Fast calculation of the exact radiological path for a three-dimensional CT array[a)]

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Ready availability has prompted the use of computed tomography (CT) data in various applications in radiation therapy. For example, some radiation treatment planning systems now utilize CT data in heterogeneous dose calculation algorithms. In radiotherapy imaging applications, CT data are projected onto specified planes, thus producing "radiographs," which are compared with simulator radiographs to assist in proper patient positioning and delineation of target volumes. All these applications share the common geometric problem of evaluating the radiological path through the CT array. Due to the complexity of the three-dimensional geometry and the enormous amount of CT data, the exact evaluation of the radiological path has proven to be a time consuming and difficult problem. This paper identifies the inefficient aspect of the traditional exact evaluation of the radiological path as that of treating the CT data as individual voxels. Rather than individual voxels, a new exact algorithm is presented that considers the CT data as consisting of the intersection volumes of three orthogonal sets of equally spaced, parallel planes. For a three-dimensional CT array of N^3 voxels, the new exact algorithm scales with $3N$, the number of planes, rather than N^3, the number of voxels. Coded in FORTRAN-77 on a VAX 11/780 with a floating point option, the algorithm requires approximately 5 ms to calculate an average radiological path in a 100^3 voxel array.

Key words: radiological path, inhomogeneity correction, CT

INTRODUCTION

In radiation therapy applications, computer tomography (CT) data are utilized in various dose calculation and imaging algorithms. For example, some radiation treatment planning systems now utilize two-dimensional CT data for pixel-based heterogeneous dose calculations. Other systems forward project three-dimensional CT data onto specified planes, thus forming "radiographs," which are compared with simulator radiographs to assist in proper patient positioning and delineation of target volumes. All such applications, whether in inhomogeneity calculations or imaging applications, essentially reduce to the same geometric problem: that of calculating the radiological path for a specified ray through the CT array.

Although very simple in principle, elaborate computer algorithms and a significant amount of computer time is required to evaluate the exact radiological path. The amount of detail involved was recently emphasized by Harauz and Ottensmeyer,[1] who stated that even for the two-dimensional case, their algorithm for calculating the exact radiological path grew more and more unwieldy and time consuming, while remaining unreliable. For three dimensions, they concluded that determining the exact radiological path is not viable. This paper describes an *exact*, efficient, and reliable algorithm for calculating the radiological path through a three-dimensional CT array.

Denoting a particular voxel density as $\rho(i,j,k)$ and the length contained by that voxel as $l(i,j,k)$, the radiological path may be written as

$$d = \sum_i \sum_j \sum_k l(i,j,k)\rho(i,j,k). \tag{1}$$

Direct evaluation of Eq. (1) entails an algorithm which scales with the number of terms in the sums, that is, the number of voxels in the CT array. The following describes an algorithm that scales with the sum of linear dimensions of the CT array.

METHOD

Rather than independent elements, the voxels are considered as the intersection volumes of orthogonal sets of equally spaced, parallel planes. Without loss of generality, Fig. 1 illustrates the two-dimensional case, where pixels are considered as the intersection areas of orthogonal sets of equally spaced, parallel lines. The intersections of the ray with the lines are calculated, rather than intersections of the ray with

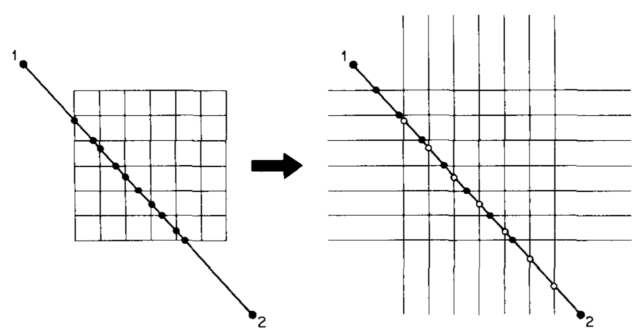

FIG. 1. The pixels of the CT array (left) may be considered as the intersection areas of orthogonal sets of equally spaced, parallel lines (right). The intersections of the ray with the pixels are a subset of the intersections of the ray with the lines. The intersections of the ray with the lines are given by two equally spaced sets: one set for the horizontal lines (filled circles) and one set for the vertical lines (open circles). The generalization to a three-dimensional CT array is straightforward.

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the individual pixels. Determining the intersections of the ray with the equally spaced, parallel lines is a particularly simple problem. As the lines are equally spaced, it is only necessary to determine the first intersection and generate all the others by recursion. As shown on the right illustration of Fig. 1, the intersections consist of two sets, one set for the intersections with the horizontal lines (closed circles) and one set for the intersections with the vertical lines (open circles). Comparing the left and right illustrations of Fig. 1, it is clear that the intersections of the ray with the pixels is a subset of the intersections with the lines. Identifying that subset allows the radiological path to be determined. The extension to the three-dimensional CT array is straightforward.

The ray from point 1 to point 2 may be represented parametrically as

$$
\begin{aligned}
X(\alpha) &= X_1 + \alpha(X_2 - X_1),\\
Y(\alpha) &= Y_1 + \alpha(Y_2 - Y_1),\\
Z(\alpha) &= Z_1 + \alpha(Z_2 - Z_1),
\end{aligned}
\tag{2}
$$

where the parameter α is zero at point 1 and unity at point 2. The intersections of the ray with the sides of the CT array are shown in Fig. 2. If both points 1 and 2 are outside the array [Fig. 2(a)], then the parametric values corresponding to the two intersection points of the ray with the sides are given by $\alpha_{\min}$ and $\alpha_{\max}$. All intersections of the ray with individual planes must have parametric values which lie in the range $(\alpha_{\min}, \alpha_{\max})$. For the case illustrated in Fig. 2(b), where point 1 is inside the array, the value of $\alpha_{\min}$ is zero. Likewise, for Fig. 2(c), if point 2 is inside, then $\alpha_{\max}$ is one. For both points 1 and 2 inside the array [Fig. 2(d)], then $\alpha_{\min}$ is zero and $\alpha_{\max}$ is one. The solution to the intersection of the ray with the CT

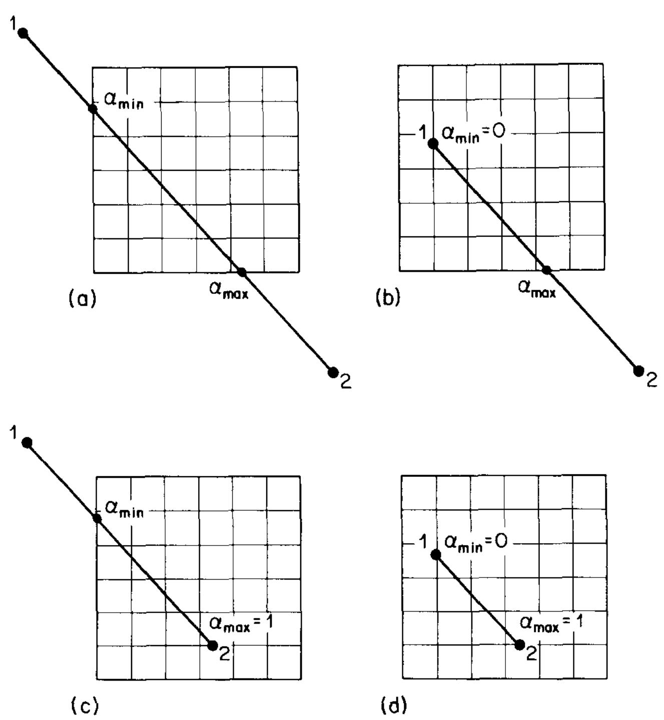

FIG. 2. The quantities $\alpha_{\min}$ and $\alpha_{\max}$ define the allowed range of parametric values for the intersections of the ray with the sides of the CT array: (a) both 1 and 2 outside the array, (b) 1 inside and 2 outside, (c) 1 outside and 2 inside, and (d) 1 inside and 2 inside.

voxels follows immediately: Determine the parametric intersection values, in the range $(\alpha_{\min}, \alpha_{\max})$, of the ray with each orthogonal set of equally spaced, parallel planes. Merge the three sets of parametric values into one set; for example, merging of the sets (1,4,7), (2,5,8), and (3,6,9) results in the merged set (1,2,3,4,5,6,7,8,9). The length of the ray contained by a particular voxel, in units of the ray length, is simply the difference between two adjacent parametric values in the merged set. For each voxel intersection length, the corresponding voxel indices are obtained, and the products of the length and density are summed over all intersections to yield the radiological path. A more detailed description of the algorithm is given in the following section.

ALGORITHM

For a CT array of $(N_x - 1, N_y - 1, N_z - 1)$ voxels, the orthogonal sets of equally spaced, parallel planes may be written as

$$
\begin{aligned}
X_{\text{plane}}(i) &= X_{\text{plane}}(1) + (i-1)d_x \quad (i = 1,\ldots,N_x),\\
Y_{\text{plane}}(j) &= Y_{\text{plane}}(1) + (j-1)d_y \quad (j = 1,\ldots,N_y),\\
Z_{\text{plane}}(k) &= Z_{\text{plane}}(1) + (k-1)d_z \quad (k = 1,\ldots,N_z),
\end{aligned}
\tag{3}
$$

where d_x, d_y, and d_z are the distances between the x, y, and z planes, respectively. The quantities d_x, d_y, and d_z are also the lengths of the sides of the voxel. The parametric values $\alpha_{\min}$ and $\alpha_{\max}$ are obtained by intersecting the ray with the sides of the CT array. From Eqs. (2) and (3), the parametric values corresponding to the sides are given by the following: If $(X_2 - X_1) \neq 0$,

$$
\begin{aligned}
\alpha_x(1) &= [X_{\text{plane}}(1) - X_1]/(X_2 - X_1),\\
\alpha_x(N_x) &= [X_{\text{plane}}(N_x) - X_1]/(X_2 - X_1),
\end{aligned}
\tag{4}
$$

with similar expressions for $\alpha_y(1)$, $\alpha_y(N_y)$, $\alpha_z(1)$, and $\alpha_z(N_z)$. If the denominator $(X_2 - X_1)$ in Eq. (4) is equal to zero, then the ray is perpendicular to the x axis, and the corresponding values of α_x are undefined, and similarly for α_y and α_z. If the α_x, α_y, or α_z values are undefined, then those values are simply excluded in all the following discussion.

In terms of the parametric values given above, the quantities $\alpha_{\min}$ and $\alpha_{\max}$ are given by

$$
\begin{aligned}
\alpha_{\min} &= \max\{0, \min[\alpha_x(1), \alpha_x(N_x)],\\
&\qquad \min[\alpha_y(1), \alpha_y(N_y)], \min[\alpha_z(1), \alpha_z(N_z)]\},\\
\alpha_{\max} &= \min\{1, \max[\alpha_x(1), \alpha_x(N_x)],\\
&\qquad \max[\alpha_y(1), \alpha_y(N_y)], \max[\alpha_z(1), \alpha_z(N_z)]\},
\end{aligned}
\tag{5}
$$

where the functions min and max select from their argument list, the minimum and maximum terms, respectively. If $\alpha_{\max}$ is less than or equal to $\alpha_{\min}$, then the ray does not intersect the CT array.

From all intersected planes, there are only certain intersected planes which will have parametric values in the range $(\alpha_{\min}, \alpha_{\max})$. From Eqs. (2), (3), and (5), the range of indices $(i_{\min}, i_{\max})$, $(j_{\min}, j_{\max})$, and $(k_{\min}, k_{\max})$, corresponding to these particular planes, are given by the following: If $(X_2 - X_1) \geqslant 0$,

$$i_{\min} = N_x - [X_{\text{plane}}(N_x) - \alpha_{\min}(X_2 - X_1) - X_1]/d_x,$$
$$i_{\max} = 1 + [X_1 + \alpha_{\max}(X_2 - X_1) - X_{\text{plane}}(1)]/d_x; \tag{6}$$

if $(X_2 - X_1) \leqslant 0$

$$i_{\min} = N_x - [X_{\text{plane}}(N_x) - \alpha_{\max}(X_2 - X_1) - X_1]/d_x,$$
$$i_{\max} = 1 + [X_1 + \alpha_{\min}(X_2 - X_1) - X_{\text{plane}}(1)]/d_x,$$

with similar expressions for $j_{\min}, j_{\max}, k_{\min}$, and $k_{\max}$.

For a given range of indices $(i_{\min}, i_{\max})$, $(j_{\min}, j_{\max})$, and $(k_{\min}, k_{\max})$, the sets parametric values $\{\alpha_x\}$, $\{\alpha_y\}$, and $\{\alpha_z\}$, corresponding to the intersections of the ray with the planes are given by the following: If $(X_2 - X_1) > 0$,

$$\{\alpha_x\} = \{\alpha_x(i_{\min}),...,\alpha_x(i_{\max})\}; \tag{7}$$

if $(X_2 - X_1) < 0$,

$$\{\alpha_x\} = \{\alpha_x(i_{\max}),...,\alpha_x(i_{\min})\},$$

where

$$\alpha_x(i) = [X_{\text{plane}}(i) - X_1]/(X_2 - X_1)$$
$$= \alpha_x(i-1) + [d_x/(X_2 - X_1)],$$

with similar expressions for $\{\alpha_y\}$ and $\{\alpha_z\}$.

As given by Eq. (7), the sets $\{\alpha_x\}$, $\{\alpha_y\}$, and $\{\alpha_z\}$ are each arranged in ascending order. Each term in each set corresponds to an intersection of the ray with a particular plane. The intersections of the ray with the voxels are found by merging the sets $\{\alpha_x\}$, $\{\alpha_y\}$, and $\{\alpha_z\}$ into one set. To include the case where one or both of the end points of the ray may be inside the CT array, the parametric values $\alpha_{\min}$ and $\alpha_{\max}$ are appended to the merged parametric sets. The terms $\alpha_{\min}$, $\alpha_{\max}$, and the merged sets $\{\alpha_x\}$, $\{\alpha_y\}$, and $\{\alpha_z\}$ are denoted by the set $\{\alpha\}$:

$$\{\alpha\} = \{\alpha_{\min}, \text{merge}[\{\alpha_x\},\{\alpha_y\},\{\alpha_z\}], \alpha_{\max}\}$$
$$= \{\alpha(0),...,\alpha(n)\}, \tag{8}$$

where the last term has an index n given by

$$n = (i_{\max} - i_{\min} + 1) + (j_{\max} - j_{\min} + 1)$$
$$+ (k_{\max} - k_{\min} + 1) + 1. \tag{9}$$

Two adjacent terms in the set $\{\alpha\}$ refer to the intersections of the ray with a particular voxel. For two intersections m and $m-1$, the voxel intersection length $l(m)$ is given by

$$l(m) = d_{1\,2}[\alpha(m) - \alpha(m-1)] \quad (m = 1,...,n), \tag{10}$$

where the quantity $d_{1\,2}$ is the distance from point 1 to point 2,

$$d_{1\,2} = [(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2]^{1/2}. \tag{11}$$

The voxel $[i(m), j(m), k(m)]$, which corresponds to intersections m and $m-1$, is that which contains the midpoint of the two intersections. From Eqs. (2), (3), and (5), the indices $[i(m), j(m), k(m)]$ are given by

$$i(m) = 1 + [X_1 + \alpha_{\text{mid}}(X_2 - X_1) - X_{\text{plane}}(1)]/d_x,$$
$$j(m) = 1 + [Y_1 + \alpha_{\text{mid}}(Y_2 - Y_1) - Y_{\text{plane}}(1)]/d_y, \tag{12}$$
$$k(m) = 1 + [Z_1 + \alpha_{\text{mid}}(Z_2 - Z_1) - Z_{\text{plane}}(1)]/d_z,$$

where α_{mid} is given by

$$\alpha_{\text{mid}} = [\alpha(m) + \alpha(m-1)]/2. \tag{13}$$

The radiological path d [Eq. (1)] may now be written as

$$d = \sum_{m=1}^{m=n} l(m)\rho[i(m), j(m), k(m)]$$
$$= d_{1\,2} \sum_{m=1}^{m=n} [\alpha(m) - \alpha(m-1)]\,\rho[i(m), j(m), k(m)], \tag{14}$$

where n is given by Eq. (9), $l(m)$ is given by Eq. (10), and the indices $[i(m), j(m), k(m)]$ are given by Eq. (12).

DISCUSSION

The new radiological path algorithm is summarized in the block diagram in Fig. 3. For a typical problem, the relative amount of computation time required in each section of the algorithm is given by the respective percentages to the right of each descriptive block. The new algorithm is coded in FORTRAN-77 and is run on a VAX 11/780 with a floating point option. At present, no attempt has been made to optimize the code in machine language or adapt the algorithm to run on an array processor. Rather, the algorithm has been coded in the straightforward manner described in the text.

The performance of the algorithm is illustrated for a typical dose calculation problem shown in Fig. 4. The CT array is taken to be a cube with N^3 voxels. Point 1 of the ray path is centered above the array. An internal calculation grid of 21^3 points, corresponding to point 2 of the ray path, is distributed uniformly within the CT array. The mean calculation time per point, t, is obtained as a function of the array size N (Fig. 5). The mean time t is the total time to calculate the

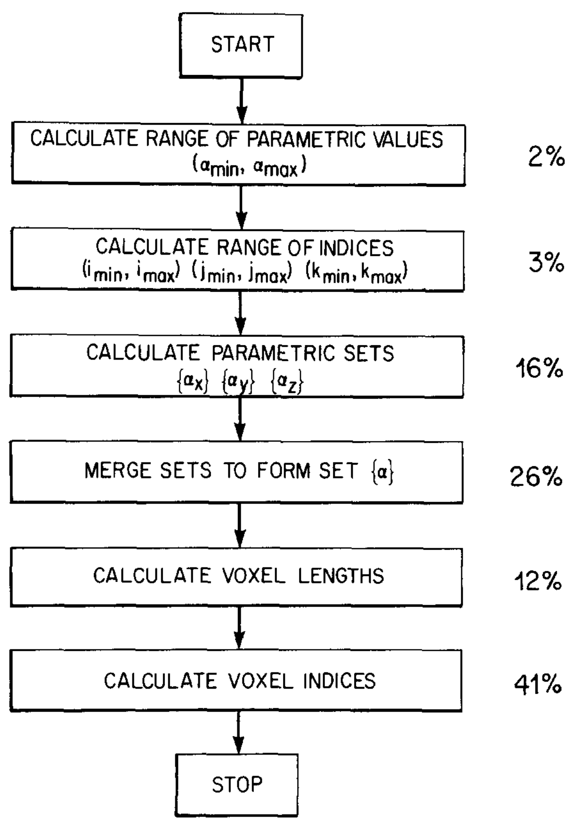

FIG. 3. Block diagram of the new algorithm to calculate the radiological path for a three-dimensional CT array. The percentages indicate the relative amount of computational time spent in various portions of the algorithm.

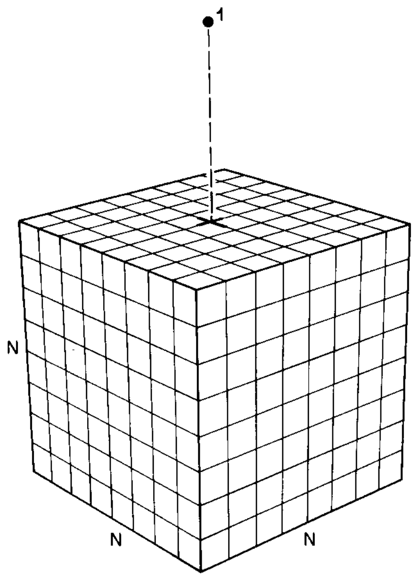

FIG. 4. The performance of the new algorithm is illustrated for the problem of a CT array of N^3 voxels. Point 1 of the ray path is centered above the CT array. A calculation grid of 21^3 points, corresponding to point 2 of the ray path, is uniformly distributed within the CT array.

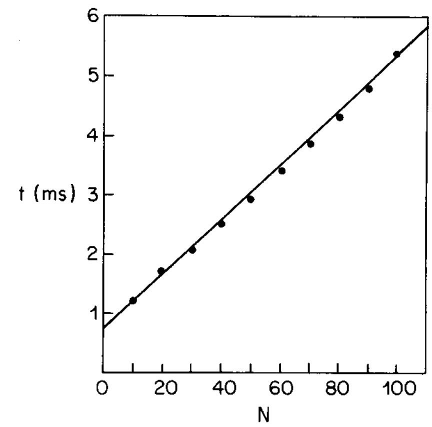

FIG. 5. The mean computational time per point, t, for the example in Fig. 4 as a function of the array size N. Note that the new algorithm scales with the linear size N and not the number of voxels N^3.

radiological path at all 21^3 points divided by the number of points. As illustrated in Fig. 5, the new algorithm scales with N for a CT array of N^3 voxels.

CONCLUSION

An algorithm has been developed which evaluates the exact radiological path of a ray through a three-dimensional CT array. Rather than consider individual voxels of the CT array, the algorithm calculates the intersections of the ray with orthogonal sets of equally spaced, parallel planes. For an array of N^3 voxels, considering the planes rather than the voxels allows the algorithm to scale with the number of planes (proportional to N), rather than the number of voxels (proportional to N^3). The intersections are described as parametric values along the ray. The intersections of the ray with the voxels are obtained as a subset of the intersections of the ray with the planes. For each voxel intersection length, the corresponding voxel indices are obtained and the products of the intersected length and particular voxel density are summed over all intersections to yield the radiological path. The algorithm is exact, efficient, reliable, and particularly straightforward to implement in computer code.

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[1] G. Harauz and F. P. Ottensmeyer, Phys. Med. Biol. 28, 1419 (1983).