**Prob-3**

In this problem we consider some numerical examples of Problem-2. Recall Eqs. (1) and (2), and assume that $\kappa_v = 3, \phi \in \{0.3, 0.6, 0.9\}$, and $\sigma_\xi^2 \in \{0.3, 0.9, 1.8\}$ (a) For each pair of $(\phi, \sigma_\xi^2)$ compute $Corr[y_t^2, y_{t-h}^2]$ with $h = 1, 2, 3, 4$ (i.e. fill out Table 1)

First, I create a placeholder matrix

```
h4=c('h=1', 'h=2', 'h=3', 'h=4') # column names
phi_sigmaSq=c('(0.3, 0.3)', '(0.6, 0.3)', '(0.9, 0.3)',  # rownames
              '(0.3, 0.9)', '(0.6, 0.9)', '(0.9, 0.9)',
              '(0.3, 1.8)', '(0.6, 1.8)', '(0.9, 1.8)')
Table1=matrix(NA, 9,4,dimnames = list(phi_sigma=phi_sigmaSq,h4))
```

Then I define a helper function for correlation computed as

$$\text{Corr}\left[y_t^2, y_{t-h}^2\right] \equiv \frac{\text{Cov}\left[y_t^2, y_{t-h}^2\right]}{\text{Var}\left[y_t^2\right]} = \frac{\exp\left[\frac{\sigma_\xi^2}{1-\phi^2} \times \phi^h\right] - 1}{\kappa_\nu \times \exp\left[\frac{\sigma_\xi^2}{1-\phi^2}\right] - 1}$$

Notice this is a fuction of $\kappa, \phi, \sigma_\xi^2$ and $h$

```
corr= function(sigmaSq, phi, h, kappa){
  num = exp(sigmaSq*(phi^h)/(1-phi^2))-1 #numerator
  denom =  kappa* (exp(sigmaSq/(1-phi^2))) -1 #denominator
  corr= num/denom
  return(corr)
}
```

Now we can plug $\kappa, \phi, \sigma_\xi^2$ and $h$ to the function defined above and fill out Table1.

```
kappa=3
sigmaSq_option=c(0.3,0.9,1.8)
phi_option=c(0.3,0.6,0.9)

for (i in 1:3){
  sigmaSq=sigmaSq_option[i]
  for(j in 1:3){
    phi=phi_option[j]
    for(h in 1:4){
      Corr=corr(sigmaSq, phi, h, kappa)
      Table1[(3*(i-1)+j),h]=Corr
    }
  }
}
knitr::kable(Table1,digits = 5)
```

| | h=1 | h=2 | h=3 | h=4 |
|---|---|---|---|---|
| (0.3, 0.3) | 0.03278 | 0.00950 | 0.00282 | 0.00084 |
| (0.6, 0.3) | 0.08561 | 0.04845 | 0.02808 | 0.01651 |
| (0.9, 0.3) | 0.23185 | 0.19136 | 0.15953 | 0.13416 |
| (0.3, 0.9) | 0.04889 | 0.01318 | 0.00383 | 0.00114 |
| (0.6, 0.9) | 0.11787 | 0.05863 | 0.03157 | 0.01778 |
| (0.9, 0.9) | 0.20525 | 0.13299 | 0.08968 | 0.06264 |

| | h=1 | h=2 | h=3 | h=4 |
|---|---|---|---|---|
| (0.3, 1.8) | 0.03917 | 0.00942 | 0.00265 | 0.00078 |
| (0.6, 1.8) | 0.09000 | 0.03580 | 0.01707 | 0.00898 |
| (0.9, 1.8) | 0.12923 | 0.05508 | 0.02555 | 0.01280 |

(b) Comment on Table 1 from a viewpoint of volatility persistence

- Larger $\phi$ leads to higher persistence (Recall $\log \sigma_t^2 = \mu + \phi\left(\log \sigma_{t-1}^2 - \mu\right) + \xi_t$)
- The level of correlation is small even when $\phi$ is large (e.g $\phi = 0.9$) and decreases relatively fast

(c) For comparison, consider a GARCH(1,1) process:

$$
\begin{aligned}
y_t = \sigma_t \nu_t, \quad \nu_t &\overset{i.d.d}{\sim} N(0,1) \\
\sigma_t^2 &= \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\omega > 0,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta &< 1
\end{aligned}
$$

Define

$$\delta(\alpha, \beta) = 3\alpha^2 + \beta^2 + 2\alpha\beta \tag{*}$$

As we learned in class, if $\delta(\alpha, \beta) < 1$,then $\text{Corr}\left[y_t^2, y_{t-h}^2\right]$ is well defined and given by

$$\text{Corr}\left[y_t^2, y_{t-h}^2\right] = \frac{\alpha[1 - \beta(\alpha + \beta)]}{1 - 2\alpha\beta - \beta^2}(\alpha + \beta)^{h-1}, \quad h \geq 1 \tag{**}$$

First, I create a placeholder matrix and helper function as I did in (a)

```
alpha_beta= c('(0.2, 0.5)', '(0.4, 0.1)', '(0.21, 0.71)', '(0.3, 0.6)','(0.5, 0.2)')
Table2= matrix(NA, 5,5,dimnames = list(alpha_beta=alpha_beta, c('delta', h4))) # placeholder

delta= function(alpha, beta){ # helper function for (*)
  return(3*(alpha^2)+ beta^2 + 2*alpha*beta)
}

Corr_garch= function(alpha, beta, h){ # helper function for (**)
  return((alpha*(1-(beta*(alpha+beta)))/(1-2*alpha*beta-beta^2))*((alpha+beta)^(h-1)))
}
```

Then plug $(\alpha, \beta) = (0.2, 0.5), (0.4, 0.1), (0.21, 0.71), (0.3, 0.6)$ and $(0.5, 0.2)$ and we get Table 2

```
alpha_option=c(0.2,0.4,0.21,0.3,0.5)
beta_option= c(0.5, 0.1,0.71,0.6,0.2)

for (i in 1:5){
  alpha=alpha_option[i]
  beta= beta_option[i]

  Table2[i, 1]= delta(alpha, beta)
```

```
  for (h in 1:4){
    corr_i = Corr_garch(alpha, beta, h)
    Table2[i, h+1]= corr_i
  }
}
knitr::kable(Table2, digits = 4)
```

| | delta | h=1 | h=2 | h=3 | h=4 |
|---|---|---|---|---|---|
| (0.2, 0.5) | 0.5700 | 0.2364 | 0.1655 | 0.1158 | 0.0811 |
| (0.4, 0.1) | 0.5700 | 0.4176 | 0.2088 | 0.1044 | 0.0522 |
| (0.21, 0.71) | 0.9346 | 0.3684 | 0.3389 | 0.3118 | 0.2869 |
| (0.3, 0.6) | 0.9900 | 0.4929 | 0.4436 | 0.3992 | 0.3593 |
| (0.5, 0.2) | 0.9900 | 0.5658 | 0.3961 | 0.2772 | 0.1941 |

(d) Compare the volatility persistence of SV and GARCH based on Tables 1 and 2

- learger value of $\alpha$ and $\beta$ result in higher persistence in GARCH
- When $\alpha + \beta$ is large, GARCH model shows high correlation and it doesn't decrease much as h increases
- Overall, persistence is higher in GARCH than in SV

### Prob-4

In this problem we study empirical examples related with Problem-3. See Federal Reserve Economic Data Download daily S&P 500 Index (not seasonally adjusted) from July 16, 2012 through July 14, 2014 (501 observations). Call it $\{x_t\}$.

We first import data from FRED using `getSymbol` function in `quantmod` library, select the time period and remove missing values

```
library(quantmod)
getSymbols('SP500', src = 'FRED', auto.assign = T)
SP500= SP500['2012-07-16/2014-07-14']
SP500= SP500[!is.na(SP500)]
```

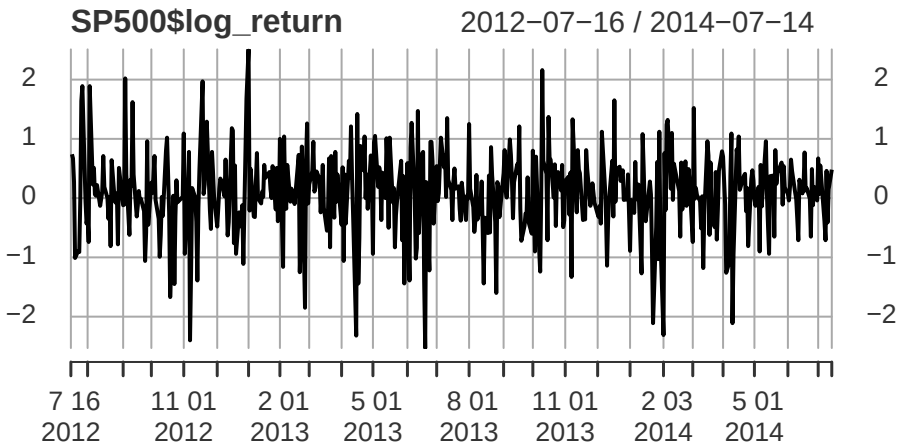
Then I add two variables, log-return and squared log-return to the dataframe. Log-return is calculated in percentage to avoid very small values. I show first six observations to make sure that transformation is correct.

```
SP500$log_return = round((log(SP500$SP500)-lag(log(SP500$SP500)))*100, 2) # per cent, two digits
SP500$log_return_sq = SP500$log_return^2
head(SP500)
```

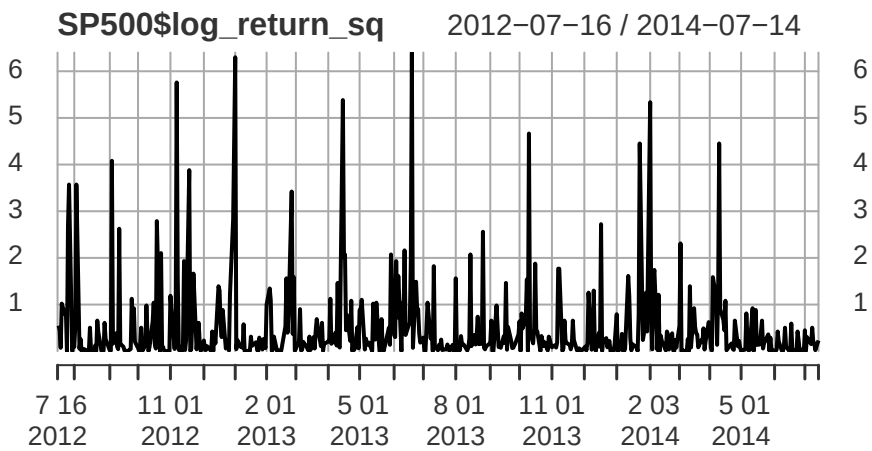
```
##              SP500 log_return log_return_sq
## 2012-07-16 1353.64         NA            NA
## 2012-07-17 1363.67       0.74        0.5476
## 2012-07-18 1372.78       0.67        0.4489
## 2012-07-19 1376.51       0.27        0.0729
## 2012-07-20 1362.66      -1.01        1.0201
## 2012-07-23 1350.52      -0.89        0.7921
```

(a) Draw a time series plot of the log-return series $y_t = \log x_t - \log x_{t-1}$

```
plot.xts(SP500$log_return)
```




```
plot.xts(SP500$log_return_sq)
```




(c) Compute sample autocorrelation of $ {y_t^2} $ at lags 1-4, denoted by $\{\hat{\rho}_{y^2,1}, \hat{\rho}_{y^2,2}, \hat{\rho}_{y^2,3}, \hat{\rho}_{y^2,4}\}$

Recall that $Var[y_t] = (1/T)E[\sum_{t=1}^{T}(y_t - E[y_t])^2], Cov[y_t, y_{t-h}] = (1/T)E[\sum_{t=1}^{T}(y_t - E[y_t])(y_{t-h} - E[y_{t-h}])]$ and $Corr[y_t, y_{t-h}] = Cov[y_t, y_{t-h}]$. Therefore sample correlation for $h = 1, 2, 3, 4$ are:

```
corr=numeric(4)
cov=numeric(4)

mean=mean(SP500$log_return_sq, na.rm=T)
var=mean((SP500$log_return_sq-mean)^2, na.rm=T)
for (h in 1:4){
  cov[h]=mean((SP500$log_return_sq-mean)*(lag(SP500$log_return_sq, k=h) - mean), na.rm=T)
}
corr=cov/var
corr
```

```
## [1] 0.10553097 0.03530987 0.07924055 0.06387127
```

(d) In view of Tables 1 and 2, which do you think is a better fit for $\{\hat{\rho}_{y^2,1}, \hat{\rho}_{y^2,2}, \hat{\rho}_{y^2,3}, \hat{\rho}_{y^2,4}\}$: SV or GARCH?

From the result of (c), we can say that $\{y_t\}$ has relatively high persistence. Recall that GARCH model better fits the series with high persistence, as we have seen in Prob3-(d). Therefore, GARCH model is the better model for our SP500 series.