STATISTICS – 2

Date .....

Repeated Independent trials :-

Single Bernoulli trial :

Occurence of Event A : Success (1)

Non- " " : Failure (0)

Let p = $P(A)$ → probability

Bernoulli trial (p) :

Sample space is {success, failure}

with $P(\text{success}) = p$

$\{0,1\}$ with $P(1) = p$, $P(0) = 1-p$

Repeated bernoulli trial :

perform bernoulli trials n times

Sample space :- 2^n

Let $n = 3$ $\{(000)\ (100)\ (110)\ (010)\ (001)\ (101)\ (011)\ (111)\}$

$P(000) = (1-p)(1-p)(1-p) = (1-p)^3$

$P(001) = (1-p)(1-p)\,p = p\,(1-p)^2$

$$\boxed{P(b_1 b_2 b_3 \ldots b_n) = p^w\,(1-p)^{n-w}}$$

w = no. of 1 in $b_1 b_2 b_3 \ldots b_n$.

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Binomial trial

Binomial (n, p)

n independent Bernoulli (p) trials

Outcome $\rightarrow$ number of successes.

Sample space : $\{0, 1, 2, 3 \ldots\ldots n\}$

no. of success

Ex :- $n = 3$

$P(B=0) = P(000) = (1-p)^3$

$P(B=1) = P(100 \text{ or } 010 \text{ or } 001) = 3p(1-p)^2$

$P(B=2) = P(110 \text{ or } 101 \text{ or } 110) = 3p^2(1-p)$

$P(B=3) = P(111) = p^3$

$$P(B(n,p) = k) = {}^nC_k \, p^k (1-p)^{n-k}$$

Geometric Distribution.

Bernoulli trial until success is there.

known 'n'.

toss a coin success $\rightarrow$ H

$P(1) = \frac{1}{2} \rightarrow$ Head

$P(2) = \underset{\text{Tail}}{\frac{1}{2}} \times \underset{\text{Head}}{\frac{1}{2}} = \frac{1}{4}$

$P(3) = \underset{\text{Tail}}{\frac{1}{2} \times \frac{1}{2}} \times \underset{\text{Head}}{\frac{1}{2}} = \frac{1}{8}$

$\vdots$

$P(k) = \left(\frac{1}{2}\right)^k$

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outcome :- no. of trials needed for 1st success.

$G(p)$ or G.

Sample space :- $\{1, 2, 3, \ldots\}$

$P(G=1) = p$

$P(G=2) = (1-p)(p)$

$P(G=3) = (1-p)^2 p$

$\vdots$

$P(G=k) = (1-p)^{k-1} p$

$$P(G \leq k) = p + (1-p)p + (1-p)^2 p \ldots (1-p)^{k-1} p$$
$$= 1 - (1-p)^k$$

Monte Carlo Simulation.

$$P(A) \approx \frac{N_A}{N}$$

N_A → no. of times A occured.

N → no. of times simulation done.

$P(A)$ → Prob of event A.

Distributions -

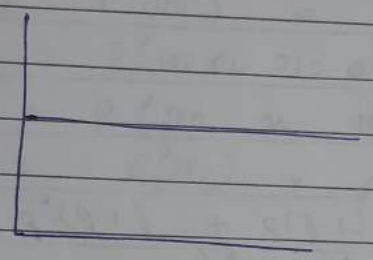
Date .....

Uniform Random variable.

$X \sim \text{Uniform}(T)$, where T is some finite set.

Range : Finite set T

PMF : $f_X(t) = \frac{1}{|T|}$ for all $t \in T$



Bernoulli random variable.

$X \sim \text{Bernoulli}(p)$, $0 \le p \le 1$

Range : $\{0,1\}$

PMF : $f_X(0) = 1-p$
$f_X(1) = p$

Binomial random Variable.

$X \sim \text{Binomial}(n,p)$ n is positive integer
$0 \le p \le 1$

Range : $\{0, 1, 2 \text{ — — } n\}$

PMF : $f_X(k) = {}^nC_k \, p^k (1-p)^{n-k}$

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Geometric Random variable -

$X \sim \text{Geometric}(p)$ $\quad 0 \leq p \leq 1$

Range : $\{1, 2, 3 \text{ ——— } \}$

PMF : $f_X(k) = (1-p)^{k-1} \, p$

Negative binomial Random Variable

$X \sim \text{Negative Binomial}(r, p)$, $\quad r$ is positive, $0 < p \leq 1$

Range : $\{r, r+1, r+2 \text{ ——— } \}$

PMF : $f_X(k) = {}^{(k-1)}C_{r-1} \, (1-p)^{k-r} \, p^r$

Poisson Random Variable

$X \sim \text{Poisson}(\lambda)$ $\quad \lambda > 0$ (Real no.)

Range : $\{0, 1, 2, 3 \text{ ——— } \}$

PMF : $f_X(k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$

Hypergeometric Random Variable.

$X \sim \text{Hypergeo}(N, r, m)$ $\quad N, r, m \geq$ positive integers

Range : $\{\max(0, m-(N-r)), \text{ ——— } \min(r, m)\}$

PMF : $f_X(k) = \dfrac{{}^{r}C_k \cdot {}^{(N-r)}C_{m-k}}{{}^{N}C_m}$

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MULTIPLE RANDOM VARIABLES. Date .....

TWO RANDOM VARIABLES :—

JOINT PMF :—

Suppose X and Y are discrete random variables defined in the same probability space.

Range of $X = T_X$

$Y = T_Y$

Joint PMF denoted by f_{XY} is a f^n from $T_X \times T_Y$ to $[0,1]$ as :-

$$f_{XY}(t_1, t_2) = P(X = t_1 \text{ and } Y = t_2), \quad t_1 \in T_X, \; t_2 = T_Y$$

usually written as a table / matrix.

$P(X = t_1, Y = t_2)$

Eg :- Toss a fair coin twice.

$X_i = 1$ if i^{th} toss is heads

$X_i = 0$ tail

$i = 1, 2$

$$f_{X_1X_2}(0,0) = P(X_1 = 0, X_2 = 0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$f_{X_1X_2}(0,1) = P(X_1 = 0, X_2 = 0) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

independent events

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$t_2 \backslash t_1$	0	1
0	1/4	1/4
1	1/4	1/4

Each entry is b/w 0 and 1
Sum of all entries = 1 } Joint PMF

Marginal PMF

X and Y are jointly distributed discrete random variables with Joint PMF f_{XY}.
The PMF of the individual random variables X and Y are called Marginal PMFs.

Marginalization = Joint PMF ⟶ Marginal PMF (individual)

$$f_X(t) = P(X=t) = \sum_{t' \in T_Y} f_{XY}(t, t')$$

$$f_Y(t) = P(Y=t) = \sum_{t' \in T_X} f_{XY}(t', t)$$

Example :- Toss of fair coin twice.

$t_2 \backslash t_1$	0	1	$f_{X_2}(t_2)$
0	1/4 f(0,0)	1/4	1/2
1	1/4 f(1,0)	1/4	1/2
$f_{X_1}(t_1)$	1/2	1/2	

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X_1 :- add over columns.

$f_{X_1}(0) = f_{X_1X_2}(0,0) + f_{X_1X_2}(0,1)$

$f_{X_1}(1) = f_{X_1X_2}(1,0) + f_{X_1X_2}(1,1)$

X_2 :- add over rows.

$f_{X_2}(0) = f_{X_1X_2}(0,0) + f_{X_1X_2}(1,0)$

$f_{X_2}(1) = f_{X_1X_2}(0,1) + f_{X_1X_2}(1,1)$

Conditional PMF given an Event.

X is a discrete random variable with range T_X, A is an event in the same probability space.

Conditional PMF of X given A :—

$Q(t) = P(X = t|A)$ $\quad t \in T_X$

notation $\longrightarrow$ $f_{X|A}(t)$

$$f_{X|A}(t) = \frac{P((X=t) \cap A)}{P(A)}$$

Suppose X and Y are jointly distributed random variables.

Conditional PMF of Y given $X = t$

$$Q(t') = P(Y = t' | X = t) = \frac{P(Y = t', X = t)}{P(X = t)}$$

$$f_{Y|X=t}(t') = \frac{f_{XY}(t, t')}{f_X(t)}$$

Joint = Conditional × Marginal

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$$\boxed{f_{XY}(t,t') = f_{Y|X=t}(t')\, f_X(t)}$$

Range of $(Y|X=t)$ can be different from range of Y and will depend on t.

$t_2 \backslash t_1$	0	1	2	$f_Y(t_2)$
0	1/4	1/8	1/8	1/2
1	1/8	1/8	1/4	1/2
$f_X(t_1)$	3/8	1/4	3/8	

divide

$X \in \{0,1,2\}$ $\quad Y = \{0,1\}$

$(Y|X=0) \in \{0,1\}$

$$f_{Y|X=0}(0) = \frac{f_{XY}(0,0)}{f_X(0)} = \frac{1/4}{3/8} = \frac{2}{3}$$

$$f_{Y|X=0}(1) = \frac{f_{XY}(0,1)}{f_X(0)} = \frac{1/8}{3/8} = \frac{1}{3}$$

$(X|Y=1) \in \{0,1,2\}$

$$f_{X|Y=1}(0) = \frac{f_{XY}(0,1)}{f_Y(1)} = \frac{1/8}{1/2} = \frac{1}{4}$$

$$f_{X|Y=1}(1) = \frac{f_{XY}(1,1)}{f_Y(1)} = \frac{1/8}{1/2} = \frac{1}{4}$$

$$f_{X|Y=1}(2) = \frac{f_{XY}(2,1)}{f_Y(1)} = \frac{1/4}{1/2} = \frac{1}{2}$$

1	1/8	1/8	1/4	1/2

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More than 2 discrete variables.

Joint PMF

$$f_{X_1 X_2 \cdots X_n} = P(X_1 = t_1, X_2 = t_2 \cdots X_n = t_n)$$

$$[0,1]$$

Marginal PMF.

$$f_{X_1}(t) = P(X_1 = t) = \sum_{t'_2 \in T_{X_2},\ t'_3 \in T_3 \cdots t'_n \in T_n} f_{X_1 \cdots X_n}(t, t'_2, t'_3 \cdots t'_n)$$

~~$f_{X_1}(t)$~~

$$f_{X_n}(t) = P(X_n = t) = \sum_{t'_1 \in T_{X_1} \cdots t'_{n-1} \in T_{X_{n-1}},\ t \in T_{X_n}} f_{X_1 \cdots X_n}(t'_1 \cdots t'_{n-1}, t)$$

Conditioning :—

$$(X_1 \mid X_2 = t_2) \sim f_{X_1 \mid X_2 = t_2}(t_1) = \frac{f_{X_1 X_2}(t_1, t_2)}{f_{X_2}(t_2)}$$

$$(X_1, X_2 \mid X_3 = t_3) \sim f_{X_1 X_2 \mid X_3 = t_3}(t_1, t_2) = \frac{f_{X_1 X_2 X_3}(t_1, t_2, t_3)}{f_{X_3}(t_3)}$$

$$(X_1 \mid X_2 = t_2, X_3 = t_3) \sim f_{X_1 \mid X_2 = t_2, X_3 = t_3}(t_1) = \frac{f_{X_1 X_2 X_3}(t_1, t_2, t_3)}{f_{X_2 X_3}(t_2, t_3)}$$

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Independence of 2 random variables.

Random variable $\rightarrow$ X with range T_X
Y T_Y

Independent if

$$f_{XY}(t_1, t_2) = f_X(t_1)\, f_Y(t_2)$$

for $t_1 \in T_X$, $t_2 \in T_Y$

Conditional PMF equals marginal PMF

$\rightarrow$ To show X and Y are independent, verify,

$$f_{XY}(t_1, t_2) = f_X(t_1) = f_Y(t_2)$$

for all $t_1 \in T_X$, $t_2 \in T_Y$

$\rightarrow$ To show X and Y are dependent, verify

$$f_{XY}(t_1, t_2) \neq f_X(t_1)\, f_Y(t_2)$$

for some $t_1 \in T_X$ and $t_2 \in T_Y$

$\rightarrow$ Special case: $f_{XY}(t_1, t_2) = 0$ when $f_X(t_1) \neq 0$, $f_Y(t_2) \neq 0$

Independence of multiple random variable.

$$F_{X_1 \ldots X_n}(t_1 \ldots t_n) = f_{X_1}(t_1)\, f_{X_2}(t_2) \ldots f_{X_n}(t_n)$$

All subsets of independent random variables are independent.

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Independent & Identically distributed (IID)

Random variable X_1 — X_n are said to be independent and identically distributed (iid) if

→ they are independent.

→ the marginal PMFs f_{X_i} are identical

Memoryless property of Geometric

$X \sim \text{Geometric}(p)$

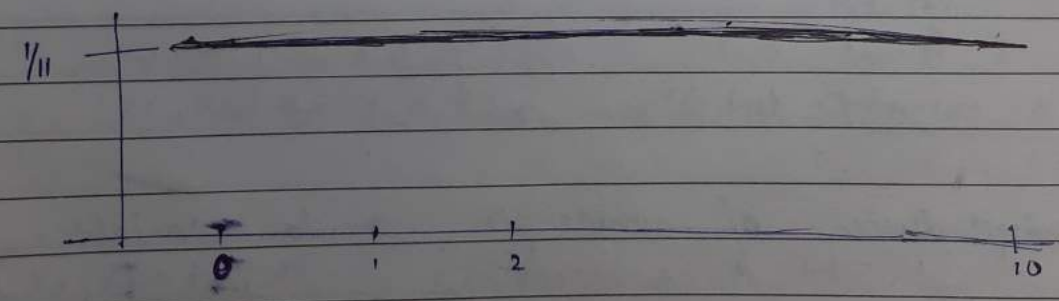
$$P(X > n) = (1-p)^n$$

$$P(X > m+n \mid X > m) = P(X > n) = (1-p)^n$$

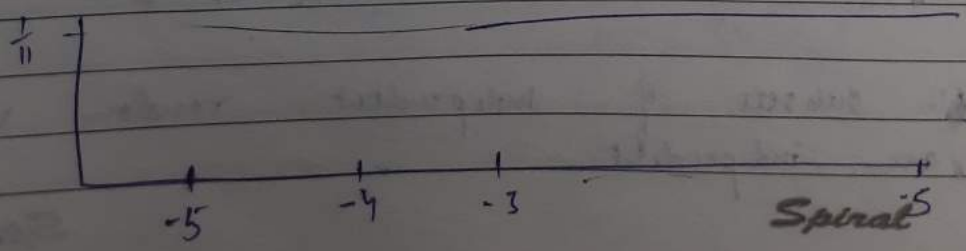
Visualizing functions of Random Variables

One random variable, one-to-one fn.

Uniform $\{0, 1, 2 \ldots 10\}$



$Y = X - 5$



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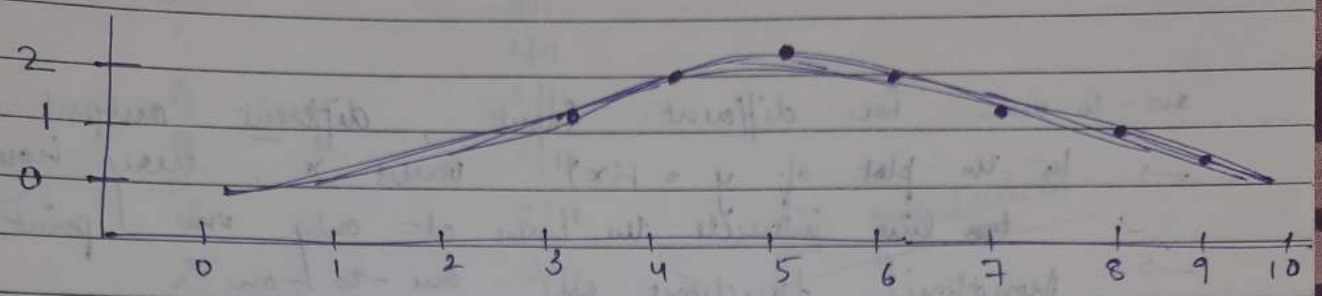
Binomial(10, 0.5) $\quad P(X=k) = \binom{10}{k} (0.5)^k (0.5)^{10-k}$



PMF of $Y = X - 5$ (table method)

uniform $\{0, 1, \ldots 10\}$

x	$P(X=x)$	$y = x-5$
0	$P(X=0)$	$P(Y=-5)$ -5
1		-4
2		-3
3		-2
4		-1
5	$P(X=5)$	$P(Y=0)$ 0
6		1
7		2
8		3
9		4
10		5

Binomial (10, 0.5)

x	$P(X=x)$	$y = x-5$
0	0.00097656	-5
1	0.00976563	-4
2	0.04394531	-3
3	0.1171875	-2
4	$P(X=4)$ ⋮ $P(X=-1)$	-1
5	⋮	⋮
6	⋮	⋮
⋮	⋮	⋮
10	⋮	5

one - to - one function.

each value of X connects to unique value of Y

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PMF of $Y = 2^X$ is also one-to-one

one-to-one: for different input, different output.

→ In the plot of $y = f(x)$ versus x, every horizontal ~~ton~~ line intersects the curve at only one point.

→ Monotonic functions are one-to-one.

If f is one-to-one.

→ Easy to find PMF of $f(x)$ using table ~~tat~~ method.

→ Easy to visualize the plot of the PMF.

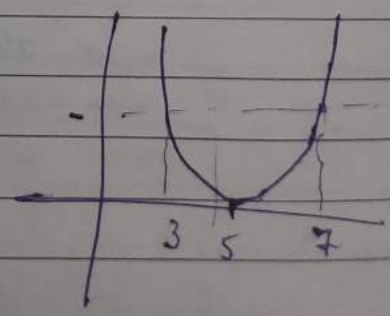
$P(Y = f(x)) = P(X = x)$.

One random variable, many-to-one.

$Y = (X-5)^2$

$X \sim$ uniform $\{0, 1, 2 \ldots\ldots 10\}$

x	$P(X=x)$	$y = (x-5)^2$
0	1/11	25
1	1/11	16
2	1	9
⋮	⋮	4
		1
		0
		1
		⋮
10		25



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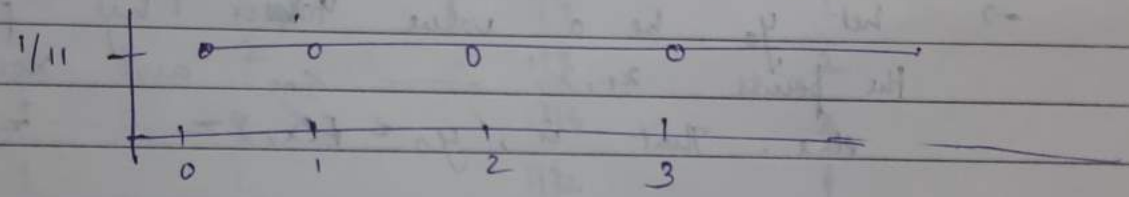
Y	$P(Y=y)$
0	1/11
1	2/11
4	2/11
9	2/11
16	2/11
25	2/11

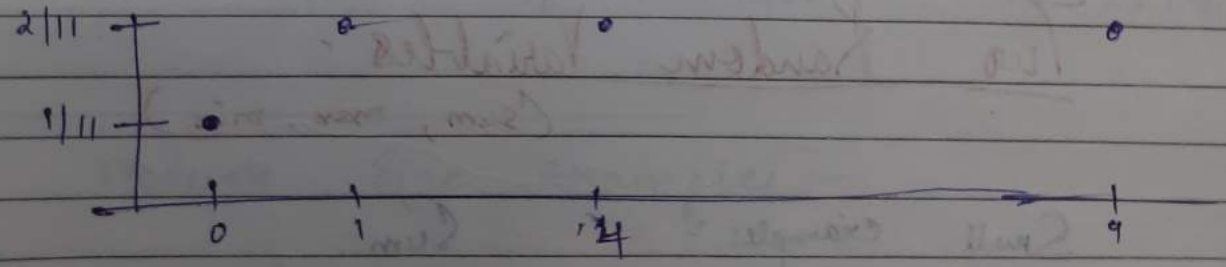
add probabilities

$$Y \in \{0, 1, 4, 9, 16, 25\}$$

$X \sim$ uniform $\{0, 1, 2, \ldots 10\}$



$Y = (X-5)^2$



Same for Binomial.

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Many - to - one : for 2 or more different ~~ty~~ input, the function should result in the same output.

→ plot of $y = f(x)$ versus X, some horizontal line intersects the curve at more than 1 point.

→ Non - monotic functions are many - to - one.

$$y = x^2, \; y = x(1-x), \; y = xe^{-x}$$

If f is many - to - one

→ The table method will generally work.

→ Let y_0 be a value taken by f at the points x_1, x_2 —— x_m and nowhere else. That is, $y_0 = f(x_1) = \text{——} = f(x_m)$.

$$P(Y = y_0) = P(X = x_1) + \text{——} P(X = x_m)$$

Two Random Variables.

(sum, max, min)

Small examples :- Sum

$X, Y \sim$ iid unif $\{0,1\}$, $Z = X + Y$

X	Y	$f_{XY}(x,y)$	Z	P(Z)
0	0	1/4	0	1/4
0	1	1/4	1	1/2
1	0	1/4	1	1/2
1	1	1/4	2	1/4

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$Z \in \{0, 1, 2\}$

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Z	$P(Z)$	
0	1/4	
1	1/2	$(1/4 + 1/4)$
2	1/4	

max.

$Z = \max(X, Y)$

X	Y	$f_{XY}(X, Y)$	Z
0	0	1/2	0
0	1	1/4	1
0	2	1/8	2
1	0	1/16	1
1	1	1/32	1
1	2	1/32	2

$P(Z=0) = 1/2$ $\quad P(Z=1) = 11/32$ $\quad P(Z=2) = 5/32$

Moderate size examples :-

Prone to errors.

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Visualisation :—— .

$g(x,y)$: fn

3D is an option, but often not very useful.

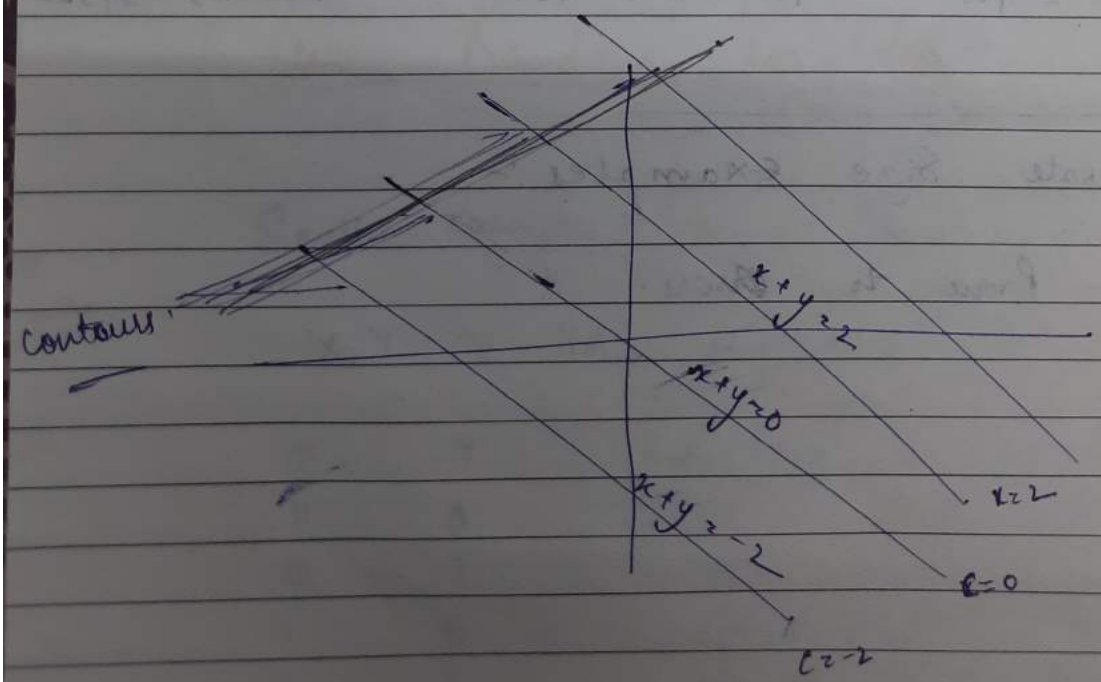
Contours : value of (x,y) that result in same $g(x,y) = c$

Make a plot of those (x,y) for different c.

Regions : values of (x,y) that result in $g(x,y) \le c$.

make a plot of those (x,y) for different c.

Sum f^n, $g(x,y) = x+y$



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Function of 2 variable.

$X, Y \sim f_{XY}$, $X \in \mathcal{X}$, $Y \in \mathcal{Y}$

Let $Z = g(X, Y)$ be a f^n of X and Y

What is PMF of Z?

Step 1 :- find range of Z

Step 2 :- Add over the counter.

Suppose z is a possible value taken by Z.

$$P(Z = z) = \sum_{(x,y) : g(x,y) = z} f_{XY}(x, y)$$

Functions of Random Variables.

Suppose $X_1 \text{ ——— } X_n$. joint PMF $f_{X_1 \text{ ——— } X_n}$

T_{X_i} denoting the range

$g : T_{X_1} \times T_{X_2} \text{ ——— } T_{X_n} \to \mathbb{R}$ be a f^n with range T_g, The PMF of $X = g(X_1 \text{ ——— } X_n)$ is given by

$$f_X(t) = P(g(X_1 \text{ ——— } X_n) = t) =$$

$$\sum_{(t_1 \text{ ——— } t_n) : g(t_1 \cdots t_n)} f_{X_1 \text{ ——— } X_n}(t_1 \text{ ——— } t_n)$$

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If ~~X~~ and Y are independent, $g(X)$ and $h(Y)$ are independent for any 2 functions g and h.

If X_1, X_2, X_3, X_4 are mutually independent,
→ $g(X_1, X_2)$ is independent of $h(X_3, X_4)$
→ $g(X_1, X_2, X_3)$ is independent of $h(X_4)$

Functions of non-overlapping sets of independent random variables are also independent.

EXPECTED VALUE

↳ Average

Discrete random variable X.
↳ Range T_X
PMF f_X.

Expected value, denoted by $E[X]$:—

$$E[X] = \sum_{t \in T_X} t \cdot f_X(t)$$

assuming above sum exist.

$E[X]$ has same units as X
↳ may or may not belong to the range of X.

Eg:-

$X \sim$ Bernulli (p)

$$E[X] = 0 \cdot (1-p) + 1p = p$$

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$X \sim$ uniform $\{1, 2, 3, 4, 5, 6\}$ $\quad \frac{1}{6} \quad \frac{1}{6}$

$$E[X] = \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right)$$

$$+ \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right) = 3.5$$

$X \sim$ uniform $\{a, a+1, a+2 \ldots \ldots b\}$

$$E[X] = a \cdot \frac{1}{b-a+1} + (a+1) \frac{1}{b-a+1} \ldots \ldots$$

$$b \cdot \frac{1}{b-a+1}$$

$$= \frac{1}{b-a+1} \left\{ a + (a+1) + (a+2) \ldots \ldots b \right\}$$

$$= (b-a+1) \left(\frac{a+b}{2}\right) \cdot \frac{1}{b-a+1}$$

$$= \boxed{\frac{a+b}{2}}$$

Exponential :— $\displaystyle\sum_{t=0}^{\infty} e^{-\lambda} \frac{\lambda^t}{t!} = 1$

Binomial $\rightarrow$ $\displaystyle\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$

$a = p$

$b = 1-p$

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$X \sim$ Geometric (p)

$$E[X] = \frac{1}{p}$$

$X \sim$ Poisson (λ)

$$E[X] = \lambda$$

$X \sim$ Binomial (n, p)

$$E[X] = np$$

$X \sim$ Bernoulli (p)

$$E[X] = p$$

Properties of Expected value.

$$E[X] = \sum_{t \in T_X} t \, P(X = t)$$

Consider a constant c as random variable X with $P(X = c) = 1$

$$E[c] = c \cdot 1 = c$$

Suppose X takes only non-negative values, i.e, $P(X \geq 0) = 1$, Then,

$$E[X] \geq 0$$

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→ $X_1 \ldots X_n$ have joint PMF $f_{X_1 \ldots X_n}$

$Y = g(X_1 \ldots X_n)$

$E[Y] = E[g(X_1 \ldots X_n)] = \sum_{t \in T_Y} t \, f_Y(t)$

$= \sum_{t_i \in T_{X_i}} g(t_1 \ldots t_n) \, f_{X_1 \ldots X_n}(t_1 \ldots t_n).$

$\sum_{t} g(t) \, P(X = t).$

→ linearity of expected value.

$E[cX] = c \, E[X]$ $c \to$ constant.

$E[X+Y] = E[X] + E[Y]$

$E[aX + bY] = a E[X] + b E[Y]$

Zero - mean random variable :-

$E[X] = 0$

Translation of random variable.

$X + c$

Range becomes $X + c$ is $\{t + c : t \in T_X\}$

$P(X + c = t + c) = P(X = t)$

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Translation by $E[X]$.
↳ constant.

$Y = \underbrace{X - E[X]}_{\downarrow}$ is translated version of ~~E~~ X and $E[Y] = 0$

0 mean translation.

VARIANCE → spread.

$E[\{10\}] = E[\{9, 11\}] = E[\{0, 20\}]$.

$Var(X) = E[(X - E[X])^2]$. ≥ 0

for every value

same constant

$SD(X) = +\sqrt{var(X)}$

Same unit as X.

↑ spread ↑ $Var(X)$

$$Var(X) = \sum_{t \in T_X} (t - E(X))^2 \, P(X = t)$$

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

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$a \rightarrow$ constant real no.

$Var(ax) = a^2 Var(x)$

$SD(ax) = |a| \, SD(x)$

$Var(X+a) = Var(x)$

$SD(X+a) = SD(x)$

for any random variables (independent or dependent)

$$E[X+Y] = E[X] + E[Y]$$

Only when X and Y are independent :-

$$E[XY] = E[X]E[Y]$$

$$Var[X+Y] = Var[X] + Var[Y]$$

Distribution.	Expected value.	Variance.
Bernoulli (p)	p	$p(1-p)$
Binomial (n, p)	np	$np(1-p)$
Geometric (p)	$1/p$	$(1-p)/p^2$
Poisson (λ)	λ	λ
Uniform $\{1, \ldots, n\}$	$(n+1)/2$	$(n^2-1)/12$

STANDARDISED Random Variable.

$$E[X] = 0$$

$$Var(X) = 1$$

X is a random variable then,

$$\frac{X - E(X)}{SD(X)} \qquad \left(\frac{X-\mu}{\sigma}\right)$$

is standardised

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COVARIANCE : Relation b/w X and Y.

$$Cov(X,Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

→ If $cov(X,Y)$ is +ve.

$(X - E[X])(Y - E[Y])$ tends to be +ve

X is ↑ avg Y is ↑ avg

X is below avg Y is ↓ avg

→ If $cov(X,Y)$ is -ve.

negative

X ↑ Y ↓

X ↓ Y ↑

→ If $Cov(X,Y) = 0$

no ~~correlation~~ correlation.

	X = -1	X = 0	X = 1	f_Y
Y = -1	1/15	2/15	2/15	1/3
Y = 0	2/15	1/15	2/15	1/3
Y = 1	2/15	2/15	1/15	1/3
f_X	1/3	1/3	1/3	

$$Cov(X,Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

$$E[X] = \cancel{-1 \times \frac{1}{3}} + 0 \times \frac{1}{3} + \cancel{1 \times \frac{1}{3}} = 0$$

$$E[Y] = \text{"} = 0$$

$$(-1)(-1)\frac{1}{15} + (-1)(1)\frac{1}{15} + (-1)(1)\frac{2}{15} + (1)(-1)\frac{1}{15} = \frac{-2}{15}$$

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$Cov(X,X) = Var(X)$

$Cov(X,Y) = Cov(Y,X)$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

Linear quantity :-

$Cov(X, aY+bZ) = a\,Cov(X,Y) + b\,Cov(X,Z)$

$Cov(aX+bY, Z) = a\,Cov(X,Z) + b\,Cov(Y,Z)$

Covariance & Independence :-

If independent, $Cov(X,Y) = 0$
↳ uncorrelated.

If $Cov(X,Y) = 0$, they may or may not be ~~dep~~ independent.

Correlation coefficient :- (summarizes trend)

$$\rho(X,Y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}$$

$$-SD(X)\,SD(Y) \leq Cov(X,Y) \leq SD(X)\,SD(Y)$$

$$-1 \leq \rho(X,Y) \leq 1$$

$\rho(X,Y)$ is close to zero :- uncorrelated

$\rho(X,Y) = 1$ or $\rho(X,Y) = -1$
Y is linear function of X
strongly correlated

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$-1/4 \le x \le 1/4$

	$X=0$	$X=1$	f_Y
$Y=0$	$1/4 - x$	$1/4 + x$	$1/2$
$Y=1$	$1/4 + x$	$1/4 - x$	$1/2$
f_X	$1/2$	$1/2$	

$E[X] = 1/2$ $\quad E[X^2] = 1/2$

$E[Y] = 1/2$ $\quad E[Y^2] = 1/2$

$E[XY] = \frac{1}{4} - x$

$Cov(X,Y) = \frac{1}{4} - x - \frac{1}{2} \cdot \frac{1}{2}$

$= -x$

$Var(X) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = Var(Y)$

$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-x}{\frac{1}{2} \cdot \frac{1}{2}} = -4x$

Bounds on prob. using mean and variance

	1 random variable	multiple.
$E[X] \longrightarrow$	μ	μ_X
$Var(X) \longrightarrow$	σ^2	σ^2_X
$SD(X) \longrightarrow$	σ	σ_X

Standard units $\longrightarrow$

$X - \mu$ measures the dist of X from the mean μ.

may be +ve or -ve.

Date .....

→ we expect $X - \mu$ to fall b/w $-c\sigma$ and $c\sigma$ for a small value of c.

→ X to fall b/w $\mu - c\sigma$ and $\mu + c\sigma$

Markov's Inequality :-

Let X be a discrete random variable taking non-negative values with a finite mean μ. Then,

$$P(X \geq c) \leq \frac{\mu}{c}$$

Chebyshev's Inequality :-

Let X be a discrete random variable with a finite mean μ and a finite variance σ^2. Then,

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

$\downarrow$

upper bound.

CONTINUOUS RANDOM VARIABLE

Date .....

CUMULATIVE DISTRIBUTION FUNCTION.

$F_X(x)$ from R to $[0,1]$

defined as :-

$$F_X(x) = P(X \leq x)$$

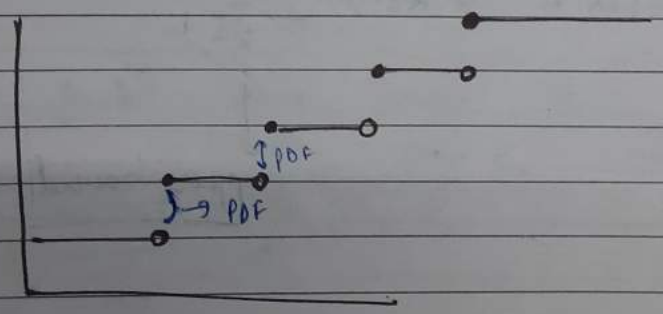
PROPERTIES :-

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

F_X : non-decreasing function taking non-negative values.

As $x \to -\infty$, F_X goes to 0

As $x \to \infty$, F_X goes to 1.



Cumulative Distribution function

$F : R \to [0,1]$ is CDF if :-

F is a non-decreasing f^n taking values b/w 0 and 1

As $x \to -\infty$ F goes to 0

As $x \to \infty$ F goes to 1

F is continous from the right.

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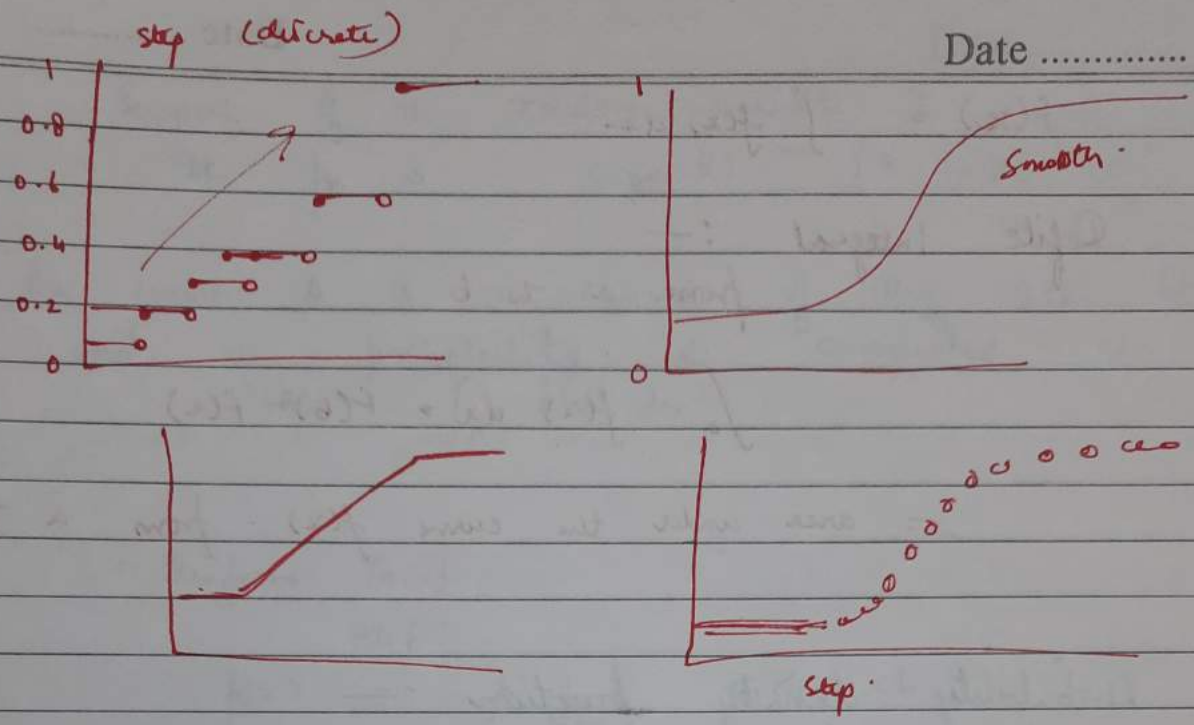

If F_X is continuous at x_0
the $P(X = x_0) = 0$

A random variable X with CDF $F_X(x)$ is said to be continuous random variable if $F_X(x)$ is continuous at every x.

Discrete CDF has steps, does not ~~jump~~ increase without jump.

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Date .....

$F(x) = \int f(x)\, dx$

Defite Integral :—

from a to b

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

= area under the curve $f(x)$ from a to b

Probability density function :—

A continuous random variable X with CDF $F_X(x)$ is said to have a PDF $f_X(x)$ if, for all x_0,

$$F_X(x) = \int_{-\infty}^{x_0} f_X(x)\, dx$$

CDF is integral of PDF.
PDF is derivative of CDF.

Properties of PDF :—

$f : \mathbb{R} \to \mathbb{R}$ is said to be density fn if :—

$f(x) \geq 0$

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

$f(x)$ is piecewise continuous—

Date .....

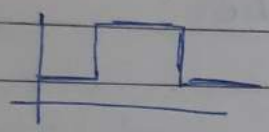
Support of the random variable X with PDF f_X is $\text{supp}(X) = \{x : f_X(x) > 0\}$

An event A is a subset of the real line and its probability is computed as
$$P(A) = \int_A f(x)\,dx$$

$X \sim \text{Uniform}\ [a,b]$

PDF :-

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$



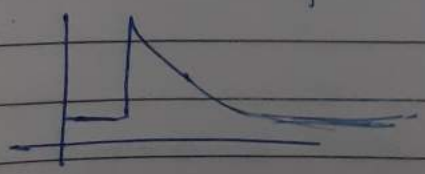
CDF :-

$$F_X(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

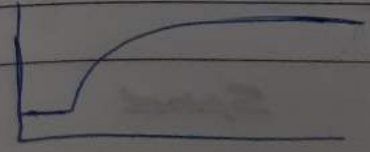
$X \sim \text{Exp}\ (\lambda)$

PDF

$$f_X(x) = \begin{cases} \lambda \exp(-\lambda x) \rightarrow \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$



CDF :- $$F_X(x) = \begin{cases} 0 & x \le 0 \\ 1 - \exp(-\lambda x) & x > 0 \end{cases}$$



Spiral

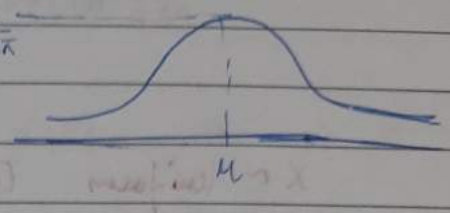
Date .....

Normal or Gaussian.

$X \sim \text{Normal}(\mu, \sigma^2)$ $\quad \mu \in \mathbb{R}$, σ : positive real no.

PDF

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



or :-

$$\frac{1}{\sigma\sqrt{2\pi}\, e^{\frac{(x-\mu)^2}{2\sigma^2}}}$$

CDF :

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

Standard Normal : Normal $(0,1)$

$$\text{Supp}(X) = \mathbb{R}$$

$$P(X < \mu) = P(X > \mu) = 1/2$$

Standardization :-

$$Z = \frac{X-\mu}{\sigma} \sim \text{Normal}(0,1)$$

PDF $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

CDF. $F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$

Spiral

Functions of Continuous Random Variable

Date .....

Example :—

Suppose $X \sim$ uniform

function $Y = 2X \in [0,2]$ is a random variable

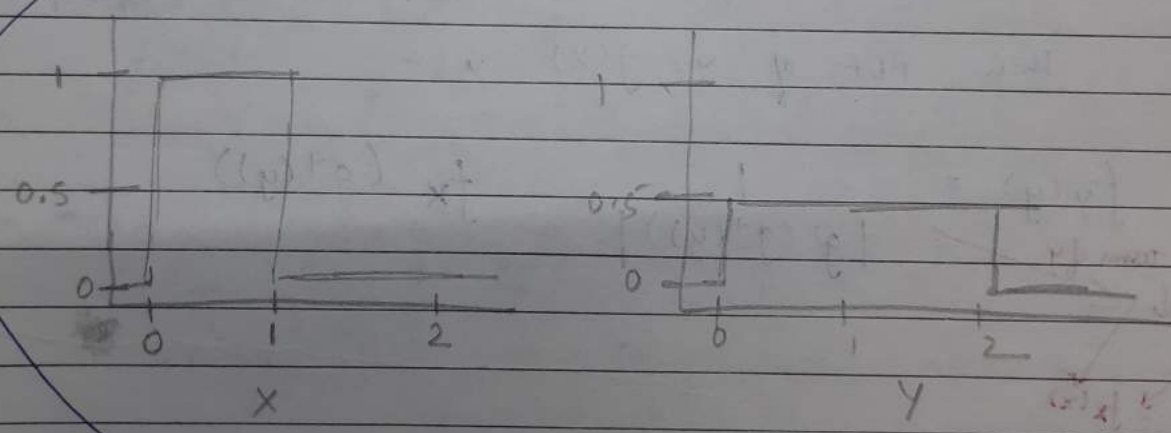
For $y \in [0,2]$,

find CDF $F_Y(y) = P(Y \le y) = P(2X \le y)$

$= P(X \le y/2) = \int_0^{y/2} f_X(x)\,dx = \frac{y}{2}$

PDF of Y, $f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2}$

$Y \sim$ uniform $[0,2]$



for any $Y = aX + b \sim$ uniform $[b, b+a]$

Suppose X is a continuous random variable with CDF F_X and PDF f_X.

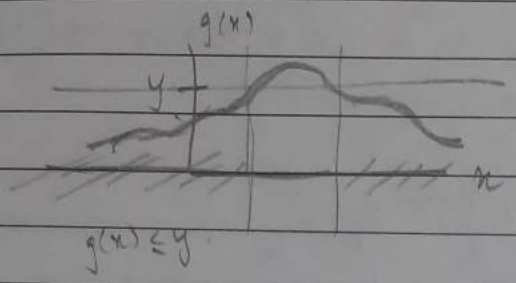
g is $\mathbb{R} \to \mathbb{R}$ fn.

Date .....

$Y = g(X)$ is a random variable with CDF F_Y :—

$$F_Y(y) = P(Y \le y) = P(g(X) \le y)$$
$$= P(X \in \{x : g(x) \le y\})$$

↳ defined using X



If F_Y has no jumps, differentiate F_Y to get f_Y.

Theorem (works only for monotonic & differentiable fn)

→ either decreasing or increasing.

Suppose X is a continuous random variable with PDF f_X. Let $g(x)$ be monotonic for $x \in supp(X)$ with derivative $g'(x) = \frac{dg(x)}{dx}$.

Then PDF of $Y = g(X)$ is :-

$$f_Y(y) = \frac{1}{|g'(g^{-1}(y))|} f_X(g^{-1}(y))$$

① x in terms of y

② $\frac{dx}{dy}$ → $f_X(x)$

③ X

$g'(x)$ → $g^{-1}(y)$

$f_X(x)$ → $g^{-1}(y)$

Eg :- Translation.

$Y = X + a$

$x = y - a$

$g^{-1}(y) = y - a$

$g(x) = X + a$

$g'(x) = 1$

$$f_Y(y) = \frac{f_X(y-a)}{1} = f_X(y-a)$$

Spiral

Date .....

Eg:- scaling

$Y = aX$ $\qquad g(x) = ax$

$g(x) = ax$ $\qquad g'(x) = a$

$y/a = x$

$g^{-1}(y) = y/a$

$$f_Y(y) = \frac{f_X(y/a)}{|a|}$$

Eg:- Affine. $\quad Y = aX + b$

$$f_Y(y) = \frac{1}{|a|} f_X((y-b)/a)$$

Affine transformation of normal distributions.

$X \sim \text{Normal}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

replace $x = \frac{y-\mu}{\sigma}$

$\Rightarrow Y = \sigma X + \mu$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

PDF of normal (μ, σ^2)

Reverse.

$\Rightarrow X \sim \text{Normal}(\mu, \sigma^2)$

$$Y = \frac{(X-\mu)}{a} \sim \text{Normal}(0,1)$$

Affine transformation of a normal random variable is normal.

Transformation Spiral

Date .....

Expectations :- Expected Value.

$X \to$ continuous with density $f_X(x)$

$g : \mathbb{R} \to \mathbb{R}$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

for supp (x).

MEAN AND VARIANCE :-

X: continuous Random variable.

$\boxed{\text{Mean}}$, $E[X]$ or μ_X or μ

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

average / expected value.

$\boxed{\text{Variance}}$, $Var(X)$ or σ_X^2 or σ^2

$$Var(X) = E[(X-\mu_X)^2] = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)\, dx$$

measure of spread of X about its mean

$$Var(X) = E[X^2] - E[X]^2$$

	$f_X(x)$	$E[X]$	$Var(X)$
$X \sim \text{uniform}(a,b)$	$\frac{1}{b-a}$, $a < x < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
$X \sim \text{Exp}(\lambda)$	$\lambda e^{-\lambda x}$, $x > 0$	$1/\lambda$	$1/\lambda^2$
$X \sim \text{Normal}(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2

Spiral

Date .....

Joint Distributions

(X, Y) : Jointly distributed

X : discrete Range $\rightarrow T_X$ PMF $p_X(x)$

for each $x \in T_X$, we have continuous Y_x with density $f_{Y_x}(y)$

Y_x : Y given $X = x$, $Y | X = x$

$f_{Y_x}(y)$: conditional density of Y given $X = x$, $f_{Y|X=x}(y)$.

Marginal density of Y:

$$f_Y(y) = \sum_{x \in T_X} p_X(x) \, f_{Y|X=x}(y)$$

Conditional prob. of discrete given continuous -

X and Y are jointly distributed.

$x \in T_X$ PMF $p_X(x)$

Conditional densities $f_{Y|X=x}(y)$ for $x \in T_X$

The conditional prob of X given $Y = y_0 \in \text{supp}(Y)$ is :-

$$P(X = x \mid Y = y_0) = \frac{p_X(x) \, f_{Y|X=x}(y_0)}{f_Y(y_0)}$$

$f_Y \rightarrow$ marginal density of Y.

Spiral

STATISTICS FROM SAMPLES AND LIMIT THEOREMS -

Date .....

I.I.D Samples (Identical and Independent distribution)

Empirical distribution :-

$X_1, X_2 \text{ ——— } X_n \sim X$ be iid sample.

Let $\#(X_i = t)$ denote no. of times t occur.

The empirical distribution is the discrete distribution with PMF

$$p(t) = \frac{\#(X_i = t)}{n}$$

$n = 15$

1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1

$p(0) = \frac{6}{15}$ $p(1) = \frac{9}{15}$

Empirical dist $\{0 \overset{6/15}{,} 1 \overset{9/15}{\}}$

Depends on the actual sample instances.

SAMPLE MEAN

$X_1, X_2 \text{ ——— } X_n$ be i.i.d samples.

$$\overline{X} = \frac{X_1 + X_2 + X_3 \text{ ——— } X_n}{n}$$

Date .....

Expected value and variance of $\boxed{\text{Sample mean}}$.

$x_1, x_2 \text{ — } x_n$ iid samples. has a finite mean μ and variance σ^2, The sample mean $\bar{x}$ has expected value and variance :-

$E[\bar{x}] = \mu$

As $n \downarrow$ or $n \uparrow$
$E[\bar{x}] = \mu$

$Var(\bar{x}) = \frac{\sigma^2}{n}$

As $n \uparrow$, spread of dist. $\downarrow$
values taken by sample mean will be closer to the dist.

Sample variance.

$$S^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 \text{ — — — } + (x_n - \bar{x})^2}{n-1}$$

where $\bar{x}$ is the sample mean.

Expected value of $\boxed{\text{Sample variance.}}$

$x_1 \text{ — — } x_n$ be IID with finite variance σ^2,

$E[S^2] = \sigma^2$

Date .....

Sample proportion.

$X_1, X_2 - - - X_n \sim X$ be iid.

Let A be an event defined using X
Example $A = (X > t)$, $A = (a < X < b)$ etc.

The sample proportion of A :-

$$S(A) = \frac{\#\ (x_i \text{ for which } A \text{ is true})}{n}$$

Expected value and variance of $\boxed{\text{Sample proportion.}}$

$P(A)$ be the probability of A.

$$E[S(A)] = P(A)$$
$$var(S(A)) = \frac{P(A)(1-P(A))}{n}$$

As $n \uparrow$ values of $S(A)$ will be close to $P(A)$.
Mean of $S(A)$ equals $P(A)$
Variance of $S(A)$ tends to 0.

Sum of Independent random variables.

$X_1 - - - X_n$ be random variables (may or may not be independent)

Sum $= S = X_1 + X_2 + X_3 - - - X_n$

Date .....

$$E[S] = E[x_1] + E[x_2] + \cdots + E[x_n]$$

If $x_1, x_2 \cdots x_n$ are pairwise uncorrelated then,

$$Var(S) = Var(x_1) + Var(x_2) \cdots Var(x_n)$$

pairwise uncorrelated $\Rightarrow$ $E[x_i x_j] = E[x_i]E[x_j]$ for all i, j ; $i \neq j$

mean of sum is sum of means.

If uncorrelated, variance of sum is sum of variances.

If x_i are independent, they are also uncorrelated.

→ Scaling and summing

- $S = a_1 X_1 + a_2 X_2 + \cdots a_n X_n$ $\quad a_j \in R$

  $E[S] = a_1 E[X_1] + a_2 E[X_2] \cdots a_n E[X_n]$

  $Var(S) = a_1^2 Var(X_1) \cdots a_n^2 Var(X_n)$, if uncorrelated.

→ IID samples : $X_1 \cdots X_n \sim X$, iid.

- $S = a_1 X_1 + \cdots a_n X_n$ $\quad a_j \in R$

  $E[S] = (a_1 + a_2 \cdots a_n) E[X]$.

  $Var(S) = (a_1^2 + a_2^2 \cdots a_n^2) Var(X)$

→ Sample mean : $X_1 \cdots X_n \sim X$, iid.

$$\bar{X} = (X_1 + X_2 \cdots X_n)/n \qquad a_i = 1/n$$

$$E[(\bar{X})] = E[X]$$

$$Var((\bar{X})) = Var(X)/n$$

Spiral

Date .....

Sample mean versus distribution mean.

$x_1, x_2 \ldots \ldots x_n \sim$ iid x.

$\mu = E[x]$ $\qquad \sigma^2 = Var(x)$.

Sample mean: $\bar{x} = (x_1 + x_2 \ldots x_n)/n$

Expected value : μ,

Variance : σ^2/n

As $n \uparrow$, spread goes to 0.

Weak law of large numbers.

(Sharper bound than Chebyshev's)

Theorem :-

$$P(|\bar{x} - \mu| > \delta) \leq \frac{\sigma^2}{n\delta^2} \rightarrow 0$$

upper bound

With probability more than $1 - \frac{\sigma^2}{n\delta^2}$, sample means lies in $[\mu - \delta, \mu + \delta]$.

lower bound

works for $\delta > \sigma/\sqrt{n}$

Spiral

Date .....

Central Limit Theorem

Moment Generating function (MGF).

centralised : $(x-\mu)$

Let X be a zero-mean random variable, The MGF of X, denoted $M_X(\lambda)$ is a function from $\mathbb{R}$ to $\mathbb{R}$ defined as

$$M_X(\lambda) = E[e^{\lambda X}]$$

X: Discrete with PMF f_X

X takes values $\{x_1, x_2 \text{ ——} \}$.

$$M_X(\lambda) = f_X(x_1)e^{\lambda x_1} + f_X(x_2)e^{\lambda x_2} \text{ ——}$$

X: continuous with PDF f_X and supp(T_X).

$$M_X(\lambda) = \int_{x \in T_X} f_X(x)\, e^{\lambda x}\, dx$$

$X \sim \{0\}$

$$M_X(\lambda) = 1 \times e^0 = 1$$

$X \sim \{\overset{1-p}{-p}, \overset{p}{1-p}\}$

$$M_X(\lambda) = (1-p)\, e^{-p\lambda} + p\, e^{(1-p)\lambda}$$

MGF to dist.

$X \sim \{\overset{1/2}{-1}, \overset{1/4}{0}, \overset{1/4}{2}\}$

$$M_X(\lambda) = 0.5e^{-\lambda} + 0.25 + 0.25\, e^{2\lambda}$$

$X \sim \text{Normal}(0, \sigma^2)$

$$M_X(\lambda) = e^{\lambda^2\sigma^2/2}$$

$M_X(\lambda) \to 1$ as n increases

Spiral

if multiplied by N, MGF → 1

Date .....

Theorem (CLT)

$X_1, X_2 - - X_n \sim X$ iid with $E[X] = 0$

$Var(X) = \sigma^2$

$Y = (X_1 \text{———} + X_n) / \sqrt{n}$, Then,

$$M_Y(\lambda) \to e^{\lambda^2 \sigma^2 / 2}$$

MGF of Normal $(0, \sigma^2)$: $e^{\lambda^2 \sigma^2 / 2}$

Using CLT to approximate probability.

$X_1, X_2 - X_n \sim X$ iid.

let $\mu = E[X]$ $\sigma^2 = Var(X)$

$Y = X_1 + X_2 - \text{——} X_n$, $E[Y] = n\mu$

$P(Y - n\mu > \delta n\mu)$?

$\frac{Y - n\mu}{\sqrt{n}}$: approximately Normal $(0, \sigma^2)$

divide by σ

$$\frac{Y - n\mu}{\sigma\sqrt{n}} \approx \text{Normal } (0,1)$$

- $F(z)$, CDF of normal $(0,1)$
- $P(Y - n\mu > \delta n\mu) = P\left(\frac{Y - n\mu}{\sigma\sqrt{n}} > \frac{\delta\sqrt{n}\mu}{\sigma}\right)$

$$\approx 1 - F\left(\frac{\delta\sqrt{n}\mu}{\sigma}\right)$$

Spiral

Distributions, Properties and connections. Date .....

Linear Combination of independent normals.

$X_1, X_2 \ldots X_n \sim$ Independent Normal

let $X_i \sim \text{Normal}(\mu_i, \sigma_i^2)$

$Y = a_1 X_1 + \ldots + a_n X_n$ (linear combination)

Then $Y \sim \text{Normal}(\mu, \sigma^2)$

where $\mu = E[Y] = a_1\mu_1 + \ldots a_n\mu_n$

$\sigma^2 = a_1^2\sigma_1^2 + \ldots a_n^2\sigma_n^2$

linear combinations of ind normals is normal.

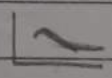
Gamma distribution.

$X \sim \text{Gamma}(\alpha, \beta)$

if PDF $f_X(x) \propto \underbrace{x^{\alpha-1} e^{-\beta x}}_{\text{controls the shape of dist.}}, \quad x > 0$

proportional.

$\alpha > 0$: shape parameter. $\alpha = 1 \Rightarrow$ Exp. function.

shape falls 

$\beta > 0$: rate parameter.

$\theta = 1/\beta$: scale parameter.

Mean : α/β Variance : α/β^2

MGF : $(1 - \lambda/\beta)^{-\alpha}, \quad \lambda < \beta$

Date .....

Sum of n iid $\exp(\beta)$ is Gamma (n, β)

Square of normal $(0, \sigma^2)$ is Gamma $(1/2, 1/2\sigma^2)$

Cauchy Distribution:

$X \sim$ Cauchy (θ, α^2)

if PDF $f_X(x) = \frac{1}{\pi} \frac{\alpha}{\alpha^2 + (x-\theta)^2}$

θ : location parameter , $\alpha > 0$; scale parameter

Mean : undefined Variance , MGF : undefined

Suppose $X, Y \sim$ iid normal $(0, \sigma^2)$ Then,

$$\frac{X}{Y} \sim \text{Cauchy } (0,1)$$

Beta distributions:

$X \sim$ Beta (α, β)

if PDF $f_X(x) \propto x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$

$\alpha > 0$, $\beta > 0$: shape parameters

Mean: $\alpha/(\alpha+\beta)$. Variance : $\alpha\beta / ((\alpha+\beta)^2 (\alpha+\beta+1))$

Beta $(\alpha, 1)$ has PDF $\propto x^{\alpha-1}$: power function dist.

Spiral

Date .....

Suppose $X \sim \text{Gamma}(\alpha, 1/\theta)$ $\quad Y \sim \text{Gamma}(\beta, 1/\theta)$

then, $\dfrac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$.

Descriptive statistics of Normal Sample.

Normal samples —

$$X_1, X_2 \text{ ——— } X_n \sim \text{iid Normal}(\mu, \sigma^2)$$

Sample mean:-

$$\bar{X} = \frac{X_1 + X_2 \text{ ——— } X_n}{n}$$

random variables.

Sample variance:

$$S^2 = \frac{(X_1 - \bar{X})^2 + \text{ ——— } + (X_n - \bar{X})^2}{n-1}$$

Sample mean:-

$$X_1, X_2 \text{ ——— } X_n \sim \text{iid normal}(\mu, \sigma^2)$$

$$\bar{X} = \frac{1}{n} X_1 + \text{ ——— } \frac{1}{n} X_n$$

Sample mean is a linear combination of iid normal random variables.

normal dist.

$$\bar{X} \sim \text{normal}(\underbrace{\mu}_{\text{mean}}, \underbrace{\sigma^2/n}_{\text{variance}})$$

Spiral

Date .....

Sum of Squares of normal Samples : Chi-square.

$$X_1 - \text{---} X_n \sim \text{iid normal } (0, \sigma^2)$$

X_i^2 : Gamma $(1/2, 1/2\sigma^2)$, independent.

~~Sa~~ Result: Sum of n ind. Gamma (α, β) is Gamma $(n\alpha, \beta)$

$$X_1^2 + X_2^2 - \text{---} X_n^2 \sim \text{Gamma}\left(n/2, 1/2\sigma^2\right)$$

$\rightarrow \sigma^2 = 1$ (standard normal)

Gamma $(n/2, 1/2)$: called Chi-square distribution with n degrees of freedom denoted χ^2_n

Theorem :-

Suppose $X_1 - - X_n \overset{iid}{\sim} \text{Normal } (\mu, \sigma^2)$. Then,

$\bar{X} \sim \text{normal } (\mu, \sigma^2/n)$

$\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$; Chi-square with $n-1$ degree of freedom.

$\bar{X}$ and $\bar{S}$ are independent.

PARAMETER ESTIMATION

Date .....

iid samples. $x_1, x_2 \text{ ---} x_n \sim \text{iid } X$.

X has a dist described by some parameters $\theta_1, \theta_2 \ldots$

Estimator for a parameter θ:

functions of the samples: $\hat{\theta}(x_1, x_2 \text{ ---} x_n)$

$\hat{\theta}$ is an estimator for parameter θ.

Parameter vs estimator

θ : constant parameter, not a random value.

$\hat{\theta}$: f^n of n random variables : random variable

↳ will have a distribution, PMF or PDF

ERROR in estimation:

$$e = \hat{\theta} - \theta$$

we may want error less than $\theta/10$ or $\theta/100$

↳ depends.

random variable

Chebyshev bound:

$$P\left(\left|\text{Error} - \underbrace{E[\text{Error}]}_{=0}\right| > \delta\right) \leq \frac{Var(\text{error})}{\delta^2}$$

$$P\left(|\text{Error}| > \delta\right) \leq \frac{Var(\text{Error})}{\delta^2}$$ upper bound.

$P \to 0$ as $n \to \infty$

Date:

Bias of an Estimator, Risk, Variance.

$$E[X] = \sum_{x \in X} x \, f_X(x)$$

Second moment : $E[X^2] = \sum_{x \in X} x^2 f_X(x)$

$$Var(X) = E[(X-\mu)^2]$$
$$Var(X) = E[X^2] - \mu^2$$
$$E[X^2] = Var(X) + \mu^2$$

Bias

$X_1, X_2 \text{ ——— } X_n \sim$ iid X parameter (θ).

Estimator for θ : $\hat{\theta}(X_1 \text{ ——— } X_n)$ or $\hat{\theta}$

$$\text{Bias}(\hat{\theta}, \theta) = E[\hat{\theta} - \theta] = \underline{E[\hat{\theta}] - \theta}$$

random variable

An estimator with bias is equal to 0, is said to be unbiased estimator.

Risk (squared error Risk)

The (squared - error) risk :-

$$\text{Risk}(\hat{\theta}, \theta) = E[(\hat{\theta} - \theta)^2]$$

also called mean squared error (MSE)

second moment of error.

Spiral

Date .....

Variance

$$\underbrace{Var(\hat{\theta})}_{\text{variance of estimator}} = E[(\hat{\theta} - E[\hat{\theta}])^2]$$

low $var(\hat{\theta})$ is good.

Variance of error :

$$\text{Variance (Error)} = Var(\hat{\theta})$$

Bias - variance trade off

$$\underbrace{Risk(\hat{\theta}, \theta)}_{\text{keep it low}} = \underbrace{Bias(\hat{\theta}, \theta)^2 + Var(\hat{\theta})}_{\text{both are positive, keep both low}}$$

Estimator design Approach : Method of Moments.

Moments $E[x]$, $E[x^2]$ etc can be expressed as functions of the parameters.

Moments of samples: $x_1, x_2 \text{ ——— } x_n \sim$ iid x

Sample moments (random variable)

$$M_k(x_1 \text{ ——— } x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i^k$$

Example :-

$x_1, x_2 \text{ ——— } x_n \sim$ iid Bernoulli(p)

$E(x) = p$

method of moment eq.

$$p = m_1$$

Spiral

Estimator

$$\boxed{\hat{p} = M_1} = \frac{1}{n}(X_1 + \cdots + X_n)$$

Example — Poisson

$$E[X] = \lambda$$

Method of moments eq.

$$\hat{\lambda} = M_1$$

Example: Exponential

$$E[X] = 1/\lambda$$

Method of moments eq:

$$1/\lambda = m_1$$

Sol. $\lambda = \frac{1}{m_1}$

Estimator:—

$$\hat{\lambda} = 1/M_1$$

Example: Normal (μ, σ^2)

2 moments because 2 parameters

$E[X] = \mu$ $\qquad$ $E[X^2] = \mu^2 + \sigma^2$

$\mu = m_1$ $\qquad$ $\mu^2 + \sigma^2 = m_2$

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Date .....

Sol:- $\mu = m_1$

$\sigma = \sqrt{m_2 - m_1^2}$

Estimator for μ -

$\hat{\mu} = M_1$

Estimator for σ

$$\hat{\sigma} = \sqrt{\frac{x_1^2 + x_2^2 - \cdots x_n^2}{n} - \frac{(x_1 - \cdots + x_n)^2}{n^2}}$$

Gamma (α, β)

$x_1 - \cdots x_n \sim$ iid Gamma (α, β)

$E[X] = \alpha/\beta$ $\qquad$ $E[X^2] = (\alpha^2/\beta^2) + (\alpha/\beta^2)$

$\alpha/\beta = m_1$

$\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta^2} = m_2$

Solve for α and β

$\Rightarrow \boxed{\alpha = \beta m_1}$

$\Rightarrow \frac{\beta^2 m_1^2}{\beta^2} + \frac{\beta m_1}{\beta^2} = m_2$

$\frac{m_1}{\beta} = m_2 - m_1^2$

$$\boxed{\beta = \frac{m_1}{m_2 - m_1^2}}$$

Spiral

Date .....

$$\boxed{\alpha = \frac{m_1^2}{m_2 - m_1^2}}$$

Estimator for α.

$$\hat{\alpha} = \frac{m_1^2}{m_2 - m_1^2}$$

for β

$$\hat{\beta} = \frac{m_1}{m_2 - m_1^2}$$

~~Bino~~ Binomial (N, p)

$E[x] = np$ $\qquad$ $E[x^2] = N^2p^2 + Np(1-p)$

~~$m_1 =$~~ $np = m_1$

$$n^2p^2 + np(1-p) = m_2$$

$$n = m_1/p$$

$$\frac{m_1^2}{\cancel{p^2}}\cancel{p^2} + \frac{m^1}{\cancel{p}} \cdot \cancel{p}(1-p) = m_2$$

$$m_1(1-p) = m_2 - m_1^2$$

$$1-p = \frac{m_2 - m_1^2}{m_1}$$

$$\boxed{p = 1 - \frac{m_2 - m_1^2}{m_1}}$$

Spiral

Date .....

$$\boxed{n = \frac{m_1^2}{m_1^2 + (m_1 - m_2)}}$$

$$\hat{p} = \frac{M_1^2 + (M_1 - M_2)}{M_1}$$

$$\hat{n} = \frac{m_1^2}{M_1^2 + (M_1 - M_2)}$$

Estimator design approach : Maximum likelihood.

likelihood of iid samples.

$X_1 \ldots X_n \sim$ iid X parameters: $\theta_1, \theta_2 \ldots$

$f_X(x)$: depends on θ_1, θ_2 ——

$f_X(x)$ as $f_X(x; \theta_1, \theta_2 \ldots)$

Example :- Normal (μ, σ^2)

$$f_X(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

likelihood of a sampling $x_1, x_2, x \ldots x_n$ denoted by $L(x_1 \ldots x_n)$.

$$L(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2 \ldots).$$

product.

Saare PDF/PMF ka multiplication

Spiral

Date .....

Maximum likelihood (ML) Estimator.

$$\theta_1^*, \theta_2^* \text{ ---- } = \underbrace{\arg}_{\text{value at which } L \text{ is max}} \max_{\theta_1, \theta_2} \underbrace{\prod_{i=1}^{n} f_X(x_i; \theta_1, \theta_2 \text{ ----})}_{\text{likelihood.}}$$

Example :-

X_1, X_2 ---- $X_n \sim$ iid Bernoulli(p)

Samples : x_1, x_2, x_3 ---- x_n

$x_i = 0$ or 1

$$L(x_1, x_2 \text{ ---- } x_n) = \prod_{i=1}^{n} f_X(x_i)$$

w is no. of 1s.

$$L(x_1, x \text{ ---}) = p^w (1-p)^{n-w}$$

To maximize, differentiate and equate to zero —

$$\cancel{\frac{dL}{dp}}$$

maximizing L is equivalent to maximize $\log L$.

$h(p)$. $\log L = w \log p + (n-w) \log (1-p)$

$$\frac{dh(p)}{dp} = \frac{w}{p} + \frac{n-w}{1-p}(-1)$$

Spiral

Date .....

$$\frac{w}{p} - \frac{(n-w)}{1-p} = 0$$

$$w(1-p) = p(n-w)$$

$$\frac{1-p}{p} = \frac{n-w}{w}$$

$$\frac{1}{p} - \cancel{1} = \frac{n}{w} - \cancel{1}$$

$$\boxed{p^* = w/n}$$

Example. Poisson (λ)

$$L = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

$$L = \frac{1}{x_1! \, x_2! \cdots x_n!} e^{-n\lambda} \lambda^{x_1 + x_2 \cdots x_n}$$

~~maxizie~~ :-

taking - log

$$\lambda^* = \arg\max_{\lambda} \left[(x_1 + x_1 \cdots x_n) \log \lambda - n\lambda \right]$$

diff w.r.t λ

$$\frac{x_1 + x_2 \cdots x_n}{\lambda} - n = 0$$

$$\lambda = \frac{x_1 + x_2 \cdots x_n}{n}$$

Spiral

Date .....

$\boxed{\text{Example normal } (\mu, \sigma^2)}$

$$L = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x_i-\mu)^2/2\sigma^2}$$

$$L = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-1/2\sigma^2 \sum_{i=1}^{n} (x_i-\mu)^2}$$

$$\mu^*, \sigma^* = \arg\max_{\mu,\sigma} \left(-\frac{1}{2\sigma^2} \sum_{1}^{n} (x_i-\mu)^2 - n \log \sigma\right)$$

for μ^*, diff with rt. μ (σ is constant)

$$\mu^* = \frac{d(x_i^2 - 2x_i\mu + \mu^2)}{d\mu}$$

$$= -2x_i + 2\mu$$

$$= -2(x_1 + x_2 + x_3 - x_n) + 2m = 0$$

$$= \cancel{2}m = \cancel{2}(x_1 - \quad)$$

$$\mu = \frac{x_1 + x_2 \quad}{n}$$

Date .....

finding MME and ML estimators :-

$\boxed{\underline{Exp(\lambda)}}$ $\quad x_1, x_2 \text{ --- } x_n \sim$ iid $Exp(\lambda)$.

$$\boxed{\hat{\lambda}_{MME} = \frac{n}{x_1 + \text{--} x_n}}$$

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^{n} e^{-\lambda (x_1 + x_2 \text{---} x_n)}$$

take log.

$$\hat{\lambda} = \arg\max \left(n \log \lambda - \lambda (x_1 + x_2 \text{---} x_n) \right)$$

diff. w.r.t λ.

$$0 = \frac{n}{\lambda} - (x_1 + x_2 \text{---} x_n)$$

$$\frac{n}{\lambda} = x_1 + x_2 \text{---} x_n$$

$$\lambda = \frac{n}{x_1 + x_2 \text{---} x_n}$$

$$\boxed{\hat{\lambda}_{ML} = \frac{n}{x_1 + x_2 \text{---} x_n}}$$

Problem.

$X_1 \text{---} x_n \sim$ iid $\{\overset{p_1}{1}, \overset{p_2}{2}, \overset{p_3}{3}\}$

$p_3 = 1 - p_1 - p_2$

p_1 and p_2 are unknown parameters

MME

$$E(x) = m_1 = p_1 + 2p_2 + 3p_3$$

Spiral

Date .....

$E[X^2]$ $\quad m_2 = p_1 + 4p_2 + 9p_3$

$p_3 = 1 - p_2 - p_3$

$m_1 = p_1 + 2p_2 + 3 - 3p_1 - 3p_2$

$m_1 = 3 - 2p_1 - p_2$

$\boxed{2p_1 + p^2 = 3 - m_1}$ —— ①

$m_2 = p_1 + 4p_2 + 9 - 9p_1 - 9p_2$

$m_2 = 9 - 8p_1 - 5p_2$

$\boxed{8p_1 + 5p_2 = 9 - m_2}$ —— ②.

$5 \times ① - 2$

$2p_1 = \cancel{15 - 5m_1}\ 6 - 5m_1 + m_2$

$p_1 = 3 - \frac{5m_1}{2} + \frac{m_2}{2}$

$p_2 = 3 - m_1 - 3 + 5m_1 + m_2$

$= 4m_1 - m_2 - 3.$

$$\boxed{\begin{aligned} \hat{p}_{1\,MME} &= 3 - \frac{5M_1}{2} + \frac{M_2}{2} \\ \hat{p}_{2\,MME} &= 4M_1 - M_2 - 3 \end{aligned}}$$

Date .....

ML

$$L = p_1^{w_1} p_2^{w_2} (1-p_1-p_2)^{n-w_1-w_2}$$

$w_1 \rightarrow$ no. of 1s

$w_2 \rightarrow$ no. of 2s.

$$p_1^*, p_2^* = \arg\max \left(w_1 \log p_1 + w_2 \log p_2 + (n-w_1-w_2) \log(1-p_1-p_2) \right)$$

$$\cancel{p_1^* = \frac{w_1}{p_1} + \frac{(n-w_1-w_2)}{1-p_1-p_2} \cdot (-1) = 0}$$

$$p_1^* = \frac{w_1}{n} \qquad p_2^* = \frac{w_2}{n}$$

$$\hat{p}_{ML} = \frac{\#1s}{n} \qquad \hat{p}_{ML} = \frac{\#2s}{n}$$

Uniform $[0, \theta]$ Continuous.

$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

MME.

$$m_1 = \frac{\theta}{2} \qquad \theta = 2m_1$$

$$\boxed{\hat{\theta}_{MME} = 2M_1}$$

~~MME~~.

$$\hat{\theta}_{MME} = 2\left(\frac{x_1 + x_2 \cdots x_n}{n}\right)$$

Spiral

Date .....

ML -

$$L = \begin{cases} \frac{1}{\theta^n} & 0 < x_1 \ldots x_n < \theta \\ 0 & \text{otherwise} \end{cases}$$

$\theta \geq \max(x_1 \ldots x_n)$

$\arg\max_{\theta} \left(\frac{1}{\theta^n}\right)$

pick least possible θ.

$\hat{\theta}_{ML} = \max(x_1, x_2 \ldots x_n)$.

Uniform $\{1, 2 \ldots N)$ (discrete).

ML.

$\hat{N}_{ML} = \max(x_1 \ldots x_n)$

MME.

$$M_1 = \frac{1}{n} + \frac{2}{N} + \frac{3}{N} \ldots \frac{N}{N}.$$

$$= \frac{1}{n}(1 + 2 \ldots N)$$

$$= \frac{1}{N} \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}.$$

Spiral

Date .....

$N = 2m_1 - 1$

$\hat{N}_{MME} = 2\bar{X} - 1$

Bayesian Estimation

Treate 'p' as random variable with a known parameter
↳ prior distribution

Prior distribution :-
Intial dist of "p".

$X_1, X_2 \ldots X_n \sim$ iid Ber (θ)

$X_i = \begin{cases} 0 & \text{Head appears on } i^{th} \text{ toss} \\ 1 & \text{tail} \quad '' \end{cases}$

Success → ~~gg~~ getting a tail.

Estimate what is the prob of getting a tail.

'θ is a parameter'

Assumption. lets Assume $\theta \sim \{0.1, 0.5, 0.9\}$
1/4 1/2 1/2

prior dist.

Conditional expected value :- (posterior mean)

given sample

$\hat{\theta} = p(a) \times p(a|s) + p(b) \times p(b|s)$

Spiral

Date .....

Step 1: Prior dist.
Samples.

Step 2: Using Sample, we will compute conditional probabilities.

Step 3: Compute posterior mean.

Example :-

prior dist. $\theta \sim \{0.1, 0.5, 0.9\}$
$1/4 \quad 1/2 \quad 1/4$

$S = (1, 0, 1, 0, 0, 0, 0, 1, 1, 1)$

Baye's Rule :-

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Conditional :-

$$P(\theta = 0.1 | S)$$

$$= \frac{P(S | \theta = 0.1) \cdot P(\theta = 0.1)}{P(S)}$$

So $P(S|\theta = 0.1) = P(x_1 = 1, x_2 = 0 \text{ ——— } x_{10} = 1 \mid \theta = 0.1)$

$P(x_1 = 1) \times P(x_2 = 0) \text{ ——— } \times P(x_{10} = 1)$ given $\theta = 0.1$

$0.1 \times 0.9 \text{ ——— } 0.1$

$= (0.1)^5 (0.9)^5$

$$\frac{(0.1)^5 (0.9)^5 \cdot (0.24)}{P(S)}$$

Spiral

Date .....

$P(\theta = 0.5 | S)$

$$= \frac{(0.5)^5 \cdot (0.5)^5 \times (0.5)}{P(S)}$$

$P(\theta = 0.9 | S)$

$$\frac{(0.9)^5 \cdot (0.1)^5 \cdot (0.25)}{P(S)}$$

max : . . $\hat{\theta} = 0.5$

As $P(\theta = 0.5) > P(\theta = 0.1)$ & $P(\theta = 0.9)$

This is called posterior mod.

$\boxed{\text{Posterior mean.}}$

$$\hat{\theta} = P(\theta = 0.1) \times P(\theta = 0.1 | S) + P(\theta = 0.5) \cdot P(\theta = 0.5 | S) + P(\theta = 0.9) \cdot P(\theta = 0.9 | S)$$

numerators ka sum

Requires $P(S)$. Total probability

$$P(S) = P(S | \theta = 0.1) \cdot P(\theta = 0.1) + P(S | \theta = 0.5) \cdot P(\theta = 0.5) + P(S | \theta = 0.9) \cdot P(\theta = 0.9)$$

$\hat{\theta}$ ——

Given S:

$$\hat{\theta} = x_1 \times P(\theta = x_1 | S) + x_2 \times P(\theta = x_2 | S) \text{ —— } x_n \times P(\theta = x_n | S).$$

$$= \frac{x_1 \times P(\theta = x_1) + x_2 \times P(\theta = x_2) \text{ —— } x_n P(\theta = x_n)}{S}$$

Spiral

Date .....

likelihood

prior

$$P(\theta = 0.1 | S) = \frac{P(S | \theta = 0.1) \; P(\theta = 0.1)}{P(S)}$$

posterior

normalization $(0 < P < 1)$

Posterior $\propto$ likelihood $\times$ prior.

$f_X(x)$

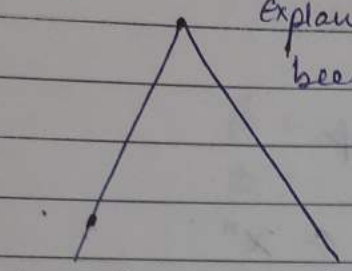
$E[X]$ of a ~~prior~~ posterior dist.

Spiral

HYPOTHESIS TESTING

Date .....

Hypothesis :- An idea that is suggested as the possible explanation of something but has not yet been proven.



Accept it Reject it.

two types of statistical hypothesis :-

(i) Null hypothesis (H_0) :
states that there is no difference b/w two parameters.
Our claim.

(ii) Alternate hypothesis (H_1) or (H_A) :
states that there is an existence of difference b/w two parameters.

Simple Hypothesis :- " = "
Complex " :- " >, <, ≥, ≤, ≠ "

Binary Hypothesis - testing :-
↳ 2 hypo. : H_0 and H_A (or H_1)
↓
verify (accept / reject H_0)

Date .....

Acceptance Set (A) & test :-

$X_1, X_2 \text{ — — } X_n \sim \text{iid } X$

H_0: null hypo $\qquad$ H_1: Alternate Hypo

$x_1 \in X$ then $(x_1, x_2 \text{ — — } x_n) \in X^n$

Any subset $A \subseteq X^n$ is an Acceptance set.
For every A, there is a corresponding hypothesis testing.

If $(x_1, x_2 \text{ — — } x_n) \in A$, then we will accept, else we reject H_0.

Total no. of subset :- $2^{\text{no. of elements}}$

ERRORS :-

Type I:
H_0 rejected when true.

Type II:
H_0 accepted when not true.

Size of a test (significance level)

denoted by α.
probability of type I error.

$\alpha = P(\text{Type I error})$
$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

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power of a test $(1-\beta)$:

β = prob. of Type II error

$\beta = P(\text{Type II error})$

$\beta = P(\text{Accept } H_0 \mid H_1 \text{ is true})$

$$\text{power} = 1-\beta = 1 - P(\text{Accept } H_0 \mid H_0 \text{ is true})$$
$$= P(\text{Reject } H_0 \mid H_1 \text{ is true})$$

Note: The test will be powerful whose α and β is as lowest as possible.

As $\alpha \uparrow$, $\beta \downarrow$ and vice-versa.

Z-test

$$\alpha = P\left(\frac{\bar{x}-\mu}{\sqrt{var}} < \frac{c-\mu}{\sqrt{var}} \;\middle|\; \mu = \mu\right)$$

$\frac{\sigma}{\sqrt{n}}$

P-value α^*

lowest significance ~~of~~ level of 'α' at which the null will be rejected for T = t (test statistic value) is called p-value of the sampling.

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P-value $= \alpha^* = P(T > t \mid H_0 \text{ is true})$

If P-value is very low, we reject H_0.

Eg:-

$X_1, X_2 \text{ ——— } X_n \sim \text{iid Bernoulli}(p)$

$H_0 : P = 0.4$ $\qquad$ $H_1 : P = 0.6$

$T = X_1 + \text{——} + X_n$ = # of heads.

$E[p] = 100p$

$Var(p) = 100\, p(1-p)$

Test: reject H_0 if $T > c$

we observed 50 heads -

$t = 50$

P value $= \alpha^* = P(T > 50 \mid P = 0.4)$

$$P\left(\frac{T - E(T)}{\sqrt{Var(T)}} > \frac{50 - E(T)}{\sqrt{Var(T)}} \;\middle|\; p = 0.4\right)$$

$$P\left(Z > \frac{50-40}{\sqrt{100 \times .4 \times .6}}\right) = P\left(Z > \frac{10}{\sqrt{24}}\right)$$

$$= 1 - F_Z\left(\frac{10}{\sqrt{24}}\right)$$

P-value $> \alpha \longrightarrow$ Accept H_0

P-value $< \alpha \longrightarrow$ Reject H_0

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Standard tests :-

(i) One Sample :-

$X_1 \text{ ---- } X_n \sim$ iid X

$E[X] = \mu$ $\qquad$ $var(X) = \sigma^2$ (known)

Testing for mean:

$H_0 = \mu = \mu_0$

$H_A = \mu > \mu_0$ $\quad$ Right tailed test

$\mu < \mu_0$ $\quad$ left tail

$\mu \neq \mu_0$ $\quad$ two tail.

} z-test

$$T = \frac{X_1 + X_2 \text{ --- } X_n}{n}$$

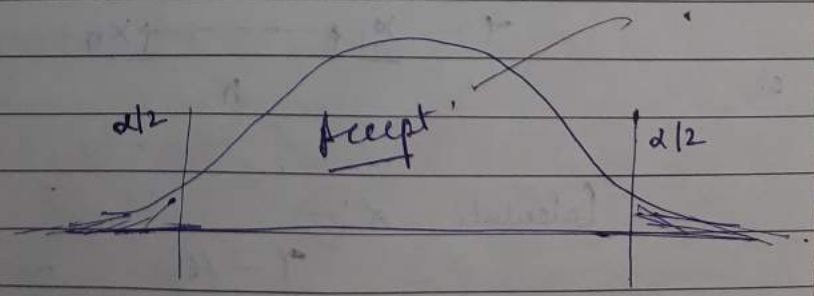
Test : Reject H_0 if $T > C$

$T < C$

$|T - \mu| > C$

$$z = \frac{T - \mu}{\sigma / \sqrt{n}}$$

for $|T - \mu| > C$



$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= P(|T - \mu| > C \mid \mu = \mu_0)$

$= P(T - \mu > C \mid \mu = \mu_0) + P(T - \mu < -C \mid \mu = \mu_0)$

$= P\left(\frac{T - \mu}{\sigma/\sqrt{n}} > \frac{C}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0\right) + P\left(\frac{T - \mu}{\sigma/\sqrt{n}} < \frac{-C}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu_0\right)$

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$$P\left(z > \frac{c}{\sigma/\sqrt{n}}\right) + P\left(z < \frac{-c}{\sigma/\sqrt{n}}\right)$$

$$1 - F_Z\left(\frac{c}{\sigma/\sqrt{n}}\right) + F_Z\left(\frac{-c}{\sigma/\sqrt{n}}\right)$$

$$\boxed{\alpha = 2 \times F_Z\left(-\frac{c\sqrt{n}}{\sigma}\right)}$$

(ii) One Sample :- $\longrightarrow$ z-test

$x_1 - \quad - x_n \sim$ iid x

$E[x] = \mu$ $\qquad$ $Var(x) = \sigma^2$ (unknown)

Testing for mean

$H_0 = \mu = \mu_0$

$H_1 = \mu > \mu_0$ right tail

$\mu < \mu_0$ left "

$\mu \neq 0$ two "

} t-test

$$\bar{x} = \frac{x_1 + \quad + x_n}{n}$$

Calculate α :-

$$\frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

$s = \sqrt{\text{Sample variance}}$

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(iii) One sample.

$$X_1 \text{ ——— } X_n \sim \text{iid } X$$

$$E[X] = \mu \qquad Var(X) = \sigma^2$$

Testing for variance

$$H_0 = \sigma = \sigma_0$$

$$H_1 = \sigma > \sigma_0$$

$$T = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

χ^2 test or chi^2-test.

Test: Reject H_0, if $S > c$ or $S^2 > c^2$

How to compute α:-

Given H_0:-

$$\frac{n-1}{\sigma_0^2} S^2 \sim \chi^2_{n-1}$$

Two Samples test:-

$$X_1 \text{ ——— } X_n \sim \text{iid } X \qquad E[X] = \mu_1 \quad var(X) = \sigma_1^2$$

$$Y_1 \text{ ——— } Y_n \sim \text{iid } Y \qquad E[Y] = \mu_2 \quad var(Y) = \sigma_2^2$$

Testing to compare mean:-

$$H_0 = \mu_1 = \mu_2$$

$$H_1 = \mu_1 \neq \mu_2$$

$$T = \bar{Y} - \bar{X}$$

Test: Reject H_0 if $|T| > c$

Two sample z-test.

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How to calculate α:

Given H_0: $T \sim N(0, \sigma_T^2)$

$$\sigma_T^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

$$\alpha = P(|T| > c \mid \mu_1 = \mu_2)$$

$$= P(T > c) + P(T < -c)$$

$$P\left(\frac{T-0}{\sigma_T} > \frac{c}{\sigma_T}\right) + P\left(\frac{T-0}{\sigma_T} < \frac{-c}{\sigma_T}\right)$$

$$P\left(z > \frac{c}{\sigma_T}\right) + P\left(z < \frac{-c}{\sigma_T}\right)$$

$$2 \times F_z\left(z < \frac{-c}{\sigma_T}\right)$$

Testing to Compare Variance:

$H_0: \sigma_1 = \sigma_2$

$H_1: \sigma_1 \neq \sigma_2$

$$T = \frac{S_X^2}{S_Y^2} \sim F(n_1 - 1, n_2 - 1)$$

Test: Reject H_0 if $T > 1 + C_R$ or
$T < 1 - C_L$

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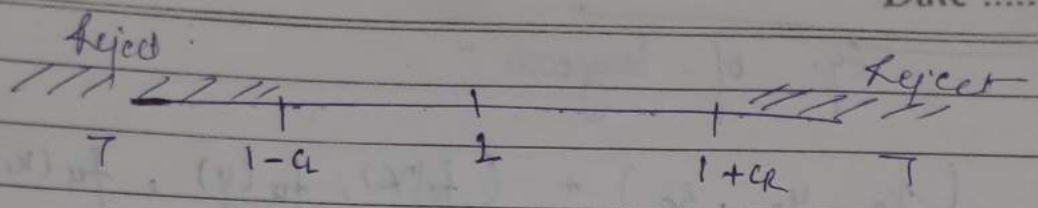

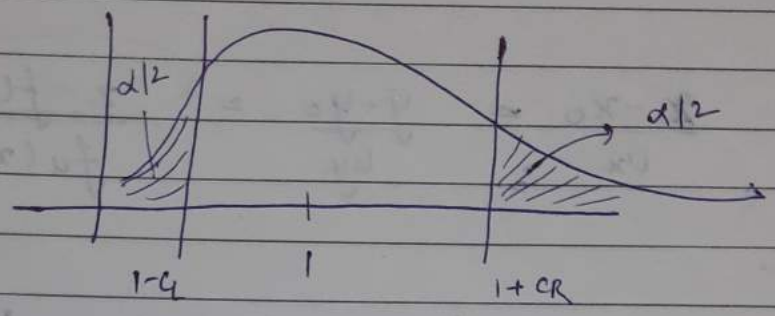

α :—

Given H_0 : $T \sim F(n_1-1, n_2-1)$

$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= P(T < 1-C_L \mid \sigma_1 = \sigma_2)$ — $\alpha/2$
$+ P(T > 1+C_R \mid \sigma_1 = \sigma_2)$

$\alpha/2 = F_{F(n_1-1, n_2-1)}(1-C_L) = 1 - F_{F(n_1-1, n_2-1)}(1+C_R)$

$n \leq 25$

$n \leq 15$ → dont use CLT

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