Correction DS LE2

Méthodes Mathématiques 01/20.

Ex I : 1 (a) $X_{n+1} = AX_n$

(b) Montrons par récurrence que $X_n = A^n X_0$.

Initialisation : $n=0$ $\quad X_0 = I_3 X_0 = A^0 X_0$.

Hérédité : Supposons qu'il existe $k \in \mathbb{N}$ tq $X_k = A^k X_0$.

$$X_{k+1} \underset{1.(a)}{=} AX_k = AA^k X_0 = A^{k+1} X_0.$$

Conclusion : $\forall n \in \mathbb{N}$, $X_n = A^n X_0$.

2. (a)
$$\chi_A(X) = \begin{vmatrix} 3-X & -1 & 1 \\ 1 & 2-X & 0 \\ 0 & 1 & 1-X \end{vmatrix} \underset{C_3 \leftarrow C_3 + C_2}{=} \begin{vmatrix} 3-X & -1 & 0 \\ 1 & 2-X & 2-X \\ 0 & 1 & 2-X \end{vmatrix} = (2-X) \begin{vmatrix} 3-X & -1 & 0 \\ 1 & 2-X & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\underset{L_2 \leftarrow L_2 - L_3}{=} (2-X) \begin{vmatrix} 3-X & -1 & 0 \\ 1 & 1-X & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \times (-1)^{3+3} (2-X) \begin{vmatrix} 3-X & -1 \\ 1 & 1-X \end{vmatrix} = (2-X)\left[(3-X)(1-X)+1\right].$$

$$= (2-X)(X^2 - 4X + 4)$$
$$= (2-X)(X-2)^2 = -(X-2)^3.$$

$Sp(A) = \{2\}$.

(b)
$$\operatorname{Ker}(A - 2I_3) = \operatorname{Ker}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \begin{array}{c} x - y + z = 0 \\ x = 0 \\ y - z = 0 \end{array} \right\} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, \begin{array}{c} x = 0 \\ y = z \end{array} \right\} = \operatorname{Vect}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$: sous-espace propre associé à la vp 2.

$\dim \operatorname{Ker}(A - 2I_3) = 1 \neq 3 =$ multiplicité de 2 dans χ_A. A n'est pas diagonalisable.

3.
$$T = \begin{array}{cc} \begin{matrix} Ae_1' & Ae_2' & Ae_3' \end{matrix} & \\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} & \begin{matrix} e_1' \\ e_2' \\ e_3' \end{matrix} \end{array}$$

$Ae_1' = 2e_1'$: on pose $e_1' = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{Ker}(A - 2I_3)$.

$Ae_2' = e_1' + 2e_2'$ où $e_2' = \begin{pmatrix} a \\ b \\ -1 \end{pmatrix}$.

$\Updownarrow$

$$\begin{cases} 3a - b - 1 = 0 + 2a \\ a + 2b + 0 = 1 + 2b \\ 0 + b - 1 = 1 - 2 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = 0 \end{cases} \Leftrightarrow \boxed{e_2' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$Ae_3' = e_2' + 2e_3' \Leftrightarrow$ $e_3' = \begin{pmatrix} a \\ b \\ 2 \end{pmatrix}$

$$\begin{cases} 3a - b + 2 = 1 + 2a \\ a + 2b + 0 = 0 + 2b \\ 0 + b + 2 = -1 + 4 \end{cases} \Leftrightarrow \begin{cases} a = 0 \\ b = 1 \end{cases} \Leftrightarrow \boxed{e_3' = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}.$$

3.(b) $T = \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{=2I_3} + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{:=N}$. On remarque $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ et $N^3 = 0$.

$$T^n = (2I+N)^n = \sum_{k=0}^{n} C_n^k (2I)^{n-k} N^k = C_n^0 (2I)^n N^0 + C_n^1 (2I)^{n-1} N + C_n^2 (2I)^{n-2} N^2$$

$$= 2^n I + n 2^{n-1} N + \frac{n(n-1)}{2} 2^{n-2} N^2$$

$$= 2^{n-2} \begin{pmatrix} 4 & 2n & \frac{n(n-1)}{2} \\ 0 & 4 & 2n \\ 0 & 0 & 4 \end{pmatrix}.$$

4. (a) $T = P^{-1}AP$ donc $A = PTP^{-1}$. puis $A^n = \underbrace{PTP^{-1}PTP^{-1}\ldots PTP^{-1}}_{n \text{ fois}} = PT^nP^{-1}$

(b) $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$. On vérifie aisement que $PP^{-1} = I_3$.
Par unicité de l'inverse. $P^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}$.

(c) $X_n = A^n X_0 \iff \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = PT^nP^{-1} \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix} = PT^nP^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = PT^n \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$$= 2^{n-2} P \begin{pmatrix} -4+2n+\frac{n(n-1)}{2} \\ 4+2n \\ 4 \end{pmatrix} = 2^{n-2} \begin{pmatrix} 4+2n \\ -4+2n+\frac{n(n-1)}{2}+4 \\ -4+2n+\frac{n(n-1)}{2}-4-2n+8 \end{pmatrix} = 2^{n-2} \begin{pmatrix} 2n+4 \\ 2n+\frac{n(n-1)}{2} \\ \frac{n(n-1)}{2} \end{pmatrix}$$

D'où $u_n = 2^{n-2}(2n+4)$
$v_n = 2^{n-2}\left(2n + \frac{n(n-1)}{2}\right)$
$w_n = 2^{n-3} n(n-1)$.

Ex II : Si $Y(p) = \mathcal{L}(y(t))(p)$ alors $p^2Y(p) + 4pY(p) + 3Y(p) = \frac{1}{p-2}$, $p \in \mathbb{C}$. car $y(0) = y'(0) = 0$

D'où $Y(p) = \frac{1}{(p-2)(p^2+4p+3)} = \frac{1}{(p-2)(p+1)(p+3)} = \frac{a}{p-2} + \frac{b}{p+1} + \frac{c}{p+3}$.

$a = Y(p)\times(p-2)\Big|_{p=2} = \frac{1}{(2+1)(2+3)} = \frac{1}{15}$ $\quad b = Y(p)\times(p+1)\Big|_{p=-1} = \frac{1}{(-1-2)(-1+3)} = \frac{1}{-6} = \frac{-1}{6}$

$c = Y(p)\times(p+3)\Big|_{p=-3} = \frac{1}{(-3-2)(-3+1)} = \frac{1}{10}$ $\quad$ d'où $Y(p) = \frac{1}{15}\frac{1}{p-2} + \frac{-1}{6}\frac{1}{p+1} + \frac{1}{10}\frac{1}{p+3}$.

par passage à $\mathcal{L}^{-1}$ on obtient $y(t) = \left(\frac{1}{15}e^{2t} - \frac{1}{6}e^{-t} + \frac{1}{10}e^{-3t}\right)U(t)$.