Prob-3

In this problem we consider some numerical examples of Problem-2. Recall Eqs. (1) and (2), and assume that $\kappa_v = 3, \phi \in \{0.3, 0.6, 0.9\}$, and $\sigma_\xi^2 \in \{0.3, 0.9, 1.8\}$
(a) For each pair of (ϕ, σ_ξ^2) compute $Corr[y_t^2, y_{t-h}^2]$ with $h = 1, 2, 3, 4$ (i.e. fill out Table 1)

First, I create a placeholder matrix

```
# set-up
h4=c('h=1', 'h=2', 'h=3', 'h=4') # column names
phi_sigmaSq=c('(0.3, 0.3)', '(0.6, 0.3)', '(0.9, 0.3)',  # rownames
              '(0.3, 0.9)', '(0.6, 0.9)', '(0.9, 0.9)',
              '(0.3, 1.8)', '(0.6, 1.8)', '(0.9, 1.8)')
Table1=matrix(NA, 9,4,dimnames = list(phi_sigma=phi_sigmaSq,h4))
```

Then I define a helper function for correlation computed as

$$\text{Corr}\left[y_t^2, y_{t-h}^2\right] \equiv \frac{\text{Cov}\left[y_t^2, y_{t-h}^2\right]}{\text{Var}\left[y_t^2\right]} = \frac{\exp\left[\frac{\sigma_\xi^2}{1-\phi^2} \times \phi^h\right] - 1}{\kappa_\nu \times \exp\left[\frac{\sigma_\xi^2}{1-\phi^2}\right] - 1}$$

Notice this is a fuction of $\kappa, \phi, \sigma_\xi^2$ and h

```
# helper function
corr= function(sigmaSq, phi, h, kappa){
  num = exp(sigmaSq*(phi^h)/(1-phi^2))-1 #numerator
  denom =  kappa* (exp(sigmaSq/(1-phi^2))) -1 #denominator
  corr= num/denom
  return(corr)}
```

Now we can plug $\kappa, \phi, \sigma_\xi^2$ and h to the function defined above and fill out Table1.

```
kappa=3
sigmaSq_option=c(0.3,0.9,1.8)
phi_option=c(0.3,0.6,0.9)

for (i in 1:3){
  sigmaSq=sigmaSq_option[i]
  for(j in 1:3){
    phi=phi_option[j]
    for(h in 1:4){
      Corr=corr(sigmaSq, phi, h, kappa)
      Table1[(3*(i-1)+j),h]=Corr}}}
knitr::kable(Table1,digits = 5)
```

	h=1	h=2	h=3	h=4
(0.3, 0.3)	0.03278	0.00950	0.00282	0.00084
(0.6, 0.3)	0.08561	0.04845	0.02808	0.01651
(0.9, 0.3)	0.23185	0.19136	0.15953	0.13416
(0.3, 0.9)	0.04889	0.01318	0.00383	0.00114
(0.6, 0.9)	0.11787	0.05863	0.03157	0.01778
(0.9, 0.9)	0.20525	0.13299	0.08968	0.06264
(0.3, 1.8)	0.03917	0.00942	0.00265	0.00078
(0.6, 1.8)	0.09000	0.03580	0.01707	0.00898
(0.9, 1.8)	0.12923	0.05508	0.02555	0.01280

(b) Comment on Table 1 from a viewpoint of volatility persistence

- Larger ϕ leads to higher persistence (Recall $\log \sigma_t^2 = \mu + \phi\left(\log \sigma_{t-1}^2 - \mu\right) + \xi_t$)
- The level of correlation is relatively small even when ϕ is large (e.g $\phi = 0.9$) and decreases relatively fast

(c) For comparison, consider a GARCH(1,1) process:

$$
\begin{aligned}
y_t = \sigma_t \nu_t, \quad \nu_t \overset{i.d.d}{\sim} N(0,1) \\
\sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2 \\
\omega > 0,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha + \beta < 1
\end{aligned}
$$

Define

$$
\delta(\alpha, \beta) = 3\alpha^2 + \beta^2 + 2\alpha\beta \tag{*}
$$

As we learned in class, if $\delta(\alpha, \beta) < 1$,then $\operatorname{Corr}\left[y_t^2, y_{t-h}^2\right]$ is well defined and given by

$$
\operatorname{Corr}\left[y_t^2, y_{t-h}^2\right] = \frac{\alpha[1-\beta(\alpha+\beta)]}{1-2\alpha\beta-\beta^2}(\alpha+\beta)^{h-1}, \quad h \geq 1 \tag{**}
$$

First, I create a placeholder matrix and helper function as I did in (a)

```
# GARCH setup
alpha_beta= c('(0.2, 0.5)', '(0.4, 0.1)', '(0.21, 0.71)', '(0.3, 0.6)','(0.5, 0.2)')
Table2= matrix(NA, 5,5,dimnames = list(alpha_beta=alpha_beta, c('delta', h4))) # placeholder

delta= function(alpha, beta){ # helper function for (*)
  return(3*(alpha^2)+ beta^2 + 2*alpha*beta)}

# GARCH funciton
Corr_garch= function(alpha, beta, h){ # helper function for (**)
  return((alpha*(1-(beta*(alpha+beta)))/(1-2*alpha*beta-beta^2))*((alpha+beta)^(h-1)))}
```

Then plug $(\alpha, \beta) = (0.2, 0.5), (0.4, 0.1), (0.21, 0.71), (0.3, 0.6)$ and $(0.5, 0.2)$ and we get Table 2

```
alpha_option=c(0.2,0.4,0.21,0.3,0.5)
beta_option= c(0.5, 0.1,0.71,0.6,0.2)

for (i in 1:5){
  alpha=alpha_option[i]
  beta= beta_option[i]

  Table2[i, 1]= delta(alpha, beta)

  for (h in 1:4){
    corr_i = Corr_garch(alpha, beta, h)
    Table2[i, h+1]= corr_i
  }
}
knitr::kable(Table2, digits = 4)
```

	delta	h=1	h=2	h=3	h=4
(0.2, 0.5)	0.5700	0.2364	0.1655	0.1158	0.0811
(0.4, 0.1)	0.5700	0.4176	0.2088	0.1044	0.0522
(0.21, 0.71)	0.9346	0.3684	0.3389	0.3118	0.2869
(0.3, 0.6)	0.9900	0.4929	0.4436	0.3992	0.3593
(0.5, 0.2)	0.9900	0.5658	0.3961	0.2772	0.1941

(d) Compare the volatility persistence of SV and GARCH based on Tables 1 and 2

- learger value of α and β result in higher persistence in GARCH
- When $\alpha + \beta$ is large, GARCH model shows high correlation and it doesn't decrease much as h increases
- Overall, persistence is higher in GARCH than in SV

Prob-4

In this problem we study empirical examples related with Problem-3. See Federal Reserve Economic Data Download daily S&P 500 Index (not seasonally adjusted) from July 16, 2012 through July 14, 2014 (501 observations). Call it $\{x_t\}$.

We first import data directly from FRED using `getSymbol` function in `quantmod` library, select the time period and remove missing values

```r
# Import data
quantmod::getSymbols('SP500', src = 'FRED', auto.assign = T)
SP500= SP500['2012-07-16/2014-07-14']
SP500= SP500[!is.na(SP500)]
```

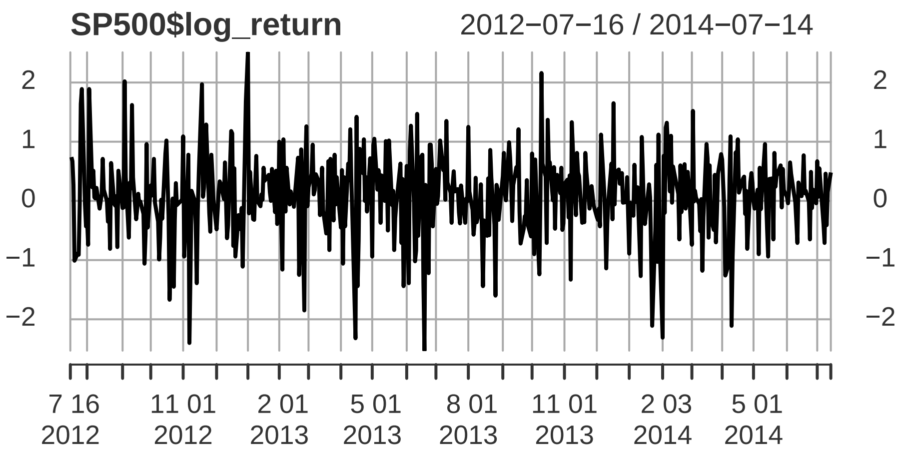
Then I add two variables, log-return and squared log-return to the dataframe. Log-return is calculated in percentage to avoid very small values. I show first six observations to make sure that transformation is correct.

```r
SP500$log_return = round((log(SP500$SP500)-lag(log(SP500$SP500)))*100, 2) # per cent, two digits
SP500$log_return_sq = SP500$log_return^2
head(SP500)
```

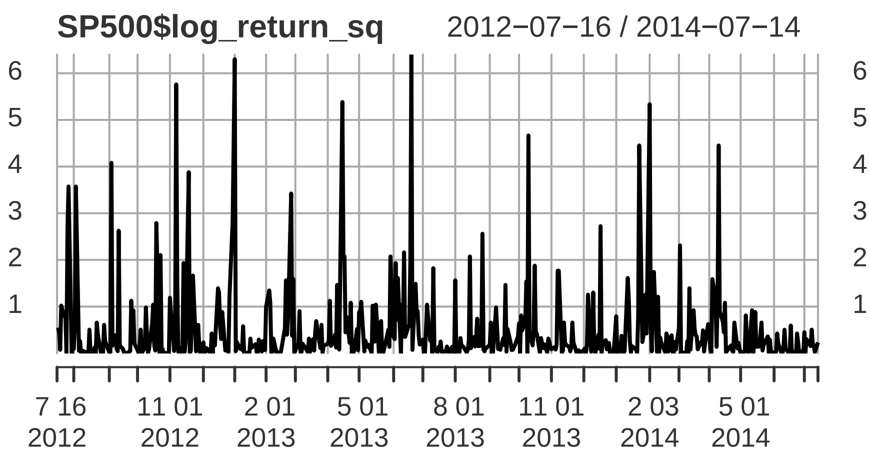
```
##              SP500 log_return log_return_sq
## 2012-07-16 1353.64         NA            NA
## 2012-07-17 1363.67       0.74        0.5476
## 2012-07-18 1372.78       0.67        0.4489
## 2012-07-19 1376.51       0.27        0.0729
## 2012-07-20 1362.66      -1.01        1.0201
## 2012-07-23 1350.52      -0.89        0.7921
```

(a) Draw a time series plot of the log-return series $y_t = \log x_t - \log x_{t-1}$

```r
xts::plot.xts(SP500$log_return)
```




```
xts::plot.xts(SP500$log_return_sq)
```



(c) Compute sample autocorrelation of $\{y_t^2\}$ at lags 1-4, denoted by $\{\hat{\rho}_{y^2,1}, \hat{\rho}_{y^2,2}, \hat{\rho}_{y^2,3}, \hat{\rho}_{y^2,4}\}$

Recall that $Var[y_t] = (1/T)E[\sum_{t=1}^{T}(y_t - E[y_t])^2], Cov[y_t, y_{t-h}] = (1/T)E[\sum_{t=1}^{T}(y_t - E[y_t])(y_{t-h} - E[y_{t-h}])]$ and $Corr[y_t, y_{t-h}] = Cov[y_t, y_{t-h}]$. Therefore sample correlation for $h = 1, 2, 3, 4$ are:

```
corr=numeric(4) # empty vector
cov=numeric(4)

mean=mean(SP500$log_return_sq, na.rm=T)
var=mean((SP500$log_return_sq-mean)^2, na.rm=T)
for (h in 1:4){
  cov[h]=mean((SP500$log_return_sq-mean)*(lag(SP500$log_return_sq, k=h) - mean), na.rm=T)
}
corr=cov/var
corr
```

```
## [1] 0.10553097 0.03530987 0.07924055 0.06387127
```

(d) In view of Tables 1 and 2, which do you think is a better fit for $\{\hat{\rho}_{y^2,1}, \hat{\rho}_{y^2,2}, \hat{\rho}_{y^2,3}, \hat{\rho}_{y^2,4}\}$: SV or GARCH?

From the result of (c), we can say that $\{y_t\}$ has relatively high persistence. Recall that GARCH model better fits the series with high persistence, as we have seen in Prob3-(d). Therefore, GARCH model is the better model for our SP500 series.