called marginalisation. We will see few examples to see how marginalisation works.

Example: Toss a fair coin twice

Suppose a fair coin is tossed twice independently and we define the random variable X_i as

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is a head} \\ 0, & \text{if the } i\text{-th toss is a tail} \end{cases}$$

for $i : 1, 2$

It is always easy to work with the table. We will make a joint PMF table for X_1 and X_2 and see how marginalisation works here.

- Marginal PMF of X_1: We will add over the columns of joint PMF table.
  $f_{X_1}(0) = f_{X_1X_2}(0, 0) + f_{X_1X_2}(0, 1)$
  $f_{X_1}(1) = f_{X_1X_2}(1, 0) + f_{X_1X_2}(1, 1)$
- Marginal PMF of X_2: We will add over the rows of joint PMF table.
  $f_{X_2}(0) = f_{X_1X_2}(0, 0) + f_{X_1X_2}(1, 0)$
  $f_{X_1}(1) = f_{X_1X_2}(0, 1) + f_{X_1X_2}(1, 1)$

X_2 \ X_1	0	1	$f_{X_2}(t_2)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$f_{X_1}(t_1)$	$\frac{1}{2}$	$\frac{1}{2}$	1

Joint PMF of X_1 and X_2

To marginalize, we just sum it over either the rows or the columns.

Note: Given the joint PMF, the marginal is unique.

1.4.3 Marginal from Joint PMF

Suppose the joint PMF of X_1 and X_2 is as follows:

X_2 \ X_1	0	1
0	0.05	0.35
1	0.25	0.35

Joint PMF of X_1 and X_2

1. Verify that it is a valid joint PMF.
   - All the entries of the table are greater than equal to 0.
   - All the values inside the table sum up to 1.
2. Find the marginal PMF.

$X_2 \backslash X_1$	0	1	$f_{X_2}(t_2)$
0	0.05	0.35	0.40
1	0.25	0.35	0.60
$f_{X_1}(t_1)$	0.30	0.70	1

Joint PMF of X_1 and X_2

Same marginal PMF from different joint PMFs

Many a times, given the marginal PMF, we tend to think that the joint PMF will be the product of the marginals. But this is not the only joint PMF which will give that marginal. We will see here how the same marginal PMF can result from different joint PMFs.

Case I:

$X_2 \backslash X_1$	0	1	$f_{X_2}(t_2)$
0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$f_{X_1}(t_1)$	$\frac{1}{2}$	$\frac{1}{2}$	1

Joint PMF of X_1 and X_2

In this case, we saw earlier that the marginals are 1/2.

Case II:

For every x between 0 and 1/2, we get a joint PMF that results in the same marginal. Therefore, from joint to marginal, we can go in a unique way, but we cannot go from marginal to joint in a unique way.

$X_2 \backslash X_1$	0	1	$f_{X_2}(t_2)$
0	x	$\frac{1}{2} - x$	$\frac{1}{2}$
1	$\frac{1}{2} - x$	x	$\frac{1}{2}$
$f_{X_1}(t_1)$	$\frac{1}{2}$	$\frac{1}{2}$	1

Joint PMF of X_1 and X_2

Example: Random 2-digit number

A 2-digit number from 00 to 99 is selected at random. Let X be the digit in units place. Let Y be the remainder obtained when the number is divided by 4.

We have already seen the joint PMF of X and Y before, here we will find their marginals.

$Y \backslash X$	0	1	2	3	4	5	6	7	8	9	$f_Y(y)$
0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{4}$
1	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	$\frac{1}{4}$
2	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{4}$
3	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	0	$\frac{1}{20}$	$\frac{1}{4}$
$f_X(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	1

Joint PMF of X and Y

Observations:

- From the table, you can observe that $X \sim \text{Uniform}\{0, 1, \ldots, 9\}$ and $Y \sim \text{Uniform}\{0, 1, 2, 3\}$.
- $f_{XY}(0,0) = \frac{1}{20} \neq f_X(x) f_Y(y) = \frac{1}{10} \times \frac{1}{4} = \frac{1}{40}$. So, the events $X = 0$ and event $Y = 0$ are not independent. Hence, X and Y are not independent.

1.4.4 Conditional distribution of a random variable given an event

Definition: Suppose X is a discrete random variable with range T_X , and A is an event in the same probability space. The conditional PMF of X given A is defined as the PMF

$$Q(t) = P(X = t \mid A), \qquad\qquad t \in T_X$$

For the above conditional PMF, we will use the notation $f_{X|A}(t)$ and $(X \mid A)$ as the conditional random variable.

$$f_{X|A}(t) = \frac{P((X = t) \cap A)}{P(A)}$$

Note: The range of $(X \mid A)$ can be different from the range of X and will depend on A.

1.4.5 Conditional distribution of one random variable given another

Definition: Suppose X and Y are jointly distributed discrete random variables with joint PMF f_{XY}. The conditional PMF of Y given $X = t$ is defined as the PMF

$$Q(t') = P(Y = t' \mid X = t) = \frac{P((X = t) \cap (Y = t'))}{P(X = t)} = \frac{f_{XY}(t, t')}{f_X(t)}, \qquad t \in T_X, t' \in T_Y$$

For the above conditional PMF, we will use the notation $f_{Y|X=t}(t')$ and $(Y \mid X = t)$ to denote the conditional random variable.

$$f_{XY}(t, t') = f_{Y|X=t}(t')$$

Note: The range of $(Y \mid X = t)$ can be different from the range of Y and will depend on t.

Example: Compute conditional PMFs from Joint PMF

Let the joint PMF of X and Y be

$Y \backslash X$	0	1	2	$f_Y(t_2)$
0	1/4	1/8	1/8	1/2
1	1/8	1/8	1/4	1/2
$f_X(t_1)$	3/8	1/4	3/8	1

Joint PMF of X and Y

Range $X = \{0, 1, 2\}$
Range $Y = \{0, 1\}$

1. Find $f_{Y|X=0}(0)$

   Solution:

$$f_{Y|X=0}(0) = \frac{f_{XY}(0,0)}{f_X(0)} = \frac{1/4}{3/8} = \frac{2}{3}$$

2. Find $f_{Y|X=0}(1)$

   Solution:

$$f_{Y|X=0}(1) = \frac{f_{XY}(0,1)}{f_X(0)} = \frac{1/8}{3/8} = \frac{1}{3}$$

   You can observe here that $f_{Y|X=0}(0)$ and $f_{Y|X=0}(1)$ sums up to 1.

3. Find $f_{X|Y=1}(0)$

   Solution:

$$f_{X|Y=1}(0) = \frac{f_{XY}(0,1)}{f_Y(1)} = \frac{1/8}{1/2} = \frac{1}{4}$$

Example: Throw a die and toss coins

Throw a die and toss a coin as many times as the number shown on die. Let X be the number shown on die. Let Y be the number of heads. What is the joint PMF of X and Y?

Solution:

The random variable X represent the number shown on the die.
Range $X = \{1, 2, 3, 4, 5, 6\}$

$f_X(t) = \dfrac{1}{6}, 1 \leq t \leq 6$
Therefore,

$$X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$$

The random variable Y represent the number of heads obtained on X number of tosses. Therefore,

$$(Y \mid X = t) \sim \text{Binomial}(t, 1/2)$$

Range $(Y \mid X = t) = \{0, 1, \ldots, t\}$

$f_{Y|X=t}(t') = \binom{t}{t'} \left(\dfrac{1}{2}\right)^{t'} \dfrac{1}{2}^{t-t'} = \binom{t}{t'} \left(\dfrac{1}{2}\right)^{t}, t' = 0, 1, 2, \ldots, t$

Now the joint distribution of X and Y is $f_{XY}(t, t') = f_X(t) f_{Y|X=t}(t') = \dfrac{1}{6}\binom{t}{t'} \left(\dfrac{1}{2}\right)^{t}$

t' \ t	1	2	3	4	5	6
0	$\frac{1}{6}.\frac{1}{2}$	$\frac{1}{6}.\left(\frac{1}{2}\right)^2$	$\frac{1}{6}.\left(\frac{1}{2}\right)^3$	$\frac{1}{6}.\left(\frac{1}{2}\right)^4$	$\frac{1}{6}.\left(\frac{1}{2}\right)^5$	$\frac{1}{6}.\left(\frac{1}{2}\right)^6$
1	$\frac{1}{6}.\frac{1}{2}$	$\frac{1}{6}.2\left(\frac{1}{2}\right)^2$	$\ldots$	$\ldots$	$\ldots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Joint PMF of X and Y

Example: Poisson number of coin tosses

Let $N \sim \text{Poisson}(\lambda)$. Given $N = n$, toss a fair coin n times and denote the number of heads obtained by X. What is the distribution of X?

Solution: N is distributed as Poisson random variable.

$$f_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, 2, \ldots$$

$(X \mid N = n) \sim \text{Binomial}(n, 1/2)$

$f_{X|N=n}(k) = \binom{n}{k} \left(\frac{1}{2}\right)^n, k = 0, 1, \ldots, n$

$$f_{NX}(n,k) = \frac{e^{-\lambda}\lambda^n}{n!}\binom{n}{k}\left(\frac{1}{2}\right)^n, n = 0, 1, \ldots; k = 0, 1, \ldots, n$$

$$\begin{aligned}
f_X(k) &= \sum_{n=0}^{\infty} f_{NX}(n,k) \\
&= \begin{cases} \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\binom{n}{k}\left(\frac{1}{2}\right)^n, & \text{for } n \geq k \\ 0, & \text{for } n < k \end{cases} \\
&= \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{k!(n-k)!}\left(\frac{1}{2}\right)^n \\
&= \frac{e^{-\lambda}\lambda^k}{k!2^k}\sum_{n=k}^{\infty} \frac{\lambda^{n-k}}{(n-k)!2^{n-k}} \\
&= \frac{e^{-\lambda/2}(\lambda/2)^k}{k!}, k = 0, 1, 2, \ldots
\end{aligned}$$

Therefore,

$$X \sim \text{Poisson}(\lambda/2)$$

Example: IPL Powerplay over

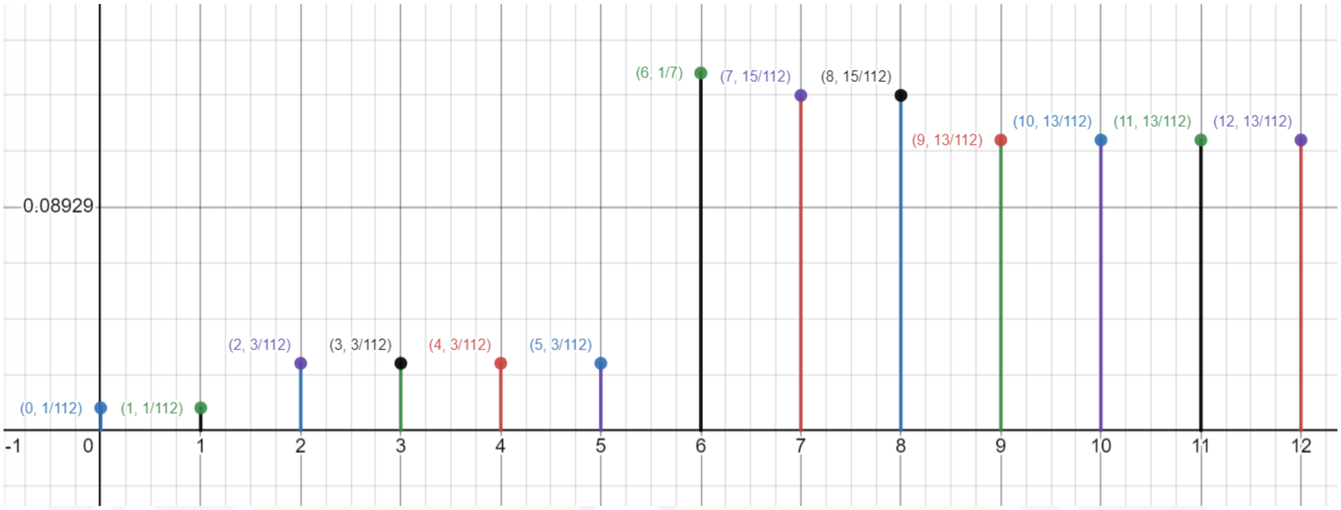
Let X = number of runs in the over. Let Y = number of wickets in the over. Assume the following:

$Y \in \{0, 1, 2\}$ and

Y	0	1	2
$f_Y(y)$	13/16	1/8	1/16

$$(X \mid Y = 0) \sim \text{Uniform}\{6, 7, 8, 9, 10, 11, 12\}$$
$$(X \mid Y = 1) \sim \text{Uniform}\{2, 3, 4, 5, 6, 7, 8\}$$
$$(X \mid Y = 2) \sim \text{Uniform}\{0, 1, 2, 3, 4, 5, 6\}$$

Calculate $f_X(t)$.



Distribution of X

Solution:

$$f_{XY}(t,t') = f_Y(t')f_{X|Y=t'}(t)$$
$$f_X(t) = f_Y(0)f_{X|Y=0}(t) + f_Y(1)f_{X|Y=1}(t) + f_Y(2)f_{X|Y=2}(t), t = 0,1,2,\ldots,12$$

$$f_X(0) = f_Y(2)f_{X|Y=2}(0) = \frac{1}{16} \times \frac{1}{7}$$
$$f_X(1) = f_Y(2)f_{X|Y=2}(1) = \frac{1}{16} \times \frac{1}{7}$$
$$f_X(2) = f_Y(1)f_{X|Y=1}(2) + f_Y(2)f_{X|Y=2}(2) = \frac{1}{8} \times \frac{1}{7} + \times \frac{1}{16} \times \frac{1}{7}$$
$$\vdots$$
$$f_X(12) = f_Y(0)f_{X|Y=0}(12) = \frac{13}{16} \times \frac{1}{7}$$

1.5 More than two random variables: Joint, marginal, conditional PMFs

1.5.1 Multiple discrete random variables: Joint PMF

Definition: Suppose $X_1, X_2, \ldots, X_n$ are discrete random variables defined in the same probability space. Let the range of X_i be T_{X_i}. The joint PMF of X_i, denoted $fX_1, \ldots, X_n$, is a function from $T_{X_1} \times \ldots \times T_{X_n}$ to $[0,1]$ defined as

$$f_{X_1,\ldots,X_n}(t_1,\ldots,t_n) = P(X_1 = t_1 \text{ and } \ldots \text{ and } X_n = t_n), t_i \in T_{X_i}$$

Example: Toss a fair coin thrice

Suppose a fair coin is tossed three times. Let us define the random variable X_i as

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is a head} \\ 0, & \text{if the } i\text{-th toss is a tail} \end{cases}$$

The joint PMF of X_1, X_2 and X_3 is given by

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	1/8
0	0	1	1/8
0	1	0	1/8
0	1	1	1/8
1	0	0	1/8
1	0	1	1/8
1	1	0	1/8
1	1	1	1/8

Example: Random 3-digit number 000 to 999

A 3-digit number from 000 to 999 is selected at random. Let X denote the first digit from the left. Let Y denote the number modulo 2. Let Z denote the first digit from the right.

<u>Solution:</u>

$$X \in \{0, 1, 2, \ldots, 9\}$$
$$Y \in \{0, 1\}$$
$$Z \in \{0, 1, 2, \ldots, 9\}$$

$f_{XYZ}(0, 0, 0)$ = P(starts with zero and the number is even and the number ends with zero)$= \dfrac{10}{1000} = \dfrac{1}{100}$

$f_{XYZ}(1, 1, 1) = \dfrac{1}{100}$

$f_{XYZ}(1, 0, 1) = 0$

$f_{XYZ}(8, 0, 6) = \dfrac{10}{1000} = \dfrac{1}{100}$

$f_{XYZ}(8, 1, 6) = 0$

Therefore, the joint distribution of X, Y and Z is

$$f_{XYZ}(t_1,t_2,t_3) = \begin{cases} 0, & \text{if } t_2 = 0, t_3 \text{ is even} \\ 0, & \text{if } t_2 = 0, t_3 \text{ is even} \\ \dfrac{1}{100}, & \text{otherwise} \end{cases}$$

Example: IPL powerplay over

We will look here at a slightly complicated situation. Consider an over from the IPL powerplay. Suppose this over has six deliveries. Let X_i denote the number of runs scored in the i-th delivery and $X_i \in \{0, 1, \ldots, 8\}$. What will be the joint PMF of $X_1, X_2, \ldots, X_6$?

It is very difficult to calculate these probabilities. Number of possibilities is going to be 8^6 in this case. We need some tools to tackle such problems. This is where marginalization and conditioning will help, which we will see in the later section.

1.5.2 Multiple discrete random variables: Marginal PMF

In the previous section, we looked at how to describe the joint distribution of multiple random variables, we looked at some simpler examples and few scenarios where computing joint PMF could not be easily describe. In such situations, marginalization and conditional distributions are nice pathways to go to. In this section, we will look at the marginal PMF. We have already seen it in the context of two random variables, here we will just increase the number of random variables from two.

Definition: Suppose $X_1, X_2, \ldots, X_n$ are jointly distributed discrete random variables with joint PMF $f_{X_1,\ldots,X_n}$. The PMF of the individual random variables $X_1, X_2, \ldots, X_n$ are called as marginal PMFs. It can be shown that

$$\begin{aligned}
f_{X_1}(t) = P(X_1 = t) &= \sum_{t_2' \in T_{X_2}, t_3' \in T_{X_3} \ldots, t_n' \in T_{X_n}} f_{X_1,\ldots,X_n}(t, t_2', t_3', \ldots, t_n') \\
f_{X_2}(t) = P(X_2 = t) &= \sum_{t_1' \in T_{X_1}, t_3' \in T_{X_3} \ldots, t_n' \in T_{X_n}} f_{X_1,\ldots,X_n}(t_1', t', t_3', \ldots, t_n') \\
&\vdots \\
f_{X_n}(t) = P(X_n = t) &= \sum_{t_1' \in T_{X_1}, t_2' \in T_{X_2} \ldots, t_{n-1}' \in T_{X_{n-1}}} f_{X_1,\ldots,X_n}(t_1', t_2', t_3', \ldots, t_{n-1}')
\end{aligned}$$

, where t_{X_i} is the range of X_i.

Example: Toss a fair coin thrice

Suppose a fair coin is tossed three times. Let us define the random variable X_i as

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th toss is a head} \\ 0, & \text{if the } i\text{-th toss is a tail} \end{cases}$$

The joint PMF of X_1, X_2 and X_3 is given by

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	1/8
0	0	1	1/8
0	1	0	1/8
0	1	1	1/8
1	0	0	1/8
1	0	1	1/8
1	1	0	1/8
1	1	1	1/8

1. Compute $f_{X_1}(0)$.

$$\begin{aligned} f_{X_1}(0) =& f_{X_1X_2X_3}(0,0,0) + f_{X_1X_2X_3}(0,0,1) + f_{X_1X_2X_3}(0,1,0) + f_{X_1X_2X_3}(0,1,1) \\ =& \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ =& \frac{1}{2} \end{aligned}$$

2. Compute $f_{X_1}(1)$.

$$\begin{aligned} f_{X_1}(1) =& f_{X_1X_2X_3}(1,0,0) + f_{X_1X_2X_3}(1,0,1) + f_{X_1X_2X_3}(1,1,0) + f_{X_1X_2X_3}(1,1,1) \\ =& \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ =& \frac{1}{2} \end{aligned}$$

Note: $f_{X_1}(0) + f_{X_1}(0) = 1$

Example: Random 3-digit number 000 to 999

A 3-digit number from 000 to 999 is selected at random. Let X denote the first digit from the left. Let Y denote the number modulo 2. Let Z denote the first digit from the right. Instead of looking at the joint PMF, we can directly look at the marginals.

- There are a total of 100 favourable outcomes out of 1000, for every first digit from the left.

$$f_X(x) = \frac{100}{1000} = \frac{1}{10}, \text{ for all } x \in \{0, 1, \dots, 9\}$$

Therefore, X is uniform in $\{0, 1, \dots, 9\}$.

- The possible values that Y can take is $\{0, 1\}$. Y will be 0, if the number is even and Y will be 1 if the number is odd.

$$f_Y(y) = \frac{500}{1000} = \frac{1}{2}, \text{ for } y \in \{0, 1\}.$$

Therefore, $Y \sim \text{Uniform}\{0, 1\}$.

- There are a total of 100 favourable outcomes out of 1000, for every first digit from the right.

$$f_Z(z) = \frac{100}{1000} = \frac{1}{10}, \text{ for all } z \in \{0, 1, \dots, 9\}$$

Therefore, Z is uniform in $\{0, 1, \dots, 9\}$.

Example: IPL powerplay Over 1

Let us consider the first over of an IPL powerplay. Assume that this over has six deliveries. Let X_i denote the number of runs scored in the i-th delivery.

$X_i \in \{0, 1, 2, \dots, 8\}$. It is very unlikely that more than eight runs will be scored in a particular delivery, so we will neglect that part. Now, the question here is how to assign the probabilities. The method that we will see here may not be the best one, but it is reasonable to do in this way.

We can collect data from the past occurrences. There are 1598 matches in the IPL where the first over has been bowled so far. Let us look at ball 1: Number of times zero runs were scored is 957. Number of times one run was scored is 429. Number of times two runs were scored is 57, three runs in 5 matches, four runs in 138 matches, five runs in 8 matches, six runs in 4 matches. We do not have any seven or eight runs so far in the first ball.

- So, one way to assign probabilities here is in proportion as data. Therefore,

$$P(X_1 = 0) = \frac{957}{1598}$$
$$P(X_1 = 1) = \frac{429}{1598}$$
$$P(X_1 = 2) = \frac{57}{1598}$$

$$P(X_1 = 3) = \frac{5}{1598}$$
$$P(X_1 = 4) = \frac{138}{1598}$$
$$P(X_1 = 5) = \frac{8}{1598}$$
$$P(X_1 = 6) = \frac{4}{1598}$$

Similarly, we can check for the other deliveries of the first over. We will get the following probabilities:

X_i	0	1	2	3	4	5	6
f_{X_1}	0.5989	0.2685	0.0357	0.0031	0.0864	0.0050	0.0025
f_{X_2}	0.5551	0.2791	0.0438	0.0031	0.1083	0.0044	0.0063
f_{X_3}	0.5338	0.2847	0.0444	0.0044	0.1139	0.0025	0.0163
f_{X_4}	0.5344	0.2516	0.0394	0.0031	0.1489	0.0038	0.0188
f_{X_5}	0.5313	0.2672	0.0407	0.0056	0.1358	0.0025	0.0169
f_{X_6}	0.5056	0.2954	0.0394	0.0050	0.1414	0.0013	0.0119

Observations:

1. A six is much more likely to be hit in the fourth ball.
2. A four is also much more likely to be hit in the fourth ball.
3. Zero is lowest in the last ball.

1.5.3 Marginalisation

Suppose $X_1, X_2, X_3 \sim f_{X_1X_2X_3}$ and $X_i \in T_{X_i}$. We have already discussed the marginal PMF of the individual random variables. How do we find $f_{X_1X_2}, f_{X_2X_3}, f_{X_3X_1}$? This can be found using the same principle of sum over everything that you do not want, which is called marginalisation. Therefore,

$$f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \sum_{t_3' \in T_{X_3}} f_{X_1X_2X_3}(t_1, t_2, t_3')$$

$$f_{X_1X_3}(t_1, t_3) = P(X_1 = t_1 \text{ and } X_3 = t_3) = \sum_{t_2' \in T_{X_2}} f_{X_1X_2X_3}(t_1, t_2', t_3)$$

$$f_{X_2X_3}(t_2, t_3) = P(X_2 = t_2 \text{ and } X_2 = t_3) = \sum_{t_1' \in T_{X_1}} f_{X_1X_2X_3}(t_1', t_2, t_3)$$

Example: $X_1, X_2, X_3 \sim f_{X_1X_2X_3}$

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	1/9
0	0	1	1/9
0	0	2	1/9
0	1	1	1/9
0	1	2	1/9
1	0	0	1/9
1	0	2	1/9
1	1	0	1/9
1	1	1	1/9

1. Find the joint PMF of X_1 and X_2.

<u>Solution:</u>

$X_1, X_2 \in \{0,1\}$

- $$\begin{aligned} f_{X_1X_2}(0,0) =& f_{X_1X_2X_3}(0,0,0) + f_{X_1X_2X_3}(0,0,1) + f_{X_1X_2X_3}(0,0,2) \\ =& \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \\ =& \frac{1}{3} \end{aligned}$$

- $$\begin{aligned} f_{X_1X_2}(0,1) =& f_{X_1X_2X_3}(0,1,1) + f_{X_1X_2X_3}(0,1,2) \\ =& \frac{1}{9} + \frac{1}{9} \\ =& \frac{2}{9} \end{aligned}$$

- $$\begin{aligned} f_{X_1X_2}(1,0) =& f_{X_1X_2X_3}(1,0,0) + + f_{X_1X_2X_3}(1,0,2) \\ =& \frac{1}{9} + \frac{1}{9} \\ =& \frac{2}{9} \end{aligned}$$

-

$$
\begin{aligned}
f_{X_1X_2}(1,1) =& f_{X_1X_2X_3}(1,1,0) + f_{X_1X_2X_3}(1,1,1)\\
=&\frac{1}{9} + \frac{1}{9}\\
=&\frac{2}{9}
\end{aligned}
$$

Therefore, the joint PMF of X_1 and X_2 is

$Y \backslash X$	0	1
0	1/3	2/9
1	2/9	2/9

Joint PMF of X_1 and X_2

2. Similarly, the joint PMF of X_2 and X_3 is

$Y \backslash X$	0	1
0	2/9	1/9
1	1/9	2/9
2	2/9	1/9

Joint PMF of X_2 and X_3

3. And, the joint PMF of X_1 and X_3 is

$Y \backslash X$	0	1
0	1/9	2/9
1	2/9	1/9
2	2/9	1/9

Joint PMF of X_1 and X_3

Marginalisation: More examples

We have already seen the marginalisation for two random variables and three random variables. Next, we will see how the same principle can be applied for four random variables.

Suppose $X_1, X_2, X_3, X_4 \sim f_{X_1X_2X_3X_4}$ and $X_i \in T_{X_i}$.

$$f_{X_1}(t_1) = P(X_1 = t_1) = \sum_{t'_2, t'_3, t'_4} f_{X_1X_2X_3X_4}(t_1, t'_2, t'_3, t'_4)$$

$$f_{X_1X_2}(t_1, t_2) = P(X_1 = t_1 \text{ and } X_2 = t_2) = \sum_{t'_3, t'_4} f_{X_1X_2X_3X_4}(t_1, t_2, t'_3, t'_4)$$

$$f_{X_2X_4}(t_2, t_4) = P(X_2 = t_2 \text{ and } X_4 = t_4) = \sum_{t'_1, t'_3} f_{X_1X_2X_3X_4}(t'_1, t_2, t'_3, t_4)$$

$$f_{X_1X_3X_4}(t_1, t_3, t_4) = P(X_1 = t_1 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4) \sum_{t'_2} f_{X_1X_2X_3X_4}(t_1, t'_2, t_3, t_4)$$

1.5.4 Multiple discrete random variables: Marginal PMF (general)

Definition: Suppose $X_1, X_2, \ldots, X_n$ are jointly distributed discrete random variables with joint PMF $f_{X_1} \ldots X_n$. The joint PMF of the random variables $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$, denoted $f_{X_{i_1},\ldots,X_{i_k}}$, is given by

$$f_{X_{i_1},\ldots,X_{i_k}}(t_{i_1}, \ldots, t_{i_k}) = \sum_{\substack{t_1,\ldots,t'_{i_1-1},t_{i_1},t'_{i_1+1},\ldots,\\ t'_{i_k-1},t_{i_k},t'_{i_k+1},\ldots,t_n}} f_{X_1,\ldots,X_n}(t_1, \ldots, t'_{i_1-1}, t_{i_1}, t'_{i_1+1}, \ldots, t'_{i_k-1}, t_{i_k}, t'_{i_k+1}, \ldots, t_n)$$

1.5.5 Conditioning with multiple discrete random variables

In the previous section, we saw how marginalisation reduces the scale of the problem from a bigger problem. Conditioning acts like a bridge between the marginal and joint, giving a picture of the whole in a sense. In case of multiple random variables, a wide variety of conditioning is possible. Lets understand this with the help of an example.

Suppose $X_1, X_2, X_3, X_4 \sim f_{X_1X_2X_3X_4}$ and $X_i \in T_{X_i}$.

1. $(X_1 \mid X_2 = t_2) \sim f_{X_1|X_2=t_2}(t_1) = \dfrac{f_{X_1X_2}(t_1, t_2)}{f_{X_2}(t_2)}$
2. $(X_1, X_2 \mid X_3 = t_3) \sim f_{X_1X_2|X_3=t_3}(t_1, t_2) = \dfrac{f_{X_1X_2X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$
3. $(X_1 \mid X_2 = t_2, X_3 = t_3) \sim f_{X_1|X_2=t_2,X_3=t_3}(t_1) = \dfrac{f_{X_1X_2X_3}(t_1, t_2, t_3)}{f_{X_2X_3}(t_2t_3)}$
4. $(X_1, X_3 \mid X_2 = t_2, X_4 = t_4) \sim f_{X_1X_3|X_2=t_2,X_4=t_4}(t_1, t_3) = \dfrac{f_{X_1X_2X_3X_4)}(t_1, t_2, t_3, t_4)}{f_{X_2X_4}(t_2t_4)}$

Example: $X_1, X_2, X_3, X_4 \sim f_{X_1X_2X_3,X_4}$

t_1	t_2	t_3	t_4	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	0	1/12
0	0	0	1	1/12
0	0	1	1	1/12
0	0	2	0	1/12
0	1	1	0	1/12
0	1	1	1	1/12
0	1	2	0	1/12
1	0	0	1	1/12
1	0	2	0	1/12
1	0	2	1	1/12
1	1	0	1	1/12
1	1	1	0	1/12

1. Find the probability distribution of $(X_1 \mid X_2 = 0)$.

   Solution:

   Step 1: Identify the range.

Range of $(X_1 \mid X_2 = 0) = \{0, 1\}$

Step 2: Calculate the conditional probabilities.

$$
\begin{aligned}
P(X_1 = 0 \mid X_2 = 0) =& \frac{P(X_1 = 0, X_2 = 0)}{P(X_2 = 0)} \\
=& \frac{4/12}{7/12} = \frac{4}{7}
\end{aligned}
$$

$$
\begin{aligned}
P(X_1 = 1 \mid X_2 = 0) =& \frac{P(X_1 = 1, X_2 = 0)}{P(X_2 = 0)} \\
=& \frac{3/12}{7/12} = \frac{3}{7}
\end{aligned}
$$

2. Find the probability distribution of $(X_1 \mid X_3 = 0, X_4 = 1)$.

Solution:

Step 1: Identify the range.

Range of $(X_1 \mid X_3 = 0, X_4 = 1) = \{0, 1\}$

Step 2: Calculate the conditional probabilities.

$$
\begin{aligned}
P(X_1 = 0 \mid X_3 = 0, X_4 = 1) =& \frac{P(X_1 = 0, X_3 = 0, X_4 = 1)}{P(X_3 = 0, X_4 = 1)} \\
=& \frac{1/12}{3/12} = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P(X_1 = 1 \mid X_3 = 0, X_4 = 1) =& \frac{P(X_1 = 0, X_3 = 0, X_4 = 1)}{P(X_3 = 0, X_4 = 1)} \\
=& \frac{2/12}{3/12} = \frac{2}{3}
\end{aligned}
$$

3. Find the joint probability distribution of $(X_3, X_4 \mid X_1 = 0)$.

Solution:

Step 1: Identify the range.

Range of $(X_3 \mid X_1 = 0) = \{0, 1, 2\}$
Range of $(X_4 \mid X_1 = 0) = \{0, 1\}$

Step 2: Calculate the conditional probabilities.

$$\begin{aligned} P(X_3 = 0, X_4 = 0 \mid X_1 = 0) =& \frac{P(X_3 = 0, X_4 = 0, X_1 = 0)}{P(X_1 = 0)} \\ =& \frac{1/12}{7/12} = \frac{1}{7} \end{aligned}$$

$$\begin{aligned} P(X_3 = 0, X_4 = 1 \mid X_1 = 0) =& \frac{P(X_3 = 0, X_4 = 1, X_1 = 0)}{P(X_1 = 0)} \\ =& \frac{1/12}{7/12} = \frac{1}{7} \end{aligned}$$

Similarly find the conditional probabilities for the other cases, i.e., $P(X_1 = 0, X_4 = 0 \mid X_1 = 0), P(X_3 = 1, X_4 = 1 \mid X_1 = 0), P(X_3 = 2, X_4 = 0 \mid X_1 = 0)$ and $P(X_3 = 2, X_4 = 1 \mid X_1 = 0)$.

Therefore, the joint PMF of X_3 and X_4 given $X_1 = 0$ is

X_4 \ X_3	0	1	2
0	1/7	1/7	2/7
1	1/7	2/7	0

1.5.6 Conditioning and factors of the joint PMF

In this section, we will look at how the conditional and marginal PMF are related to the factoring of joint PMF. So, the first thing to understand here is what is factoring? If we take any big number, we can keep factoring it and write it as a product of various small numbers. For example, number like 1024 seems very large, but it can be written as 2^{10}; which seems a bit simpler than 1024 in some sense. Similarly, any complicated big joint PMF can be factored into product of several terms. It is highly used in understanding stochastic phenomena, with many different random variables.

So, we want to see if we can factor the joint PMF, and marginal and conditioning turn out to be very crucial in factoring. Factoring can be done in many different ways, but the fundamental idea is very simple, which we have seen so far. Lets understand this with the help of an example.

Example: Suppose $X_1, X_2, X_3, X_4 \sim f_{X_1X_2X_3X_4}$ and $X_i \sim T_{X_i}$

$$
\begin{aligned}
f_{X_1X_2X_3X_4}(t_1, , t_4) =& P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4) \\
=& P(X_1 = t_1 \text{ and } (X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4)) \\
=& P(X_1 = t_1 \mid (X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4))P(X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4) \\
=& P(X_1 = t_1 \mid X_2 = t_2, X_3 = t_3, X_4 = t_4)P(X_2 = t_2 \mid X_3 = t_3, X_4 = t_4) \\
& P(X_3 = t_3, X_4 = t_4) \\
=& P(X_1 = t_1 \mid X_2 = t_2, X_3 = t_3, X_4 = t_4)P(X_2 = t_2 \mid X_3 = t_3, X_4 = t_4) \\
& P(X_3 = t_3 \mid X_4 = t_4)P(X_4 = t_4) \\
=& f_{X_1|X_2=t_2,X_3=t_3,X_4=t_4}(t_1)f_{X_2|X_3=t_3,X_4=t_4}(t_2)f_{X_3|X_4=t_4}(t_3)f_{X_4}(t_4)
\end{aligned}
$$

So, here I could write my joint PMF as a product of four factors. Lets look at an example.

Example: $X_1, X_2, X_3 \sim f_{X_1X_2X_3}$ $f_{X_1X_2X_3}$ can be written as $f_{X_3}f_{(X_2|X_3)}f_{(X_1|X_2,X_3)}$. Therefore,

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	1/9
0	0	1	1/9
0	0	2	1/9
0	1	1	1/9
0	1	2	1/9
1	0	0	1/9
1	0	2	1/9
1	1	0	1/9
1	1	1	1/9

$$
\begin{aligned}
f_{X_1X_2X_3}(0, 0, 0) =& f_{X_3}(0)f_{X_2|X_3=0}(0)f_{X_1|X_2=0,X_3=0}(0) \\
=& \frac{3}{9} \times \frac{2}{3} \times \frac{1}{2} \\
=& \frac{1}{9}
\end{aligned}
$$

Factoring can be done in any sequence

In the previous section, we saw the factoring of joint PMF. It turns out that it can be done in any sequence. We will look at two others forms of it here.

Suppose $X_1, X_2, X_3, X_4 \sim f_{X_1X_2X_3X_4}$ and $X_i \in T_{X_i}$

$$\begin{aligned}
f_{X_1X_2X_3X_4}(t_1,t_2,t_3,t_4) =& P(X_1 = t_1 \text{ and } X_2 = t_2 \text{ and } X_3 = t_3 \text{ and } X_4 = t_4) \\
=& P(X_4 = t_4 \text{ and } X_3 = t_3 \text{ and } X_2 = t_2 \text{ and } X_1 = t_1) \\
=& f_{X_4|X_3=t_3,X_2=t_2,X_1=t_1}(t_4) f_{X_3|X_2=x_2,X_1=t_1}(t_3) f_{X_2|X_1=t_1}(t_2) f_{X_1}(t_1)
\end{aligned}$$

$$\begin{aligned}
f_{X_1X_2X_3X_4}(t_1,t_2,t_3,t_4) =& P(X_3 = t_3 \text{ and } X_2 = t_2 \text{ and } X_1 = t_1 \text{ and} X_4 = t_4) \\
=& f_{X_3|X_2=t_2,X_1=t_1,X_4=t_4}(t_3) f_{X_2|X_1=t_1,X_4=t_4}(t_2) f_{X_1|X_4=t_4}(t_1) f_{X_4}(t_4)
\end{aligned}$$

Example: $X_1, X_2, X_3 \sim f_{X_1X_2X_3}$

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1,t_2,t_3)$
0	0	0	1/9
0	0	1	1/9
0	0	2	1/9
0	1	1	1/9
0	1	2	1/9
1	0	0	1/9
1	0	2	1/9
1	1	0	1/9
1	1	1	1/9

$f_{X_1X_2X_3}$ can also be written as $f_{X_1} f_{(X_2|X_1)} f_{(X_3|X_1,X_2)}$. Therefore,

$$\begin{aligned}
f_{X_1X_2X_3}(0,0,0) =& f_{X_1}(0) f_{X_2|X_1=0}(0) f_{X_3|X_1=0,X_2=0} \\
=& \frac{5}{9} \times \frac{3}{5} \times \frac{1}{3} \\
=& \frac{1}{9}
\end{aligned}$$

1.6 Problems

1. Let X and Y be two random variables with joint distribution given in Table 1.1.P, where a and b are two unknown values.

Y \ X	0	1	2
0	$\frac{1}{12}$	$\frac{3}{12}$	a
1	$\frac{2}{12}$	b	$\frac{1}{12}$
2	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Table 1.1.P: Joint distribution of X and Y.

i) Find $P(Y = 1)$.

a) $\frac{4}{12}$
b) $\frac{3}{12}$
c) $\frac{5}{12}$
d) $\frac{1}{12}$

ii) Find $P(Y = 1 \mid X = 2)$.

a) $\frac{1}{12}$
b) $\frac{1}{4}$
c) $\frac{1}{3}$
d) $\frac{1}{2}$

iii) Find $P(X = 0, Y \geq 1)$.

a) $\frac{4}{12}$
b) $\frac{3}{12}$
c) $\frac{5}{12}$
d) $\frac{1}{12}$

2. Let $X \sim \text{Uniform}(\{1,2,3,4,5,6\})$ and let Y be the number of times 2 occurs in X throws of a fair die. Choose the **incorrect** option(s) among the following.

   a) $P(Y = 2 \mid X = 2) = \dfrac{1}{6}$

   b) $P(Y = 2 \mid X = 4) = \dfrac{5^2}{6^3}$

   c) $P(Y = 5 \mid X = 6) = \dfrac{5}{6^5}$

   d) $P(Y = 6 \mid X = 5) = \dfrac{5}{6^6}$

3. Let the random variables X and Y each have range $\{1, 2, 3\}$. The following formula gives the joint PMF

$$P(X = i, Y = j) = \frac{i + 2j}{c},$$

   where c is an unknown value. Find $P(1 \le X \le 3, 1 < Y \le 3)$.

   a) $\dfrac{5}{9}$

   b) $\dfrac{7}{9}$

   c) $\dfrac{2}{9}$

   d) $\dfrac{4}{9}$

4. The joint PMF of the random variables X and Y is given in Table 1.2.P.

Y \ X	1	2	3
1	k	k	$2k$
2	$2k$	0	$4k$
3	$3k$	k	$6k$

Table 1.2.P: Joint distribution of X and Y.

Consider the random variable $Z = X^2 Y$.
i) Find the range of $Z \mid Y = 2$.

a) $\{1, 4, 9\}$

b) $\{4, 8, 18\}$

c) $\{1, 9\}$

d) $\{2, 18\}$

e) $\{2, 8, 18\}$

ii) Find the value of $P(Z = 18 \mid Y = 2)$.

a) $\frac{1}{3}$

b) $\frac{2}{3}$

c) $\frac{3}{4}$

d) $\frac{1}{4}$

5. From a sack of fruits containing 3 mangoes, 2 kiwis, and 3 guavas, a random sample of 4 pieces of fruit is selected. If X is the number of mangoes and Y is the number of kiwis in the sample, then find the joint probability distribution of X and Y.

$Y \backslash X$	0	1	2	3
0	0	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$
1	$\frac{2}{70}$	$\frac{18}{70}$	$\frac{2}{70}$	$\frac{18}{70}$
2	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$	0

a)

$Y \backslash X$	0	1	2	3
0	0	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$
1	$\frac{2}{70}$	$\frac{18}{70}$	$\frac{18}{70}$	$\frac{2}{70}$
2	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$	0

b)

Y \ X	0	1	2	3
0	0	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$
1	$\frac{2}{70}$	$\frac{18}{70}$	$\frac{18}{70}$	$\frac{2}{70}$
2	$\frac{9}{70}$	$\frac{3}{70}$	$\frac{3}{70}$	0

c)

Y \ X	0	1	2	3
0	0	$\frac{3}{70}$	$\frac{3}{70}$	$\frac{9}{70}$
1	$\frac{2}{70}$	$\frac{18}{70}$	$\frac{18}{70}$	$\frac{2}{70}$
2	$\frac{3}{70}$	$\frac{9}{70}$	$\frac{3}{70}$	0

d)

6. Suppose you flip a fair coin. If the coin lands heads, you roll a fair six-sided die 50 times. If the coin lands tails, you roll the die 51 times. Let X be 1 if the coin lands heads and 0 if the coin lands tails. Let Y be the total number of times you get the number 5 while throwing the dice. Find $P(X = 1|Y = 10)$.

   a) $\frac{85}{157}$
   b) $\frac{82}{167}$
   c) $\frac{72}{157}$
   d) $\frac{85}{167}$

7. Three balls are selected at random from a box containing five red, four blue, three yellow and six green coloured balls. If X, Y and Z are the number of red balls, blue balls and green balls respectively, choose the correct option(s) among the following.

   a) $P(X = 1, Y = 0, Z = 2) = \frac{25}{272}$

b) $P(X=1, Y=1, Z=1) = \dfrac{5}{34}$

c) $P(X=1, Y=0 \mid Z=2) = \dfrac{1}{4}$

d) $P(X=0, Y=0, Z=3) = \dfrac{5}{204}$

8. A computer system receives messages over three communications lines. Let X_i be the number of messages received on line i in one hour. Suppose that the joint pmf of X_1, X_2, and X_3 is given by
$f_{X_1X_2X_3}(x_1, x_2, x_3) = (1-p_1)(1-p_2)(1-p_3)p_1^{x_1}p_2^{x_2}p_3^{x_3}$ for $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ and $0 < p_i < 1$.
i) Find $f_{X_1X_2}(x_1, x_2)$.

   a) $(1-p_1)(1-p_2)$
   b) $(1-p_1)(1-p_2)(1-p_3)p_1^{x_1}p_2^{x_2}$
   c) $(1-p_1)(1-p_2)p_1^{x_1}p_2^{x_2}$
   d) $p_1^{x_1}p_2^{x_2}$

   ii) Find $f_{X_2}(x_2)$.

   a) $(1-p_1)$
   b) $(1-p_1)(1-p_2)p_2^{x_2}$
   c) $(1-p_1)p_1^{x_1}$
   d) $(1-p_2)p_2^{x_2}$

   iii) Find $P(X_1 = 2, X_3 = 5)$.

   a) $p_1^2p_3^5$
   b) $(1-p_1)(1-p_3)p_1^2p_3^5$
   c) $(1-p_1)(1-p_2)p_1^2p_2^5$
   d) $(1-p_1)(1-p_3)$

9. A coin is tossed twice. Let X denote the number of heads on the first toss and Y denote the total number of heads on the 2 tosses. If the coin is biased and a head has a 40% chance of occurring,
i) Find $P(Y = 2)$. Enter your answer correct to two decimals accuracy.

   ii) Find $P(X = 1)$. Enter your answer correct to two decimals accuracy.

   iii) Find $P(X = 1, Y = 1)$. Enter your answer correct to two decimals accuracy.

10. Let $X_1, X_2, X_3 \sim f_{X_1X_2X_3}$ where $X_i \in \{-1, 1\}$ for each i. If $f_{X_1X_2X_3}(-1,-1,1) = \frac{1}{8}, f_{X_2}(1) = \frac{1}{6}, f_{X_3|X_2=-1}(1) = \frac{1}{5}$, find $f_{X_1|X_2=-1,X_3=1}(-1)$. Enter your answer correct to two decimals accuracy.

11. Suppose that the number of people who visit a yoga academy each day is a Poisson random variable with mean 30. Suppose further that each person who visits is, independently, a girl with probability 0.5 or a boy with probability 0.5. Find the joint probability that exactly 10 boys and 15 girls visit the yoga academy on any given day.

a) $\dfrac{e^{-30}30^{25}}{15!10!}$

b) $\dfrac{e^{-15}30^{25}}{15!10!}$

c) $\dfrac{e^{-8}15^{25}}{15!10!}$

d) $\dfrac{e^{-30}15^{25}}{15!10!}$

Chapter 2

2 Independence and Functions of discrete random variables

In this chapter, we will look at the independence of random variables. First, we will start with the independence of two random variables and then we will move to the independence of multiple random variables. We know that the two events A and B are independent if

$$P(A \cap B) = P(A)P(B)$$

Now, we can extend this definition to random variables. We know that the events can be defined using the random variables.

Definition: Let X and Y be two random variables defined in a probability space with ranges T_X and T_Y, respectively. X and Y are said to be independent if any event defined using X alone is independent of any event defined using Y alone. Equivalently, if the joint PMF of X and Y is f_{XY}, X and Y are independent if

$$f_{XY}(t_1, t_2) = f_X(t_1)f_Y(t_2)$$

for $t_1 \in T_X$ and $t_2 \in T_Y$

- General: $f_{XY}(t_1, t_2) = f_X(t_1)f_{Y|X=t_1}(t_2)$
- Independent: $f_{Y|X=t_1}(t_2) = f_Y(t_2)$
- If X and Y are independent, joint PMF equals the product of marginals and conditional PMF equals the marginal PMF.

2.1 Examples

1. To check for independence given the joint PMF.
   The joint PMF of two random variables X and Y with range t_1 and t_2, respectively is given below:

t_2 \ t_1	0	1	f_Y
0	1/4	1/4	1/2
1	1/4	1/4	1/2
f_X	1/2	1/2	1

We can see that $f_{XY}(t_1, t_2) = f_X(t_1) f_Y(t_2)$, for all t_1, t_2.

2. Consider another example. Let the joint PMF of X and Y be given as

$t_2 \backslash t_1$	0	1	f_Y
0	1/2	1/4	3/4
1	1/8	1/8	1/4
f_X	5/8	3/8	1

We can observe here that $f_{XY}(0,0) = \frac{1}{2} \neq f_X(0) f_Y(0)$. So, X and Y are not independent.

3. Let the joint PMF of X and Y be given by

$t_2 \backslash t_1$	0	1	f_Y
0	x	$1/2 - x$	1/2
1	$1/2 - x$	x	1/2
f_X	1/2	1/2	1

For $x = \frac{1}{4}$, X and Y are independent.

For $x \neq \frac{1}{4}$, X and Y are not independent.

4. A 2-digit number from 00 to 99 is selected at random. Partial information is available about the number as two random variables. Let X be the digit in the units place. Let Y be the remainder obtained when the number is divided by 4. Are X and Y

independent?

Solution:

$X \sim \text{Uniform}\{0, 1, 2, \ldots, 9\}$
$Y \sim \text{Uniform}\{0, 1, 2, 3\}$

$f_{XY}(1, 0) = 0 \neq f_X(1) f_Y(0)$

Therefore, X and Y are not independent.

5. Let X =number of runs in the over. Let Y = number of wickets in the over. Are X and Y independent? No. The question like this will appear very often when you model a real life scenario.

Try it yourself: Check for the independence of random variables X and Y.

$Y \backslash X$	0	1	2
0	1/9	1/9	1/9
1	1/9	1/9	1/9
2	1/9	1/9	1/9

1.

$Y \backslash X$	0	1	2
0	1/6	1/12	1/12
1	1/4	1/8	1/8
2	1/12	1/24	1/24

2.

Y \ X	0	1	2
0	0	1/8	1/8
1	1/8	1/8	1/8
2	1/8	1/8	1/8

3.

Y \ X	0	1	2
0	1/6	1/12	1/8
1	1/4	1/8	1/8
2	1/24	1/24	1/24

4.

Independent vs dependent random variables

- To show X and Y are independent, verify that the joint PMF is the product of marginals, for all values of X and Y.

  $$f_{XY}(t_1, t_2) = f_X(t_1)f_Y(t_2)$$

  for **all** $t_1 \in T_X, t_2 \in T_Y$.

- To show X and Y are dependent, it is enough to verify the inequality for some particular values of X and Y.

  $$f_{XY}(t_1, t_2) \neq f_X(t_1)f_Y(t_2)$$

  for **some** $t_1 \in T_X, t_2 \in T_Y$.

  – Special case: $f_{XY}(t_1, t_2) = 0$ when $f_X(t_1) \neq 0, f_Y(t_2) \neq 0$

2.2 Independence of multiple random variables

Definition: Let $X_1, X_2, \dots, X_n$ be random variables defined in a probability space with range of X_i denoted T_{X_i}. $X_1, X_2, \dots, X_n$ are said to be independent if events defined using different X_i are mutually independent. Equivalently, $X_1, X_2, \dots, X_n$ are independent iff

$$f_{X_1 \dots X_n}(t_1, t_2, \dots, t_n) = f_{X_1}(t_1) f_{X_2}(t_2) \dots f_{X_n}(t_n)$$

for all $t_i \in T_{X_i}$

Examples:

1. **Toss a fair coin thrice**

   Let $X_i = 1$ if i-th toss is heads and $X_i = 0$ if i-th toss is tails, $i = 1, 2, 3$.

   The joint PMF of X_1, X_2 and X_3 is given by $f_{X_1 X_2 X_3}(t_1, t_2, t_3) = \dfrac{1}{8}$ for $t_i \in \{0, 1\}$.

   Also, $X_i \sim \text{Uniform}\{0, 1\}$. Therefore, $f_{X_1 X_2 X_3}(t_1, t_2, t_3) = f_{X_1}(t_1) f_{X_2}(t_2) f_{X_3}(t_3)$

   Hence, X_1, X_2, X_3 are independent.

2. **Random 3-digit number 000 to 999**

   X = first digit from the left, Y = number modulo 2, Z = first digit from right. Are X, Y and Z independent?

   Solution:

   $X \sim \text{Uniform}\{0, 1, 2, \dots, 9\}$
   $Y \sim \text{Uniform}\{0, 1\}$
   $Z \sim \text{Uniform}\{0, 1, 2, \dots, 9\}$

   Consider the pair X and Z:

   - $f_{XZ}(t_1, t_3) = \dfrac{1}{100}$, for all $t_1 \in X, t_3 \in Z$
     $f_{XZ}(t_1, t_3) = f_X(t_1) f_Z(t_3) = \dfrac{1}{100}$

     Therefore, X and Z are independent.

- $f_{XY}(t_1, t_2) = \frac{1}{100}$, for all $t_1 \in X, t_2 \in Y$
  $f_{XY}(t_1, t_2) = f_X(t_1)f_Y(t_2) = \frac{1}{100}$

  Therefore, X and Y are independent.
- $f_{YZ}(1, 2) = 0 \neq f_Y(1)f_Z(2)$

  Therefore, Y and Z are not independent.

Hence, the random variables, X, Y and Z are not independent.

3. **Even parity**

Let the joint PMF be

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	1/4
0	1	1	1/4
1	0	1	1/4
1	1	0	1/4

The above joint PMF is called the even parity because in every case, the number of 1s is even.

We can observe that $X_i \sim \text{Uniform}\{0, 1\}$.

Independence:

Consider the pair X_1 and X_2, their joint PMF will be given by

t_2 \ t_1	0	1	f_{X2}
0	1/4	1/4	1/2
1	1/4	1/4	1/2
f_{X1}	1/2	1/2	1

$f_{X_1X_2}(t_1, t_2) = f_{X_1}(t_1)f_{X_2}(t_2)$ for all $t_1 \in X_1, t_2 \in X_2$.

Consider the pair X_1 and X_3, their joint PMF will be given by

$t_3 \backslash t_1$	0	1	f_{X3}
0	1/4	1/4	1/2
1	1/4	1/4	1/2
f_{X1}	1/2	1/2	1

$f_{X_1X_3}(t_1, t_3) = f_{X_1}(t_1)f_{X_3}(t_3)$ for all $t_1 \in X_1, t_3 \in X_3$.

Consider the pair X_2 and X_3, their joint PMF will be given by

$t_3 \backslash t_2$	0	1	f_{X3}
0	1/4	1/4	1/2
1	1/4	1/4	1/2
f_{X2}	1/2	1/2	1

$f_{X_2X_3}(t_2, t_3) = f_{X_2}(t_2)f_{X_3}(t_3)$ for all $t_2 \in X_2, t_3 \in X_3$.

We saw that all the pairs of random variables X_1, X_2 and X_3 are independent. Now, we need to check if all the three random variables together are independent.

Consider the case, where $X_1 = 0, X_2 = 0, X_3 = 1$.
$P(X_1 = 0, X_2 = 0, X_3 = 1) = 0 \neq P(X_1 = 0)P(X_2 = 0)P(X_3 = 1)$

2.3 Functions of random variables

2.3.1 One random variable

So far, we have learnt about multiple random variables, their PMF and so on. In this section, we will see how to visualize these functions a little better and work with it. Many a times in Statistics, you will encounter the situations where you will have to model using random variables. Whenever you have a random variables, you will have a distribution for it, and you would want to use that to find out the distribution of the functions.

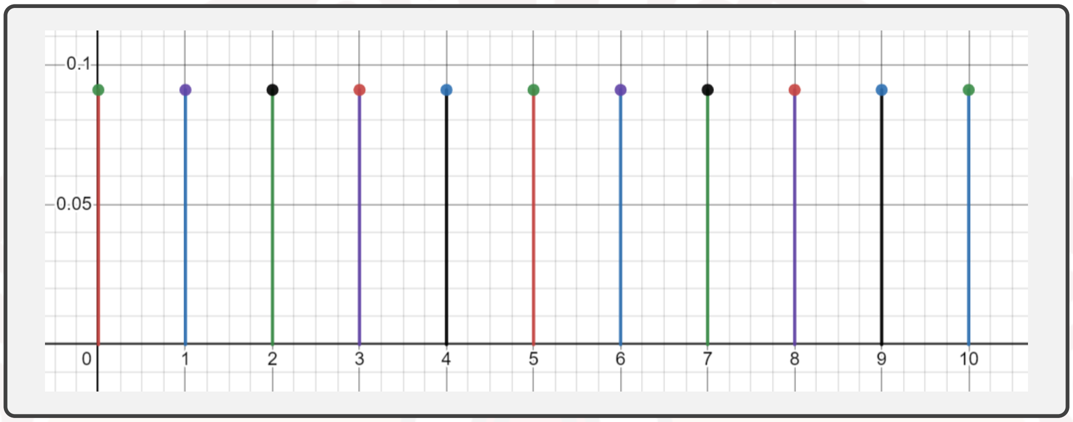
One random variable, one-to-one functions

1. Find the plot of the PMFs.

(a) Let $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$. Find the plot of $f_X(x)$.

Solution:

PMF of X is $P(X = x) = \dfrac{1}{11}, x \in \{0, 1, 2, \ldots, 10\}$.



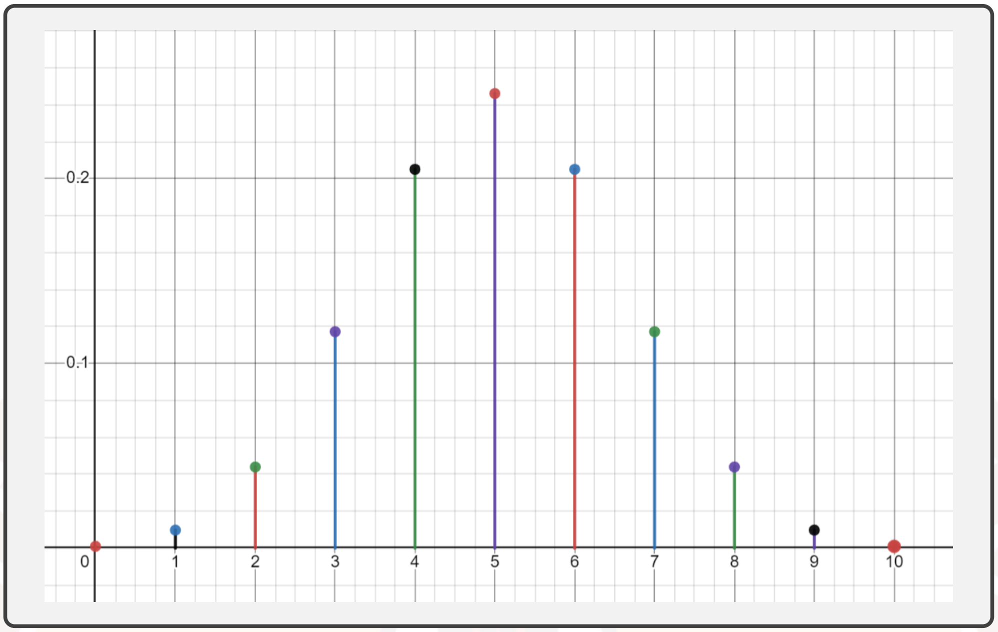
The height of the plot is $1/11$ for each of the x_i's. Such kind of plots are called the stem plots. They are very useful in the visualisation of PMF of a random variable. From this plot, it is quite clear that the random variable is taking the values from 0 to 10 with the probabilities $1/11$ each.

(b) Let $X \sim \text{Binomial}(10, 0.5)$. Find the plot of $f_X(x)$.

Solution:

PMF of Y is given by

$$P(Y = k) = \binom{10}{k}(0.5)^{10}$$



In this case, the probabilities are different for different values of Y. You can observe that, in this case the value $Y = 5$ is the most frequent, then we have $Y = 4$ and $Y = 6$. The values $Y = 0, 10$ is the least likely.

So, given a PMF, you can do the stem plot and see how the distribution behaves. Later we will see how the distribution looks like for the functions of a random variable.

2. Find the PMF of $Y = X - 5$ and draw its stem plot.

   (a) $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$

   (b) $X \sim \text{Binomial}(10, 0.5)$

   Solution:

   Step I: Given any distribution, we can always make a simple table like below, which we refer to as the table method and it is a very powerful method to compute the distributions.

x	$P(X = x)$	$y = x - 5$
0	1/11	-5
1	1/11	-4
2	1/11	-3
3	1/11	-2
4	1/11	-1
5	1/11	0
6	1/11	1
7	1/11	2
8	1/11	3
9	1/11	4
10	1/11	5

Uniform$\{0, 1, \ldots, 10\}$

x	$P(X = x)$	$y = x - 5$
0	1/1024	-5
1	10/1024	-4
2	45/1024	-3
3	120/1024	-2
4	210/1024	-1
5	252/1024	0
6	210/1024	1
7	120/1024	2
8	45/1024	3
9	10/1024	4
10	1/1024	5

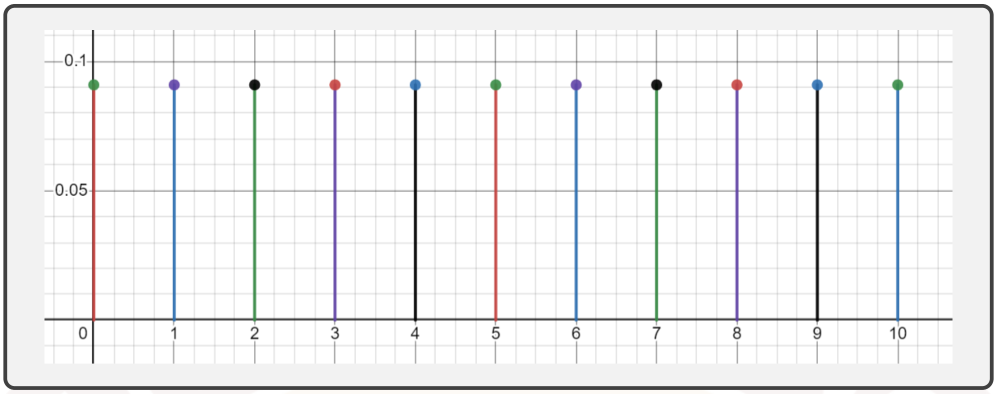
Binomial$(10, 0.5)$

From the table, we can see the list of values that Y is taking and it is in one-to-one correspondence to X. Different values of X go to different values of Y. Since there is no repetition in the values, the distribution of Y is easy to compute here.

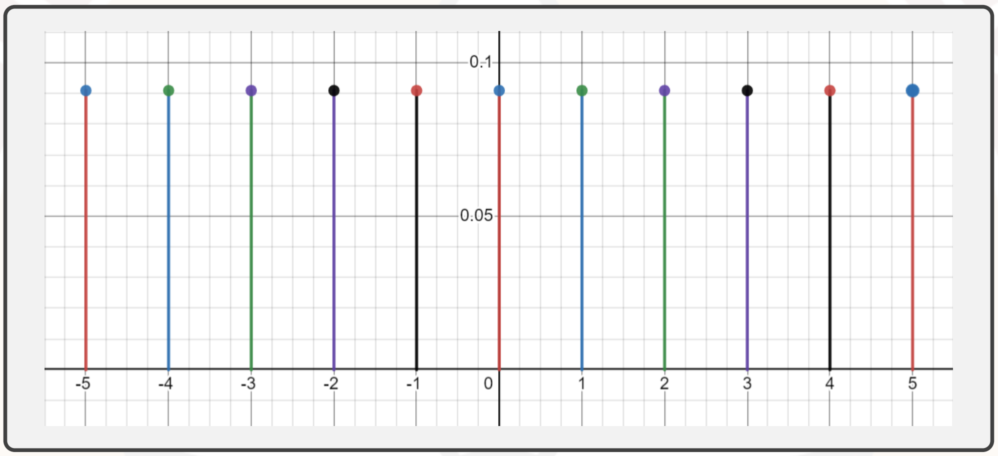
Step II: Range of $Y = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

In the case of Uniform distribution, $P(Y = y) = \dfrac{1}{11}$ for $y \in Y$. While, in case of Binomial distribution, the probabilities vary for different values of X.

Plots of Y

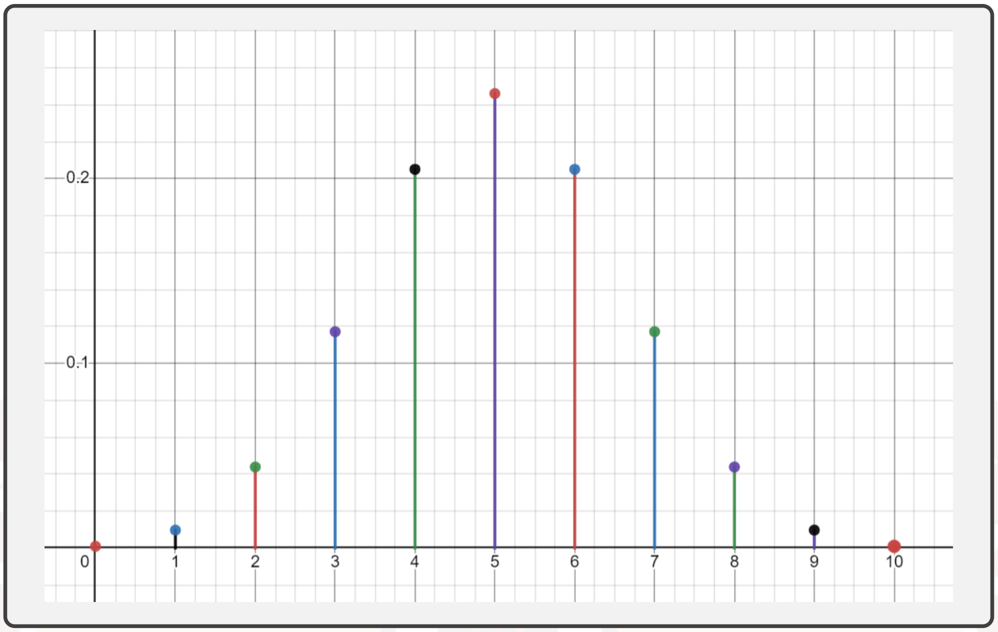


$$X \sim \text{Uniform}\{0, 1, \ldots, 100\}$$

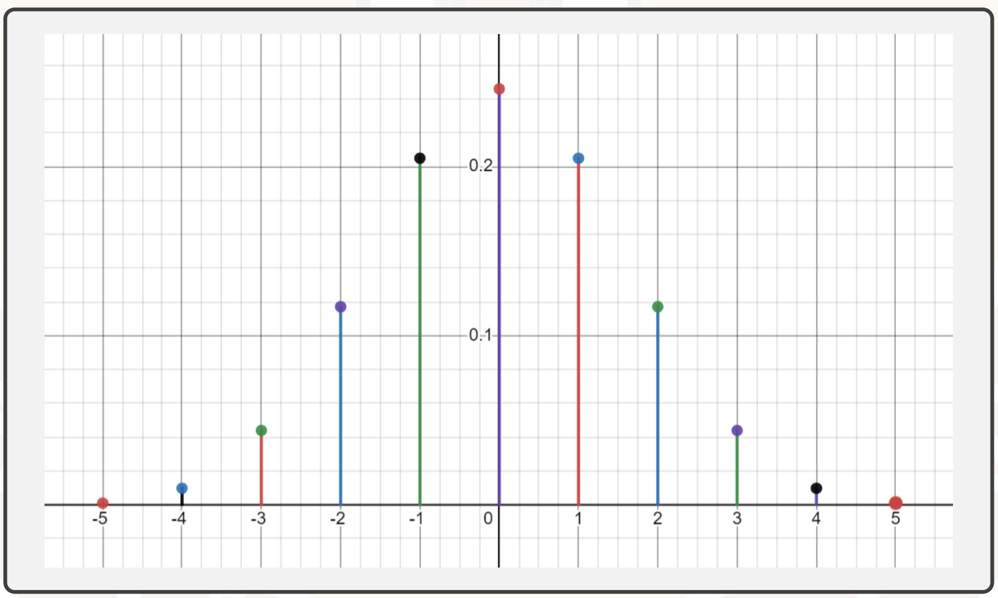


$$Y = X - 5$$

i.



$X \sim \text{Binomial}(10, 0.5)$



$Y = X - 5$

ii.

3. Find the PMF of $Y = 2^X$ and draw its stem plot.

   (a) $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$

(b) $X \sim \text{Binomial}(10, 0.5)$

<u>Solution:</u>

Step I:

x	$P(X=x)$	$y=2^x$
0	1/11	1
1	1/11	2
2	1/11	4
3	1/11	8
4	1/11	16
5	1/11	32
6	1/11	64
7	1/11	128
8	1/11	256
9	1/11	512
10	1/11	1024

$\text{Uniform}\{0, 1, \ldots, 10\}$

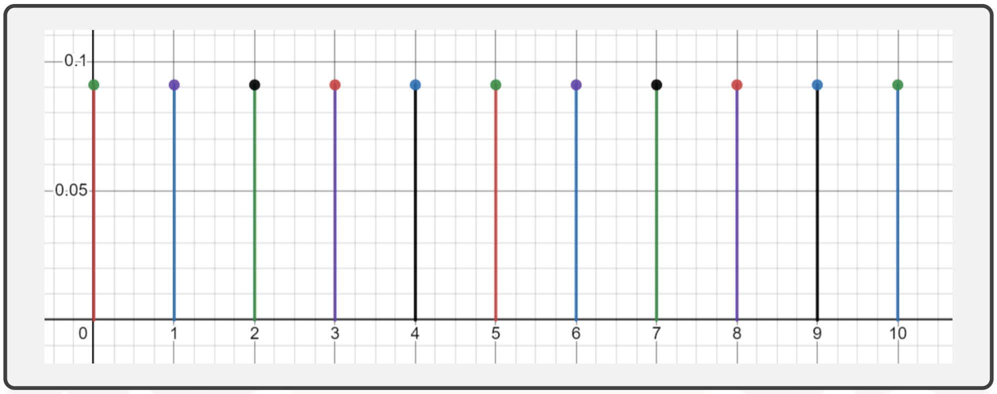
x	$P(X=x)$	$y=2^x$
0	1/1024	1
1	10/1024	2
2	45/1024	4
3	120/1024	8
4	210/1024	16
5	252/1024	32
6	210/1024	64
7	120/1024	128
8	45/1024	256
9	10/1024	512
10	1/1024	1024

$\text{Binomial}(10, 0.5)$

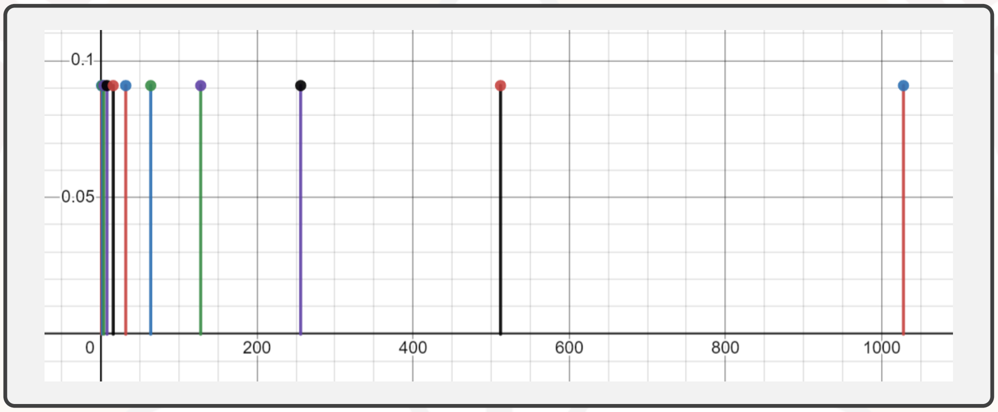
Step II: Range of $Y = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\}$

In the case of Uniform distribution, $P(Y=y) = \dfrac{1}{11}$ for $y \in Y$. While, in case of Binomial distribution, the probabilities vary for different values of X.

<u>Plots of Y</u>

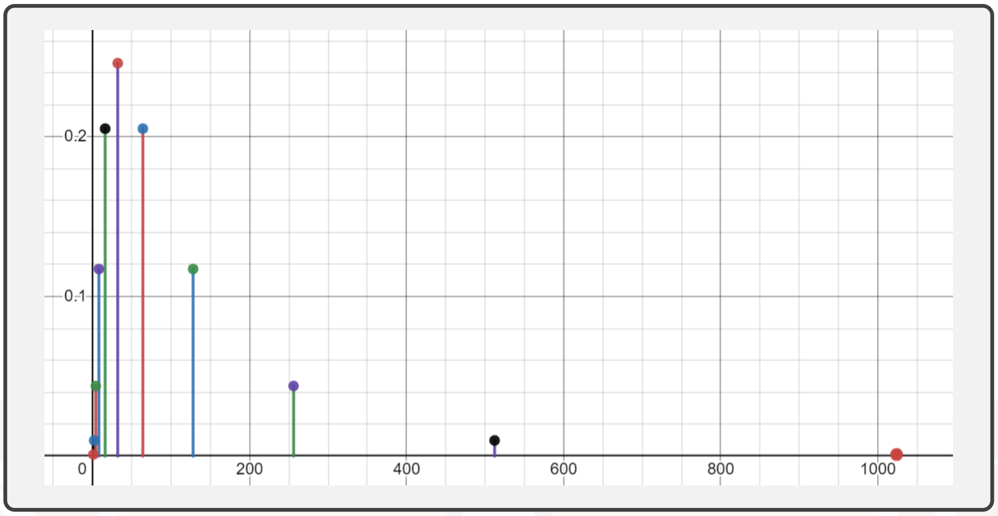


$$X \sim \text{Uniform}\{0, 1, \ldots, 100\}$$

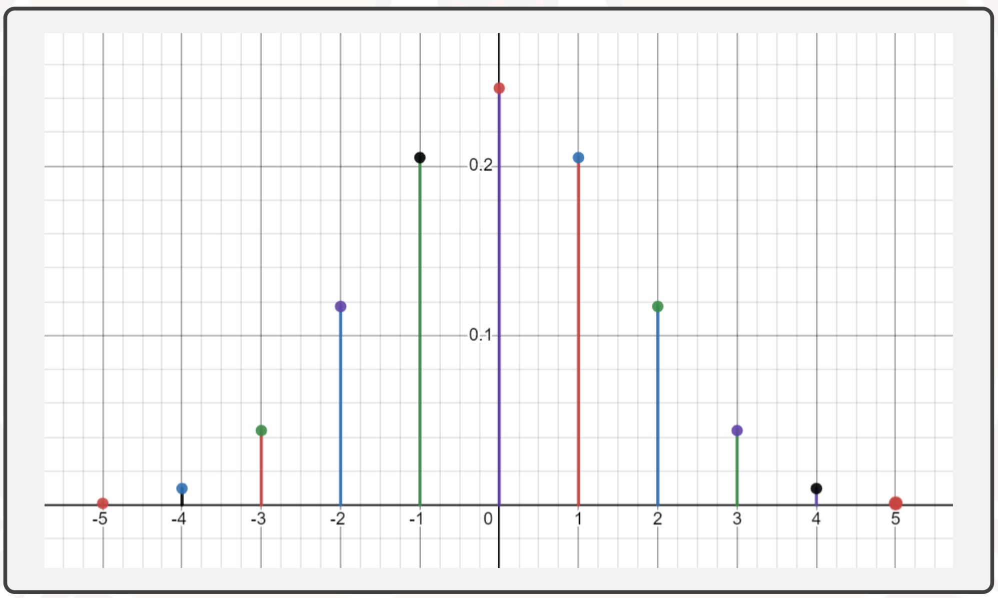


$$Y = 2^X$$

i.



$X \sim \text{Binomial}(10, 0.5)$



$Y = 2^X$

ii.

One random variable, many-to-one functions

1. Find the PMF of $Y = (X - 5)^2$ and draw its stem plot.

   (a) $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$

(b) $X \sim \text{Binomial}(10, 0.5)$

Solution:

Step I: Given any distribution, we can always make a simple table like below, which we refer to as the table method and it is a very powerful method to compute the distributions.

x	$P(X=x)$	$y=(x-5)^2$
0	1/11	25
1	1/11	16
2	1/11	9
3	1/11	4
4	1/11	1
5	1/11	0
6	1/11	1
7	1/11	4
8	1/11	9
9	1/11	16
10	1/11	25

$\text{Uniform}\{0, 1, \ldots, 10\}$

x	$P(X=x)$	$y=(x-5)^2$
0	1/1024	25
1	10/1024	16
2	45/1024	9
3	120/1024	4
4	210/1024	1
5	252/1024	0
6	210/1024	1
7	120/1024	4
8	45/1024	9
9	10/1024	16
10	1/1024	25

$\text{Binomial}(10, 0.5)$

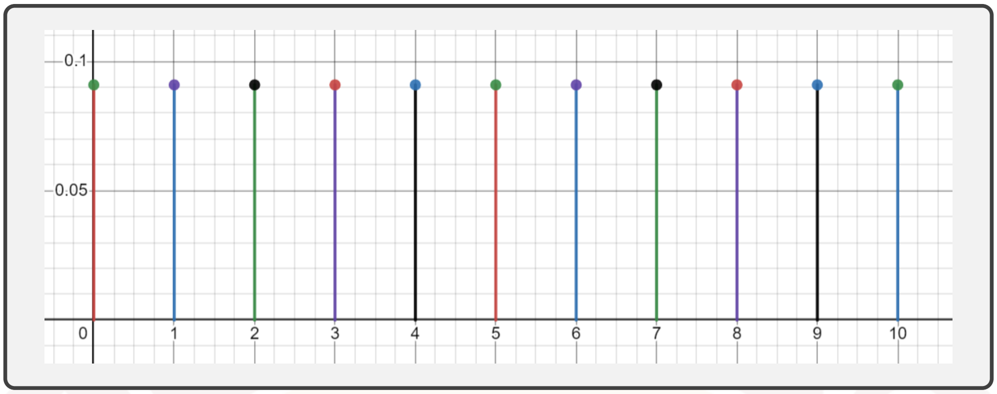
From the table, we can see that the values of Y are repeating. There is no one-to-one correspondence here.

Step II: Range of $Y = \{0, 1, 4, 9, 16, 25\}$

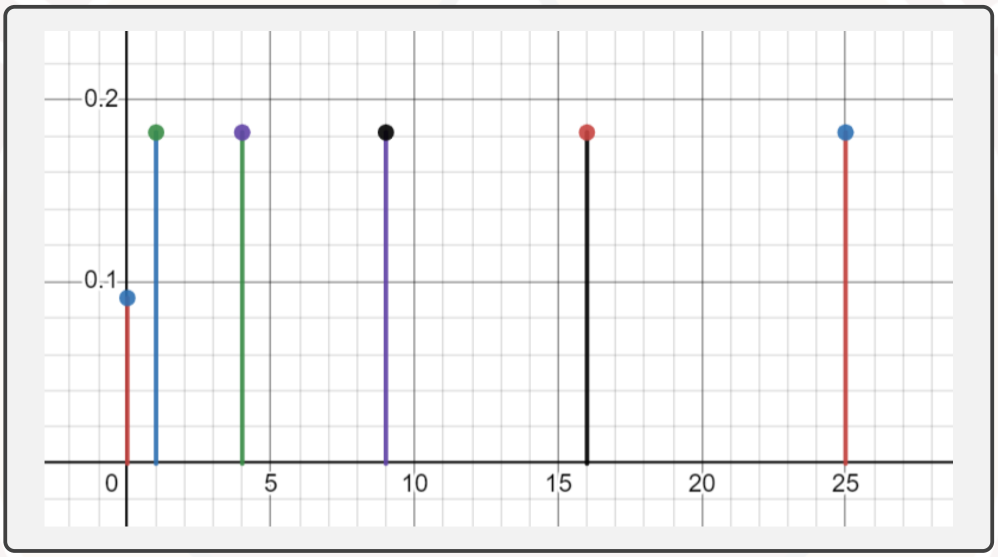
Plots of Y

(a) In case of $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$, the distribution of Y is

y	0	1	4	9	16	25
$f(y)$	1/11	2/11	2/11	2/11	2/11	2/11



$$X \sim \text{Uniform}\{0, 1, \ldots, 100\}$$

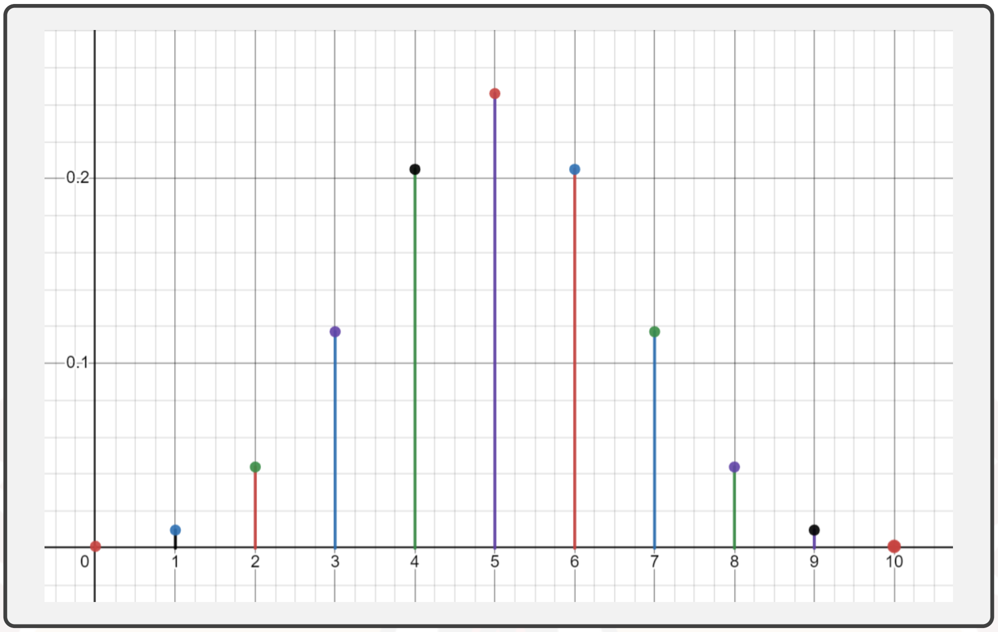


$$Y = (X - 5)^2$$

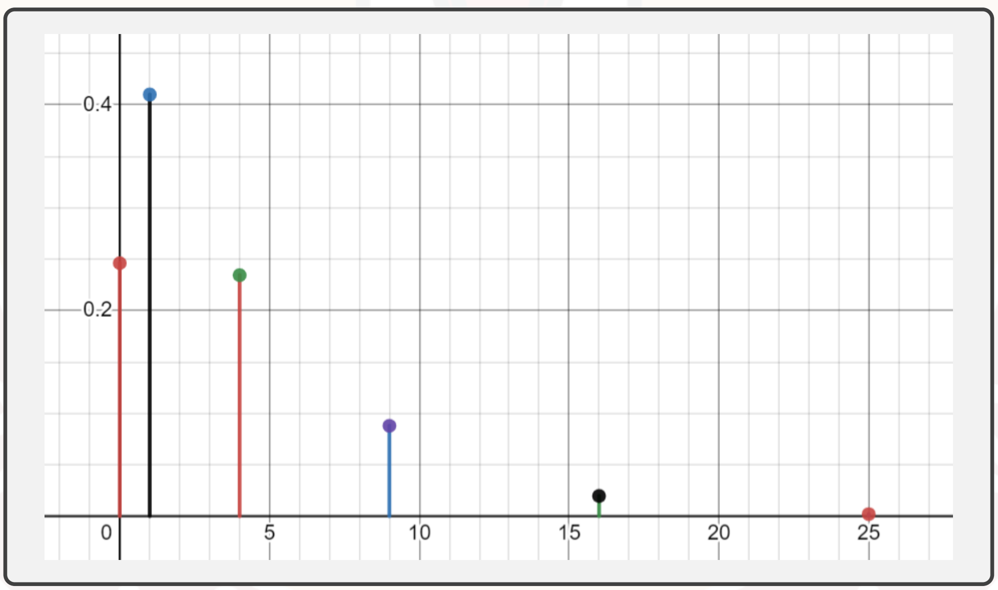
Notice that in case of $X \sim \text{Uniform}\{0, 1, 2, \ldots, 10\}$, the graph was flat. The probabilities were $1/11$ for each x. Now, in case of $Y = (X - 5)^2$, the PMF have changed. So, in case of many-to-one functions, all this sort of things can happen.

(b) In case of $X \sim \text{Binomial}(10, 0.5)$, the distribution of Y is

y	0	1	4	9	16	25
$f(y)$	252/1024	420/1024	240/1024	90/1024	20/1024	2/1024



$X \sim \text{Binomial}(10, 0.5)$



$Y = (X-5)^2$

Problems

1. Let $X \sim \text{Uniform}\{-5, -4, \ldots, 5\}$. Let

$$f(x) = \begin{cases} x, x > 0 \\ 0, x \leq 0 \end{cases}$$

Find the distribution of $Y = f(X)$.

<u>Solution:</u> Using the table method:

x	$P(X = x)$	$y = f(x)$
-5	$1/11$	0
-4	$1/11$	0
-3	$1/11$	0
-2	$1/11$	0
-1	$1/11$	0
0	$1/11$	0
1	$1/11$	1
2	$1/11$	2
3	$1/11$	3
4	$1/11$	4
5	$1/11$	5

Therefore, the distribution of Y is

y	0	1	2	3	4	5
$P(Y = y)$	$6/11$	$1/11$	$1/11$	$1/11$	$1/11$	$1/11$

2. Let $X \sim \text{Uniform}\{-500, -499, \ldots, 500\}$. Let $f(x) = \max(x, 5)$. Find the distribution of $Y = f(X)$.

<u>Solution:</u> Using the table method:

x	$P(X = x)$	$y = f(x)$
-500	$1/1001$	5
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
4	$1/1001$	5
5	$1/1001$	5
6	$1/1001$	6
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
500	$1/1001$	500

Therefore, the distribution of Y is

y	5	6	7	...	...	500
$P(Y=y)$	506/1001	1/1001	1/1001	...	...	1/1001

2.3.2 Two random variables

In this section, we will look at function of two random variables. Functions like sum, maximum, minimum occur very commonly while dealing with random data.

Small examples and table method

When the random variable take very few values, the table method that we saw for one random variable case, will continue to work in this case too.

1. Sum

   $X, Y \sim$ i.i.d. Uniform$\{0, 1\}, Z = X + Y$.

x	y	$f_{XY}(x, y)$	z
0	0	1/4	0
0	1	1/4	1
1	0	1/4	1
1	1	1/4	2

   Therefore, the distribution of Z is

z	0	1	2
$P(Z=0)$	1/4	1/2	1/4

2. Maximum

   Let the joint PMF of X and Y be

$X \backslash Y$	0	1	2
0	1/2	1/4	1/8
1	1/16	1/32	1/32

   Let $Z = \max(X, Y)$.

x	y	$f_{XY}(x,y)$	z
0	0	1/2	0
0	1	1/4	1
0	2	1/8	2
1	0	1/16	1
1	1	1/32	1
1	2	1/32	2

Therefore, the distribution of Z is

z	0	1	2
$P(Z=0)$	1/2	11/32	5/32

Moderate size examples: Too cumbersome

The table method becomes very cumbersome as the size of the possible values taken by random variables increases. For example, a pair of fair dice is thrown, what is the distribution of sum or max or min? You will see that you already have 36 possible values, and the table method can be very prone to errors.

Let's say we have two random variables, each taking the values from 1 to 100. So, we will have a total of 10^4 possibilities, and the table method is not going to be very efficient in this case.
So, when the size of this two random variables become very large, we will need something different and that is what we are going to see in the next section.

Visualizing functions of two variables

When we have a function of two random variables $g(x,y)$, we can do a 3D plot. For example, suppose $g(x,y) = x + y$ for every (x,y). For (x_1,y_1), we will have $g(x_1,y_1)$, and for (x_2,y_2), we will have $g(x_2,y_2)$. In this case, we will need a third axis to denote the function value which can be done using computers, but such graphs do not help us in solving problems in exam.

In reality, it is very difficult to visualize such functions, but contours can be of great use in problem solving.

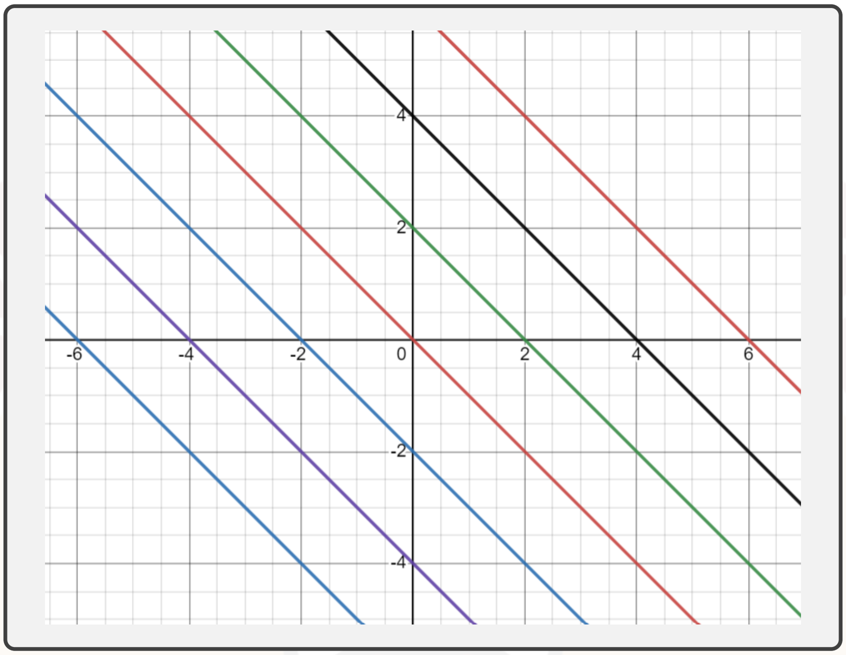
- Contours I: All the values of (x,y) that will result in $g(x,y) = c$.
  Make a plot of those (x,y) for different c.

- Regions II: All the values of (x,y) that will result in $g(x,y) \leq c$.
  Make a plot of those (x,y) for different c.

The contours and region are very useful in visualizing the $g(x,y)$. Let us start with a very simple function, i.e., sum of two random variables.

Sum function, $g(x, y) = x + y$

Contours for $g(x, y)$ will be the set of all values of (x, y) for which $x + y$ equals to some value.



Notice that the contours are set of straight lines here. All the lines have the same slope as -1.

2.3.3 Distribution of functions of random variables

In this section, we will look at how to find the distribution of function of two random variables. First, let's start with the general idea and later we will see how to apply it in several cases.

Suppose you have two random variables, X and Y, and their joint PMF is given by f_{XY}. Let $Z = g(X, Y)$. Find the PMF of Z.

- Step I: Find the range of Z.
- Step II: Add over the contours.
  Suppose z is a possible value taken by Z. So, we will add over the contours $g(x, y) = z$.

$$P(Z = z) = \sum_{(x,y):g(x,y)=z} f_{XY}(x, y)$$

Examples

1. Throw a die twice

   A fair die is thrown twice. What is the probability that the sum of the two numbers seen is 6? What is the PMF of the sum?

   <u>Solution:</u>

   Let X_1 be the first number observed and X_2 be the second number observed.
   Let $S = X_1 + X_2$

   (a) To find $P(X_1 + X_2 = 6)$

   Sum of the two numbers is 6 for $(1,5),(2,4),(3,3),(4,2),(5,1)$.
   Total number of possible outcomes $= 36$
   Therefore, the required probability is $\frac{5}{36}$.

   (b) PMF of S

   - $S \in \{2,3,4,5,6,7,8,9,10,11,12\}$
   - $S = 2$ for $(1,1)$, $\mathrm{P}(S=2) = \frac{1}{36}$

     $S = 3$ for values $(1,2),(2,1)$, $\mathrm{P}(S=3) = \frac{2}{36}$

     $S = 4$ for values $(1,3),(2,2),(3,1)$, $\mathrm{P}(S=4) = \frac{3}{36}$

     $\vdots$

     $S = 12$ for $(6,6)$, $\mathrm{P}(S=12) = \frac{1}{36}$

2. Area of a random rectangle

   The length of a rectange $L \sim \text{Uniform}\{5,7,9,11\}$. Given $L = l$, the breadth $B \sim \text{Uniform}\{l-1, l-2, l-3\}$. Find the joint PMF of area of the rectangle.

   <u>Solution:</u>

   It is given that $L \sim \text{Uniform}\{5,7,9,11\}$. Therefore, $f_L(l) = \frac{1}{4}$

   $(B \mid L) \sim \text{Uniform}\{l-1, l-2, l-3\}$. Therefore, $f_{B|L}(b,l) = \frac{1}{3}$

   We know that for any two random variables X and Y,

$$f_{XY} = f_X f_{Y|X}$$

Therefore, for every (l, b), $f_{LB}(l, b) = f_L(l) f_{B|L}(b, l) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

Hence, the joint PMF of LB is

l	b	LB	$f_{LB}(l, b)$
5	4	20	1/12
5	3	15	1/12
5	2	10	1/12
7	4	28	1/12
7	3	21	1/12
7	2	14	1/12
9	4	36	1/12
9	3	27	1/12
9	2	18	1/12
11	4	44	1/12
11	3	33	1/12
11	2	22	1/12

Therefore, Area $\sim$ Uniform$\{10, 14, 15, 18, 20, 21, 22, 27, 28, 33, 36, 44\}$.

PMF of $g(X_1, \ldots, X_n)$

Definition: Suppose $X_1, \ldots, X_n$ have the joint PMF $f_{X_1,\ldots,X_n}$ with T_{X_i} denoting the range of X_i. Let $g : T_{X_1} \times \ldots \times T_{X_n} \to \mathbb{R}$ be a function with range T_g. The PMF of $X = g(X_1, \ldots, X_n)$ is given by

$$f_X(t) = P(g(X_1, \ldots, X_n) = t) = \sum_{t_1,\ldots,t_n : g(t_1,\ldots,t_n)=t} f_{X_1,\ldots,X_n}(t_1, \ldots, t_n)$$

Examples

1. Given a small table for joint PMF

   Let the joint PMF of three random variables X_1, X_2 and X_3 be given in the following table.

t_1	t_2	t_3	$f_{X_1X_2X_3}(t_1, t_2, t_3)$
0	0	0	1/9
0	0	1	1/9
0	0	2	1/9
0	1	1	1/9
0	1	2	1/9
1	0	0	1/9
1	0	2	1/9
1	1	0	1/9
1	1	1	1/9

Let $X = g(X_1, X_2, X_3) = X_1 + X_2 + X_3$ and $Y = h(X_1, X_2, X_3) = X_2X_3$.

(a) What is the PMF of X?

Solution:

t_1	t_2	t_3	$f_{X_1X_2X_3}$	$g = t_1 + t_2 + t_3$
0	0	0	1/9	0
0	0	1	1/9	1
0	0	2	1/9	2
0	1	1	1/9	2
0	1	2	1/9	3
1	0	0	1/9	1
1	0	2	1/9	3
1	1	0	1/9	2
1	1	1	1/9	3

Therefore, the distribution of $X = g(X_1, X_2, X_3)$ is

X	0	1	2	3
$P(X = x)$	1/9	2/9	3/9	3/9

(b) What is the PMF of Y?

Solution:

t_1	t_2	t_3	$f_{X_1X_2X_3}$	$h = t_2t_3$
0	0	0	1/9	0
0	0	1	1/9	0
0	0	2	1/9	0
0	1	1	1/9	1
0	1	2	1/9	2
1	0	0	1/9	0
1	0	2	1/9	0
1	1	0	1/9	0
1	1	1	1/9	1

Therefore, the distribution of $Y = h(X_1, X_2, X_3)$ is

Y	0	1	2
$P(Y = y)$	6/9	2/9	1/9

(c) What is the joint PMF of X and Y?

Solution:

t_1	t_2	t_3	$f_{X_1X_2X_3}$	$g = t_1 + t_2 + t_3$	$h = t_2t_3$
0	0	0	1/9	0	0
0	0	1	1/9	1	0
0	0	2	1/9	2	0
0	1	1	1/9	2	1
0	1	2	1/9	3	2
1	0	0	1/9	1	0
1	0	2	1/9	3	0
1	1	0	1/9	2	0
1	1	1	1/9	3	1

Therefore, the joint PMF of X and Y is

Y \ X	0	1	2	3
0	1/9	2/9	2/9	1/9
1	0	0	1/9	1/9
2	0	0	0	1/9

2. Sum of two uniforms

Let $X \sim \text{Uniform}\{0,1,2,3\}$ and $Y \sim \text{Uniform}\{0,1,2,3\}$ be independent. What is the PMF of $Z = X + Y$?

Solution:

- Find the range of Z.

  Since $X \in \{0,1,2,3\}$ and $Y \in \{0,1,2,3\}$, $Z \in \{0,1,2,3,4,5,6\}$.
- Calculate the probabilities

$$
\begin{aligned}
&P(Z=0) = P(X=0, Y=0) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \\
&P(Z=1) = P(X=0, Y=1) + P(X=1, Y=0) = \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{2}{16} \\
&\vdots \\
&\vdots \\
&P(Z=5) = P(X=2, Y=3) + P(X=3, Y=2) = \left(\frac{1}{4} \times \frac{1}{4}\right) + \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{2}{16} \\
&P(Z=6) = P(X=3, Y=3) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}
\end{aligned}
$$

Therefore, the distribution of Z is

z	0	1	2	3	4	5	6
$f_Z(z)$	1/16	2/16	3/16	4/16	3/16	2/16	1/16

Result: Let $X_1, \ldots, X_n$ be the results of n i.i.d. Bernoulli(p) trials. The sum of the n random variables $X_1 + \ldots + X_n$ is Binomial(n, p).

2.3.4 Sum of two random variables taking integer values

Suppose X and Y are two random variables taking integer values and let their joint PMF be f_{XY}. Let $Z = X + Y$. Find the PMF of Z.

Solution:

Let z be an integer.

$$\begin{aligned} P(Z=z) =& P(X+Y=z) \\ =& \sum_{x=-\infty}^{\infty} P(X=x, Y=z-x) \\ =& \sum_{x=-\infty}^{\infty} f_{XY}(x, z-x) \\ =& \sum_{y=-\infty}^{\infty} f_{XY}(z-y, y) \end{aligned}$$

Convolution: If X and Y are independent, $f_{X+Y}(z) = \sum_{x=-\infty}^{\infty} f_X(x) f_Y(z-x)$

Example: Sum of two independent Poissons

Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$ be independent.

1. Find the PMF of $Z = X + Y$.

Solution:

$X \in \{0, 1, 2, \ldots\}$
$Y \in \{0, 1, 2, \ldots\}$

$Z = X + Y$ $Z \in \{0, 1, 2, \ldots\}$

$$\begin{aligned} f_Z(z) &= \sum_{x=0}^{\infty} f_X(x) f_Y(z-x) \\ &= \sum_{x=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!} \end{aligned}$$

$$f_Z(z) = \begin{cases} 0, & \text{if} x < 0 \\ 0, & \text{if} x > z \end{cases}$$

Therefore,

$$\begin{aligned} f_Z(z) = &\sum_{x=0}^{z} \frac{e^{-\lambda_1}\lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2}\lambda_2^{z-x}}{(z-x)!} \\ &\frac{e^{-\lambda_1}e^{-\lambda_2}}{z!} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{z-x} \end{aligned}$$

Now, we know that $\sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{z-x} = (\lambda_1 + \lambda_2)^z$

Therefore,

$$f_Z(z) = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!}$$
$$Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

2. Find the conditional distributin of $X \mid Z$.

Solution:

$$\begin{aligned}
P(X=k \mid Z=n) =& \frac{P(X=k, Z=n)}{P(Z=n)} \\
=& \frac{P(X=k)P(Z=n \mid X=k)}{P(Z=n)} \\
=& \frac{P(X=k)P(Y=n-k)}{P(Z=n)} \\
=& \frac{\left(\frac{e^{-\lambda_1}\lambda_1^k}{k!}\right)\left(\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\right)}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}} \\
=& \frac{n!}{k!(n-k)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}$$

Therefore, $X \mid Z \sim \text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

Similarly, $Y \mid Z \sim \text{Binomial}\left(n, \frac{\lambda_2}{\lambda_1+\lambda_2}\right)$

Try it Yourself

1. Sum of n **independent** Bernoulli(p) trials is Binomial(n, p).
2. Sum of 2 **independent** Uniform random variables is not Uniform.
3. Sum of **independent** Binomial(n, p) and Binomial(m, p) is Binomial($n + m, p$).
4. Sum of r **i.i.d.** Geometric(p) is Negative-Binomial(r, p).
5. Sum of **independent** Negative-Binomial(r, p) and Negative-Binomial(s, p) is Negative-Binomial($r + s, p$)
6. If X and Y are independent, then $g(X)$ and $h(Y)$ are also independent.

2.3.5 Minimum and Maximum of two random variables

We saw the general functions of random variables and their sum and how they results in many interesting relationship. Another function which occur quite often is maximum and minimum. For example, Suppose you throw a die twice and you are interested in finding the maximum or minimum of the two numbers seen. There are many situations where you want to track how low or how high some value can go and this will show quite often in practice.

Given the joint PMF, the distribution of minimum or maximum can be written quite easily. Let's look at how to do it.

Suppose X and Y have the joint PMF as f_{XY}. Let $Z = \min(X, Y)$.
Find the PMF of Z.

Solution:

$$\begin{aligned}
f_Z(z) = P(Z = z) &= P(\min\{X, Y\} = z) \\
&= P(X = z, Y = z) + P(X = z, Y > z) + P(X > z, Y = z) \\
&= f_{XY}(z, z) + \sum_{t_2 > z} f_{XY}(z, t_2) + \sum_{t_1 > z} f_{XY}(t_1, z)
\end{aligned}$$

Suppose X and Y have the joint PMF as f_{XY}. Let $Z = \max(X, Y)$.
Find the PMF of Z.

Solution:

$$\begin{aligned}
F_Z(z) = P(Z \leq z) &= P(\min\{X, Y\} \leq z) \\
&= 1 - P(\min\{X, Y\} > z) \\
&= 1 - [P(X > z, Y > z)]
\end{aligned}$$

Example: Throw a die twice

Two fair die is tossed. Define Z as the minimum of the two numbers seen. Find the PMF of Z.

$X \backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

$P(Z=1) = P(\min(X,Y)=1) = \frac{11}{36}$
$P(Z=2) = P(\min(X,Y)=2) = \frac{9}{36}$
$P(Z=3) = P(\min(X,Y)=2) = \frac{7}{36}$
$P(Z=4) = P(\min(X,Y)=2) = \frac{5}{36}$
$P(Z=5) = P(\min(X,Y)=2) = \frac{3}{36}$
$P(Z=6) = P(\min(X,Y)=2) = \frac{1}{36}$

Independent case: Cumulative distributin function (CDF) of maximum and minimum

<u>Definition:</u> Cumulative distribution function of a random variable X is a function $F_X : \mathbb{R} \to [0,1]$ defined as

$$F_X(x) = P(X \leq x)$$

CDF of maximum Suppose X and Y are independent random variables and $Z = \max(X,Y)$. Observe that for any $z \in Z$, the events $[(X \leq z)$ and $(Y \leq z)]$ and $\max(X,Y)$ are same.

Now,

$$\begin{aligned} F_Z(z) &= P(\max(X,Y) \leq z) \\ &= P((X \leq z) \text{ and } (Y \leq z)) \\ &= P(X \leq z)P(Y \leq z) \qquad \text{Since } X \text{ and } Y \text{ are independent} \\ &= F_X(z)F_Y(z) \end{aligned}$$

Therefore, CDF of maximum of two random variables is the product of CDFs of X and Y.

What about the CDF of minimum of two random variables? Now here, instead of less than equal to we can go with greater than equal to. This is called the complementary CDF. So, in this case the events $[(X \geq z)$ and $(Y \geq z)]$ and $\min(X, Y)$ are same, for some $z \in \min(X, Y)$.

Problem: Min and max of i.i.d. sequences

Let $X_1, \ldots, X_n \sim$ i.i.d. X. Find the distribution of the following:

1. $\min(X_1, \ldots, X_n)$

   Solution:

$$\begin{aligned}
P(\min(X_1, \ldots, X_n) \leq z) =& 1 - P(\min(X_1, \ldots, X_n) \geq z) \\
=& 1 - P(X_1 \geq z, X_2 \geq z, \ldots, X_n \geq z) \\
=& 1 - [P(X_1 \geq z) \ldots P(X_n \geq z)] \qquad X_i's \text{ are independent} \\
=& 1 - [P(X \geq z) \ldots P(X \geq z)] \qquad X_i's \text{ are identically distributed} \\
=& 1 - (P(X \geq z))^n
\end{aligned}$$

2. $\max(X_1, \ldots, X_n)$

   Solution:

$$\begin{aligned}
P(\max(X_1, \ldots, X_n) \leq z) =& P(\max(X_1, \ldots, X_n) \leq z) \\
=& P(X_1 \leq z, X_2 \leq z, \ldots, X_n \leq z) \\
=& P(X_1 \leq z) \ldots P(X_n \leq z)] \qquad X_i's \text{ are independent} \\
=& P(X \leq z) \ldots P(X \leq z)] \qquad X_i's \text{ are identically distributed} \\
=& (P(X \leq z))^n \\
=& [F_Z(z)]^n
\end{aligned}$$

Problem: Min of two independent Geometrics

Let $X \sim$ Geometric(p) and $Y \sim$ Geometric(p) be independent. Find the distribution of $\min(X, Y)$.

Solution:

$X, Y \sim$ i.i.d Geometric(p)

$$P(\min(X, Y) \geq k) = P(X \geq k)P(Y \geq k)$$

For any random variable X,

$$\begin{aligned}
P(X \geq k) =& \sum_{x=k}^{\infty} P(X = k) \\
=& P(X = k) + P(X = k+1) + P(X = k+2) + \ldots \\
=& p(1-p)^{k-1} + p(1-p)^{k} + p(1-p)^{k+1} + \ldots \\
=& p(1-p)^{k-1}[1 + (1-p) + (1-p)^2 + \ldots] \\
=& p(1-p)^{k-1} \times \frac{1}{p} \\
=& (1-p)^{k-1}
\end{aligned}$$

Therefore, $P(\min(X, Y) \geq k) = (1-p)^{k-1}(1-p)^{k-1} = [(1-p)^2]^{k-1}$

$P(\min(X, Y) \geq k+1) = (1-p)^{k}(1-p)^{k} = 1 - [(1-p)^2]^{k}$

Now, $P(\min(X, Y) = k) = P(\min(X, Y) \geq k) - P(\min(X, Y) \geq k+1)$

Let $q = (1-p)^2$

$P(\min(X, Y) = k) = q^{k-1} - q^k = q^{k-1}(1-q)$

Therefore, $\min(X, Y) \sim \text{Geometric}(1-q)$.

Result: In general, if $X_1 \sim \text{Geometric}(p_1), X_2 \sim \text{Geometric}(p_2)$ and X_1 and X_2 are independent, then

$$\min(X_1, X_2) \sim \text{Geometric}[1 - (1-p_1)(1-p_2)]$$

Try it Yourself: Maximum of two independent Geometric is not geometric.

2.4 Problems

1. The joint distribution of random variables X_1, X_2 and X_3 each taking values in $\{0, 1\}$ is uniform with joint PMF denoted $f_{X_1X_2X_3}$.
   Define $g(X_1, X_2, X_3) = X_1X_2 + 2X_3$ and $h(X_1, X_2, X_3) = 2X_1 + X_2$. Find the joint distribution of g and h.

2. Let $X \sim \text{Uniform}\{0, 1, 4, 5\}$ and $Y \sim \text{Uniform}\{0, 2, 8, 10\}$ be independent random variables. Find the distribution of $X + Y$.

3. Let $X_1, X_2, \ldots, X_5 \sim$ i.i.d. Binomial$(4, \frac{1}{4})$. Define $Z_1 = \max(X_1, X_2, \ldots, X_5)$ and $Z_2 = \min(X_1, X_2, \ldots, X_5)$.
   (a) Calculate $P(Z_1 \leq 2)$.

(b) Calculate $P(Z_1 \leq 2)$.

4. Suppose that the random variables X, Y and Z are independent and are equally likely to be either 0 or 1.
   (a) Find the probability mass function of $X + Y + Z$.
   (b) Find the probability mass function of $U = XY + YZ + ZX$.

5. Let X and Y be two independent Geometric(p) random variables. Assume that $Z = X - Y$. Find the value of $f_Z(k)$ where k is a natural number.

6. Suppose that the number of people who visit a dance academy each day is a Poisson random variable with mean λ. Suppose further that each person who visits is, independently, female with probability p or male with probability $1 - p$. Find the joint probability that exactly m men and w women visit the dance academy on any particular day.

7. A random experiment consists of rolling a fair die until six appears. Let X denote the number of times the die is rolled. Suppose six does not appear until the sixth throw, find the probability that six will appear after the eighth throw of die.
   A. $\left(\frac{1}{6}\right)^2$
   B. $\left(\frac{5}{6}\right)^2$
   C. $\left(\frac{1}{6}\right)^8$
   D. $\left(\frac{5}{6}\right)^8$

8. Let $X_1, X_2, \cdots, X_{10} \sim i.i.d.$ Geometric($\frac{1}{5}$). Find the probability that $(X_1 > 10, X_2 > 10, \cdots, X_{10} > 10)$.
   A. $(0.8)^{10}$
   B. $(0.8)^{100}$
   C. $(0.2)^{10}$
   D. $(0.8)^{20}$

9. Let the random variables X and Y, which represent the number of people visiting shopping malls in city 1 and city 2 in an one hour interval, respectively, follow the Poisson distribution. The average number of people visiting the shopping malls in city 1 and city 2 is 10 per hour and 20 per hour, respectively. Assume that X and Y are independent.
   (a) Let Z denote the total number of people visiting shopping malls in city 1 and city 2. Find the pmf of Z, $f_Z(z)$.

A. $\dfrac{e^{-\frac{1}{30}}\left(\frac{1}{30}\right)^z}{z!}$

B. $\dfrac{e^{-30}(30)^z}{z!}$

C. $\dfrac{e^{-\frac{1}{20}}\left(\frac{1}{20}\right)^z}{z!}$

D. $\dfrac{e^{-20}(20)^z}{z!}$

(b) Find the conditional distribution of Y given that the total number of people visiting shopping malls in city 1 and city 2 is 30.

A. $(Y \mid Z = 30) \sim \text{Binomial}\left(30, \dfrac{2}{3}\right)$

B. $(Y \mid Z = 30) \sim \text{Binomial}\left(30, \dfrac{1}{3}\right)$

C. $(Y \mid Z = 30) \sim \text{Poisson}\left(\dfrac{2}{3}\right)$

D. $(Y \mid Z = 30) \sim \text{Poisson}(30)$