IIT MADRAS **BS DEGREE**

STATISTICS

STATISTICS FOR DATA SCIENCE - II

Continuous random variables

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About the authors

Nikita Kumari and Mayur Gundal, both course instructors for Statistics in the IITM BS degree program, have played a vital role in sharing their knowledge with students from diverse backgrounds. Their experience in teaching Statistics to a wide range of learners has sharpened their skills in simplifying complex concepts, making them easily understandable.

Nikita holds a Masters degree in Mathematics, while Mayur has an Mtech in Ocean Engineering, both from the esteemed Indian Institute of Technology, Madras. Their passion for continuous learning and teaching shines through as they approach each topic with enthusiasm, aiming to foster a deep understanding of Statistics among their students.

With their expertise, love for learning, and dedication to teaching, Nikita Kumari and Mayur Gundal contribute a wealth of knowledge and enthusiasm to this Statistics textbook.

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Chapter 4

1 Continuous random variable

1.1 Introduction

So far, we have looked at discrete random variables, where X usually take values like $\{1, 2, 3, \ldots\}$. We also saw the notion of PMF. Now we are going to see what is called a continuous random variable. We will see how the numbers can become very unwieldy if we want to stick to the discrete domain. Continuous random variable domain gives very easy ways to deal with such situations.

Consider the following two situations:

- **Situation I:** Suppose we have the weight of different meteorites[1] that enters the earth.
  - We have the data for $45,000+$ meteorites and the range of weights vary from 0.01 grams to 60 tons.
  - If we want to do some statistical study on this vast data, we will think of meteorite weight as the random variable. If we stick with the discrete random variable, it becomes really difficult to gain any insight from it.
- **Situation II:** Suppose we have Binomial(n, p) distribution, where p is fixed and n is growing very large. If we look at Bernoulli trials where population is very large and p is a constant, we will have to deal with big calculations in combinatorics.

In such random-like phenomena, the notion of continuous random variable enters the picture. The number of outcomes is so large that dealing with discrete PMF becomes difficult. The core idea is when we have a lot of data and that data takes lot of different values within the same range, we should use continuous model rather than discrete model.

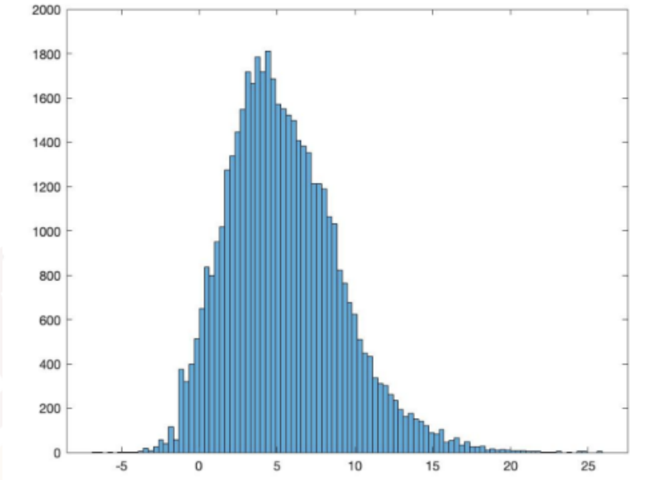
A trick that can be used when we model data is taking logarithm to it. This will help in making the range smaller and make data fall into a certain range. Now we will be focusing on intervals rather than the individual values that the random variable is taking. We will see how to do it in the above two situations.

(i) **Meteorite data**

- Preprocessing: Take ($\log_2$) to the data. Now, the range is reduced from [0.01, 60000000] to $[-6.6, 25.8]$. But still, we have 45000+ data.
- Main idea: Divide $[-6.6, 25.8]$ into 100 intervals of the size 0.3. We will have

$$[-6.6, -6.3], [-6.3, -6], \ldots, [25.5, 25.8]$$

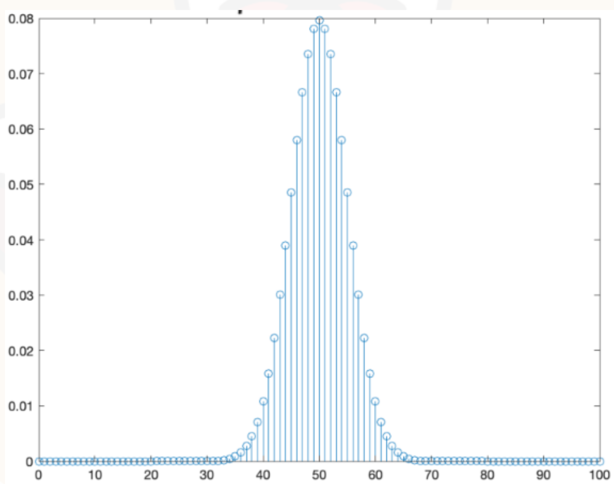
Now count the number of values falling inside each interval.



Histogram for the log of weight of meteorite data

The x-axis represents the bin size and the y-axis gives the count of number of values that lie in each bin.

(ii) **Binomial**(n, p)



PMF of a Binomial$(100, p)$ distribution

We can see here the pmf has a nice shape but dealing with calculation is not easy. So, we can give up the precision of individual values and focus on the shapes and come with some alternative model.

[1]

[1] Meteorite data has been taken from NASA's open data portal.

1.2 Cumulative Distribution function

1.2.1 CDF of a random variable

Definition: The Cumulative Distribution Function (CDF) of a random variable X, denoted $F_X(x)$, is a function from $\mathbb{R}$ to [0, 1], defined as

$$F_X(x) = P(X \leq x)$$

CDF is a very important bridge between the discrete world and the continuous world.

1.2.2 Properties of CDF

i) $F_X(b) - F_X(a) = P(a < X \leq b)$

ii) F_X : non-decreasing function taking non-negative values.

iii) As $x \to -\infty, F_X$ goes to 0.

iv) As $x \to \infty, F_X$ goes to 1.

1.2.3 Examples

i) Bernoulli random variable
Consider a Bernoulli random variable X with X taking the values 0 and 1 with probabilities $(1-p)$ and p, respectively.

For $x < 0, F_X(x) = P(X \leq x) = 0$
For $0 \leq x < 1, F_X(x) = P(X \leq x) = P(X = 0) = 1 - p$
For $x > 1, F_X(x) = P(X \leq x) = P(X = 0) + P(X = 1) = p + (1 - p) = 1$

Therefore, CDF of X is given by

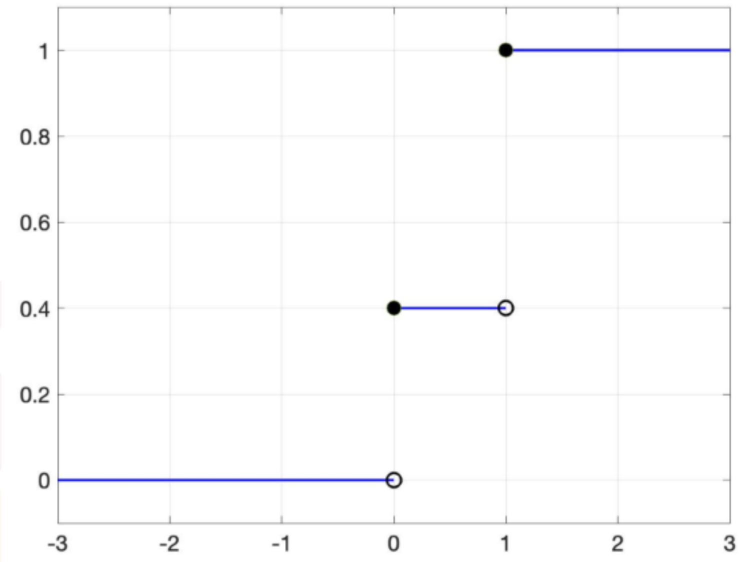
$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Compute: (a)$F_X(-1), (b)\ F_X(x),$ where $0 \leq x < 1$.

Solution:

(a) $F_X(-1) = P(X \leq -1) = 0$

(b) $F_X(x)_{0 \leq x < 1} = P(X \leq x) = P(X = 0) = 1 - p$



ii) Throw a die

Consider a random variable X that represent the outcomes on throwing a fair die. The outcomes are $\{1, 2, 3, 4, 5, 6\}$.

$$X \sim \text{Uniform}\{1, 2, 3, 4, 5, 6\}$$

For $x < 1, F_X(x) = P(X \leq x) = 0$

For $1 \leq x < 2, F_X(x) = P(X \leq x) = P(X = 1) = \dfrac{1}{6}$

For $2 \leq x < 3, F_X(x) = P(X \leq x) = P(X = 1) + P(X = 2) = \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{1}{3}$

For $3 \leq x < 4, F_X(x) = P(X \leq x) = \sum\limits_{k=1,2,3} P(X = k) = \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{1}{2}$

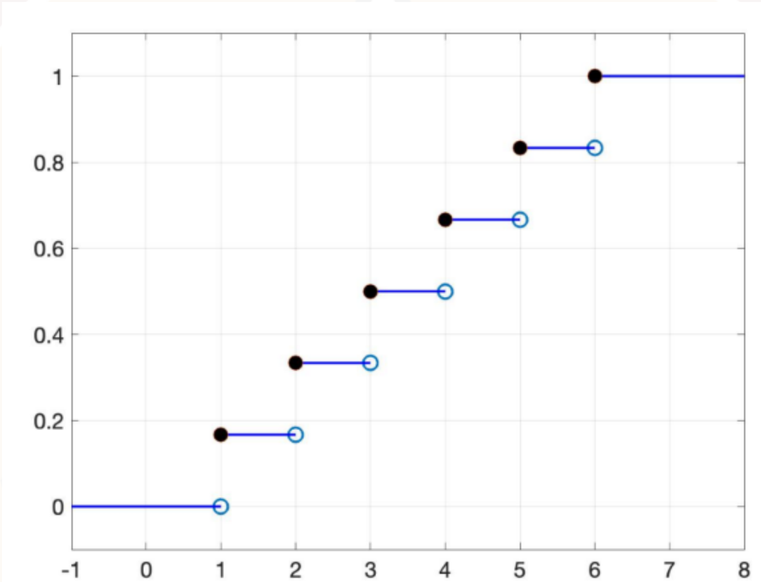
For $4 \leq x < 5, F_X(x) = P(X \leq x) = \sum\limits_{k=1,2,3,4} P(X = k) = \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{2}{3}$

For $5 \leq x < 6, F_X(x) = P(X \leq x) = \sum\limits_{k=1,2,3,4,5} P(X = k) = \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} + \dfrac{1}{6} = \dfrac{5}{6}$

For $x \geq 6, F_X(x) = P(X \leq x) = \sum\limits_{k=1,2,3,4,5,6} P(X = k) = 6.\dfrac{1}{6} = 1$

Therefore, the CDF of X is given by

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 1/6, & 1 \leq x < 2 \\ 2/6, & 2 \leq x < 3 \\ 3/6, & 3 \leq x < 4 \\ 4/6, & 4 \leq x < 5 \\ 5/6, & 5 \leq x < 6 \\ 1, & x \geq 6 \end{cases}$$



Compute: $P(X = 4.5)$

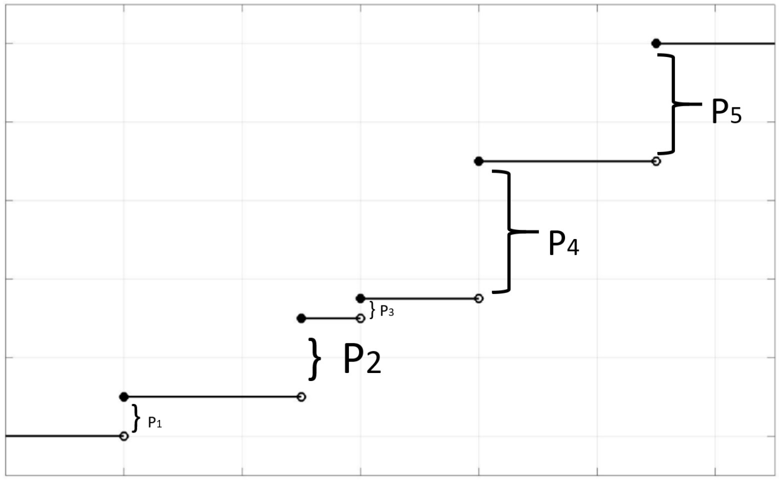
$P(X = 4.5) = 0$ (since there is no jump at $x = 4.5$)

iii) CDF of a discrete random variable

Let random variable X has the following PMF:

x	x_1	x_2	x_3	x_4	x_5
$f_X(x)$	p_1	p_2	p_3	p_4	p_5

Remark: CDF of a discrete random variable have a step like structure.



iv) Computing probability of intervals using CDF

Let $X \sim \text{Uniform}\{1, 2, \ldots, 100\}$
The CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < 1 \\ k/100 & k \leq x < k+1, k = 1, 2, \ldots, 99 \\ 1 & x \geq 100 \end{cases}$$

Compute: (a) $P(3 < X \leq 10)$, (b) $P(3.2 < X \leq 10.6)$, (c) $P(X \leq 17)$, (d) $P(X \leq 17.3)$, (e) $P(X > 87)$, (f) $P(X > 87.4)$

Solution:

(a) $P(3 < X \leq 10) = F_X(10) - F_X(3) = \dfrac{10}{100} - \dfrac{3}{100} = \dfrac{7}{100}$

(b) $P(3.2 < X \leq 10.6) = F_X(10.6) - F_X(3.2) = \dfrac{10}{100} - \dfrac{3}{100} = \dfrac{7}{100}$

(c) $P(X \leq 17) = F_X(17) = \dfrac{17}{100}$

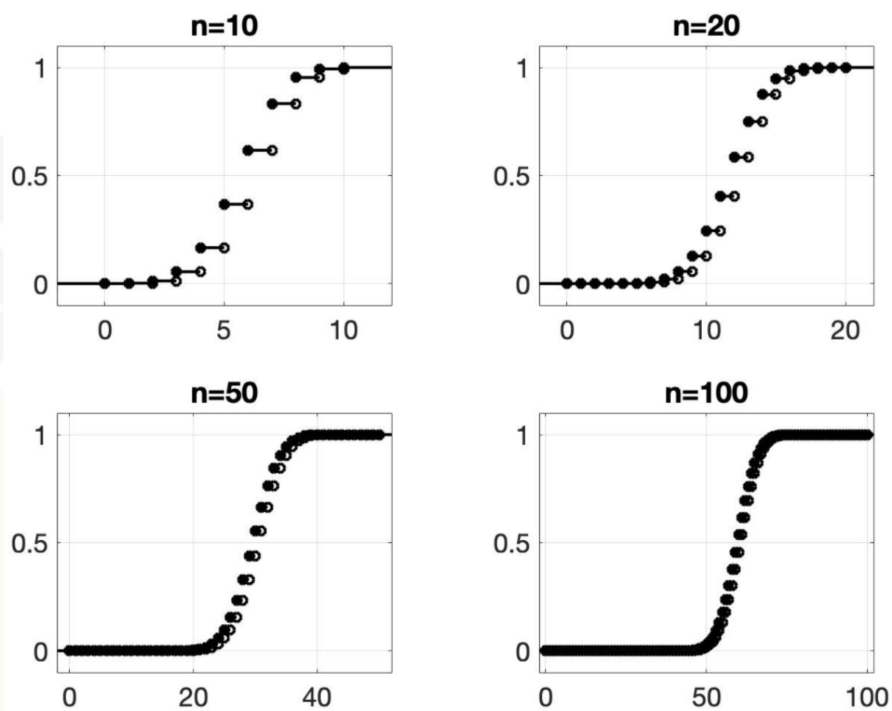
(d) $P(X \leq 17.3) = F_X(17.3) = \dfrac{17}{100}$

(e) $P(X > 87) = 1 - F_X(87) = 1 - \dfrac{87}{100} = \dfrac{13}{100}$

(f) $P(X > 87.4) = 1 - F_X(87.4) = 1 - \dfrac{87}{100} = \dfrac{13}{100}$

1.3 Continuous Random Variable: Approximation of CDF from Discrete to Continuous

Consider the plot of CDF of Binomial random variable $(n, 0.6)$. Keep the scale of the picture same and increase the value of n.



Notice that the CDFs start to look like a continuous line with the increase in the value of n.

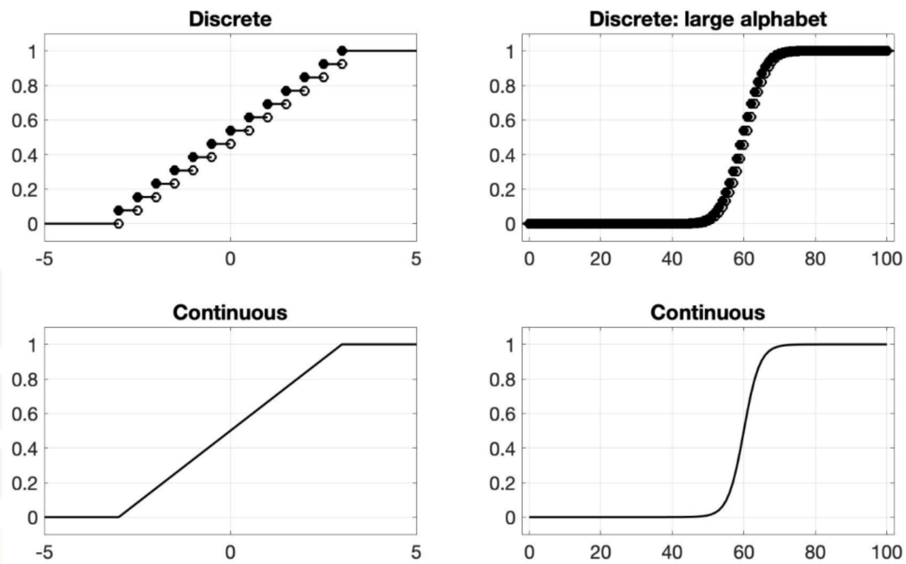
1.4 Cumulative Distribution Functions

Definition: A function $F : \mathbb{R} \to [0, 1]$ is said to be a Cumulative Distribution Function (CDF) if it satisfies the following four properties:

1. F is a non-decreasing function taking values between 0 and 1.
2. As $x \to -\infty, F$ goes to 0.
3. As $x \to \infty, F$ goes to 1.
4. Technical: F is continuous from the right.

The functions defined in the above manner mirror the properties of CDF of a random variable. If we take any arbitrary CDF, it does not have to be a step like structure, it can also be smooth and continuous.

1.4.1 Examples of valid CDF's



- We can observe that all the CDF's are non-decreasing, starts at 0 and ends at 1, so they are valid CDF's.
- Continuous CDFs appear to be close approximations to CDFs of discrete random variables, particularly when alphabet grows.
- When we want to move towards a continuous random variable, we should have a distribution function which is continuous. The graphs on the bottom seem like a continuous here, but it sort of mirrors the discrete picture on the top very closely.
- We can describe the continuous curves in many interesting ways. Also, the calculations with probabilities of intervals become much simpler if we have a continuous model.

1.4.2 Probability of intervals using continuous CDF

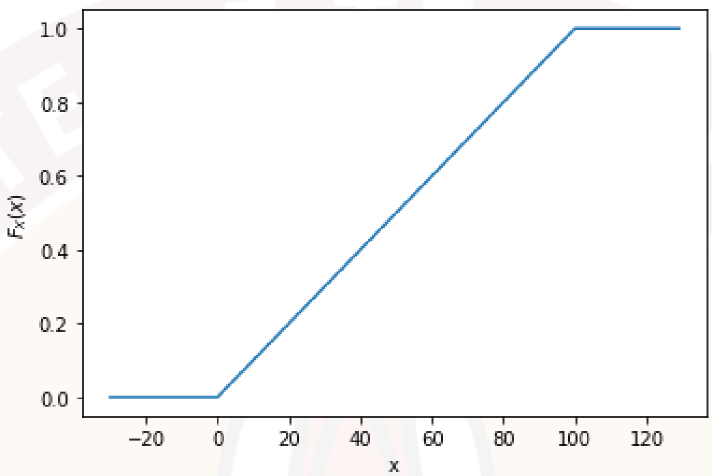
Using continuous CDF, we will now see how to find probabilities for discrete uniform distribution.

1. Let $X \sim$ Uniform $\{1, 2, \ldots, 100\}$.

   Let $F_X(k)$ denotes the CDF of X and $F(x)$ is the approximate CDF of X.

$$F_X(k) = \begin{cases} 0 & x \leq 0 \\ k/100 & k \leq x < k+1, k = 1, 2, \ldots, 99 \\ 1 & x \geq 100 \end{cases}$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x/100 & 0 \leq x \leq 100 \\ 1 & x \geq 100 \end{cases}$$



CDF of X

(a) $P(3 < X \leq 10) = F(10) - F(3) = \dfrac{10}{100} - \dfrac{3}{100} = \dfrac{7}{100}$

(b) $P(3.2 < X \leq 10.6) = F(10.6) - F(3.2) = \dfrac{10.6}{100} - \dfrac{3.2}{100} = \dfrac{7.4}{100}$

(c) $P(X \leq 17) = F(17) = \dfrac{17}{100}$

(d) $P(X \leq 17.3) = F(17.3) = \dfrac{17.3}{100}$

(e) $P(X > 87) = 1 - F(87) = 1 - \dfrac{87}{100} = \dfrac{13}{100}$

(f) $P(X > 87.4) = 1 - F(87.4) = 1 - \dfrac{87.4}{100} = \dfrac{12.6}{100}$

Observations:

- We will get the same value for $P(3 < x \leq 10)$ even if we use the exact CDF $F_X(k)$ or the approximated CDF $F(x)$.
- We will get different values for $P(3.2 < X \leq 10.6)$ if we use $F_X(k)$ and if we use $F(x)$. There is a small difference in the values.

2. Binomial using continuous CDF

Let $X \sim \text{Binomial}(100, 0.6)$.

Let $F_X(k)$ denotes the CDF of X and $F(x)$ is the approximate CDF of X.

$$F_X(k) = \sum_{j=0}^{k} \binom{100}{j} (0.6)^j (0.4)^{n-j}$$

$$F(x) = \frac{1}{1 + \exp\left(\dfrac{-1.65451(x-60)}{\sqrt{24}}\right)}$$

Here 24 is the variance of $\text{Binomial}(100, 0.6)$ and 60 is the mean of $\text{Binomial}(100, 0.6)$.

(a) $P(40 < X \leq 50) = F(50) - F(40) = 0.0318$

(b) $P(50 < X \leq 60) = F(60) - F(50) = 0.4670$

(c) $P(60 < X \leq 70) = F(70) - F(60) = 0.4670$

(d) $P(70 < X \leq 80) = F(80) - F(70) = 0.0318$

$F(x)$ is a good approximation for $F_X(k)$. To check, we will compute the probabilities using both $F_X(k)$ and $F(x)$. We will see that both gives a very close value. We are not losing much.

Remark: Better approximations are possible, particularly as n grows. Continuous CDF turns out to be a valuable tool in our hands.

1.5 General random variables and continuous random variables

We saw that the discrete CDF came from an actual probability space. The probability space had outcomes, events, PMF and then we got the CDF. Now the question to ask here is from where this continuous CDF coming from.

In this section, we will sort of accept the continuous CDF as corresponding to a random variable and start studying what kind of random variable it is, how to deal with it, etc. This brings us into this wonderful possibility of general random variables, continuous random variables. The continuous random variable has very interesting new properties, and it is very useful in modeling, that we have already seen in section 1.1.

1.5.0.1 CDFs and random variables

Theorem (Random variable with CDF $F(x)$) Given a valid CDF $F(x)$, there exists a random variable X taking values in $\mathbb{R}$ such that

$$P(X \leq x) = F(x)$$

Remarks:

- This theorem allows us to define a CDF first, a valid CDF that can be defined in any way we want. It assures that there is a random variable in some probability space.
- The value of the CDF at a particular input x, $F(x)$ is $P(X \leq x)$. This connection between the random variable and the CDF is very important, and it also allows us to use the CDF directly to compute probabilities involved in the random variable.
- Any event we define using the random variable X, for example, $X > a$ or $X < a$, etc. we can use this connection to derive the probabilities.

1.5.1 Properties of CDF:

i) $P(a < X \leq b) = F(b) - F(a)$.

ii) If $F(x)$ rises from F_1 to F_2 at $x_1, P(X = x_1) = F_2 - F_1$.

iii) If $F(x)$ is continuous at $x_0, P(X = x_0) = 0$ (non-intuitive!)

Example: Let X be a random variable with CDF $F(x)$

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.5 + 0.1x & 0 \leq x < 5 \\ 1 & x > 5 \end{cases}$$

Find:

(i) $P(X = 0)$

(ii) $P(1.99 < X \leq 2.01)$

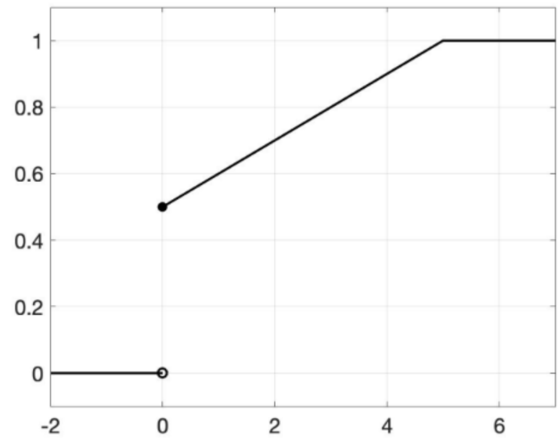
(iii) $P(1.9999999 < X \leq 2.0000001)$

(iv) $P(X = 2.00000\ldots)$

Solution:

(i) $P(X = 0) = 0.5$

(ii) $P(1.99 < X \leq 2.01) = F(2.01) - F(1.99)$[Using the properties of CDF]



(iii) $P(1.9999999 < X \leq 2.0000001) = 0.00000002$
We can observe that as the precision increases, probability decreases.

(iv) $P(X = 2.00000\ldots) = 0$
Here X is taking value with infinite precision and $F(x)$ is continuous at $x = 2$, so the probability is 0.

Remark:

- If $F(x)$ jumps at a point, then it takes that value with non-zero probability.
- If there is no jump in $F(x)$, if it is smooth and continuous at that point, it takes that value with probability 0.

1.5.2 Continuous random variable

Definition: A random variable X with CDF $F_X(x)$ is said to be a continuous random variable if $F_X(x)$ is continuous at every x.

Remarks:

- CDF has no jumps or steps.
- $P(X = x) = 0$ for all x.
- Probability of X falling in an interval will be nonzero.

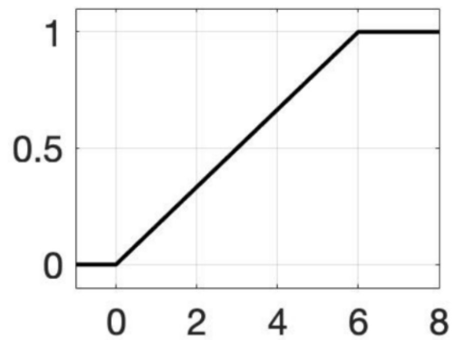
$$P(a < X \leq b) = F(b) - F(a)$$

- Since $P(X = a) = 0$ and $P(X = b) = 0$, we have

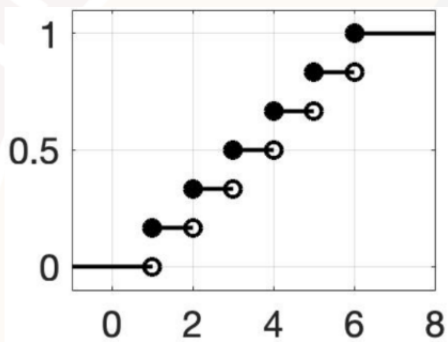
$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Examples:

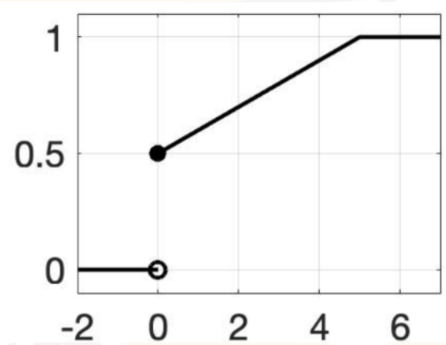
1. Given below are the plot of few CDFs. Identify the kind of distribution from the following:



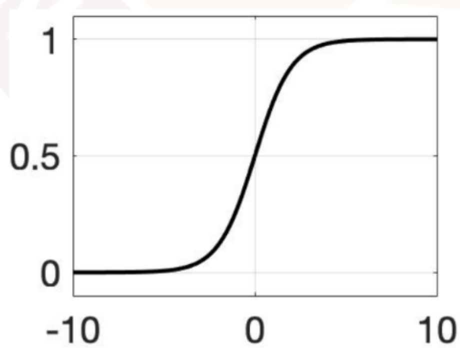
(a)



(b)



(c)



(d)

Solution:

(a) Since the CDF is continuous, it has a continuous distribution.

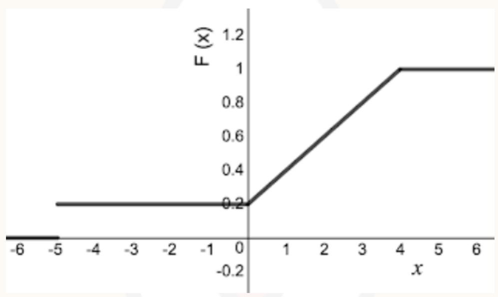
(b) The CDF has a step-like structure, so it a discrete distribution.

(c) At 0, there is a jump in the CDF, and then it is a continuous curve, so it has mixture (both discrete and continuous) distribution.

(d) Since the CDF is continuous, it has a continuous distribution.

2. Consider a random variable X with CDF

$$F(x) = \begin{cases} 0 & x < -5 \\ 0.2 & -5 \leq x < 0 \\ 0.2 + 0.2x & 0 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$



CDF of X

i) Find $P(X < -3), P(-3 < X < -1), P(-1 < X < 1), P(X \leq 3), P(0 \leq X < 3)$.

ii) Is there an x_0 for which $P(X = x_0) > 0$?

iii) Is X a continuous random variable?

<u>Solution:</u>

(i) $P(X < -3)$

- $$P(X < -3) = F(-3) = 0.2$$

- $$\begin{aligned} P(-3 < X < -1) &= F(-1) - F(-3) \quad ; \quad \text{(Using properties of CDF)} \\ &= 0 \end{aligned}$$

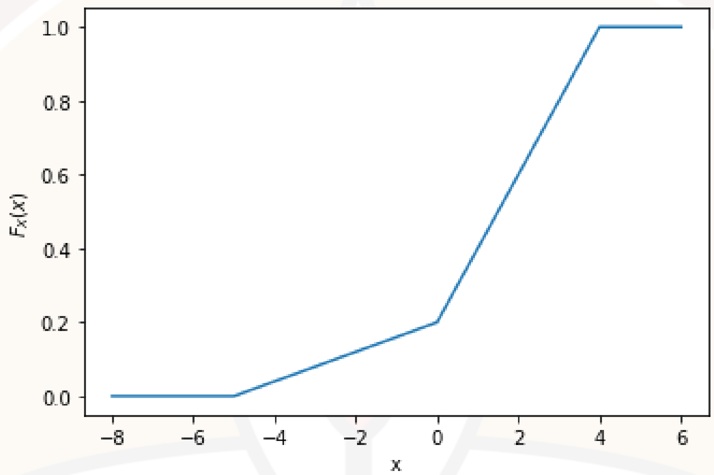
- $$\begin{aligned} P(-1 < X < 1) &= F(1) - F(-1) \quad ; \quad \text{(Using properties of CDF)} \\ &= (0.2 + 0.2) - (0.2) \\ &= 0.2 \end{aligned}$$

(ii) As we can observe from the figure that $P(X = -5) = 0.2$, so there is an x_0 for which $P(X = x_0) = 0$.

(iii) Since there is a jump in the CDF at $x = -5$, therefore, it has a mixed distribution.

3. Consider a random variable X with CDF

$$F(x) = \begin{cases} 0 & x < -5 \\ 0.04x + 0.2 & -5 \le x < 0 \\ 0.2 + 0.2x & 0 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$



CDF of X

i) Find $P(X < -3), P(-3 < X < -1), P(-1 < X < 1), P(X \le -3), P(0 \le X < 3)$.

ii) Is there an x_0 for which $P(X = x_0) > 0$?

iii) Is X a continuous random variable?

<u>Solution:</u>

(i)

- $P(X < -3) = F(-3) = (0.04 \times -3) + 0.2 = 0.08$

- $$\begin{aligned} P(-3 < X < -1) &= F(-1) - F(-3) \quad ; \quad \text{(Using properties of CDF)} \\ &= 0 \end{aligned}$$

- $$\begin{aligned} P(-1 < X < 1) &= F(1) - F(-1) \quad ; \quad \text{(Using properties of CDF)} \\ &= (0.2 + 0.2) - (0.2 - 0.04) \\ &= 0.16 \end{aligned}$$

(ii) The CDF is continuous for all x, so there is not any x_0 for which $P(X = x_0) > 0$.

(iii) Since the CDF is continuous, random variable X is continuous .

1.5.3 Some scenarios for continuous models

Given below are few scenarios, where the number of values that are taken in that random phenomenon is such that discrete may not really be very useful. So, we may want to use the continuous random variable to approximate situation.

- Throw a dart onto a circular board - distance of the point of impact from the center of the board.
- Weight of a meteorite hitting earth.
- Weight of a human being, height of a human being Speed of a delivery in cricket.
- Price of a stock.

1.6 Probability Density Function

For the discrete random variable, we had looked at the probability mass function, probability that discrete random variable takes a particular value. In case of continuous random variable, PMF will give us the probabilities zeroes. So, we need something called the density.

Continuous random variables will take values over an interval and they will have a certain density over that interval but not over a particular value. A random variable whose CDF is continuous at every point is termed as the continuous random variable.

Definition: A continuous random variable X with CDF $F_X(x)$ is said to have a PDF $f_X(x)$ if, for all x_0,

$$F_X(x_0) = \int_{-\infty}^{x_0} f_X(x_0)\, dx$$

Derivative of the CDF (if exists) is given by PDF.

$$\int_{-\infty}^{x_0} f_X(x_0)\, dx = F_X(x_0) - F_X(-\infty)$$

Why do we need PDF when we already have a CDF?

1. Whenever we want to have a measure of some distribution, we describe it with the CDF. If PDF is high, the probability of X taking a value there is high or if it is low, then probability is low.

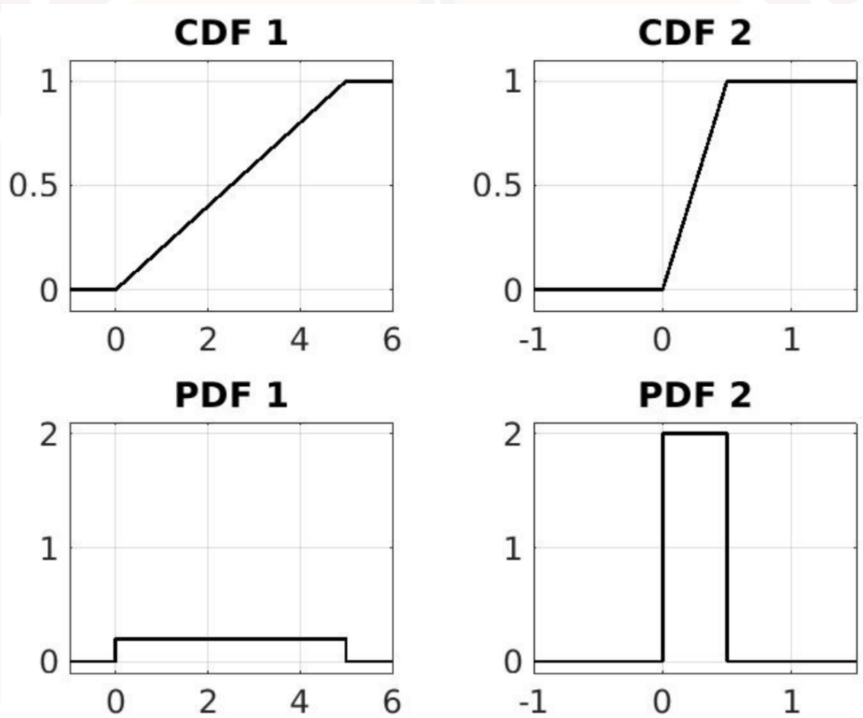
2. CDF is an increasing function. CDF being higher at some x does not mean that X takes more values there. On the other hand, if the density is higher, then X takes mores values around those points.

3. Density gives a clear picture of how the distribution looks like, but in case of CDF, we only see how the probability increases.

4. PDF is much easier in probability computations.

Examples:

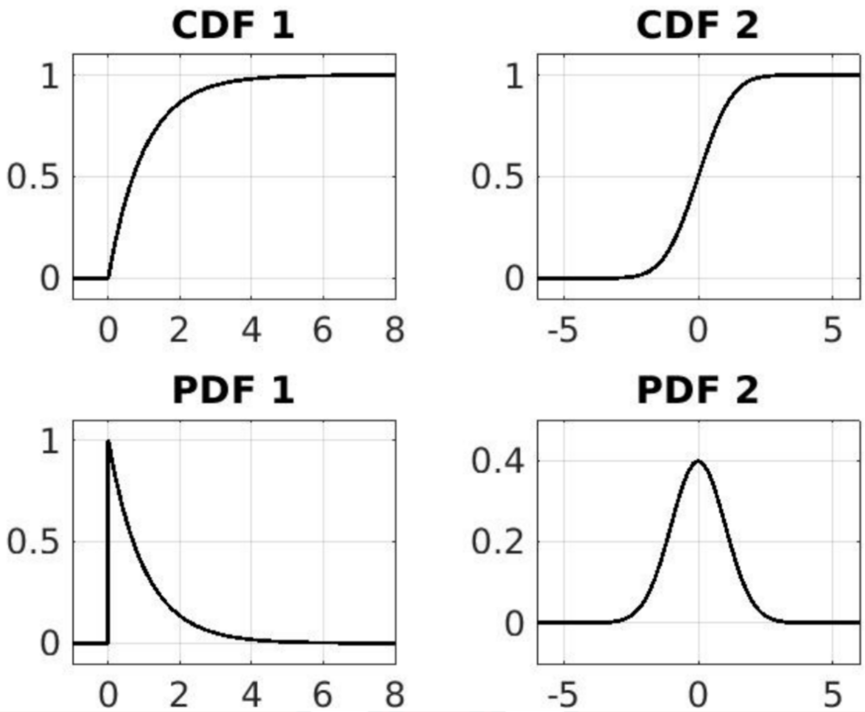
1. CDF and PDF of Uniform distributions



   The above figure is for the Uniform distribution. The image on the left hand side is for Uniform$[0, 5]$ and the image on the right hand side is for Uniform$[0, 1/2]$.

2. CDF and PDF of Exponential and Normal distribution

   The images on the left hand side is for the Exponential distribution and the image on the right hand side is for the Normal distribution.



1.6.1 Properties of PDF

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a density function if

1. $f(x) \geq 0$
2. $\int\limits_{-\infty}^{\infty} f_X(x)\ dx = 1$
3. $f(x)$ is piecewise continuous.

Remark: Given a density function f , there is a continuous random variable X with PDF as f.

Support of a random variable: It is defined as the points where the density function is strictly greater than 0. Mathematically, for any random variable X with density $f_X(x)$

$$\text{Supp}(X) = \{x : f_X(x) > 0\}$$

Note: Supp(X) contains intervals in which X can fall with positive probability.
For any event A defined using the random variable X, probability of event is computed as

$$P(A) = \int_A f(x)\ dx$$

Examples:

i) Consider the function

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that f is a density function.

- $f(x) \geq 0$.
-

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx &= \int_{0}^{1} 3x^2\ dx \\ &= x^3 \Big|_0^1 \\ &= 1 \end{aligned}$$

Since $f(x)$ satisfies all the properties, it is a valid density function.

(b) Consider a random variable X with density f. Find $P(X = 1/5), P(X = 2/5), P(X \in [1/5 - \epsilon, 1/5 + \epsilon]), P(X \in [2/5 - \epsilon, 2/5 + \epsilon])$.

Solution

- $P(X = 1/5) = 0$; (Since X is continuous)
- $P(X = 2/5) = 0$; (Since X is continuous)
- $P(X \in [1/5 - \epsilon, 1/5 + \epsilon])$

$$\begin{aligned} P\left(\frac{1}{5} - \epsilon < X < \frac{1}{5} + \epsilon\right) &= \int_{1/5-\epsilon}^{1/5+\epsilon} 3x^2\ dx \\ &= x^3 \Big|_{1/5-\epsilon}^{1/5+\epsilon} \\ &= \left(\frac{1}{5} + \epsilon\right)^3 - \left(\frac{1}{5} - \epsilon\right)^3 \\ &= \frac{6}{25}\epsilon + 2\epsilon^3 \quad \text{, where} \quad \epsilon << 0 \end{aligned}$$

- Similarly, $P(X \in [2/5 - \epsilon, 2/5 + \epsilon]) = \left(\frac{2}{5} + \epsilon\right)^3 - \left(\frac{2}{5} - \epsilon\right)^3 = \frac{24}{25}\epsilon + 2\epsilon^3$, where $\epsilon << 0$

ii) Consider a random variable X with density

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(X \in [0.1, 0.3]), P(X \in (0.1, 0.03]), P(X \in [0.1, 0.03)), P(X \in (0.1, 0.03))$.

Solution:

- $P(X \in [0.1, 0.3])$

$$\begin{aligned} P(0.1 \leq X \leq 0.3) &= \int_{0.1}^{0.3} 2x \ dx \\ &= x^2 \Big|_{0.1}^{0.3} \\ &= (0.3)^2 - (0.1)^2 \\ &= 0.08 \end{aligned}$$

- $P(X \in (0.1, 0.3])$

$$\begin{aligned} P(0.1 < X \leq 0.3) &= \int_{0.1}^{0.3} 2x \ dx \\ &= x^2 \Big|_{0.1}^{0.3} \\ &= (0.3)^2 - (0.1)^2 \\ &= 0.08 \end{aligned}$$

- $P(X \in [0.1, 0.3))$

$$\begin{aligned} P(0.1 \leq X < 0.3) &= \int_{0.1}^{0.3} 2x \ dx \\ &= x^2 \Big|_{0.1}^{0.3} \\ &= (0.3)^2 - (0.1)^2 \\ &= 0.08 \end{aligned}$$

- $P(X \in (0.1, 0.03))$

$$\begin{aligned}
P(0.1 < X < 0.3) &= \int_{0.1}^{0.3} 2x\ dx \\
&= x^2 \Big|_{0.1}^{0.3} \\
&= (0.3)^2 - (0.1)^2 \\
&= 0.08
\end{aligned}$$

Observe that in all the cases, probabilities are same.

iii) Consider the function

$$f(x) = \begin{cases} k & 0 < x < 1/4 \\ 2k & 1/4 < x < 3/4 \\ 3k & 3/4 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find k such that $f(x)$ is a valid density function.

<u>Solution:</u>

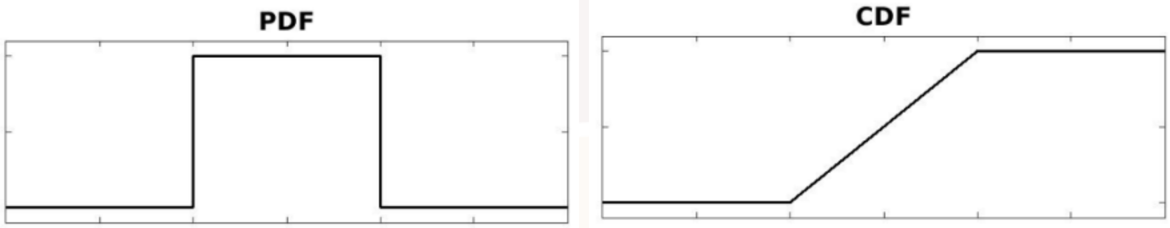
For $f(x)$ to be a valid density function, $\int\limits_{-\infty}^{\infty} f(x)\ dx$ should be 1. Therefore,

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\ dx = 1 \\
\implies &\int_{-\infty}^{0} f(x)\ dx + \int_{0}^{1/4} f(x)\ dx + \int_{1/4}^{3/4} f(x)\ dx + \int_{3/4}^{1} f(x)\ dx + \int_{1}^{\infty} f(x)\ dx = 1 \\
\implies &\int_{-\infty}^{0} 0\ dx + \int_{0}^{1/4} k\ dx + \int_{1/4}^{3/4} 2k\ dx + \int_{3/4}^{1} 3k\ dx + \int_{1}^{\infty} f(x)\ dx = 1 \\
\implies &k(x)\Big|_{0}^{1/4} + 2k(x)\Big|_{1/4}^{3/4} + 3k(x)\Big|_{3/4}^{1} = 1 \\
\implies &k = \frac{1}{2}
\end{aligned}$$

1.7 Common distributions

1.7.1 Uniform distribution

A continuous random variable X is said to be uniform in $[a, b]$, if it has a flat PDF in the range $[a, b]$.



PDF of X is given by:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

CDF of X:

$$F_X(x) = \int_{-\infty}^{a} 0 \ dx + \int_{a}^{x} \frac{1}{b-a} \ dx + \int_{b}^{\infty} 0 \ dx$$
$$= 0 + \frac{x-a}{b-a} + 0$$
$$= \frac{x-a}{b-a}$$

$$F_X(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

Example: Suppose $X \sim \text{Uniform}[-10, 10]$.

Find: $P(-3 \leq X \leq 2), P(5 <| X |< 7), P(1 - \epsilon < X < 1 + \epsilon), P(9 - \epsilon < X < 9 + \epsilon)$, $P(X > 7 | X > 3)$.

Solution: (Using the PDF)

Since $X \sim \text{Uniform}[-10, 10]$, PDF of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{20}, & -10 \leq x \leq 10 \\ 0, & \text{elsewhere} \end{cases}$$

Now,

- $P(-3 \leq X \leq 2) = \int\limits_{-3}^{2} \frac{1}{20}\ dx = \frac{5}{20} = \frac{1}{4}$
- $P(5 <\mid X \mid< 7)$

$$\begin{aligned} P(5 <\mid X \mid< 7) =& P(5 < X < 7) + P(-7 < X < -5) \\ =& \int\limits_{5}^{7} \frac{1}{20}\ dx + \int\limits_{-7}^{-5} \frac{1}{20}\ dx \\ =& \frac{1}{5} \end{aligned}$$

- $P(1 - \epsilon < X < 1 + \epsilon)$

$$\begin{aligned} P(1 - \epsilon < X < 1 + \epsilon) =& \int\limits_{1-\epsilon}^{1+\epsilon} \frac{1}{20}\ dx \\ =& \frac{2\epsilon}{20}, \quad \text{where } \epsilon << 0 \end{aligned}$$

For any x_0 in the ϵ interval of $[-1, 1], P(x_0 - \epsilon < X < x_0 + \epsilon) = \frac{2\epsilon}{20}$.

- $P(9 - \epsilon < X < 9 + \epsilon)$
Similarly, for any x_0 in the ϵ interval of $[-9, 9], P(x_0 - \epsilon < X < x_0 + \epsilon) = \frac{2\epsilon}{20}$.
- $P(X > 7|X > 3)$

$$\begin{aligned} P(X > 7|X > 3) =& \frac{P(X > 7, X > 3)}{P(X > 3)} \\ =& \frac{P(X > 7)}{P(X > 3)} \\ =& \frac{\int\limits_{7}^{10} \frac{1}{20}\ dx}{\int\limits_{3}^{10} \frac{1}{20}\ dx} \\ =& \frac{3/20}{7/20} = \frac{3}{7} \end{aligned}$$

Solution: (Using the CDF)

Since $X \sim \text{Uniform}[-10, 10]$, CDF of X is given by

$$F_X(x) = \begin{cases} 0, & -10 < x \\ \dfrac{x+10}{20}, & -10 \le x < 10 \\ 1, & x \ge 10 \end{cases}$$

Now,

- $P(-3 \le X \le 2) = F_X(2) - F_X(-3) = \left(\dfrac{2+10}{20}\right) - \left(\dfrac{-3+10}{20}\right) = \dfrac{5}{20}$
- $P(5 <\mid X \mid< 7)$

$$\begin{aligned} P(5 <\mid X \mid< 7) =& P(5 < X < 7) + P(-7 < X < -5) \\ =& [F_X(7) - F_X(5)] + [F_X(-5) - F_X(-7)] \\ =& \left[\left(\frac{7+10}{20}\right) - \left(\frac{5+10}{20}\right)\right] + \left[\left(\frac{-5+10}{20}\right) - \left(\frac{-7+10}{20}\right)\right] \\ =& \frac{2}{20} + \frac{2}{20} \\ =& \frac{4}{20} \end{aligned}$$

- $P(1 - \epsilon < X < 1 + \epsilon)$

$$\begin{aligned} P(1 - \epsilon < X < 1 + \epsilon) =& F_X(1+\epsilon) - F_X(1-\epsilon) \\ =& \left(\frac{1+\epsilon+10}{20}\right) - \left(\frac{1-\epsilon+10}{20}\right) \\ =& \frac{2\epsilon}{20}, \quad \text{where } \epsilon << 0 \end{aligned}$$

  For any x_0 in the ϵ interval of $[-1, 1], P(x_0 - \epsilon < X < x_0 + \epsilon) = \dfrac{2\epsilon}{20}$.

- $P(9 - \epsilon < X < 9 + \epsilon)$
  Similarly, for any x_0 in the ϵ interval of $[-9, 9], P(x_0 - \epsilon < X < x_0 + \epsilon) = \dfrac{2\epsilon}{20}$.

- $P(X > 7|X > 3)$

$$
\begin{aligned}
P(X > 7|X > 3) =& \frac{P(X > 7, X > 3)}{P(X > 3)} \\
=& \frac{P(X > 7)}{P(X > 3)} \\
=& \frac{1 - P(X \leq 7)}{1 - P(X \leq 3)} \\
=& \frac{1 - F_X(7)}{1 - F_X(3)} \\
=& \frac{1 - \left(\dfrac{7 + 10}{20}\right)}{1 - \left(\dfrac{3 + 10}{20}\right)} = \frac{3/20}{7/20} = \frac{3}{7}
\end{aligned}
$$

1.7.2 Exponential distribution

$X \sim$ Exponential(λ), where $\lambda > 0$

If X is a random variable with pdf given by

$$
f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}
$$

then, X is exponentially distributed with parameter λ.

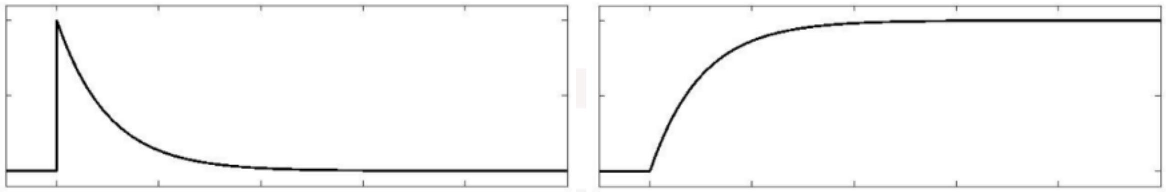
<u>Check:</u> $f_X(x)$ is a valid PDF.

1. Clearly $f_X(x) \geq 0$.
2. Support of X is $\{x : x > 0\}$

$$
\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\ dx =& \int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} \lambda e^{-\lambda x}\ dx \\
=& 0 + \lambda \frac{e^{-\lambda x}}{-\lambda} \\
=& -e^{-\lambda x}\Big|_0^{\infty} = 1
\end{aligned}
$$

Therefore, $f_X(x)$ is a valid PDF.

Now, $P(X \leq x) = \int\limits_{-\infty}^{x} f_X(x)\ dx = \int\limits_{-\infty}^{x} \lambda e^{-\lambda x}\ dx = -e^{-\lambda x} \mid_0^x = 1 - e^{-\lambda x}$



Therefore, the CDF of exponential distribution is

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x > 0 \end{cases}$$

Remarks:

1. In case of Uniform random variable, support is finite, whereas in case of exponential random variable, support is infinite. Since the support is very large, it is much more likely that it take values close to 0 than say close to 100.
2. $f_X(x)$ never becomes 0 when x is positive.
3. In practice, it is a good model for many situations.

Example: Suppose $X \sim$ Exponential(2).

Find: $P(5 < X < 7), P(X > 4), P(1-\epsilon < X < 1+\epsilon), P(9-\epsilon < X < 9+\epsilon), P(X > 7|X > 3)$.

Solution: (Using the PDF)

Since $X \sim$ Exponential(2), PDF of X is given by

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Now,

- $P(5 < X < 7) = \int\limits_{5}^{7} 2e^{-2x}\ dx = 2\left(\dfrac{e^{-2x}}{-2}\right)\Bigg|_5^7 = e^{-10} - e^{-14}$

- $P(1-\epsilon < X < 1+\epsilon)$

$$\begin{aligned}
P(1-\epsilon < X < 1+\epsilon) =& \int\limits_{1-\epsilon}^{1+\epsilon} 2e^{-2x}\ dx \\
=& 2\left(\frac{e^{-2x}}{-2}\right)\Bigg|_{1-\epsilon}^{1+\epsilon} \\
=& e^{-2(1-\epsilon)} - e^{-2(1+\epsilon)}, \quad \text{where } \epsilon << 0
\end{aligned}$$

<u>Exercise:</u> Find: $P(9-\epsilon < X < 9+\epsilon)$

- $P(X>7|X>3)$

$$\begin{aligned}
P(X>7|X>3) =& \frac{P(X>7, X>3)}{P(X>3)} \\
=& \frac{P(X>7)}{P(X>3)} \\
=& \frac{\int_7^\infty 2e^{-2x}\ dx}{\int_3^\infty 2e^{-2x}\ dx} \\
=& \frac{2\left(\frac{e^{-2x}}{-2}\right)\Big|_7^\infty}{2\left(\frac{e^{-2x}}{-2}\right)\Big|_3^\infty} = e^{-8}
\end{aligned}$$

<u>Solution</u>: (Using the CDF)

Since $X \sim$ Exponential(2), PDF of X is given by

$$f_X(x) = \begin{cases} 1-e^{-2x}, & x>0 \\ 0, & \text{elsewhere} \end{cases}$$

Now,

- $P(5<X<7) = F_X(7) - F_X(5) = (1-e^{-7\times 2}) - (1-e^{-5\times 2}) = e^{-10} - e^{-14}$
- $P(1-\epsilon < X < 1+\epsilon)$

$$\begin{aligned}
P(1-\epsilon < X < 1+\epsilon) =& F_X(1+\epsilon) - F_X(1-\epsilon) \\
=& (1-e^{-2(1+\epsilon)}) - (1-e^{-2(1-\epsilon)}) \\
=& e^{-2(1-\epsilon)} - e^{-2(1+\epsilon)}, \quad \text{where } \epsilon << 0
\end{aligned}$$

- $P(X > 7|X > 3)$

$$\begin{aligned}P(X > 7|X > 3) &= \frac{P(X > 7, X > 3)}{P(X > 3)}\\ &= \frac{P(X > 7)}{P(X > 3)}\\ &= \frac{1 - F_X(7)}{1 - F_X(3)}\\ &= \frac{1 - (1 - e^{-14})}{1 - (1 - e^{-6})}\\ &= e^{-8}\end{aligned}$$

1.7.2.1 Memoryless property of Exponential

If $X \sim \text{Exponential}(\lambda)$, then for any $s, t > 0$, we have

$$P(X > s + t \mid X > s) = P(X > t)$$

Proof:

$$\begin{aligned}P(X > s + t \mid X > s) &= \frac{P(X > s + t, X > s)}{P(X > s)}\\ &= \frac{P(X > s + t)}{P(X > s)}\\ &= \frac{1 - F_X(s + t)}{1 - F_X(s)}\\ &= \frac{1 - (1 - e^{-\lambda(s+t)})}{1 - (1 - e^{-\lambda s})} = e^{-\lambda t} = P(X > t)\end{aligned}$$

Suppose you are waiting for a bus at a bus stop, it is a random waiting time, so that random waiting time is very commonly modeled as an exponential random variable. No matter at what time you go to the bus stop, the waiting time is going to be same. For any real life situation, memoryless property is very useful.

1.7.3 Normal distribution

The most common distribution among all the models is Normal distribution. It is also termed as the Gaussian distribution.

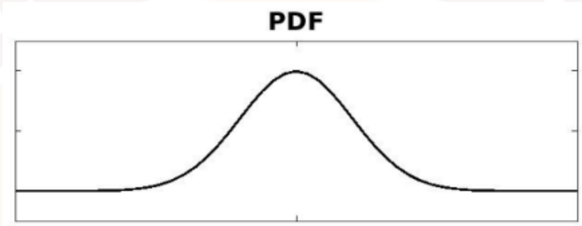
Normal distribution has two parameters; the first parameter is denoted as μ which represent the mean and the the second parameter is denoted as σ^2 which represent the variance. It is denoted as $\text{Normal}(\mu, \sigma^2)$.

1.7.3.1 PDF of Normal distribution

The PDF of normal distribution is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}}, \text{where } Supp(X) = \mathbb{R}$$

$\mu \in \mathbb{R}, \sigma$ is a positive real number.



Observations:

- As the value of X becomes much larger from μ, $f_X(x)$ goes to 0. Similarly as the value of X becomes much smaller from μ, $f_X(x)$ goes to 0.
- $f_X(x)$ takes the maximum value at $x = \mu$.
- Height of the curve is $\dfrac{1}{\sigma\sqrt{\pi}}$.
- X take the values closer to μ with higher probability and as it move away from μ, the probabilities decrease.

Exercise: Show $f_X(x)$ is a valid PDF.

1. $f_X(x)$ is non-negative.
2. $\int\limits_{-\infty}^{\infty} \dfrac{1}{\sigma\sqrt{2\pi}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}} \, dx = 1.$

   Proof: Let $\left(\dfrac{x-\mu}{\sigma}\right) = z$
   Now, $x = \sigma z + \mu \implies dx = \sigma dz$

$$\int\limits_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}} \, dx = \int\limits_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma \, dz \tag{1}$$

Now, substitute $\dfrac{z^2}{2} = y^2$ in (1), we will have

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \sigma \, dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} \, dy$$

We know that $\int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi}$

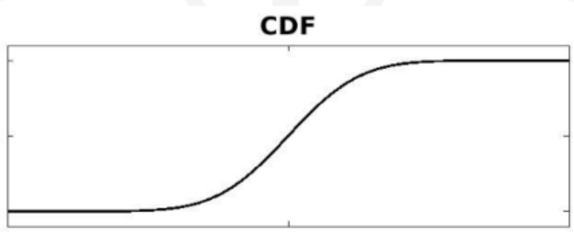
Therefore, $\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}} \, dx = 1$

Exercise: Show that $P(X < a) = P(X > a) = 1/2$.

1.7.3.2 CDF of Normal distribution

$$F_X(x) = \int_{u=\infty}^{x} f_X(u) \, du$$

There is no closed form expression for CDF of normal distribution.



1.7.3.3 Standard normal distribution

Normal distribution with mean 0 and variance 1 is called standard normal distribution. It is denoted as $\text{Normal}(0, 1)$.

Standardization: If $X \sim \text{Normal}(\mu, \sigma^2)$, then

$$Z = \frac{X - \mu}{\sigma} \sim \text{Normal}(0, 1).$$

PDF:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

CDF:

$$F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du$$

1.7.3.4 Probability computations with normal distribution

1. First convert the probability computation to the standardized case.
2. There are many scientific computing devices available which can be used to calculate $F_Z(z)$. Standard normal table can also be used.

Example 1: Suppose $X \sim \text{Normal}(2, 5)$. Find:

i) $P(X < 5)$
Solution:

$$\begin{aligned} P(X < 5) =& P\left(\frac{X-2}{\sqrt{5}} < \frac{5-2}{\sqrt{5}}\right) \\ =& P(Z < 3/\sqrt{5}) \\ =& F_Z(3/\sqrt{5}) \end{aligned}$$

ii) $P(X < -5)$
Solution:

$$\begin{aligned} P(X < -5) =& P\left(\frac{X-2}{\sqrt{5}} < \frac{-5-2}{\sqrt{5}}\right) \\ =& P(Z < -7/\sqrt{5}) \\ =& F_Z(-7/\sqrt{5}) \end{aligned}$$

iii) $P(X > 5)$
Solution:

$$\begin{aligned} P(X > 5) =& P\left(\frac{X-2}{\sqrt{5}} > \frac{5-2}{\sqrt{5}}\right) \\ =& P(Z > 3/\sqrt{5}) \\ =& 1 - F_Z(3/\sqrt{5}) \end{aligned}$$

Similarly try to find $P(X < 10), P(X < -10), P(X > 10)$.

Example 2: Suppose $X \sim \text{Normal}(3, 1)$. Find:

i) $P(5 < X < 7)$
Solution:

$$
\begin{aligned}
P(5 < X < 7) =& P\left(\frac{5-3}{\sqrt{1}} < \frac{X-3}{\sqrt{1}} > \frac{7-3}{\sqrt{1}}\right) \\
=& P(2 < Z < 4) \\
=& F_Z(4) - F_Z(2)
\end{aligned}
$$

ii) $P(1-\epsilon < X < 1+\epsilon)$
Solution:

$$
\begin{aligned}
P(1-\epsilon < X < 1+\epsilon) =& P\left(\frac{1-\epsilon-3}{\sqrt{1}} < \frac{X-3}{\sqrt{1}} > \frac{1+\epsilon-3}{\sqrt{1}}\right) \\
=& P(-2-\epsilon < Z < -2+\epsilon) \\
=& F_Z(-2+\epsilon) - F_Z(-2-\epsilon)
\end{aligned}
$$

iii) $P(X > 7 | X > 3)$
Solution:

$$
\begin{aligned}
P(X > 7 | X > 3) =& \frac{P(X > 7, X > 3)}{P(X > 3)} \\
=& \frac{P(X > 7)}{P(X > 3)} \\
=& \frac{P(X-3 > 7-3)}{P(X-3 > 3-3)} \\
=& \frac{P(Z > 4)}{P(Z > 0)} \\
=& \frac{1 - F_Z(4)}{1 - F_Z(0)}
\end{aligned}
$$

Similarly, try to find $P(9-\epsilon < X < 9+\epsilon)$, $P(-5 < X < 5)$, $P(X > 4)$.

1.8 Problems

1. If

$$
f_X(x) = \begin{cases} \frac{1}{18}(x^2 - 8x + 16) & 1 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}
$$

What is the value of $P(X \leq 4)$? Enter the answer correct to one decimal accuracy. ($\int x^a dx = \frac{x^{a+1}}{a+1}$)

2. If $X \sim$ Normal$(10, 25)$, what is the value of $E[2X^2]$?

3. If $X \sim$ Normal$(10, 4)$, then what is the value of $P(X \geq 8 | X \leq 9)$?

4. The time taken by Rohith to complete a race follows the exponential distribution with an expected time of completion of 10 minutes. What is the probability that Rohith takes less than 20 minutes but more than 10 minutes to complete the race? Enter the answer correct to 2 decimals accuracy. $(\int e^{-ax} dx = \dfrac{e^{-ax}}{-a})$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} k\sqrt{x} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the value of k for which f is a valid PDF.

Hint: Use $\int\limits_a^b \sqrt{x}\, dx = \dfrac{2}{3}(b^{\frac{3}{2}} - a^{\frac{3}{2}})$

6. Suppose that X is a continuous random variable with PDF

$$f_X(x) = \begin{cases} c(6x - 2x^2) & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the value of c.
   (b) Find $P(X > 1)$.
   (c) Find $P(X = 1/10)$.

Hint: Use $\int\limits_a^b x^r\, dr = \dfrac{1}{r+1}\left[b^{r+1} - a^{r+1}\right]$

7. Let X be a random variable with CDF $F(x)$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } 0 \leq x < \frac{1}{2} \\ x - \frac{1}{4} & \text{for } \frac{1}{2} \leq x < \frac{3}{4} \\ \sqrt{x - \frac{1}{2}} & \text{for } \frac{3}{4} \leq x < \frac{3}{2} \\ 1 & \text{for } x \geq \frac{3}{2} \end{cases}$$

   (i) $P\left(X < \dfrac{1}{4}\right)$
   (ii) $P\left(\dfrac{1}{2} \leq X < \dfrac{3}{4}\right)$

(iii) $P(X \geq 1)$

(iv) $P\left(-1 \leq X < \dfrac{1}{4}\right)$

8. The amount of time in hours that a machine works before breaking down is a continuous random variable with PDF is given by,

$$f_X(x) = \begin{cases} \lambda e^{-x/50} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

   (i) What is the probability that a machine will work between 30 and 70 hours before breaking down?

   (ii) What is the probability that machine will work for less than 50 hours?

   (iii) What is the probability that a machine will work more than 60 hours given that it is worked more than 40 hours?
   (Hint: Use memoryless property)

Chapter 5

2 Functions of a continuous random variable

2.1 Why functions?

Why functions when we already have a random variable and its PDF? We may model one quantity as a random variable X and we may have to work with another closely related quantity. There are many instances where the functions of random variable shows off very naturally. Here are the few examples:

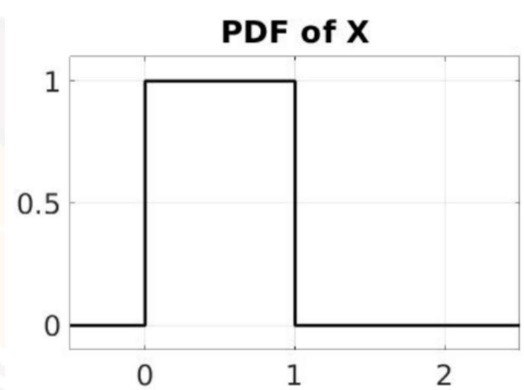
1. Suppose we model the length of a square as a random variable X. If we want to model the area of a square, it is going to be X^2, it is a function of X. We don't need to have a new random variable to model this.

2. Suppose the density, ρ is constant for some fluid. It's volume is given by the random variable X. Then, weight can be modelled as ρX.

Given the PDF and CDF of X, we want to know what is the PDF and CDF of $g(X)$. It is useful to have a method for finding the distribution of a function of X. There are simple and straightforward methods to find this, that we will see in the next section.

Example: $X \sim \text{Uniform}[0, 1]$. Let $Y = 2X$. Find the distribution of Y.

Solution:



Step 1: Find the range of Y.
$Y \in [0, 2]$.

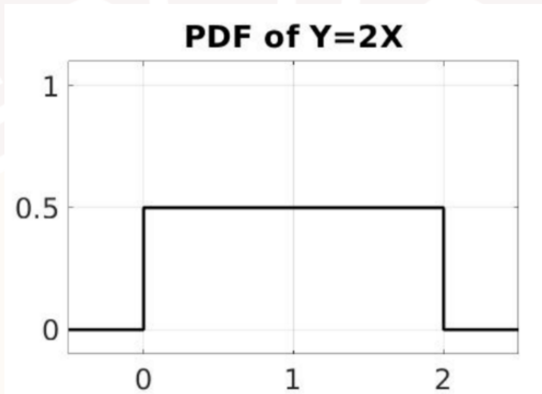
Step 2: Find the CDF of Y.
For $0 \leq y \leq 2$,

$$F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq y/2) = F_X(y/2)$$

We know that if $X \sim \text{Uniform}[a,b], F_X(x) = \dfrac{x-a}{b-a}$, for $0 \leq x \leq 1$.

Therefore, $F_Y(y) = F_X(y/2) = \dfrac{(y/2)-0}{1} = \dfrac{y}{2}$, for $0 \leq y \leq 2$.

Now, PDF of $Y, f_Y(y) = \dfrac{dF_Y(y)}{dy} = \dfrac{1}{2}$



Hence, $Y \sim \text{Uniform}[0,2]$.

Remark: Using the CDF method, it is easier to find the CDF and PDF of functions of random variable.

Exercise: Similarly try for $Y = X + 2, Y = aX + b$.

2.2 General Case: CDF of g(X)

Suppose X is a continuous random variable with CDF $F_X(x)$ and PDF $f_X(x)$. Now, we want to find the CDF of a function of X, say $g(X)$. The function should be reasonable. Let $Y = g(X)$ is a random variable. Then, the CDF of Y will be determined as follows:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \in \{x : g(X) \leq y\})$$

Now, it has been converted into the problem of finding the probability. An event is defined using X, and we have to find its probability.

To find the above probability, we will follow the following steps:

1. Convert the subset $A_y = \{x : g(X) \leq y\}$ into intervals in real line.
2. Find the probability that X falls in those intervals.
3. $F_Y(y) = P(X \in A_y) = \int\limits_{A_y} f_X(x)\, dx$

If g is monotonic and differentiable, we can use the following formula to find the CDF of $g(X)$.

2.3 Monotonic, differentiable functions

Suppose X is a continuous random variable with PDF f_X . Let $g(x)$ be monotonic for $x \in supp(X)$ with derivative $g'(x) = \frac{dg(x)}{dx}$. Then, the PDF of $Y = g(X)$ is

$$f_Y(y) = \frac{1}{\mid g'(g^{-1}(y)) \mid} f_X(g^{-1}(y))$$

1. Translation: $Y = X + a$

   We know that $X + a$ is monotonic (increasing function) in $\mathbb{R}$.
   Therefore, we can use the formula, $f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$

   Let $g(x) = x + a$. That implies $g'(x) = 1$.

   $g^{-1}(y) = y - a$
   $g'(g^{-1}(y)) = g'(y - a) = 1$

   Now,

$$\begin{aligned} f_Y(y) =& \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y)) \\ =& f_X(y - a) \end{aligned}$$

2. Scaling: $Y = aX$

   We know that aX is monotonic in $\mathbb{R}$.
   Therefore, we can use the formula, $f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$

   Let $g(x) = ax$. That implies $g'(x) = a$.

   $g^{-1}(y) = y/a$
   $g'(g^{-1}(y)) = g'(y/a) = 1/a$

   Now,

$$\begin{aligned} f_Y(y) =& \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y)) \\ =& \frac{1}{\mid a \mid} f_X(y/a) \end{aligned}$$

3. Affine: $Y = aX + b$

   We know that $aX + b$ is monotonic in $\mathbb{R}$.
   Therefore, we can use the formula, $f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$

   Let $g(x) = ax + b$. That implies $g'(x) = a$.

   $g^{-1}(y) = (y - b)/a$
   $g'(g^{-1}(y)) = g'\left(\dfrac{y-b}{a}\right) = 1/a$

   Now,

$$
\begin{aligned}
f_Y(y) =& \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y)) \\
=& \frac{1}{\mid a \mid} f_X((y-b)/a)
\end{aligned}
$$

2.4 Affine transformation of Normal distributions:

1. Let $X \sim \text{Normal}(0, 1)$.

$$
f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)
$$

   Suppose $Y = \sigma X + \mu$
   Since Y is an affine transformation of X, we can write

$$
\begin{aligned}
f_Y(y) =& \frac{1}{\mid \sigma \mid} f_X((y-\mu)/\sigma) \\
=& \frac{1}{\sigma\sqrt{2\pi}} \exp(-((y-\mu)/\sigma)^2/2)
\end{aligned}
$$

   Therefore, $Y \sim \text{Normal}(\mu, \sigma^2)$.

2. Let $X \sim \text{Normal}(\mu, \sigma^2)$.

$$
f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-((x-\mu)/\sigma)^2/2)
$$

   Suppose $Y = \dfrac{X - \mu}{\sigma}$

Since Y is an affine transformation of X, we can write

$$\begin{aligned}
f_Y(y) =& \frac{1}{1/\mid \sigma \mid} f_X\left(\frac{\frac{x-\mu}{\sigma}+\frac{\mu}{\sigma}}{1/\sigma}\right)\\
=&\sigma f_X(x)\\
=&\frac{1}{\sqrt{2\pi}}\exp(-((x-\mu)/\sigma)^2/2)\\
=&\frac{1}{\sqrt{2\pi}}\exp(-y^2/2)
\end{aligned}$$

Therefore, $Y \sim \text{Normal}(0,1)$.

Remark: The affine transformation of a normal random variable is normal.

2.5 Problems:

1. Let $X \sim \text{Exp}(\lambda)$. Find the PDF of X^2.

   Solution:

   The PDF of X is given by

   $$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

   $\text{Supp}(X) = \{x : x > 0\}$

   (a) Using the CDF approach:

   $Y = X^2$, $y > 0$.

   For $y > 0$,

   $$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}$$

   Now, PDF of Y, $f_Y(y) = \dfrac{dF_Y(y)}{dy} = \dfrac{1}{2\sqrt{y}}\lambda e^{-\lambda\sqrt{y}}, y > 0$.

   (b) Monotonic, differentiable functions

   We know that X^2 is monotonic for $x > 0$.
   Therefore, we can use the formula, $f_Y(y) = \dfrac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$

Let $g(x) = x^2$. That implies $g'(x) = 2x$.

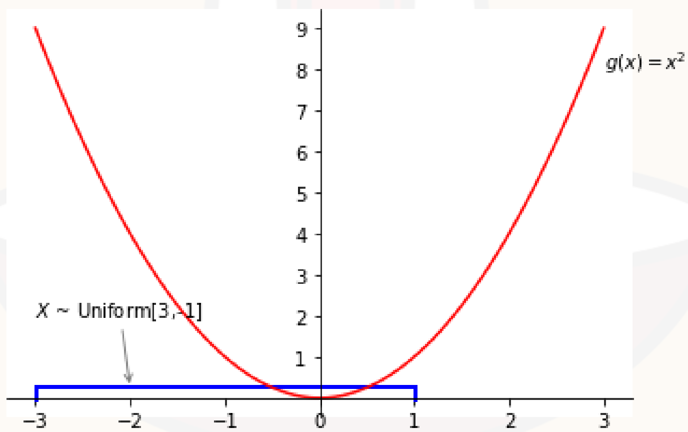
$g^{-1}(y) = \sqrt{y}$
$g'(g^{-1}(y)) = g'(\sqrt{y}) = 2\sqrt{y}$

Now,

$$\begin{aligned} f_Y(y) =& \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y)) \\ =& \frac{1}{|\ 2\sqrt{y}\ |} f_X(\sqrt{y}) \\ =& \frac{1}{2\sqrt{y}} \lambda e^{-\lambda\sqrt{y}} \end{aligned}$$

2. Let $X \sim \text{Uniform}[-3, 1]$. Find the PDF of X^2.

<u>Solution:</u>



The PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{4}, & -3 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$\text{Supp}(X) = [-3, 1]$
$Y = X^2$, $y \in [0, 9]$.

For $0 < y < 1$,

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{4}\, dy = \frac{2\sqrt{y}}{4}$$

For $1 < y < 9$,

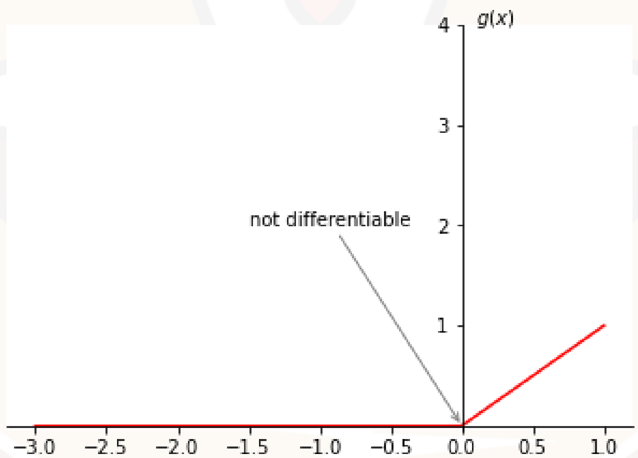
$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{1} \frac{1}{4}\, dy = \frac{1+\sqrt{y}}{4}$$

Now, PDF of Y, $f_Y(y) = \begin{cases} \sqrt{y}/4, & 0 < y < 1 \\ \sqrt{y}/8, & 1 < y < 9 \end{cases}$

X^2 is not monotonic in the given support and hence the second method is not applicable here.

3. Let $X \sim \text{Uniform}[-3, 1]$. Find the PDF of $\max(X, 0)$.

Solution: Let $g(x) = \max(X, 0)$, where $X \sim \text{Uniform}[-3, 1]$



$g(x)$ is defined as

$$g(x) = \begin{cases} 0, & -3 \leq x < 0 \\ x, & 0 \leq x < 1 \end{cases}$$

The PDF of X is given by

$$f_X(x) = \begin{cases} \frac{1}{4}, & -3 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Supp$(X) = [-3, 1]$

$Y = \max(X, 0)$, $y \in [0, 1]$.

For $y < 0$, $F_Y(y) = 0$.
For $y = 0$,

$$F_Y(0) = P(Y \le 0) = P(\max(X,0) = 0) = P(-3 \le X \le 0) = \frac{3}{4}$$

For $0 < y < 1$,

$$F_Y(y) = P(Y \le y) = P(\max(X,0) \le y) = \frac{3}{4} + P(0 \le X \le y) = \frac{3}{4} + \int_0^y \frac{1}{4}\, dy = \frac{3}{4} + \frac{y}{4}$$

Therefore, CDF of $Y, F_Y(y) = \begin{cases} 0, & 0 < y \\ \dfrac{3}{4} + \dfrac{y}{4}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$.

$\max(X, 0)$ is not monotonic in the given support and hence the second method is not applicable here.

2.6 Expected value

As we have seen previously in case of discrete random variable, expected value or expected value of a function of a random variable gives useful information about the distribution. Mean value is like the central value of the distribution. Variance give the spread around the central value. In many cases, we may not know the distribution, we will only have the data from the sample, and that data may not be rich enough to learn the entire distribution. So, to an extent expected value can be very useful when we observe a random phenomenon like this.

So, in theory, whatever we studied for the discrete random variable, we will extend it here to the continuous context.

Theorem: Let X be a continuous random variable with density $f_X(x)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a function. The expected value of $g(X)$, denoted $E[g(X)]$, is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx,$$

whenever the above integral exists.

It is enough if we restrict the integration limit to the support of X, because outside the support, the PDF is 0.

2.6.1 Mean and variance

Let X be a continuous random variable with PDF $f_X(x)$.

- Mean, denoted $E[X]$ or μ_X or μ.

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

  Here, $g(X) = X$.

- Variance, denoted $\text{Var}(X)$ or σ_X^2 or σ^2.

$$\text{Var}(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx$$

  Here, $g(X) = (X - \mu_X)^2$

  Result: $\text{Var}(X) = E[X^2] - (E[X])^2$

  Proof:

$$\begin{aligned}\text{Var}(X) =& E[(X - \mu)^2] \\ =& E[X^2 + \mu^2 - 2\mu X] \\ =& E[X^2] + \mu^2 - 2\mu E[X] \\ =& E[X^2] - (E[X])^2\end{aligned}$$

2.6.2 Examples of mean and variance

1. Uniform distribution:
   $X \sim \text{Uniform}[a, b], f_X(x) = \dfrac{1}{b-a},\ \ a \leq x \leq b.$

   - $E[X]$

$$\begin{aligned}E[X] =& \int_a^b x \frac{1}{b-a}\, dx \\ =& \frac{1}{b-a}.\frac{x^2}{2}\Big|_a^b \\ =& \frac{1}{b-a}.\frac{(b^2-a^2)}{2} \\ =& \frac{1}{b-a}.\frac{(b-a)(b+a)}{2} \\ =& \frac{a+b}{2}\end{aligned}$$

- $\text{Var}(X) = E[X^2] - (E[X])^2$

$$
\begin{aligned}
E[X^2] &= \int_a^b x^2 \frac{1}{b-a}\ dx \\
&= \frac{1}{b-a} . \frac{x^3}{3}\bigg|_a^b \\
&= \frac{1}{b-a} . \frac{(b^3 - a^3)}{3} \\
&= \frac{1}{b-a} . \frac{(b-a)(b^2 + a^2 + ab)}{3} \\
&= \frac{b^2 + a^2 + ab}{3}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\text{Var}(X) &= \frac{b^2 + a^2 + ab}{3} - \left(\frac{a+b}{2}\right)^2 \\
&= \frac{b^2 + a^2 + ab}{3} - \frac{a^2 + b^2 + 2ab}{4} \\
&= \frac{4b^2 + 4a^2 + 4ab - 3b^2 - 3a^2 - 6ab}{12} \\
&= \frac{b^2 + a^2 - 2ab}{12} = \frac{(b-a)^2}{12}
\end{aligned}
$$

2. <u>Exponential distribution:</u>
$X \sim \text{Exponential}(\lambda), f_X(x) = \lambda e^{-\lambda x},\ \ x \geq 0.$

- $E[X]$

$$
\begin{aligned}
E[X] &= \int_0^\infty x \lambda e^{-\lambda x}\ dx \\
&= \lambda \left(x \int_0^\infty e^{-\lambda x}\ dx + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x}\ dx \right) \\
&= \lambda \left(-\frac{x}{\lambda} e^{-\lambda x} \bigg|_0^\infty - \frac{1}{\lambda^2} e^{-\lambda x} \bigg|_0^\infty \right) && \dots (2) \\
&= 0 - \frac{1}{\lambda}(0 - 1) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

- $\text{Var}(X) = E[X^2] - (E[X])^2$

$$\begin{aligned}
E[X^2] = &\int_0^\infty x^2 \lambda e^{-\lambda x}\ dx \\
=&\lambda \left(x^2 \int_0^\infty e^{-\lambda x}\ dx + \frac{1}{\lambda} \int_0^\infty 2x.e^{-\lambda x}\ dx \right) \\
=&\lambda \left[-x^2 e^{-\lambda x} \Big|_0^\infty + \frac{2}{\lambda} \left(\frac{1}{\lambda^2} \right) \right] && \text{[Using (2)]} \\
=&\frac{2}{\lambda^2}
\end{aligned}$$

Now,

$$\begin{aligned}
\text{Var}(X) =&\frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 \\
=&\frac{1}{\lambda^2}
\end{aligned}$$

3. Normal distribution:

$$X \sim \text{Normal}(\mu, \sigma^2), f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2$$

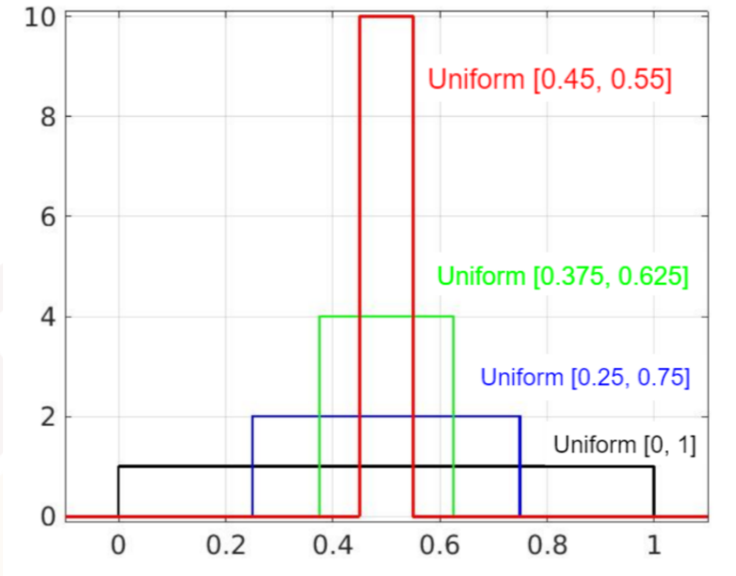
- $E[X] = \mu$
- $\text{Var}[X] = \sigma^2$

2.6.3 Uniform distribution with different variances

We will see here how the variance of the Uniform distribution decreases with the decrease in the width of the distribution, where the mean is the same for all the distributions.

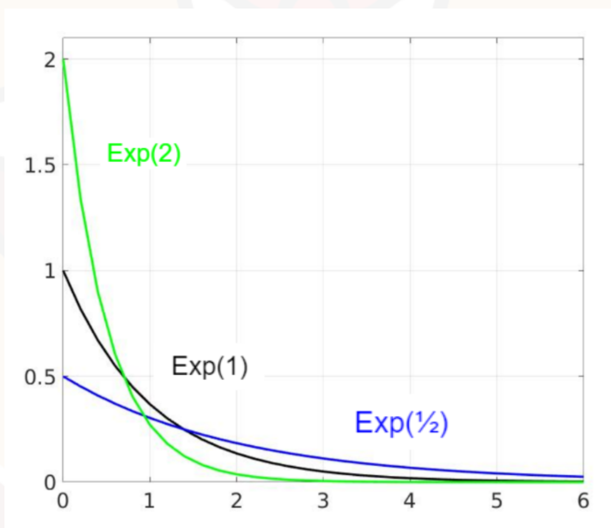
For the black one, the width is 1 and the height of the distribution is 1. For the blue one, the width is 0.5 and the height of the distribution is 2. For the green one, the width is 0.25 and the height of the distribution is 4. For the red one, the width is 0.125 and the height of the distribution is 10.

$$\text{Var}(\text{Uniform}[0, 1]) = 1/12$$
$$\text{Var}(\text{Uniform}[0.25, 0.75]) = 0.25/12$$
$$\text{Var}(\text{Uniform}[0.375, 0.625]) = 0.0625/12$$
$$\text{Var}(\text{Uniform}[0.45, 0.55]) = 0.01/12$$



Uniform distribution with different variances

2.6.4 Exponential distribution with different λ

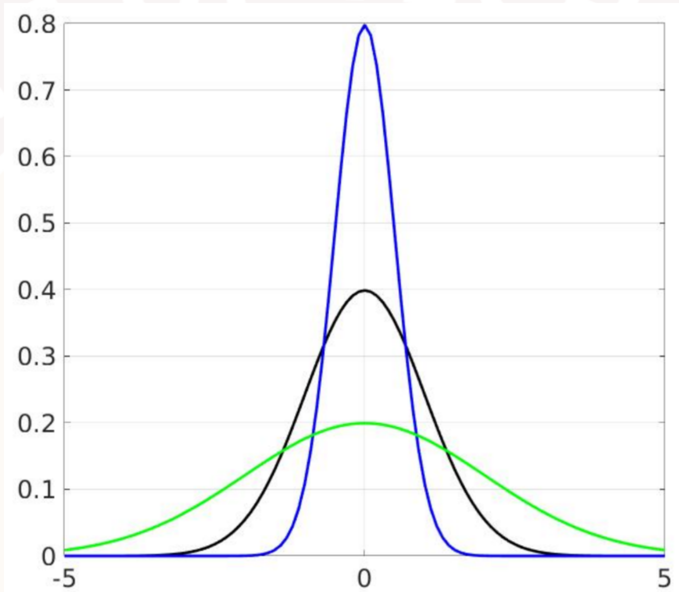


We can observe that for Exp(2), the mean has shifted to the left and it takes lower values with higher probability, while for Exp(1/2), the mean has shifted to the right and it takes higher values with low probability with respect to Exp(1).

For Exp(2), Exp(1) and Exp(1/2), the variances are $\frac{1}{2}$, 1 and 2, respectively.

2.6.5 Normal distribution with different σ

We have fixed the mean at 0 and varying σ. The distribution in black is Normal(0, 1). For the green one, variance is greater than 1. For the blue one, variance is less than 1.



2.6.6 Markov and Chebyshev's inequalities

We use Markov and Chebyshev inequality to compute the bounds on the probabilities that a random variable deviates too much from the mean. We get the proabability bound using the mean and variance. Markov and Chebyshev's inequalities carry over to an arbitrary distribution as long as mean and variance exists for that random variable. This holds for any random variable in general.

1. Markov inequality
   Let X be a continuous random variable with mean μ, where Supp(X) is non-negative, i.e., $P(X < 0) = 0$. Then,
   $$P(X > c) \leq \frac{\mu}{c}$$

2. Chebyshev inequality
   Let X be a continuous random variable with mean μ and variance σ^2. Then,
   $$P(\mid X - \mu \mid \geq k\sigma) \leq \frac{1}{k^2}$$

This can also be written as

$$P(| X - \mu | \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

<u>Proof:</u>

$$P(| X - \mu | \leq k\sigma) = 1 - P(| X - \mu | \geq k\sigma) \qquad \ldots(1)$$

From Chebyshev inequality, we know that

$$P(| X - \mu | \geq k\sigma) \leq \frac{1}{k^2}$$

$$\implies -P(| X - \mu | \geq k\sigma) \geq -\frac{1}{k^2}$$

Therefore, (1) implies $P(| X - \mu | \leq k\sigma) \geq 1 - \frac{1}{k^2}$.

2.6.7 Probability space and its axioms

1. Discrete case
   - Sample space: finite or countable set.
   - Events: power set of sample space.
   - Probability function: PMF
2. Continuous case
   - Sample space: interval of real line.
   - Events: intervals in the sample space along with their complements and countable unions.
   - Probability function: function from intervals inside sample space to [0, 1].

2.6.8 Examples:

(a) A continuous random variable X has PDF

$$f_X(x) = \begin{cases} 1 - | x |, & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the CDF of $X, E[X]$ and $\text{Var}(X)$.

<u>Solution:</u>

For $x < -1$, $F_X(x) = 0$

For $-1 < x < 0$,

$$F_X(x) = \int_{-1}^{x} (1 - \mid u \mid)\ du$$
$$= \int_{-1}^{x} (1 + u)\ du$$
$$= u + \frac{u^2}{2}\Big|_{-1}^{x}$$
$$= x + \frac{x^2}{2} - (-1 + (1/2)) = x + \frac{x^2}{2} + \frac{1}{2}$$

For $0 < x < 1$,

$$F_X(x) = \int_{-1}^{0} (1 - \mid u \mid)\ du + \int_{0}^{x} (1 - \mid u \mid)\ du$$
$$= \int_{-1}^{0} (1 + u)\ du + \int_{0}^{x} (1 - u)\ du$$
$$= \frac{1}{2} + \left(u - \frac{u^2}{2}\Big|_{0}^{x} \right)$$
$$= \frac{1}{2} + x - \frac{x^2}{2}$$

(b) A continuous random variable X has PDF

$$f_X(x) = \begin{cases} \frac{1}{2}\cos x, & -\pi/2 \leq x \leq \pi/2 \\ 0, & \text{otherwise} \end{cases}$$

Find the CDF of $X, E[X]$ and $\text{Var}(X)$.

Solution:

For $x < -\pi/2$, $F_X(x) = 0$

For $-\pi/2 < x < \pi/2$,

$$
\begin{aligned}
F_X(x) &= \int_{-\pi/2}^{x} \frac{1}{2}\cos u \; du \\
&= \left.\frac{\sin u}{2}\right|_{-\pi/2}^{x} \\
&= \frac{\sin x}{2} - \frac{\sin(-\pi/2)}{2} = \frac{1+\sin x}{2}
\end{aligned}
$$

Therefore, the CDF of X is

$$
F_X(x) = \begin{cases} 0, & \text{for} x < -\pi/2 \\ \dfrac{1+\sin x}{2}, & \text{for} -\pi/2 \leq x < \pi/2 \\ 1, & \text{for} x \geq \pi/2 \end{cases}
$$

Expected value of X is

$$
\begin{aligned}
E[X] &= \int_{-\pi/2}^{\pi/2} \frac{1}{2} x\cos x \; dx \\
&= \frac{1}{2}\left(x \int_{-\pi/2}^{\pi/2} \cos x \; dx - 1. \int_{-\pi/2}^{\pi/2} \sin x \; dx \right) \\
&= \frac{1}{2}\left(x\sin x \Big|_{-\pi/2}^{\pi/2} + \cos x \Big|_{-\pi/2}^{\pi/2} \right) \\
&= \frac{1}{2}[\pi/2 - (\pi/2)] = 0
\end{aligned}
$$

Expected value of X^2 is

$$
\begin{aligned}
E[X^2] =& \int_{-\pi/2}^{\pi/2} \frac{1}{2}x^2 \cos x \; dx \\
=& \frac{1}{2}x^2 \int_{-\pi/2}^{\pi/2} \cos x \; dx - \frac{1}{2} \int_{-\pi/2}^{\pi/2} 2x.\sin x \; dx \\
=& \frac{1}{2}x^2 \sin x \Big|_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} x.\sin x \; dx \\
=& \frac{1}{2}x^2 \sin x \Big|_{-\pi/2}^{\pi/2} - \left(x \int_{-\pi/2}^{\pi/2} \sin x \; dx + 1. \int_{-\pi/2}^{\pi/2} \cos x \; dx \right) \\
=& \frac{1}{2}x^2 \sin x \Big|_{-\pi/2}^{\pi/2} + x \cos x \Big|_{-\pi/2}^{\pi/2} - \sin x \Big|_{-\pi/2}^{\pi/2} \\
=& \frac{1}{2}\left(\pi^2/4 + \pi^2/4\right) + 0 - 2 \\
=& \frac{\pi^2}{4} - 2
\end{aligned}
$$

Therefore, $\mathrm{Var}(X) = \dfrac{\pi^2}{4} - 2$

2.7 Joint distributions: Discrete and continuous

In this section we are going to look at how to describe joint distributions between two random variables, one of them is discrete and the other is continuous. There are various methods to describe the joint distribution between a discrete and a continuous random variable but we are going to look one of the methods here, which is simple to use in practice.

Let us say X and Y are jointly distributed, where X is discrete with range T_X and PMF $p_X(x)$. To understand the joint distribution for every value that X take, we can actually have a continuous random variable Y_x with density $f_{Y_x}(y)$, where Y_x can be read as Y given x, denoted by $Y|X = x$.

Now, marginal density of Y can be given as,

$$f_Y(y) = \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y)$$

2.7.0.1 Solved Examples:

Q1. Let $X \sim \text{Uniform}\{0, 1, 2\}$. Let $Y|X = 0 \sim \text{Normal}(5, 0.4^2)$, $Y|X = 1 \sim \text{Normal}(6, 0.5^2)$ and $Y|X = 2 \sim \text{Normal}(7, 0.6^2)$. What is the marginal density of Y?

Solution:
Given, $X \sim \text{Uniform}\{0, 1, 2\}$, which means each values that X take has a probability of $\frac{1}{3}$.
Given, $Y|X = 0 \sim \text{Normal}(5, 0.4^2)$, the distribution can be written as

$$Y|(X = 0) = \frac{1}{\sqrt{2\pi \times 0.4^2}} e^{\frac{-(y-5)^2}{2\times 0.4^2}}$$

Similarly, for $Y|X = 1 \sim \text{Normal}(6, 0.5^2)$, the distribution can be written as

$$Y|(X = 1) = \frac{1}{\sqrt{2\pi \times 0.5^2}} e^{\frac{-(y-6)^2}{2\times 0.5^2}}$$

Similarly, for $Y|X = 2 \sim \text{Normal}(7, 0.6^2)$, the distribution can be written as

$$Y|(X = 2) = \frac{1}{\sqrt{2\pi \times 0.6^2}} e^{\frac{-(y-7)^2}{2\times 0.6^2}}$$

Now, the marginal density of Y is

$$f_Y(y) = \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y)$$

$$f_Y(y) = \frac{1}{3}\frac{1}{\sqrt{2\pi \times 0.4^2}} e^{\frac{-(y-5)^2}{2\times 0.4^2}} + \frac{1}{3}\frac{1}{\sqrt{2\pi \times 0.5^2}} e^{\frac{-(y-6)^2}{2\times 0.5^2}} + \frac{1}{3}\frac{1}{\sqrt{2\pi \times 0.6^2}} e^{\frac{-(y-7)^2}{2\times 0.6^2}}$$

Remember, the conditional densities are all Gaussian, but the marginal is not Gaussian. Gaussian cannot have three different peaks. This kind of a distribution is called a mixture Gaussian. So, this is called mixture of Gaussians. The commonly used terms for this kind of distribution are, mixture of Gaussian or Gaussian mixture or mixture Gaussian. The Gaussian mixtures quite often in practice end up being very popular.

2.8 Conditional probability of discrete given continuous

Suppose X and Y are jointly distributed with $X \in T_X$ being discrete with PMF $p_X(x)$ and conditional densities $f_{Y|X=x}(y)$ for $x \in T_X$. The conditional probability of X given $Y = y_0 \in \text{supp}(Y)$ is defined as

$$P(X = x|Y = y_0) = \frac{p_X(x) f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

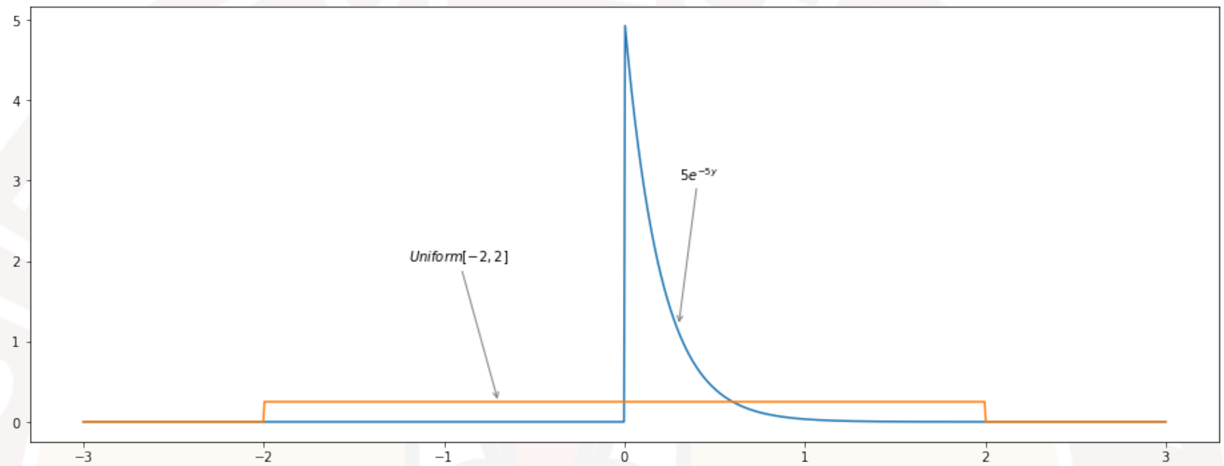
where f_Y is the marginal density of Y and $f_{Y|X}$ is the conditional density.

Since Y is a continuous random variable, we can think of $Y = y_0$ as Y taking the values around y_0.

2.8.0.1 Solved Examples:

Q1. Let $X \sim \text{Uniform}\{-1, 1\}$. Let $Y|X = -1 \sim \text{Uniform}[-2, 2]$, $Y|X = 1 \sim \text{Exp}(5)$. Find the distribution of X given $Y = -1$, $Y = 1$, $Y = 3$.

Solution:



Given, $X \sim \text{Uniform}\{-1, 1\}$, which means each value in the range of X has a probability of $\frac{1}{2}$.
Given, $(Y|X = -1) \sim \text{Uniform}[-2, 2]$, the distribution can be written as

$$P(Y|X = -1) = \frac{1}{2 - (-2)} = \frac{1}{4}$$

Given, $(Y|X = 1) \sim \text{Exp}(5)$, the distribution can be written as

$$P(Y|X = 1) = 5e^{-5y}$$

Now, the marginal density of Y is

$$f_Y(y) = \sum_{x \in T_X} p_X(x) f_{Y|X=x}(y)$$
$$f_Y(y) = \frac{1}{2} f_{Y|X=-1}(y) + \frac{1}{2} f_{Y|X=1}(y)$$
$$f_Y(y) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right) 5e^{-5y}$$

The marginal density can be written as,

$$f_Y(y) = \begin{cases} 0 & y < -2 \\ 1/8 & -2 < y < 0 \\ 1/8+(1/2)5e^{-5y} & 0 < y < 2 \\ (1/2)5e^{-5y} & y > 2 \end{cases}$$

The conditional density of X given $Y = -1$,

$$P(X = -1|Y = -1) = \frac{p_X(-1)f_{Y|X=-1}(-1)}{f_Y(-1)} = \frac{1/2 \times 1/4}{1/2 \times 1/4} = 1$$

$$P(X = 1|Y = -1) = \frac{p_X(1)f_{Y|X=1}(-1)}{f_Y(-1)} = \frac{1/2 \times 0}{1/2 \times 1/4} = 0$$

Notice that $P(X = 1|Y = -1)$ is $1 - P(X = -1|Y = -1)$.

Similarly, the conditional density of X given $Y = 1$,

$$P(X = -1|Y = 1) = \frac{p_X(-1)f_{Y|X=-1}(1)}{f_Y(1)} = \frac{1/2 \times 1/4}{1/2 \times 1/4 + 1/2 \times 5e^{-5}} = 0.8812$$

$$P(X = 1|Y = 1) = \frac{p_X(1)f_{Y|X=1}(1)}{f_Y(1)} = \frac{1/2 \times 5e^{-5}}{1/2 \times 1/4 + 1/2 \times 5e^{-5}} = 0.1188$$

Notice that $P(X = 1|Y = 1)$ is $1 - P(X = -1|Y = 1)$.

Q2. Suppose 60% of adults in the age group of 45-50 in a country are male and 40% are female. Suppose the height (in *cm*) of adult males in that age group in the country is Normal(160, 100), and that of females is Normal(150, 25). A random person is found to have a height of 155 *cm*. Is that person more likely to be male or female?

Solution:

Suppose X denote the proportion of adults in the age group of 45-50 in a country.
Let male is represented by M and female represented by F.
Let Y represent the height of an adult.

The PMF of X is given by,

$$f_X(x) = \begin{cases} 0.6 & \text{if } M \\ 0.4 & \text{if } F \end{cases}$$

From the information given in the question, $Y|(X = M) \sim \text{Normal}(160, 100)$ whose distribution can be written as,

$$f_{Y|X=M}(y) = \frac{1}{10\sqrt{2\pi}} e^{\frac{-(y-160)^2}{2\times 100}}$$

Similarly, $Y|(X = F) \sim \text{Normal}(150, 25)$ whose distribution can be written as

$$f_{Y|X=F}(y) = \frac{1}{5\sqrt{2\pi}} e^{\frac{-(y-150)^2}{2\times 25}}$$

Now, the marginal density of Y can be written as

$$f_Y(y) = \sum_{x\in T_X} p_X(x) f_{Y|X=x}(y)$$
$$f_Y(y) = 0.6 f_{Y|X=M}(y) + 0.4 f_{Y|X=F}(y)$$
$$f_Y(y) = 0.6 \times \frac{1}{10\sqrt{2\pi}} e^{\frac{-(y-160)^2}{2\times 100}} + 0.4 \times \frac{1}{5\sqrt{2\pi}} e^{\frac{-(y-150)^2}{2\times 25}}$$

We have to find the conditional probability that given a height of a randomly chosen person is 155 cm, the person is male or female.
So, we can find a conditional probability for either a male or female.

$$\begin{aligned}
P(X = M|Y = 155) &= \frac{p_X(M) f_{Y|X=M}(155)}{f_Y(155)} \\
&= \frac{0.6 \times \frac{1}{10\sqrt{2\pi}} e^{\frac{-(155-160)^2}{2\times 100}}}{0.6 \times \frac{1}{10\sqrt{2\pi}} e^{\frac{-(155-160)^2}{2\times 100}} + 0.4 \times \frac{1}{5\sqrt{2\pi}} e^{\frac{-(155-150)^2}{2\times 25}}} \\
&= 0.6857
\end{aligned}$$

Now, $P(X = F|Y = 155) = 1 - P(X = M|Y = 155) = 1 - 0.6857 = 0.3143.$

So, the randomly chosen person with height of 155 cm is more likely to be male.

Q3. Let $Y = X + Z$, where $X \sim \text{Uniform}\{-3, -1, 1, 3\}$ and $Z \sim \text{Normal}(0, \sigma^2)$ are independent. What is the distribution of Y? Find the distribution of $(X|Y = 0.5)$.

Solution:

Given, $Y = X + Z$, $X \sim \text{Uniform}\{-3, -1, 1, 3\}$ and $Z \sim N(0, \sigma^2)$
where X and Z are independent.
To find:

(a) The distribution of Y i.e.,$f_Y(y)$.

(b) The distribution of $(X|Y = 0.5)$ i.e., $f_{X|Y=0.5}$

$(Y|X = -3) = -3 + Z$, where Z have a normal distribution with mean equal to 0 and variance σ^2.
Using the property of translation, Y will also have a normal distribution with mean equal to -3 and variance 0.

Therefore, the distribution of $Y|X = -3$ i.e. $f_{Y|X=-3}(y) \sim \text{Normal}(-3, \sigma^2)$

Similarly, the distribution of $Y|X = -1$ i.e. $f_{Y|X=-1}(y) \sim \text{Normal}(-1, \sigma^2)$

Distribution of $Y|X = 1$ i.e. $f_{Y|X=1}(y) \sim \text{Normal}(1, \sigma^2)$

Distribution of $Y|X = 3$ i.e. $f_{Y|X=3}(y) \sim \text{Normal}(3, \sigma^2)$

Hence,

$$
\begin{aligned}
f_Y(y) =& f_{Y|X=-3}(y)f_X(-3) + f_{Y|X=-1}(y)f_X(-1) + f_{Y|X=1}(y)f_X(1) + f_{Y|X=3}(y)f_X(3) \\
=& \frac{1}{4}\frac{1}{\sigma\sqrt{2\pi}}\left(e^{-\dfrac{(y+3)^2}{2\sigma^2}} + e^{-\dfrac{(y+1)^2}{2\sigma^2}} + e^{-\dfrac{(y-1)^2}{2\sigma^2}} + e^{-\dfrac{(y-3)^2}{2\sigma^2}}\right)
\end{aligned}
$$

Distribution of $(X|Y = 0.5)$ is

$$f_{X|Y=0.5}(x) = \frac{f_{Y|X=x}(0.5)f_X(x)}{f_Y(0.5)}$$

where $x \in \{-3, -1, 1, 3\}$ and

$$f_Y(0.5) = \frac{1}{\sigma\sqrt{32\pi}} \exp\left(\frac{-21}{2 \times \sigma^2}\right)$$

2.9 Problems

1. Let $X \sim \text{Exp}[2]$. Define $Y = X + 5$. Find the PDF of Y.

(a)

$$f_Y(y) = \begin{cases} 2e^{-2(y+5)} & \text{for } 5 < y \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} 2e^{-2(y-5)} & \text{for } 5 < y \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} e^{-2(y-5)} & \text{for } 5 < y \\ 0 & \text{otherwise} \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} 5e^{-5(y-2)} & \text{for } 5 < y \\ 0 & \text{otherwise} \end{cases}$$

2. Let $X \sim$ Uniform[0, 2]. Define $Y = 2X$. What is the distribution of Y?

   (a) $Y \sim$ Uniform[0, 2]
   (b) $Y \sim$ Uniform[0, 1]
   (c) $Y \sim$ Uniform[0, 4]
   (d) $Y \sim \text{Uniform}\left[0, \frac{1}{2}\right]$

3. Let $X \sim \text{Uniform}(0, 1)$. Let $Y = a + (b - a)X$, where $a < b$. Find the distribution of Y.

   (a) $Y \sim \text{Uniform}(a, b - a)$
   (b) $Y \sim \text{Uniform}(0, 1)$
   (c) $Y \sim \text{Uniform}(a, b)$
   (d) $Y \sim \text{Uniform}(b - a, b)$

4. Let $X \sim$ Uniform $(0, 1)$. Let $Y = -\frac{1}{2}\ln(1 - X)$. Find the distribution of Y.

   (a) $Y \sim \text{Exp}(1)$
   (b) $Y \sim \text{Exp}\left(\frac{1}{2}\right)$
   (c) $Y \sim \text{Exp}(2)$
   (d) $Y \sim \text{Exp}(4)$

5. Let $X \sim \text{Normal}(1, 9)$. Let $Z = \frac{X-1}{3}$. Find $P(X^2 > 9)$.

   (a) $1 - \left[F_Z\left(\frac{4}{3}\right) - F_Z(-\frac{2}{3})\right]$
   (b) $F_Z\left(\frac{2}{3}\right) + F_Z\left(\frac{4}{3}\right)$

(c) $F_Z\left(\frac{2}{3}\right) - F_Z\left(-\frac{4}{3}\right)$

(d) $1 - \left[F_Z\left(\frac{2}{3}\right) - F_Z\left(-\frac{4}{3}\right)\right]$

6. Let $X \sim$ Uniform [0, 20]. Find the CDF of $Y = \max\{4, \min\{8, X\}\}$.

(a)

$$F_Y(y) = \begin{cases} 0 & \text{for } -\infty < y < 4 \\ \frac{y}{20} + \frac{4}{20} & \text{for } 4 \leq y < 8 \\ 1 & \text{for } y \geq 8 \end{cases}$$

(b)

$$F_Y(y) = \begin{cases} 0 & \text{for } -\infty < y < 4 \\ \frac{y}{20} - \frac{4}{20} & \text{for } 4 \leq y < 8 \\ 1 & \text{for } y \geq 8 \end{cases}$$

(c)

$$F_Y(y) = \begin{cases} 0 & \text{for } -\infty < y < 4 \\ \frac{y}{20} & \text{for } 4 \leq y < 8 \\ 1 & \text{for } y \geq 8 \end{cases}$$

7. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} ax + bx^2 & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X) = 0.5$, find $\text{Var}(X)$.

(a) $\text{Var}(Y) = 0.10$

(b) $\text{Var}(Y) = 0.05$

(c) $\text{Var}(Y) = 0.3$

(d) $\text{Var}(Y) = 0.5$

8. Let a random variable X has the following PDF:

$$f_X(x) = \frac{1}{2}(\lambda e^{-\lambda|x|})$$

for $-\infty < x < \infty$. Find the CDF of X.

(a)

$$F_X(x) = \begin{cases} \dfrac{e^{\lambda x}}{2} & \text{for } x < 0 \\ 1 - \dfrac{e^{-\lambda x}}{2} & \text{for } x \geq 0 \end{cases}$$

(b)

$$F_X(x) = \begin{cases} \dfrac{e^{\lambda x}}{2} & \text{for } x < 0 \\ \dfrac{e^{-\lambda x}}{2} & \text{for } x \geq 0 \end{cases}$$

(c)

$$F_X(x) = \begin{cases} \dfrac{e^{\lambda x}}{2} & \text{for } x < 0 \\ \dfrac{1}{2} + \dfrac{e^{-\lambda x}}{2} & \text{for } x \geq 0 \end{cases}$$

9. The waiting time (in minutes) of a bus at a certain bus stop is uniformly distributed between 9:00 PM to 9:30 PM. You arrive at the bus stop at 9:00 PM.

   (i) What is the probability that you have to wait longer than 5 minutes?(Correct to 2 decimal palces)

   (ii) What is the expected waiting time(in minutes)?

10. Let $X \sim$ Uniform$[0, 1]$.
    Find a bound on $P\left(\mid X - \frac{1}{2} \mid\geq 2\sqrt{\frac{1}{12}}\right) = p$ using Chebyshev's inequality.

    (a) $p < \dfrac{1}{4}$

    (b) $p \leq \dfrac{1}{4}$

    (c) $p \geq \dfrac{1}{4}$

    (d) $p > \dfrac{3}{4}$

11. Let $X \sim$ Uniform[0,2]. Define $Y = e^X$. Find the PDF of Y. (Hint: Use monotonic differentiable functions)

    (a)

$$f_Y(y) = \begin{cases} e^y & \text{for } 1 < x < e^2 \\ 0 & \text{Otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} \dfrac{1}{2y} & \text{for } 1 < x < e^2 \\ 0 & \text{Otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} \dfrac{1}{2y} & \text{for } 0 < x < 2 \\ 0 & \text{Otherwise} \end{cases}$$

(d)

$$f_Y(y) = \begin{cases} e^2 & \text{for } 1 < x < e^2 \\ 0 & \text{Otherwise} \end{cases}$$

12. Let $Y = XZ$ where $X \sim \text{Uniform}\{-1, 1\}$ and $Z \sim \text{Normal}\ (10, 4)$ are independent.

    (i) Find $f_Y(1)$.

    (ii) Find $f_{X|Y=1}(1)$.

13. Let $X \sim \text{Binomial}(2, 0.5), (Y \mid X = 0) \sim \text{Exp}(2), (Y \mid X = 1) \sim \text{Normal}(0, 4), (Y \mid X = 2) \sim \text{Uniform}(0, 1)$. Find $f_Y(y)$.

Chapter 6

3 Jointly continous random variables

For every joint density $f(x, y)$, there exists two jointly distributed continuous random variables X and Y such that, for any two-dimensional region A,

$$P((X, Y) \in A) = \iint_A f(x, y)\, dx\, dy$$

where $f(x, y)$, also denoted $f_{XY}(x, y)$, is called the joint density of X and Y
And the support of X, Y will be $\text{supp}(X, Y) = \{(x, y) : f_{XY}(x, y) > 0\}$

3.1 2D uniform distribution

Fix some (reasonable) region D in R^2 with total area $|D|$. We say that $(X, Y) \sim \text{Uniform}(D)$ if they have the joint density

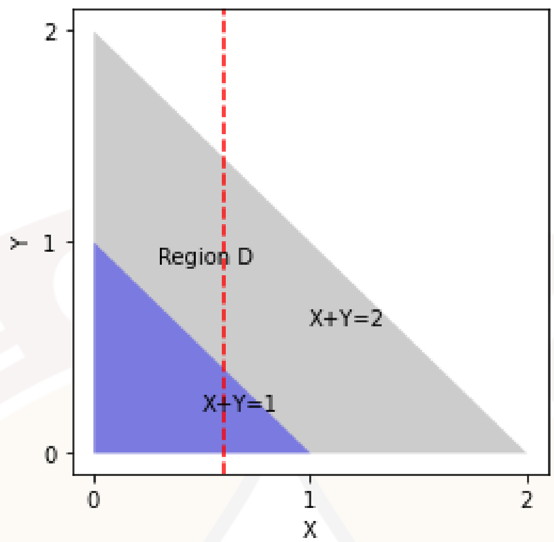
$$f_{XY}(x, y) = \begin{cases} \dfrac{1}{|D|} & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

- Rectangle: $D = [a, b] \times [c, d] = \{(x, y) : a < x < b, c < y < d\}$
- Circle: $D = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 \leq r^2\}$
- Multiple disjoint areas and so many other possibilities
- For any sub-region A of D, $P((X, Y) \in A) = |A|/|D|$
- Uniform distribution is a good approximation for flat histograms

3.1.0.1 Solved Examples:

Q1. Let $(X, Y) \sim \text{Uniform}(D)$, where $D = \{(x, y) : x + y < 2, x > 0, y > 0\}$. Sketch the support and compute $P(X + Y < 1), P(X + 2Y > 1)$.
Solution:

The grey shaded region represents support for region D and the blue shaded region shows the sub-region A where we have to compute the probability.
Method 1:



Support for region D

Since X and Y are uniform in the region D,

$$\begin{aligned}
P((X,Y) \in A) &= |A|/|D| \\
&= \frac{\frac{1}{2} \times 1 \times 1}{\frac{1}{2} \times 2 \times 2} \\
&= \frac{1}{4} \\
&= 0.25
\end{aligned}$$

<u>Method 2:</u>
The blue shaded region is the region of interest, A. The dotted vertical line represents the y-strip, from which we can find the limits of integration.
Limits of integration for y are from 0 to $1 - x$ where the dotted line cuts the line, $x + y = 1$ and the limits of integration for x are from 0 to 1.
The joint density, f_{XY} of X and Y is given by,

$$\begin{aligned}
f_{XY}(x,y) &= \frac{1}{|D|} \\
&= \frac{1}{\frac{1}{2} \times 2 \times 2} \\
&= \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
P((X,Y) \in A) &= \int_{x=0}^{1} \int_{y=0}^{y=1-x} f_{XY}(x,y)\,dy\,dx \\
&= \int_{x=0}^{1} \int_{y=0}^{y=1-x} \frac{1}{2}\,dy\,dx \\
&= \int_{x=0}^{1} \frac{1}{2} \left(y \Big|_{0}^{1-x} \right) \; dx \\
&= \int_{x=0}^{1} \frac{1}{2}(1-x)\; dx \\
&= \frac{1}{2} \int_{x=0}^{1} (1-x)\; dx \\
&= \frac{1}{2} \left(x - \frac{x^2}{2} \right) \Bigg|_{0}^{1} \\
&= \frac{1}{2} \left[(1-0) - (\frac{1^2}{2} - 0) \right] \\
&= \frac{1}{4} \\
&= 0.25
\end{aligned}
$$

Q2. Let (X,Y) have joint density

$$
f_{XY}(x,y) = \begin{cases} x+y, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases}
$$

Show that the above is a valid density. Find $P(X < 1/2 \text{ and } Y < 1/2)$, $P(X+Y < 1)$.

Solution:
We know, any PDF will be valid if,

$$
\int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} f_{XY}(x,y)\,dy\,dx = 1
$$

$$
\begin{aligned}
\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}(x+y)\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{y=1}(x+y)\,dy\,dx \\
&= \int_{x=0}^{1}\left(xy+\frac{y^2}{2}\right)\Bigg|_0^1 dy\,dx \\
&= \int_{x=0}^{1}\left(x+\frac{1}{2}\right)dx \\
&= \left(\frac{x^2}{2}+\frac{x}{2}\right)\Bigg|_0^1 \\
&= 1
\end{aligned}
$$

Hence, given joint density is a valid density.

- Find $P(X < 1/2 \text{ and } Y < 1/2)$.
  Solution:

$$
\begin{aligned}
P(X < 1/2 \text{and} Y < 1/2) &= \int_{x=0}^{1/2}\int_{y=0}^{y=1/2}(x+y)\,dy\,dx \\
&= \int_{x=0}^{1/2}\left(xy+\frac{y^2}{2}\right)\Bigg|_0^{1/2} dy\,dx \\
&= \int_{x=0}^{1/2}\left(\frac{x}{2}+\frac{1}{8}\right)dx \\
&= \left(\frac{x^2}{2}+\frac{x}{8}\right)\Bigg|_0^{1/2} \\
&= \left(\frac{1}{8}+\frac{1}{16}\right) \\
&= \left(\frac{3}{16}\right)
\end{aligned}
$$

- Find $P(X+Y<1)$.
  Solution:

$$
\begin{aligned}
P(X+Y<1) &= \int_{x=0}^{1}\int_{y=0}^{y=1-x}(x+y)\,dy\,dx \\
&= \int_{x=0}^{1}\int_{y=0}^{y=1-x}(x+y)\,dy\,dx \\
&= \int_{x=0}^{1}\left(xy+\frac{y^2}{2}\right)\Big|_0^{1-x}\,dx \\
&= \int_{x=0}^{1}\left(xy+\frac{y^2}{2}\right)\Big|_0^{1-x}\,dx \\
&= \int_{x=0}^{1} x(1-x)+\frac{(1-x)^2}{2}\,dx \\
&= \int_{x=0}^{1} x-x^2+\frac{1-2x+x^2}{2}\,dx \\
&= \int_{x=0}^{1} \frac{1-x^2}{2}\,dx \\
&= \frac{1}{2}\int_{x=0}^{1} 1-x^2\,dx \\
&= \frac{1}{2}\left(x-\frac{x^3}{3}\right)\Big|_0^{1}\,dx \\
&= \frac{1}{2}\left(1-\frac{1}{3}\right) \\
&= \frac{1}{3}
\end{aligned}
$$

3.2 Marginal density

Suppose (X, Y) have joint density $f_{XY}(x, y)$. Then,

- X has the marginal density $= f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y)dy$
- Y has the marginal density $= f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x, y)dx$
- The PDF of X and Y individually are called marginal densities.
- The joint density exactly determines both the marginal densities.

3.2.0.1 Examples

Q1. Consider X and Y are uniform in the unit square. Find the marginal densities of X and Y.

Solution

Since X and Y are uniform in unit square, the joint density of X and Y is given by,

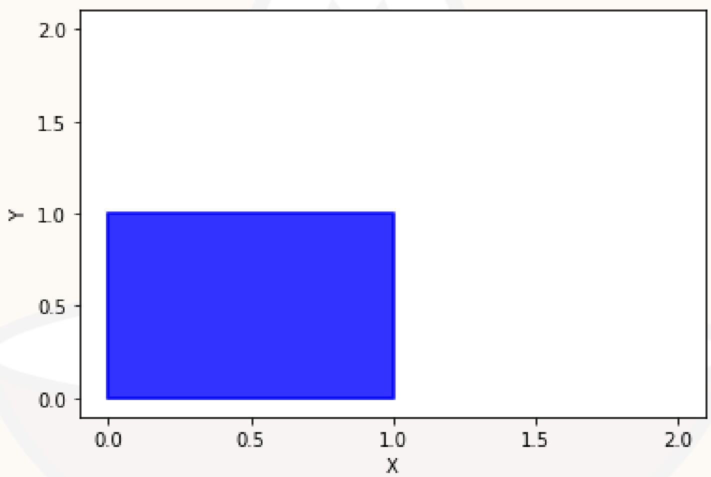
$$f_{XY}(x,y) = \begin{cases} 1 & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The marginal density of X is given by,

$$f_X(x)$$

1. Consider $(X,Y) \sim \text{Uniform}(D)$, where,

The support for region D is shown by the blue shaded region in the following figure.



Support for region D

Q2. $(X,Y) \sim \text{Uniform}(D)$, where $D = [0,1/2] \times [0,1/2] \cup [1/2,1] \times [1/2,1]$

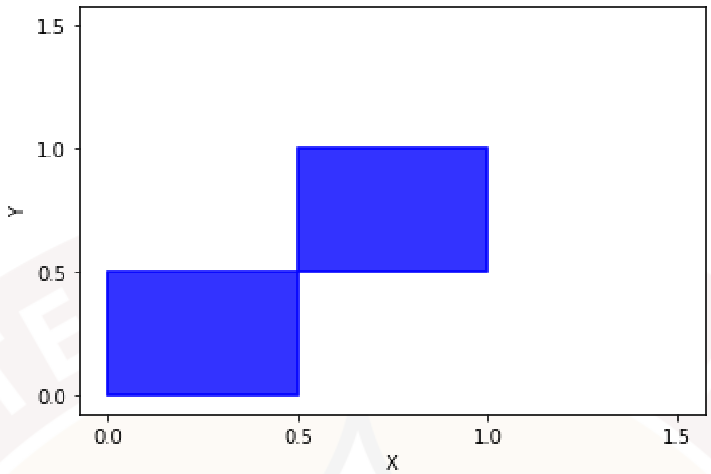
Find the marginal density of X and Y.

Solution: From the information given in the question, the support for region D is shown by the blue shaded region in the following figure. The area of the region D is

$$|D| = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

Let X and Y have joint density

$$f_{XY}(x,y) = \begin{cases} 2 & (x,y) \in D \\ 0 & \text{otherwise} \end{cases}$$



Support for region D

The marginal density of X can be given as

$$f_X(x) = 2 \times \frac{1}{2} = 1 \text{ for } \quad 0 < x < 0.5$$

Similarly,

$$f_X(x) = 2 \times \frac{1}{2} = 1 \text{ for } \quad 0.5 < x < 1$$

$$f_X(x) = \begin{cases} 1 & \text{for } 0 < x < 0.5 \\ 1 & \text{for } 0.5 < x < 1 \end{cases}$$

So, $X \sim \text{Uniform}[0, 1]$

The marginal density of Y can be given as

$$f_Y(y) = 2 \times \frac{1}{2} = 1 \text{ for } \quad 0 < x < 0.5$$

Similarly,

$$f_Y(y) = 2 \times \frac{1}{2} = 1 \text{ for } \quad 0.5 < x < 1$$

$$f_Y(y) = \begin{cases} 1 & \text{for } 0 < x < 0.5 \\ 1 & \text{for } 0.5 < x < 1 \end{cases}$$

So, $Y \sim \text{Uniform}[0, 1]$

3.3 Independence

(X, Y) with joint density $f_{XY}(x, y)$ are independent if

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

where $f_X(x)$ and $f_Y(y)$ are the marginal densities.

- Given the joint density, the marginals can be computed
- If the joint density is the product of the marginal densities, then X and Y are independent
- So, if independent, the marginals determine the joint density.

The two variables X and Y are independent if the joint density $f_{XY}(x, y)$ becomes the product of the marginal densities i.e. should be equal to $f_X(x)f_Y(y)$.They are statistically independent if this condition is satisfied.
Once you complete the marginals you check if it is the product of the marginals. If you have the marginals, you can find the joint density if you know X and Y are independent.

3.3.0.1 Examples

1. Suppose $X \sim \text{Exp}(\lambda_1)$, $Y \sim \text{Exp}(\lambda_2)$ are independent random variables. Find their joint density and compute $P(X > Y)$.
   Solution:
   Given that, X and Y are exponentially distributed. So, the marginal densities of X and Y are given as,

   $$f_X(x) = \lambda_1 e^{-\lambda_1 x} \text{ where } x > 0$$

   $$f_Y(y) = \lambda_2 e^{-\lambda_2 y} \text{ where } y > 0$$

   Since, X and Y are independent, the joint density of X and Y is given by,

   $$f_{XY}(x, y) = \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \text{ where } x, y > 0$$

   The region D in Figure 1.1 shows the support for X and Y.
   The blue shaded region shows the region where $X > Y$.
   Now,

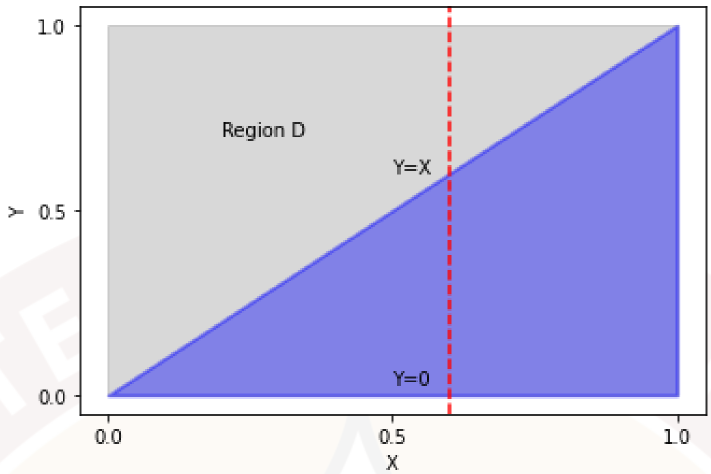


Figure 1.1

Note: To find the limits of integration:

Consider a red vertical strip which shows the slice of y

$$
\begin{aligned}
P(X > Y) &= \int_{x=0}^{\infty} \int_{y=0}^{y=x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \, dy \, dx \\
&= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \left(\int_{y=0}^{y=x} \lambda_2 e^{-\lambda_2 y} \, dy \right) dx \\
&= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \left(-e^{-\lambda_2 y} \Big|_0^x \right) \, dx \qquad \because \int_a^b e^{ny} \, dy = \frac{e^n b - e^n a}{n} \\
&= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \left(1 - e^{-\lambda_2 x}\right) \, dx \\
&= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} - \lambda_1 e^{-(\lambda_1+\lambda_2)x} \, dx \\
&= \int_{x=0}^{\infty} \lambda_1 e^{-\lambda_1 x} \, dx - \int_{x=0}^{\infty} \lambda_1 e^{-(\lambda_1+\lambda_2)x} \, dx \\
&= \left(-e^{-\lambda_1 x} \Big|_0^{\infty} \right) - \left(-\frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)x} \Big|_0^{\infty} \right) \qquad \because \int_a^b e^{nx} \, dx = \frac{e^n b - e^n a}{n} \\
&= -(0-1) - \left(-\frac{\lambda_1}{\lambda_1+\lambda_2}(0-1) \right) \\
&= 1 - \frac{\lambda_1}{\lambda_1+\lambda_2} \\
&= \frac{\lambda_2}{\lambda_1+\lambda_2}
\end{aligned}
$$

3.4 Conditional density

Let (X, Y) be random variables with joint density $f_{XY}(x, y)$. Let $f_X(x)$ and $f_Y(y)$ be the marginal densities. For a such that $f_X(a) > 0$, the conditional density of Y given $X = a$, denoted $f_{Y|X=a}(y)$, is defined as

$$f_{Y|X=a}(y) = \frac{f_{XY}(a, y)}{f_X(a)}$$

For b such that $f_Y(b) > 0$, the conditional density of X given $Y = a$, denoted $f_{X|Y=b}(x)$, is defined as

$$f_{X|Y=b}(x) = \frac{f_{XY}(x, b)}{f_Y(b)}$$

3.4.0.1 Properties of conitional density

- Both the conditional densities are valid in one dimension. So, the "conditional" random variables $(Y|X = a)$ and $(Y|X = b)$ are well-defined.
- Joint = Marginal $\times$ Conditional, for $x = a$ and $y = b$ such that $f_X(a) > 0$ and $f_Y(b) > 0$

$$f_{XY}(a, b) = f_X(a)f_{Y|X=a}(b) = f_Y(b)f_{X|Y=b}(a)$$

- The above is usually written as

$$f_{XY}(x, y) = f_X(x)f_{Y|X=x}(y) = f_Y(y)f_{X|Y=y}(x)$$

3.4.0.2 Solved Examples:

1. Let $X, Y \sim \text{Uniform}(D)$. Where D is a unit square. Find the distribution of $Y|X = a$, $X|Y = b$, where $0 < a < 1$, $0 < b < 1$.
Solution:

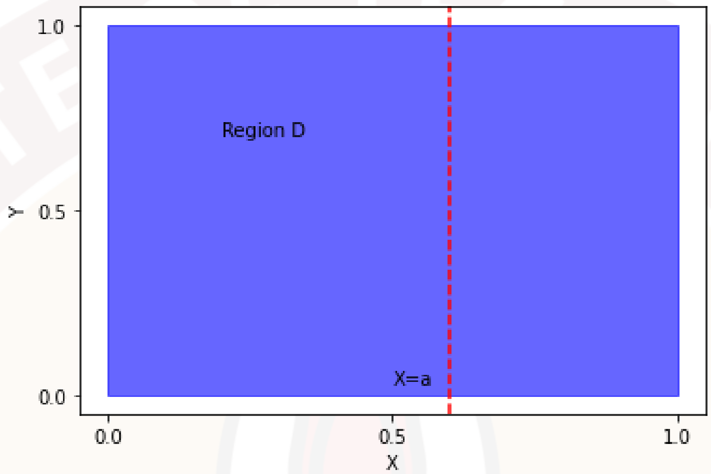


Figure 1.2

The joint density of X, Y will be given by,

$$f_{X,Y}(x, y) = \begin{cases} 1 & \text{for } \ 0 < x, y < 1 \\ 0 & \text{for } \ \text{otherwise} \end{cases}$$

To find the distribution of $Y|X = a$,

$$\begin{aligned} P(Y|X = a) =& \frac{f_{X,Y}(a, y)}{f_X(a)} \\ =& \frac{1}{\int_{y=0}^{1} f_{X,Y}\, dy} \\ =& \frac{1}{\int_{y=0}^{1} 1\, dy} \\ =& 1 \end{aligned}$$

Therefore, the distribution of $Y|X = a$ is Uniform$\{0, 1\}$.

Similarly, to find the distribution of $X|X = b$,

$$
\begin{aligned}
P(X|Y = b) =& \frac{f_{X,Y}(x, b)}{f_Y(b)} \\
=& \frac{1}{\int_{x=0}^{1} f_{X,Y}\, dy} \\
=& \frac{1}{\int_{x=0}^{1} 1\, dy} \\
=& 1
\end{aligned}
$$

Therefore, the distribution of $X|Y = b$ is Uniform$\{0, 1\}$.

2. Consider the joint density

$$
f_{XY}(xy) = \begin{cases} x + y & \text{for } 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}
$$

Find the conditionals.
Solution:
We have given, joint density of X and Y i.e. $f_{XY}(x, y)$.
And asked to find the conditional densities, i.e. $f_{Y|X=a}(y)$ and $f_{X|Y=b}(x)$, where $0 < a < 1$ and $0 < b < 1$.
To find the conditional densities, we have to find the marginal densities.
Now, marginal density of X is given by,

$$
\begin{aligned}
f_X(x) &= \int_{y=0}^{y=1} f_{XY}(x, y)\ dy \\
&= \int_{y=0}^{y=11} (x + y)\ dy \\
&= \left(xy + \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=1} \\
&= x(1 - 0) + (\frac{1^2}{2} - 0) \\
&= x + \frac{1}{2}
\end{aligned}
$$

Similarly, marginal density of Y is given by,

$$\begin{aligned}
f_Y(y) &= \int_{x=0}^{x=1} f_{XY}(x,y)\ dx \\
&= \int_{x=0}^{x=1} (x+y)\ dx \\
&= \left(\frac{x^2}{2} + xy\right)\Bigg|_{x=0}^{x=1} \\
&=(\frac{1^2}{2} - 0) + y(1-0) \\
&=y + \frac{1}{2}
\end{aligned}$$

Now, we will find the conditional density $f_{Y|X=a}(y)$.

$$\begin{aligned}
f_{Y|x=a}(y) &= \frac{f_{XY}(Y=y, X=a)}{f_X(a)} \\
&= \frac{x+y}{f_X(a)} \\
&= \frac{a+y}{a+\frac{1}{2}}
\end{aligned}$$

Similarly, the conditional density $f_{X|Y=b}(x)$ is given by,

$$\begin{aligned}
f_{X|Y=b}(x) &= \frac{f_{XY}(X=x, Y=b)}{f_Y(b)} \\
&= \frac{x+y}{f_Y(b)} \\
&= \frac{x+b}{b+\frac{1}{2}}
\end{aligned}$$

3.5 Problems

(a) Let the joint PDF of two random variables X and Y be given by:

$$f_{XY}(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the marginal of X.

(a)

$$f_X(x) = \begin{cases} \dfrac{6x}{5}\left(x + \dfrac{1}{3}\right) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_X(x) = \begin{cases} \dfrac{6}{5}\left(x - \dfrac{1}{3}\right) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_X(x) = \begin{cases} \dfrac{6}{5}\left(x + \dfrac{1}{3}\right) & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Find the marginal of Y.

(a)

$$f_Y(y) = \begin{cases} \dfrac{6y}{5}\left(y + \dfrac{1}{3}\right) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_Y(y) = \begin{cases} \dfrac{6}{5}\left(\dfrac{1}{2} + y^2\right) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$f_Y(y) = \begin{cases} \dfrac{6}{5}\left(y + \dfrac{1}{3}\right) & \text{for } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let $(X, Y) \sim \text{Uniform}(D)$, where $D := \{0 < x < 1, y \leq x\} \cup [1, 2] \times [1, 2] \cup \{2 < x < 3, x + y < 3\}$.

   (i) Find the marginal of X.

   (ii) Find the marginal of Y.

4. Let the random variables X and Y be uniformly distributed in the triangular region D, where D is bounded by $x = 0, y = 0, x = 2y, x = 2$. Find $f_{Y|X=1}(1/4)$.

   (a) $\dfrac{1}{2}$

   (b) 2

   (c) 1

(d) $\dfrac{1}{4}$

5. Let the joint PDF of two random variables X and Y be given by:

$$f_{XY}(x,y) = \begin{cases} \dfrac{6}{5}(x+y^2) & \text{for } 0<x<1, 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P\left(\dfrac{1}{5} < X < \dfrac{2}{5} \mid Y = \dfrac{1}{2}\right)$.

6. Let X and Y have the the joint PDF

$$f_{XY}(x,y) = \begin{cases} \dfrac{1}{8} & \text{for } x>y, 0<y<4 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent?

(a) Yes

(b) No

7. Let the joint PDF of two random variables X and Y be given by

$$f_{XY}(x,y) = \begin{cases} ye^{-y(x+1)} & \text{for } x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find the value of $P(X \mid Y = 1)$.

(a) e^{-x}

(b) e^{x}

(c) $\dfrac{x}{e^{-x}}$

(d) $\dfrac{e^{-x}}{x}$

8. Let $(X,Y) \sim \text{Uniform}(D)$, where $D := \{3X > Y, 0 < x < 1, y > 0\}$. Choose the correct option(s) from the following:

(a) $(Y \mid X = a) \sim \text{Uniform}\left[0, \dfrac{a}{3}\right]$

(b) $(Y \mid X = a) \sim \text{Uniform}[0, 3a]$

(c) $(X \mid Y = b) \sim \text{Uniform}\left[\dfrac{b}{3}, 1\right]$

(d) $(X \mid Y = b) \sim \text{Uniform}\left[0, \dfrac{b}{3}\right]$

9. Let $(X, Y) \sim \text{Uniform}(D)$, where $D := [0,1] \times [1,2] \cup [1,2] \times [0,1] \cup [2,3] \times [0,2]$. Are X and Y independent?

   (a) Yes

   (b) No

10. Let the joint PDF of two random variables X and Y be given by

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

   Choose the correct option(s) from the following:

   (a) $f_{XY}(x,y) \neq f_X(x) f_Y(y)$

   (b) $f_{XY}(x,y) = f_X(x) f_Y(y)$

   (c) $P(X < 2) = 1 - e^{-2}$

   (d) $P(X < Y) = \dfrac{1}{2}$