Syllabus

Contents

1 Objective

The aim of this course is to acquaint students with mathematical methods to study dynamic systems, in discrete time and in continuous time. Dynamic systems are systems that involve more than one time-period. (On the other hand, static systems involve only one time-period.) These methods are useful in many areas of macroeconomics such as growth theory, business-cycle theory, monetary economics, and the macroeconomics of the labor market. The course covers three topics:

1. Dynamic programming: recursive method to solve discrete-time dynamic optimization problems

2. Optimal control: method to solve continuous-time dynamic optimization problems

3. First-order differential equations: methods to characterize dynamic systems

2 Tentative Schedule

- Lecture 1: dynamic programming
- Lecture 2: optimal control
- Lecture 3: first-order differential equations

3 Useful References

There is no set text for this part of the course. The lecture notes for the course will be self-contained. But the following books are useful reference to study the methods introduced in this course:

- K. Sydsaeter, P. Hammond, A. Seierstad, and A. Strom, *Further Mathematics for Economics Analysis*, 2nd edition, Prentice Hall, 2008
- A.Chiang and K.Wainwright, *Fundamental Methods of Mathematical Economics*, 4th edition, McGraw-Hill, 2004
- L. Ljungqvist and T. Sargent, *Recursive Macroeconomic Theory*, 2nd edition, Cambridge, MA: MIT Press, 2004
- J. Adda and R. Cooper, *Dynamics Economics: Quantitative Methods and Applications*, Cambridge, MA: MIT Press, 2003
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Cambridge, UK: Cambridge University Press, 2000
- D. Romer, *Advanced Macroeconomics*, 4th edition, McGraw-Hill, 2011

These books also contain exercises on which you could work to improve your understanding of the methods introduced in the course. Sydsaeter covers all three topics. Chapter 12 covers dynamic programming. Chapter 11 covers first-order differential equations. Chapter 9 covers optimal control. For A. de la Fuente, Chapter 12 and 13 cover dynamic programming. Chiang and Wainwright cover first-order differential equations in Chapter 15 and optimal control theory in Chapter 20. A. de la FuenteThe other reference books are tex books for macroeconomics or dynamic economics. They apply the mathematical tools to study macro models. Adda and Cooper contain many applications and is very accessible.

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Dynamic Programming

Contents

1 A Simple Deterministic Problem

This section introduces the key concepts of dynamic programming in a deterministic problem (a problem without randomness). Section 2 show how the key concepts apply to a stochastic problem. Section 4 proposes a more formal treatment of dynamic programming.

1.1 A Consumption-Saving Problem

You start life with wealth $a_0 > 0$. Each period $t = 0, 1, \ldots, +\infty$, you consume a quantity $c_t \geq 0$ of your wealth, which provides utility $u(c_t)$. You choose consumption to maximize your lifetime utility

$$\sum_{t=0}^{+\infty} \beta^t \cdot u(c_t),$$

where $\beta \in [0, 1)$ is the discount factor. Assume that $\lim_{c \to 0} u'(c) = +\infty$, which implies that you choose to consume at least some of your wealth $c_t > 0$ each period. You save the amount of wealth that you do not consume. The wealth at the beginning of period t is a_t. Wealth is invested at a constant interest rate r, paid at the beginning of period t; hence, wealth evolves according to the law of motion

$$a_{t+1} = (1 + r) \cdot a_t - c_t.$$

1.2 Example: Solving a Two-Period Consumption-Saving Problem with Kuhn-Tucker Conditions

In this section, we will solve the following consumption-saving problem using Kuhn-Tucker conditions.

$$\begin{aligned} \max_{\substack{\{c_0, c_1\} \\ \{a_1, a_2\}}} \quad & \sum_{t=0}^{1} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t + a_{t+1} = (1+r)a_t, \ \text{for } t = 0, 1 \\ & c_0, c_1, a_1, a_2 \geq 0 \end{aligned}$$

Because the objective function is concave and the constraints are linear, we know there exists a unique consumption and saving profile, $\{c_0, c_1, a_1, a_2\}$, that solves the constrained optimization problem. What we will do in this section is to use Kuhn-Tucker conditions to find the solution (without checking second order conditions).

The Lagrangian for the problem is

$$\begin{aligned} \mathcal{L} = & \, u(c_0) + \lambda_0 \left[(1+r)a_0 - c_0 - a_1\right] + \mu_0 a_1 + \gamma_0 c_0 \\ & + \beta \left\{u(c_1) + \lambda_1 \left[(1+r)a_1 - c_1 - a_2\right] + \mu_1 a_2 + \gamma_1 c_1\right\} \end{aligned}$$

The Kuhn-Tucker conditions are

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \lambda_0 + \gamma_0 = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial a_1} = -\lambda_0 + \mu_0 + \beta\lambda_1(1+r) = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta \left[u'(c_1) - \lambda_1 + \gamma_1\right] = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial a_2} = \beta \left(-\lambda_1 + \mu_1\right) = 0 \tag{4}$$

and the complementary slackness conditions

$$\begin{gathered}
\lambda_0 \geq 0, (1+r)a_0 - c_0 - a_1 \geq 0, \lambda_0 \left[(1+r)a_0 - c_0 - a_1\right] = 0, \\
\mu_0 \geq 0, a_1 \geq 0, \mu_0 a_1 = 0, \\
\gamma_0 \geq 0, c_0 \geq 0, \gamma_0 c_0 = 0, \\
\lambda_1 \geq 0, (1+r)a_1 - c_1 - a_2 \geq 0, \lambda_1 \left[(1+r)a_1 - c_1 - a_2\right] = 0, \\
\mu_1 \geq 0, a_2 \geq 0, \mu_1 a_2 = 0, \\
\gamma_1 \geq 0, c_1 \geq 0, \gamma_1 c_1 = 0.
\end{gathered}$$

Conjecture (and verify) that $\lambda_0, \lambda_1 > 0$, and $\gamma_0 = \gamma_1 = 0$. From (4), $\mu_1 = \lambda_1 > 0$. So, $a_2 = 0$. From (1) and (3), we have

$$\begin{aligned}
\lambda_0 &= u'(c_0) \\
\lambda_1 &= u'(c_1)
\end{aligned}$$

Substitute these expression into eq. (2),

$$u'(c_0) = \mu_0 + \beta(1+r)u'(c_1).$$

If $\mu_0 > 0$, then $c_1 = 0$, $u'(c_1) = \infty$ and $c_0 = (1+r)a_0$, $u'(c_0) < \infty$. This equation cannot hold. Therefore, $\mu_0 = 0$, and we obtain the following Euler equation,

$$u'(c_0) = \beta(1+r)u'(c_1).$$

1.3 Direct Approach

Take wealth $a_0 > 0$ at $t = 0$ as given. The direct approach is to choose two sequences of variables $\{c_t\}_{t=0}^{+\infty}$ and $\{a_t\}_{t=0}^{+\infty}$ to maximize

$$\sum_{t=0}^{+\infty} \beta^t \cdot u(c_t)$$

subject to the constraints that for all $t \geq 0$,

$$a_{t+1} = (1+r) \cdot a_t - c_t \tag{5}$$
$$a_{t+1} \geq 0. \tag{6}$$

To find the two sequences $\{c_t\}_{t=0}^{+\infty}$ and $\{a_t\}_{t=0}^{+\infty}$, we write down the Lagrangian associated with the problem:

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t \cdot \{u(c_t) - \lambda_t \cdot [a_{t+1} - (1+r) \cdot a_t + c_t] + \mu_t \cdot a_{t+1}\},$$

where $\{\lambda_t\}_{t=1}^{+\infty}$ and $\{\mu_t\}_{t=1}^{+\infty}$ are the sequences of Lagrange multipliers associated with the sequences of constraints (5) and (6). For all $t \geq 0$, the first-order condition of the problem with respect to c_t is

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{du}{dc}(c_t) - \lambda_t = 0,$$

which yields a first optimality condition:

$$\frac{du}{dc}(c_t) = \lambda_t.$$

For all $t \geq 0$, the first-order condition of the problem with respect to a_{t+1} is

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = -\lambda_t + \mu_t + \beta \cdot (1+r) \cdot \lambda_{t+1} = 0,$$

which yields a second optimality condition:

$$\lambda_t = \mu_t + (1+r) \cdot \beta \cdot \lambda_{t+1}.$$

In addition, the complementary slackness conditions impose that for all $t \geq 0$, $\mu_t \geq 0$ and

$$\mu_t \cdot a_{t+1} = 0.$$

These optimality conditions and the complementary slackness conditions are necessary and sufficient if the problem is well behaved. Since $c_t \geq 0$, the wealth a_{t+1} never falls to zero because if $a_T = 0$, then $c_t = 0$ for all $t \geq T$, which cannot be optimal because $u'(c_t) = +\infty$ for all $t \geq T$. Hence for all $t \geq 0$, $a_{t+1} > 0$ and $\mu_t = 0$. We infer that

$$\lambda_t = (1 + r) \cdot \beta \cdot \lambda_{t+1}$$

for all $t \geq 0$ such that consumption c_t satisfies the following intertemporal condition for all $t \geq 0$:

$$\frac{du}{dc}(c_t) = \beta \cdot (1 + r) \cdot \frac{du}{dc}(c_{t+1}). \tag{7}$$

This intertemporal condition characterizes the optimal consumption path. It is called the **Euler equation**. For all $t \geq 0$, wealth a_t satisfies

$$a_{t+1} = (1 + r) \cdot a_t - c_t.$$

1.4 Dynamic-Programming Approach

Instead of looking for two infinite sequences $\{c_t\}_{t=0}^{+\infty}$ and $\{a_t\}_{t=0}^{+\infty}$, dynamic programming is looking for a time-invariant **policy function** h mapping wealth at the beginning of period t, a_t, into optimal consumption in period t, c_t, such that the sequence $\{c_t\}_{t=0}^{+\infty}$ generated by iterating

$$c_t = h(a_t) \tag{8}$$

$$a_{t+1} = (1 + r) \cdot a_t - c_t, \tag{9}$$

starting from initial wealth a_0 solves the consumption-saving problem. Finding the policy function allows us to determine recursively the optimal sequence of consumption $\{c_t\}_{t=0}^{+\infty}$.

Why do we want to find a policy function h instead of an infinite sequence $\{c_t\}_{t=0}^{+\infty}$? A priori, it is unclear that finding a function is easier than finding an infinite sequence. But it turns out that dynamic programming has three desirable properties:

1. Sometimes, dynamic programming allows us to find closed-form solutions for the policy function h.

2. Sometimes, dynamic programming allows us to characterize theoretical properties of the policy function h.

3. Various numerical methods are available to solve dynamic programs.

1.5 Value Function

To determine the policy function h and solve this problem, we first need to solve for an auxiliary function that we call **value function**. The value function $V(a)$ measures the optimal lifetime utility from consumption, starting with an initial wealth a. The value function is defined by

$$V(a) \equiv \max_{\{c_t, a_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \cdot u(c_t) \tag{10}$$

subject to for all $t \geq 0$

$$\begin{aligned} a_0 &= a \\ a_{t+1} &= (1+r) \cdot a_t - c_t \\ a_{t+1} &\geq 0. \end{aligned}$$

To determine the value function, we use a theorem that says that the value function V is the solution to a functional equation called the **Bellman equation**:

$$V(a) = \max_{c \in [0,(1+r)\cdot a]} u(c) + \beta \cdot V((1+r) \cdot a - c). \tag{11}$$

Not all optimization problems can be represented with a Bellman equation. The theorem applies only if the problem satisfies the **Principle of Optimality**. If a problem satisfies the Principle of Optimality, we say that it has a **recursive structure**. Section 4 characterizes problems with a recursive structure. For instance, the consumption-saving problem has a recursive structure.

To understand where the Bellman equation comes from and what it means, we manipulate the value function. The value function can be expressed as

$$V(a_0) = \max_{\{0 \leq c_t \leq (1+r)\cdot a_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \cdot u(c_t),$$

subject to for all $t \geq 0$

$$a_{t+1} = (1+r) \cdot a_t - c_t.$$

Eliminating c_t, we rewrite the value function as

$$V(a_0) = \max_{\{0 \leq a_{t+1} \leq (1+r)\cdot a_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \cdot u((1+r) \cdot a_t - a_{t+1}).$$

Separating the first term in the utility function from the rest of the sum, we obtain

$$V(a_0) = \max_{\{0 \leq a_{t+1} \leq (1+r)\cdot a_t\}_{t=0}^{+\infty}} \left\{ u((1+r) \cdot a_0 - a_1) + \beta \cdot \sum_{t=1}^{+\infty} \beta^{t-1} \cdot u((1+r) \cdot a_t - a_{t+1}) \right\}.$$

We re-index the terms in the sum:

$$V(a_0) = \max_{\{0 \leq a_{t+1} \leq (1+r)\cdot a_t\}_{t=0}^{+\infty}} \left\{ u((1+r) \cdot a_0 - a_1) + \beta \cdot \sum_{t=0}^{+\infty} \beta^{t} \cdot u((1+r) \cdot a_{t+1} - a_{t+2}) \right\}.$$

We separate the maximization process in two stages: choose consumption in period 0 given wealth in period 0, and choose all future consumptions given wealth in period 1. Thus, we rewrite the value function as

$$V(a_0) = \max_{0 \leq a_1 \leq (1+r)\cdot a_0} \left\{ u((1+r) \cdot a_0 - a_1) + \beta \max_{\{0 \leq a_{t+2} \leq a_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^{t} \cdot u((1+r) \cdot a_{t+1} - a_{t+2}) \right\}.$$

By definition, the second term is exactly the value function $V(a_1)$ so we can simplify the equation to

$$V(a_0) = \max_{0 \leq a_1 \leq (1+r) \cdot a_0} \left\{ u((1+r) \cdot a_0 - a_1) + \beta \cdot V(a_1) \right\}.$$

Since $c_0 = (1+r) \cdot a_0 - a_1$, we obtain

$$V(a_0) = \max_{0 \leq c_0 \leq (1+r) \cdot a_0} \left\{ u(c_0) + \beta \cdot V((1+r) \cdot a_0 - c_0) \right\}.$$

This last equation is the Bellman equation. For any problem with a recursive structure, we can apply this procedure and obtain a Bellman equation.

1.6 Policy Function

With the definition proposed in equation (11), we gave a **recursive formulation** to our optimization problem. Once we have determined the value function $V(a)$ for all a, we can easily solve for the optimal consumption level

$$c^* = \text{argmax}_{c \in [0,(1+r) \cdot a]} u(c) + \beta \cdot V((1+r) \cdot a - c).$$

c^* is a function of initial wealth a. We define the policy function h by

$$h(a) \equiv c^*.$$

The policy function provides a mapping from state to actions. It tells us how much should be consumed in the current period if initial wealth is a. The policy function allows us to determine the optimal path of consumption by iterating equations (8) and (9). Therefore, it allows us to solve the optimization problem defined in Section 1.1.

To solve the optimization problem defined in Section 1.1 using dynamic programming, we proceed in five steps:

1. write down the Bellman equation
2. write down the first-order conditions of the optimization program

3. write down the Benveniste-Scheinkman equation to determine the derivative of the value function with respect to wealth, a_t

4. apply the Benveniste-Scheinkman equation to next period's wealth, a_{t+1}, and plug into the first-order conditions

5. derive the Euler equation, which summarizes the optimal intertemporal behavior of consumption, c_t

1.7 Step 1: Bellman Equation

Because of the recursive structure of the consumption-saving problem, the value function, V, satisfies the **Bellman equation**:

$$V(a) = \max_{c \in [0,(1+r)\cdot a]} u(c) + \beta \cdot V((1+r) \cdot a - c). \tag{12}$$

- Equation (12) is a **functional equation**: the unknown is the function V itself. We assume for now that a solution to this equation does exist.

- a is a **state variable**. It summarizes completely the information from the past that is needed to solve the forward-looking optimization problem.

- c is a **control variable**. It is the variable to be chosen in the current period. It determines the value a' of the state variable next period according to the **transition equation**

$$a' = (1+r) \cdot a - c.$$

- *Notation:* next period's variables are denoted using a "prime". So next period's consumption is c' and next period's wealth is a'.

1.8 Step 1': Rewrite the Bellman Equation

We rewrite the Bellman equation to get rid of the state variable, a, in the term $\beta \cdot V((1+r) \cdot a - c)$. To do so, we use tomorrow's state variable, a', instead of today's consumption, c, as a control variable. We can substitute a' for c as a control variable because a' and c are directly related by $a' = (1+r) \cdot a - c$; given a, choosing a' is equivalent to choosing c.

Substituting a' for c, the Bellman equation becomes

$$V(a) = \max_{a' \in [0,(1+r)\cdot a]} u((1+r) \cdot a - a') + \beta \cdot V(a'). \tag{13}$$

After this manipulation, the term $\beta \cdot V((1+r) \cdot a - c)$ has become $\beta \cdot V(a')$; it now only depends on the control variable. This property will be convenient to apply the envelope theorem later on.

1.9 Step 2: First-Order Condition

For now, let's assume that V does exit and is differentiable. The first step is to take the first-order condition in the optimization program:

$$\max_{a' \in [0,(1+r)\cdot a]} u((1+r) \cdot a - a') + \beta \cdot V(a')$$

The first-order condition with respect to a' is

$$\begin{aligned} 0 =& (-1) \cdot \frac{\partial u}{\partial c}((1+r) \cdot a - a') + \beta \cdot \frac{dV}{da}(a') \\ \frac{du}{dc}(c) =& \beta \cdot \frac{dV}{da}(a') \end{aligned} \tag{14}$$

1.10 Step 3: Benveniste-Scheinkman Equation

In the first-order condition (14), we do not know the derivative dV/da of the value function. Hence, the next step is to determine what the derivative dV/da of the value function is. To do so, we apply the envelope theorem to the Bellman equation (13), which yields

$$\frac{dV}{da}(a) = (1+r) \cdot \frac{du}{dc}(c) \tag{15}$$

This equation is often referred to as the **Benveniste-Scheinkman equation**. It holds for any a.

1.11 Step 4: One Step Forward

Equation (15) is valid for any state variable a. In particular, it is valid for next period's state variable:

$$\frac{dV}{da}(a') = (1+r) \cdot \frac{du}{dc}(c'). \tag{16}$$

1.12 Step 5: Euler equation

If we plug equation (16) into equation (14), we obtain the Euler equation in terms of current period's consumption c and next period's consumption c':

$$\frac{du}{dc}(c) = (1+r) \cdot \beta \cdot \frac{du}{dc}(c'). \tag{17}$$

Note that this Euler equation is exactly the same as that obtained with the Lagrangian method (see equation (7)). Indeed, the Lagrangian method and the dynamic programming method are equivalent.

Using the Euler equation, we could reduce the optimization problem of Section 1.1 to a policy function problem: for all a, determine the policy function $h(a)$ such that

$$\frac{du}{dc}(h(a)) = (1+r) \cdot \beta \cdot \frac{du}{dc}(h((1+r) \cdot a - h(a))).$$

1.13 A Closed-Form Solution

One of the advantage of using dynamic programing is that we can sometimes find closed-form solutions for the value function and the policy function. Here, we can find a closed-form solution if we assume $u(c) = \ln(c)$. To find a closed-form solution, we use the **method of undetermined coefficients**. We conjecture that the value function takes the form

$$V(a) = A + B \cdot \ln(a), \tag{18}$$

where A and B are constants. The Bellman equation (13) becomes

$$A + B \cdot \ln(a) = \max_{a' \in [0,(1+r)\cdot a]} \ln((1+r) \cdot a - a') + \beta \cdot [A + B \cdot \ln(a')] . \tag{19}$$

We first express the solution a' of the maximization problem as a function of parameters $A,\ B,\ \beta$ and the state variable a. The first-order condition with respect to a' yields

$$\frac{1}{(1+r) \cdot a - a'} = \frac{\beta \cdot B}{a'}$$

$$a' = (1+r) \cdot \frac{\beta \cdot B}{1 + \beta \cdot B} \cdot a. \tag{20}$$

The optimal level of consumption, $c = (1+r) \cdot a - a'$, satisfies

$$c = (1+r) \cdot \frac{1}{1 + \beta \cdot B} \cdot a.$$

We plug the expression (20) for the optimal a' in the functional equation (19). We obtain

$$A + B \cdot \ln(a) = \ln\left(\frac{(1+r) \cdot a}{1 + \beta B}\right) + \beta \cdot \left[A + B \cdot \ln\left(\frac{\beta \cdot B \cdot (1+r) \cdot a}{1 + \beta \cdot B}\right)\right]$$

$$A + B \cdot \ln(a) = \left[\beta \cdot A + (1 + \beta \cdot B) \cdot \ln\left(\frac{1+r}{1 + \beta \cdot B}\right) + \beta \cdot B \cdot \ln(\beta \cdot B)\right] + [1 + \beta \cdot B] \cdot \ln(a).$$

The above equation must hold for all a, so we must have

$$B = 1 + \beta \cdot B$$

$$B = \frac{1}{1 - \beta}, \tag{21}$$

and

$$A = \beta \cdot A + (1 + \beta \cdot B) \cdot \ln\left(\frac{1+r}{1+\beta \cdot B}\right) + \beta \cdot B \cdot \ln(\beta \cdot B).$$

Using the fact that $1 + \beta \cdot B = \frac{1}{1-\beta}$ and $\beta \cdot B = \frac{\beta}{1-\beta}$, we infer that A satisfies

$$\begin{aligned}(1-\beta) \cdot A =& (1 + \beta \cdot B) \cdot \ln\left(\frac{1+r}{1+\beta \cdot B}\right) + \beta \cdot B \cdot \ln(\beta \cdot B)\\ (1-\beta) \cdot A =& \frac{1}{1-\beta} \cdot [\ln(1-\beta) + \ln(1+r)] + \frac{\beta}{1-\beta} \cdot [\ln(\beta) - \ln(1-\beta)]\\ A =& \frac{(1-\beta) \cdot \ln(1-\beta) + \beta \cdot \ln(\beta) + \ln(1+r)}{(1-\beta)^2}. \qquad (22)\end{aligned}$$

Equations (22) and (21) define the parameters of the value function we were solving for. With the values for the parameters A and B, the functional form proposed in equation (18) actually solves the functional equation (19). Our guess was correct. Since the value function is unique (by theorem), we have found the value function.

Notice that we did not use the Benveniste-Scheinkman equation. The reason is that we assumed a functional form for the value function V, so we could compute the derivative dV/da directly, without resorting to the Benveniste-Scheinkman equation.

Using equation (21), we can rewrite equation (20)

$$a' = \beta \cdot (1+r) \cdot a.$$

That is, the optimal behavior is to save a constant fraction β of the invested wealth $(1+r) \cdot a$ and consume what is left. Let c^* be the optimal consumption this period and $(a')^* = \beta \cdot (1+r) \cdot a$ be the optimal wealth to save for next period. The policy function is

$$\begin{aligned}h(a) &= c^* = (1+r) \cdot a - (a')^*\\ h(a) &= (1-\beta) \cdot (1+r) \cdot a.\end{aligned}$$

In this simple problem, we have been able to find a closed-form solution for the policy function. Unfortunately, it is usually not possible to find closed-form solutions to a dynamic program, and we must resort to numerical methods.

2 A Simple Stochastic Problem

In this section, we introduce randomness in the example of Section 1 and show how the techniques that we developed there can be applied.

2.1 Taste Shocks

We assume that the utility of consumption fluctuates randomly over time. The utility of consuming c_t in period t is given by

$$\epsilon_t \cdot u(c_t),$$

where ϵ_t is a taste shock in period t. The taste shock is determined at the beginning of the period and observed before the consumption decision. The shock can take only two values: $\epsilon_t \in \{\epsilon_h, \epsilon_l\}$ with $\epsilon_h > \epsilon_l > 0$. The shock follows a **Markov process**; therefore, the distribution of ϵ_t only depends on the realization ϵ_{t-1} of ϵ in the previous period.

The problem can be solved as before. The major difference is that the value function is not only a function of current wealth, a, but also a function of the current realization of the taste shock, ϵ. In other words, there are two state variables: a and ϵ.

2.2 Step 1: Bellman Equation

The Bellman equation becomes

$$V(a, \epsilon) = \max_{a' \in [0,(1+r)\cdot a]} \left\{ \epsilon \cdot u((1+r) \cdot a - a') + \beta \cdot \mathbb{E}_{\epsilon'|\epsilon} \left[V(a', \epsilon') \right] \right\}$$

2.3 Step 2: First-order condition

Taking the first-order condition with respect to a' in the Bellman equation yields

$$\epsilon \cdot \frac{du}{dc}((1+r) \cdot a - a') = \beta \cdot \mathbb{E}_{\epsilon'|\epsilon}\left[\frac{\partial V}{\partial a}(a', \epsilon')\right], \tag{23}$$

where

$$\frac{\partial V}{\partial a}(a', \epsilon')$$

designates the partial derivative of the value function $V(a, \epsilon)$ with respect to the first variable, a, evaluated at the pair (a', ϵ').

2.4 Step 3: Benveniste-Scheinkman Equation

In the first-order condition (23), we do not know the derivative $\partial V/\partial a$ of the value function. We determine $\partial V/\partial a$ by applying the Benveniste-Scheinkman equation:

$$\frac{\partial V}{\partial a}(a, \epsilon) = (1+r) \cdot \epsilon \cdot \frac{du}{dc}((1+r) \cdot a - a') \tag{24}$$

2.5 Step 4: One step forward

Equation (24) is valid for any vector (a, ϵ) of state variables. In particular, it is valid for the vector (a', ϵ') of next period's state variables. Hence,

$$\frac{\partial V}{\partial a}(a', \epsilon') = (1+r) \cdot \epsilon' \cdot \frac{du}{dc}((1+r) \cdot a' - a'') \tag{25}$$

2.6 Step 5: Euler Equation

Plugging equation (25) into the first-order condition (23) yields:

$$\begin{aligned}\epsilon \cdot \frac{du}{dc}((1+r)\cdot a - a') &= (1+r)\cdot\beta\cdot\mathbb{E}_{\epsilon'|\epsilon}\left[\epsilon'\cdot\frac{du}{dc}((1+r)\cdot a' - a'')\right]\\ \epsilon\cdot\frac{du}{dc}(c) &= (1+r)\cdot\beta\cdot\mathbb{E}_{\epsilon'|\epsilon}\left[\epsilon'\cdot\frac{du}{dc}(c')\right]. \qquad (26)\end{aligned}$$

This is the Euler equation.

The policy function gives the optimal level of consumption in any state of the world. The policy function now depends on the realization ϵ of the shock in the current period:

$$c^* = h(a,\epsilon).$$

The policy function specifies a **contingent plan of consumption**, which depends on the state variable a and on the realization of the shock variable ϵ. We can rewrite the Euler equation with the policy function:

$$\epsilon\cdot\frac{du}{dc}(h(a,\epsilon)) = (1+r)\cdot\beta\cdot\mathbb{E}_{\epsilon'|\epsilon}\left[\frac{du}{dc}(h(a',\epsilon'))\right].$$

3 Equilibrium of the Real Business Cycle Model

Section 1 and Section 2 apply dynamic programming to very simple optimization problems. However, dynamic programming has a wide range of applications and can be used to solve complex problems in macroeconomics. To illustrate how to use dynamic programming in a macroeconomic model, we solve for the equilibrium of a real business cycle model using dynamic programming.

3.1 Model

- Preferences of the representative household:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\cdot u\left(C_t,L_t\right)\right]$$

  with

$$u\left(C_t,L_t\right)=\ln\left(C_t\right)+\theta\cdot\frac{\left(1-L_t\right)^{1-\gamma}}{1-\gamma}.$$

- Technology: we introduce labor-augmenting technology so that we have a balanced growth path.
  - Production function: $Y_t=K_t^{\alpha}\cdot\left(A_t\cdot L_t\right)^{1-\alpha}$ with $0<\alpha<1$.
  - Capital accumulation: $K_{t+1}=(1-\delta)\cdot K_t+I_t$.
- National accounts identity: $Y_t=C_t+I_t$.
- Technology shock: $\ln A_t=\rho_A\cdot\ln A_{t-1}+\varepsilon_t^A$, $|\rho_A|<1$, $\varepsilon_t^A\sim N(0,\sigma_A^2)$, $\sigma_A>0$.

3.2 Optimal Allocation

DEFINITION 1. The **optimal allocation** is the collection of stochastic processes $\{C_t,I_t,Y_t,K_t,A_t,L_t\}_{t=0}^{\infty}$ that solves:

$$\max_{\{C_t,L_t\}_{t=0}^{\infty}}\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\cdot u\left(C_t,L_t\right)$$

subject to

$$K_{t+1} = (1-\delta)\cdot K_t + I_t \tag{27}$$

$$Y_t = K_t^{\alpha}\cdot (A_t\cdot L_t)^{1-\alpha} \tag{28}$$

$$Y_t = C_t + I_t \tag{29}$$

$$\ln A_t = \rho_A \cdot \ln A_{t-1} + \varepsilon_t^A,\ \varepsilon_t^A \sim N(0,\sigma_A^2).$$

The welfare theorems imply that the allocation in the competitive equilibrium coincides with the optimal allocation; thus, we focus on the optimal allocation.

3.3 Characterization of the Optimal Allocation

We use dynamic programming to characterize the optimal allocation. This stochastic problem and admits a recursive structure with

- **control** $= [C, L]$
- **state** $= [K]$
- **shock** $= [A]$.

Furthermore, the transition equation for the state variable K is obtained by combining equations (27), (28), and (29). The three equations can be aggregated into a single transition equation for capital:

$$K' = (1-\delta)\cdot K + K^{\alpha}\cdot (A\cdot L)^{1-\alpha} - C. \tag{30}$$

Step 1: Bellman Equation. The Bellman equation is

$$V(K,A) = \max_{C,L} \left\{u(C,L) + \beta \cdot \mathbb{E}_{A'|A}\left[V(K',A')\right]\right\}$$

We plug the transition equation for capital, given by (30), into the value function:

$$V(K,A) = \max_{C,L} \left\{ u(C,L) + \beta \cdot \mathbb{E}_{A'|A} \left[V\left((1-\delta) \cdot K + K^{\alpha} \cdot (A \cdot L)^{1-\alpha} - C, A' \right) \right] \right\}.$$

Step 2: First-Order Conditions. We derive the first-order conditions with respect to C and L in the Bellman equation:

$$\begin{aligned} 0 &= \frac{\partial u}{\partial C}(C,L) + \beta \cdot \mathbb{E}_{A'|A} \left[\frac{\partial V}{\partial K}(K',A') \cdot \frac{\partial K'}{\partial C} \right] \\ 0 &= \frac{\partial u}{\partial L}(C,L) + \beta \cdot \mathbb{E}_{A'|A} \left[\frac{\partial V}{\partial K}(K',A') \cdot \frac{\partial K'}{\partial L} \right]. \end{aligned}$$

Equation (30) implies that

$$\begin{aligned} \frac{\partial K'}{\partial C} &= -1 \\ \frac{\partial K'}{\partial L} &= (1-\alpha) \cdot \frac{Y}{L}. \end{aligned}$$

The assumption that $u(C,L) = \ln C + \theta \cdot \frac{(1-L)^{1-\gamma}}{1-\gamma}$ implies that

$$\begin{aligned} \frac{\partial u}{\partial C} &= \frac{1}{C} \\ -\frac{\partial u}{\partial L} &= \theta \cdot (1-L)^{-\gamma}. \end{aligned}$$

Thus, we obtain

$$\frac{1}{C} = \beta \cdot \mathbb{E}_{A'|A} \left[\frac{\partial V}{\partial K}(K',A') \right] \tag{31}$$

$$\theta \cdot (1-L)^{-\gamma} = \beta \cdot (1-\alpha) \cdot \frac{Y}{L} \cdot \mathbb{E}_{A'|A} \left[\frac{\partial V}{\partial K}(K',A') \right]. \tag{32}$$

By taking the ratio of equations (31) and (32), we obtain the following intratemporal optimality condition:

$$\frac{\theta \cdot C}{(1-L)^{\gamma}} = (1-\alpha) \cdot \frac{Y}{L}. \tag{33}$$

This condition simply says that the marginal rate of substitution between leisure and consumption (the left-hand side of the equation) equals the marginal product of labor (the right-hand side of the equation) in the optimal allocation.

Step 3: Benveniste-Scheinkman Equation. We use the Benveniste-Scheinkman equation to determine $\partial V / \partial K$ and make progress on (31):

$$\frac{\partial V}{\partial K}(K, A) = \beta \cdot \mathbb{E}_{A'|A}\left[\frac{\partial V}{\partial K}(K', A') \cdot \frac{\partial K'}{\partial K}\right].$$

Equation (30) implies that

$$\frac{\partial K'}{\partial K} = \left[(1-\delta) + \alpha \cdot \frac{Y}{K}\right] \equiv R.$$

Thus, we obtain

$$\frac{\partial V}{\partial K}(K, A) = \beta \cdot R \cdot \mathbb{E}_{A'|A}\left[\frac{\partial V}{\partial K}(K', A')\right]. \tag{34}$$

Step 4: Euler Equation. Combining the first-order condition (31) with the Benveniste-Scheinkman equation (34) yields

$$\frac{\partial V}{\partial K}(K, A) = R \cdot \frac{1}{C}.$$

Moving one period ahead yields

$$\frac{\partial V}{\partial K}(K', A') = R' \cdot \frac{1}{C'}. \tag{35}$$

Finally, we combine the first-order condition (31) with equation (35) to obtain the Euler equation:

$$\frac{1}{C} = \beta \cdot \mathbb{E}_{A'|A}\left[R' \cdot \frac{1}{C'}\right]. \tag{36}$$

Our goal is to solve explicitly for the stochastic processes of the key variables in this model (consumption, leisure, capital, etc.). There are two approaches to solve for these stochastic processes literature: (1) simplify the economic environment; (2) find an approximate analytical solution by log-linearizing the model. Here we follow approach (1).

3.4 Simplifying Assumptions

The model contains a mixture of linear and log-linear elements which make it impossible to find a closed-form solution. To eliminate the linear elements in the model and only keep log-linear elements, we assume full capital depreciation: $\delta = 1$. This assumption imply that the capital-accumulation equation simplifies to

$$K' = I = Y - C = s \cdot Y$$

where s is the current saving rate. Under these assumptions, we simplify the Euler equation as follows:

$$\begin{aligned}
\frac{1}{(1-s)\cdot Y} &= \beta \cdot \mathbb{E}_{A'|A}\left[\alpha \cdot \frac{Y'}{K'} \cdot \frac{1}{(1-s')\cdot Y'}\right] \\
\frac{1}{(1-s)\cdot Y} &= \alpha \cdot \beta \cdot \mathbb{E}_{A'|A}\left[\frac{1}{s \cdot Y} \cdot \frac{1}{1-s'}\right] \\
\frac{s}{(1-s)} &= \alpha \cdot \beta \cdot \mathbb{E}_{A'|A}\left[\frac{1}{1-s'}\right].
\end{aligned}$$

We solve this equation by guessing and verifying. We guess that the saving rate remains constant over time: $s = s' = s^*$. In that case,

$$\frac{s^*}{(1-s^*)} = \alpha \cdot \beta \cdot \mathbb{E}_{A'|A}\left[\frac{1}{(1-s^*)}\right] = \frac{\alpha \cdot \beta}{(1-s^*)}$$

Thus $s^* = \alpha \cdot \beta$. Using this result in the first-order condition (33) yields

$$\frac{\theta \cdot (1 - s^*) \cdot Y}{(1-L)^\gamma} = (1-\alpha) \cdot \frac{Y}{L}$$
$$\frac{\theta \cdot (1 - s^*)}{1-\alpha} = \frac{(1-L)^\gamma}{L}.$$

Therefore employment L is constant over time. The two first-order conditions hold for each point in time, and not just along the balanced growth path. Hence the optimal allocation is characterized by a constant saving rate and a constant employment, even in presence of transitory technology shocks.

4 Theory of Deterministic Problems

So far, we have only applied dynamic programming to specific problems. In this section we propose a general treatment of dynamic programming for deterministic problems. The goal is to show you the type of problems that can be solved with dynamic programming.

Consider the following problem: Given initial condition a_0, choose $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \cdot u(a_t, c_t)$$

subject to the law of motion

$$a_{t+1} = g(a_t, c_t).$$

$u(a_t, c_t)$ is a concave function.

Dynamic programming seeks a time-invariant policy function h mapping the state a_t into the control c_t such that the sequence $\{c_t\}_{t=0}^{\infty}$ generated by iterating the two functions

$$c_t = h(a_t)$$
$$a_{t+1} = g(a_t, c_t)$$

solves the original problem.

4.1 Step 1: Bellman Equation

The Principle of Optimality allows us to write the value function V as the solution of a functional equation

$$V(a) = \max_c \{u(a,c) + \beta \cdot V(g(a,c))\} \tag{37}$$

This functional equation is the Bellman equation.[1]

The optimal consumption is given by the policy function: $c = h(a)$. Another representation of the Bellman equation is

$$V(a) = u(a, h(a)) + \beta \cdot V(g(a, h(a))). \tag{38}$$

To highlight the recursive structure of the problem, we can write the symbolic representation of the Bellman equation:

$$V(\mathbf{state_t}) = \max_{\mathbf{control_t}} \{u(\mathbf{control_t}, \mathbf{state_t}) + \beta \mathrm{V}(\mathbf{state_{t+1}})\}$$

subject to

$$\mathbf{state_{t+1}} = g(\mathbf{control_t}, \mathbf{state_t}),$$

[1]The proof of the Principle of Optimality is due to Bellman. The formal derivation and proof of this result, as well as the conditions under which this result holds, are omitted here.

which is equivalent to

$$V(\mathbf{state_t}) = \max_{\mathbf{control_t}} \left\{ u(\mathbf{control_t}, \mathbf{state_t}) + \beta \cdot \mathrm{V}(\mathrm{g}(\mathbf{control_t}, \mathbf{state_t})) \right\},$$

where $\mathbf{control_t}$ and $\mathbf{state_t}$ are vectors of control variables and state variables.

4.2 Step 2: First-Order Condition

Taking the first-order condition with respect to c of the optimization problem (37) yields

$$\frac{\partial u}{\partial c}(a, h(a)) + \beta \cdot \frac{\partial g}{\partial c}(a, h(a)) \cdot \frac{dV}{da}(g(a, h(a))) = 0, \tag{39}$$

where

$$\frac{\partial g}{\partial c}(a, h(a))$$

designates the partial derivative of the function $g(a, c)$ with respect to the second variable, c, evaluated at the pair $(a, h(a))$;

$$\frac{\partial u}{\partial c}(a, h(a))$$

designates the partial derivative of the function $u(a, c)$ with respect to the second variable, c, evaluated at the pair $(a, h(a))$; and

$$\frac{dV}{da}(g(a, h(a)))$$

designates the derivative of the function $V(a)$ with respect to the variable a evaluated at $g(a, h(a))$.

4.3 Step 3: Benveniste-Scheinkman Equation

In the first-order condition (39), we do not know the derivative dV/da of the value function (because we do not know the value function). Hence, the next step is to determine what the derivative dV/da of the value function is. To do so, we apply the Benveniste-Scheinkman theorem. This theorem says that under some regularity conditions,

$$\frac{dV}{da}(a) = \frac{\partial u}{\partial a}(a, h(a)) + \beta \cdot \frac{\partial g}{\partial a}(a, h(a)) \cdot \frac{dV}{da}(g(a, h(a))) \tag{40}$$

The theorem is a version of the envelope theorem applied to the Bellman equation (37).

4.4 Step 3': A Combination

The first-order condition (39) yields

$$\frac{dV}{da}(g(a, h(a))) = -\frac{1}{\beta} \cdot \frac{\partial u(a, h(a))/\partial c}{\partial g(a, h(a))/\partial c}. \tag{41}$$

Combining this equation with the Benveniste-Scheinkman equation (40) yields

$$\frac{dV}{da}(a) = \frac{\partial u}{\partial a}(a, h(a)) - \frac{\partial u}{\partial c}(a, h(a)) \cdot \frac{\partial g(a, h(a))/\partial a}{\partial g(a, h(a))/\partial c}. \tag{42}$$

This step is necessary when

$$\frac{\partial g(a, c)}{\partial a} \neq 0.$$

In the consumption-saving problem, we picked a control variable—next period's wealth, a'—such that the state variable, a, does not enter the transition function, g. Therefore, we could bypass this step.

4.5 Step 4: One Step Forward

Equation (42) is true for any value of the state variable a. In particular, it is true for $a' = g(a, h(a))$. Therefore

$$\frac{dV}{da}(a') = \frac{\partial u}{\partial a}(a', h(a')) - \frac{\partial u}{\partial c}(a', h(a')) \cdot \frac{\partial g(a', h(a'))/\partial a}{\partial g(a', h(a'))/\partial c}. \tag{43}$$

4.6 Step 5: Euler Equation

We plug equation (43) into equation (39) to get the Euler equation

$$\frac{\partial u}{\partial c}(a,h(a))+\beta\cdot\frac{\partial g}{\partial c}(a,h(a))\cdot\left\{\frac{\partial u}{\partial a}(a',h(a'))-\frac{\partial u}{\partial c}(a',h(a'))\cdot\frac{\partial g(a',h(a'))/\partial a}{\partial g(a',h(a'))/\partial c}\right\}=0.$$

The equation characterizes the optimal intertemporal behavior of the control variable c.

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Mathematical Methods for Macroeconomics

Optimal Control

Contents

1 Some Mathematical Results

Before presenting the techniques of optimal control, we need to review two mathematical results: **L'Hôpital's rule** and **Leibniz's rule**. These results are used in Section 4 for the derivation of some of the results presented in this note.

L'Hôpital's rule states that if

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0,$$

then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Leibniz's rule states that if

$$I(z) \equiv \int_{a(z)}^{b(z)} f(x, z)\, dx$$

where x is the integration variable, z is a parameter and $f(x, z)$ is assumed to have a continuous derivative $\partial f(x, z)/\partial z$ in the interval $[a(z), b(z)]$, then the effect of change in z on the integral is given by

$$\frac{dI}{dz}(z) = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}(x, z) dx + \frac{db}{dz}(z) \cdot f(b(z), z) - \frac{da}{dz}(z) \cdot f(a(z), z).$$

2 A Consumption-Saving Problem in Continuous Time

We consider the same consumption-saving problem as in the notes on dynamic programming, except that the problem is now in continuous time. Taking initial wealth a_0 as given, the problem is to choose the consumption path $\{c(t)\}_{t \geq 0}$ to maximise the lifetime utility

$$\int_0^\infty e^{-\rho \cdot t} \cdot u\left(c(t)\right) dt, \tag{1}$$

subject to the law of motion

$$\dot{a}(t) = r \cdot a(t) - c(t). \tag{2}$$

The parameter $r > 0$ is the constant interest rate at which wealth is invested. The parameter $\rho > 0$ is the discount factor. The notation $\dot{a}$ denotes the derivative of wealth a with respect to time t:

$$\dot{a}(t) \equiv \frac{\partial a(t)}{\partial t}.$$

2.1 Optimal-Control Approach

To solve this problem, it is inconvenient to use the Lagrangian technique or dynamic programming technique because they are designed for discrete-time optimisation problems. Instead, we use a technique called **optimal control**. We will see in Section 4 that optimal control is related both to the Lagrangian technique and the dynamic programming technique. But optimal control is designed for continuous-time optimisation problems.

There are two ways to apply the optimal-control approach: one by forming a **present-value Hamiltonian**, the other by forming an **current-value Hamiltonian**. These two ways are roughly equivalent, but using the current-value Hamiltonian is usually simpler.

2.2 Present-Value Hamiltonian

Here are the main steps of the optimal-control approach with present-value Hamiltonian:

1. Identify the **state variables** and the **control variables**. By definition, a

control variable can be adjusted at any time t whereas the evolution of a state variable follows a law of motion such as (2). Here, the control variable is consumption $c(t)$ and the state variable is wealth $a(t)$.

2. Write down the present-value Hamiltonian

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot u\left(c(t)\right) + \lambda(t) \cdot \left(r \cdot a(t) - c(t)\right)$$

To form the Hamiltonian, we introduce a new variable $\lambda(t)$, which we call the **co-state variable** associated with the state variable $a(t)$. In general, we introduce as many co-state variables as there are state variables. As a consequence, we introduce as many co-state variables as there are laws of motion, because each state variable is associated with one law of motion.

3. Write down the optimality conditions, which derive from the Maximum Principle:

$$\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0 \tag{3}$$

$$\frac{\partial \mathcal{H}(t)}{\partial a(t)} = -\dot{\lambda}(t). \tag{4}$$

The optimality conditions can be rewritten as

$$0 = e^{-\rho \cdot t} \cdot u'\left(c(t)\right) - \lambda(t) \tag{5}$$

$$-\dot{\lambda}(t) = \lambda(t) \cdot r. \tag{6}$$

4. Impose a **transversality condition**:

$$\lim_{t \to +\infty} \lambda(t) \cdot a(t) = 0. \tag{7}$$

The transversality condition prevents the state variable to grow without bounds. In our consumption-saving problem, it prevents the consumer from running a Ponzi scheme by accumulating debt. It makes sure that the discounted present value of consumption does not exceed the discounted present value of the income from investment.

We can solve explicitly for the optimal consumption path by eliminating the co-state variable $\lambda(t)$ using (5) and (6). To eliminate $\lambda(t)$, we first take log in (5):

$$-\rho \cdot t + \ln\left(u'\left(c(t)\right)\right) = \ln\left(\lambda(t)\right).$$

We then take time derivatives in this equation:

$$\rho + \left[\frac{-u''(c(t)) \cdot c(t)}{u'(c(t))}\right] \cdot \left[\frac{\dot{c}(t)}{c(t)}\right] = -\frac{\dot{\lambda}(t)}{\lambda(t)}.$$

Equation (6) can be rewritten as

$$-\frac{\dot{\lambda}(t)}{\lambda(t)} = r.$$

Combining these two equations, we obtain the Euler equation for optimal consumption:

$$\frac{\dot{c}(t)}{c(t)} \cdot \left[\frac{-u''\left(c(t)\right) \cdot c(t)}{u'\left(c(t)\right)}\right] = r - \rho. \tag{8}$$

The term

$$\frac{-u''\left(c(t)\right) \cdot c(t)}{u'\left(c(t)\right)}$$

measures relative risk aversion. The coefficient of relative risk aversion also corresponds to the inverse of the intertemporal elasticity of substitution.

Consider the following class of utility function:

$$u\left(c\right) = \frac{c^{1-\gamma} - 1}{1-\gamma}.$$

This class of utility function is know as Constant Relative Risk Aversion utility, or CRRA utility. This class of function is characterised by a constant coefficient of relative risk aversion γ as

$$\frac{-u''\left(c\right) \cdot c}{u'\left(c\right)} = \gamma.$$

With CRRA utility, the Euler equation simplifies to

$$\frac{\dot{c}(t)}{c(t)} = \frac{r - \rho}{\gamma}.$$

2.3 Current-Value Hamiltonian

The present-value Hamiltonian $\mathcal{H}$ depends on t because of the discounting $e^{-\rho \cdot t}$, which might create some difficulties in deriving and analysing solutions to the problem. Multiplying $\mathcal{H}$ by $e^{\rho \cdot t}$ addresses this problem. We denote the resulting Hamiltonian $\mathcal{H}^*$ as current-value Hamiltonian. The current-value Hamiltonian is given by

$$\mathcal{H}^*(t) \equiv e^{\rho \cdot t} \cdot \mathcal{H}(t) = u\left(c(t)\right) + e^{\rho \cdot t} \cdot \lambda(t) \cdot \left(r \cdot a(t) - c(t)\right).$$

We define a new co-state variable $q(t)$ as

$$q(t) \equiv e^{\rho \cdot t} \cdot \lambda(t).$$

The current-value Hamiltonian becomes

$$\mathcal{H}^*(t) = u\left(c(t)\right) + q(t) \cdot \left(r \cdot a(t) - c(t)\right).$$

This is the expression of the current-value Hamiltonian that we use in practice.

The optimality conditions are slightly different. The optimality condition become

$$\begin{aligned} \frac{\partial \mathcal{H}^*(t)}{\partial c(t)} =& 0 \\ \frac{\partial \mathcal{H}(t)^*}{\partial a(t)} =& \rho \cdot q(t) - \dot{q}(t). \end{aligned}$$

Compared to the optimality conditions (3) and (4) with the present-value Hamiltonian, there is an extra term $+\rho \cdot q(t)$ in the second condition. The extra term arises because the co-state variable $q(t)$ is defined differently form the co-state variable $\lambda(t)$ that we used in the present-value Hamiltonian. The optimality conditions can be rewritten as

$$0 = u'(c(t)) - q(t) \tag{9}$$
$$\rho \cdot q(t) - \dot{q}(t) = q(t) \cdot r. \tag{10}$$

We also impose a transversality condition:

$$\lim_{t \to +\infty} e^{-\rho \cdot t} \cdot q(t) \cdot a(t) = 0.$$

Compared to the transversality condition (7) with the present-value Hamiltonian, there is an extra factor $e^{-\rho \cdot t}$ in this condition. The extra factor arises because the co-state variable $q(t)$ is defined differently form the co-state variable $\lambda(t)$ that we used in the present-value Hamiltonian.

By combining equations (9) and (10), we obtain exactly the same condition (11) as with the present-value Hamiltonian. We take the log of equation (9) to obtain

$$\ln(u'(c(t))) = \ln(q(t)).$$

We then take time derivatives in this equation:

$$\left[\frac{-u''(c(t)) \cdot c(t)}{u'(c(t))}\right] \cdot \left[\frac{\dot{c}(t)}{c(t)}\right] = -\frac{\dot{q}(t)}{q(t)}.$$

Equation (10) can be rewritten

$$-\frac{\dot{q}(t)}{q(t)} = r - \rho.$$

Combining these two equations, we obtain the same Euler equation for optimal consumption as the one we obtained with the present-value Hamiltonian:

$$\frac{\dot{c}(t)}{c(t)} \cdot \left[\frac{-u''(c(t)) \cdot c(t)}{u'(c(t))}\right] = r - \rho. \tag{11}$$

The two approaches—the approach with the present-value Hamiltonian and the approach with the current-value Hamiltonian—are exactly equivalent. They lead to the same Euler equation. However, it is often more convenient to work with the current-value Hamiltonian.

3 Theory of Optimal Control

Optimal control is used to solve continuous-time optimisation problems. The general problem is to choose $\{c(t)\}_{t\geq 0}$ to maximise

$$\int_0^\infty e^{-\rho \cdot t} \cdot u\left(a(t), c(t)\right) \tag{12}$$

given the constraint that for all t

$$\dot{a}(t) = g\left(a(t), c(t)\right), \tag{13}$$

and taking a_0 as given. The parameter $\rho > 0$ is the discount rate. The functions u and g are concave and twice differentiable.

3.1 Present-Value Hamiltonian

An important result in optimal control theory is the Maximum Principle. It is due to Pontryagin. In addition to the control and state variables, we introduce a co-state variable $\lambda(t)$ associated with the state variable. The co-state variable measures the shadow price of the associated state variable. The co-state variable enters the optimal control problem through the present-value Hamiltonian, defined as

$$\mathcal{H}(t) = e^{-\rho \cdot t} \cdot u\left(a(t), c(t)\right) + \lambda(t) \cdot g\left(a(t), c(t)\right). \tag{14}$$

The Maximum Principle gives necessary conditions for optimality. There are three conditions. The first two conditions are

$$\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0 \tag{15}$$

$$\frac{\partial \mathcal{H}(t)}{\partial a(t)} = -\dot{\lambda}(t) \tag{16}$$

Condition (15) implies that the Hamiltonian must be maximised with respect to the control variable at any point in time. Condition (16) says that the marginal change of the Hamiltonian associated with a unit change of the state variable is equal to minus the rate of change of the co-state variable. The optimal solution must also satisfy a third condition, which we call transversality condition:

$$\lim_{t\to\infty} \lambda(t) \cdot a(t) = 0. \tag{17}$$

The transversality condition implies the product of co-state and state must be converging to zero as time goes to infinity.

3.2 Current-Value Hamiltonian

We can reformulate the results from the Maximum Principle with the current-value Hamiltonian which is often easier to manipulate. The current-value Hamiltonian is defined as

$$\mathcal{H}^*(t) = u\left(a(t), c(t)\right) + q(t) \cdot g\left(a(t), c(t)\right),$$

where $q(t)$ is the co-state variable associated with the state variable $a(t)$. The three necessary conditions (15), (16), and (17) for optimality given by the Maximum Principle become

$$\frac{\partial \mathcal{H}^*(t)}{\partial c(t)} = 0$$

$$\frac{\partial \mathcal{H}^*(t)}{\partial a(t)} = \rho \cdot q(t) - \dot{q}(t)$$

$$\lim_{t\to\infty} e^{-\rho \cdot t} \cdot q(t) \cdot a(t) = 0.$$

4 Heuristic Derivation of the Maximum Principle

In this section, we provide an heuristic derivation of the necessary conditions for optimality provided by the Maximum Principle. One way to derive the optimality conditions (15) and (16) is to apply informally the results from dynamic programming. Formally, many of the claims below are imprecise or inappropriate, but they will serve their purpose of providing intuition for the Maximum Principle.

We first define the value function of the problem, which the maximised value of the objective function as a function of the state variable $a(t)$ and time t:

$$V\left(a(t), t\right) = \max_{\{c(s)\}_{s \geq t}} \int_t^{\infty} e^{-\rho\cdot(s-t)} \cdot u\left(a(s), c(s)\right) ds,$$

where the maximisation is subject for all $s \geq t$ to the law of motion of the state variable

$$\dot{a}(s) = g\left(a(s), c(s)\right). \tag{18}$$

Since the problem has a recursive structure, we can apply the Principle of Optimality and write the value function as the solution to a Bellman equation:

$$V\left(a(t), t\right) = \max_{\{c(s)\}_{t \leq s \leq t+\Delta t}} \left\{ \int_t^{t+\Delta t} e^{-\rho\cdot(s-t)} \cdot u\left(a(s), c(s)\right) ds + e^{-\rho\cdot\Delta t} \cdot V\left(a(t+\Delta t), t+\Delta t\right) \right\},$$

where the maximisation is subject for all $t \leq s \leq t+\Delta t$ to (18).

Subtract $V\left(a(t), t\right)$ from both side and divide by Δt:

$$0 = \max_{\{c(s)\}_{t \leq s \leq t+\Delta t}} \left[\frac{\int_t^{t+\Delta t} e^{-\rho\cdot(s-t)} \cdot u\left(a(s), c(s)\right) ds}{\Delta t} + \frac{e^{-\rho\cdot\Delta t} \cdot V\left(a(t+\Delta t), t+\Delta t\right) - V\left(a(t), t\right)}{\Delta t} \right], \tag{19}$$

where the maximisation is subject for all $t \leq s \leq t+\Delta t$ to (18).

We now take the limit as $\Delta t \to 0$. We start with the first term in the curly brackets. Since numerator and denominator of the first term approach zero as $\Delta t \to 0$, we apply L'Hôpital's rule. The derivative of the denominator with respect to Δt is 1. We apply Leibniz's rule to determine the derivative with respect to Δt of the integral in the numerator. Leibniz's rule tells us that the derivative of the integral with respect to $\triangle t$ is

$$e^{-\rho\cdot\triangle t}\cdot u\left(a(t+\triangle t),c(t+\triangle t)\right).$$

Therefore, the limit as $\triangle t\to 0$ for the first term in the bracket is

$$u\left(a(t),c(t)\right). \tag{20}$$

We move on to the second term. Since

$$\lim_{\triangle t\to 0} e^{-\rho\cdot\triangle t}=1$$

and

$$\lim_{\triangle t\to 0} V\left(a(t+\triangle t),t+\triangle t\right)=V\left(a(t),t\right),$$

both numerator and denominator approach zero as $\triangle t\to 0$. Therefore we apply L'Hôpital's rule. The derivative of the denominator with respect to $\triangle t$ is 1. The derivative of the numerator with respect to $\triangle t$ is

$$\begin{aligned}&-\rho\cdot e^{-\rho\cdot\triangle t}\cdot V\left(a(t+\triangle t),t+\triangle t\right)+e^{-\rho\cdot\triangle t}\cdot\frac{\partial V}{\partial a}\left(a(t+\triangle t),t+\triangle t\right)\cdot\dot{a}(t+\triangle t)\\&+e^{-\rho\cdot\triangle t}\cdot\frac{\partial V}{\partial t}\left(a(t+\triangle t),t+\triangle t\right).\end{aligned}$$

We have the following limits:

$$\begin{aligned}\lim_{\triangle t\to 0}\rho\cdot e^{-\rho\cdot\triangle t}\cdot V\left(a(t+\triangle t),t+\triangle t\right)&=\rho\cdot V\left(a(t),t\right)\\\lim_{\triangle t\to 0}e^{-\rho\cdot\triangle t}\cdot\frac{\partial V}{\partial t}\left(a(t+\triangle t),t+\triangle t\right)&=\frac{\partial V}{\partial t}\left(a(t),t\right)\\\lim_{\triangle t\to 0}e^{-\rho\cdot\triangle t}\cdot\frac{\partial V}{\partial a}\left(a(t+\triangle t),t+\triangle t\right)\cdot\dot{a}(t+\triangle t)&=\frac{\partial V}{\partial a}\left(a(t),t\right)\cdot\dot{a}(t)=\frac{\partial V}{\partial a}\left(a(t),t\right)\cdot g(a(t),c(t)),\end{aligned}$$

where the last equality results from the law of motion (13) of state variable $a(t)$.

Therefore, the limit as $\triangle t \to 0$ for the first term in the bracket is

$$-\rho \cdot V\left(a(t), t\right) + \frac{\partial V}{\partial a}\left(a(t), t\right) \cdot g(a(t), c(t)) + \frac{\partial V}{\partial t}\left(a(t), t\right). \tag{21}$$

Combining equations (19), (20), and (21), we obtain a version of the Bellman equation for the continuous-time optimisation problem. This equation is called the **Hamilton-Jacobi-Bellman equation**. The equation is

$$\rho V\left(a(t), t\right) = \max_{c(t)} \left[u\left(a(t), c(t)\right) + \frac{\partial V}{\partial a(t)}\left(a(t), t\right) \cdot g\left(a(t), c(t)\right) + \frac{\partial V}{\partial t}\left(a(t), t\right) \right], \tag{22}$$

where $a(t)$ is given. We define

$$\lambda(t) \equiv e^{-\rho \cdot t} \cdot \frac{\partial V}{\partial a(t)}\left(a(t), t\right).$$

We can rewrite the Hamilton-Jacobi-Bellman equation as

$$\rho V\left(a(t), t\right) = \max_{c(t)} \left[u\left(a(t), c(t)\right) + e^{\rho \cdot t} \cdot \lambda(t) \cdot g\left(a(t), c(t)\right) + \frac{\partial V}{\partial t}\left(a(t), t\right) \right]. \tag{23}$$

Taking the first-order condition with respect to $c(t)$ in the Hamilton-Jacobi-Bellman equation (23) implies

$$\frac{\partial u}{\partial c(t)}\left(a(t), c(t)\right) + e^{\rho \cdot t} \cdot \lambda(t) \cdot \frac{\partial g}{\partial c(t)}\left(a(t), c(t)\right) = 0.$$

Furthermore, the envelope theorem implies

$$\rho \frac{\partial V}{\partial a(t)}\left(a(t), t\right) = \frac{\partial u}{\partial a(t)}\left(a(t), c(t)\right) + e^{\rho \cdot t} \cdot \lambda(t) \cdot \frac{\partial g}{\partial a(t)}\left(a(t), c(t)\right) + \frac{\partial^2 V}{\partial t \partial a(t)}\left(a(t), t\right).$$

The last two equations become equivalent to optimality conditions (15) and (16). This is the case because, using the definition of the present-value Hamiltonian, equation (15) can be written

$$\frac{\partial H(t)}{\partial c(t)}(a(t), c(t)) = 0$$

and equation (16) can be written

$$\frac{\partial u}{\partial a(t)}(a(t), c(t)) + e^{\rho \cdot t} \cdot \lambda(t) \cdot \frac{\partial g}{\partial a(t)}(a(t), c(t)) = -e^{\rho \cdot t} \cdot \frac{\partial \lambda(t)}{\partial t},$$

which implies

$$\frac{\partial H(t)}{\partial a(t)}(a(t), c(t)) = -\frac{\partial \lambda(t)}{\partial t}.$$

Note that we consider that the Hamiltonian is a function of a_t, t, and λ_t, and we only take the partial derivative with respect to a_t, thus keeping λ_t constant.

5 The Hamilton-Jacobi-Bellman Equation

The Hamilton-Jacobi-Bellman equation (22) is an equilibrium condition that equates flow costs with flow benefits. In practice, we write it down without going through all the algebra relating to $\triangle t$.

This equation is commonly used in macroeconomics. For instance, it is frequently used in search-and-matching models of the labour market. In a search-and-matching model, a vacant job costs c per unit time and becomes occupied according to a Poisson process with arrival rate q. In the labour market, the occupied job yields net returns $p - w$, where p is real output and w is the cost of labour. The job runs a risk λ of being destroyed.

Let V be the value of the vacant job and J be the value of occupied job. Let r be the discount factor. In steady state, the Hamilton-Jacobi-Bellman equations are

$$\begin{aligned} r \cdot V &= -c + q \cdot (J - V) \\ r \cdot J &= p - w + \lambda \cdot (0 - J) = p - w - \lambda \cdot J \end{aligned}$$

There is no maximisation on the right-hand-side in this particular example. These equations simply describe the relationship between the equilibrium values V, J and w.

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Mathematical Methods for Macroeconomics

First-Order Differential Equations

Contents

1 First-Order Differential Equations

1.1 Constant Growth Rate

Let $x(t)$ be a function of time $t \in \mathbb{R}$. Let

$$\dot{x}(t) \equiv dx/dt$$

denote the derivative of $x(t)$ with respect to time. Consider the equation

$$\dot{x}(t) - \lambda \cdot x(t) = 0 \tag{1}$$

where $\lambda \in \mathbb{R}$ is a constant. Equation (1) is a **first-order differential equation** (FODE), because it involves $x(t)$ and the first-order derivative of $x(t)$ with respect to time: $\dot{x}(t)$. Equation (1) is a functional equation: the unknown is the function $x(t)$ rather than a number or a vector. Solving equation (1) means finding the functions $x(t)$ that, together with their derivative $\dot{x}(t)$, satisfy equation (1) for all $t \in \mathbb{R}$.

Equation (1) is an especially simple differential equation. It can be rewritten as

$$\frac{\dot{x}(t)}{x(t)} = \lambda,$$

so it imposes that $x(t)$ has a constant growth rate λ over time. It admits a simple class of functions as solution:

$$x(t) = A \cdot e^{\lambda \cdot t}, \tag{2}$$

for any constant $A \in \mathbb{R}$. Furthermore, the constant A can be determined by an additional boundary condition because

$$A = x(0) = x(t_0) \cdot e^{-\lambda \cdot t_0}$$

for any date $t_0 \in \mathbb{R}$.

It is clear that functions of the type (2) satisfy equation (1). We now show that if a function $x(t)$ solves equation (1), it is necessarily of the type (2). Observe that

$$\frac{\dot{x}(t)}{x(t)} = \frac{d\ln(x(t))}{dt},$$

which allows us to rewrite differential equation (1) as

$$\frac{d\ln x(t)}{dt} = \lambda$$

Let $t_0 \in \mathbb{R}$. Integrating the equation from t_0 to t, $x(t)$ necessarily satisfies

$$\begin{aligned} \int_{t_0}^{t} d\ln x(t) &= \int_{t_0}^{t} \lambda \cdot dt \\ \ln(x(t)) - \ln(x(t_0)) &= \lambda(t - t_0) \\ x(t) &= x(t_0) \cdot e^{\lambda \cdot (t - t_0)} = \left[x(t_0) \cdot e^{-\lambda \cdot t_0}\right] \cdot e^{\lambda \cdot t}. \end{aligned}$$

Therefore, if $x(t)$ solves equation (1), it is necessarily of the type (2).

1.2 Constant Coefficient

We have solved the simplest FODE, which takes the form of equation (1). We now study a more general FODE:

$$\dot{x}(t) - \lambda \cdot x(t) = f(t), \tag{3}$$

where $f(t) \in \mathbb{R}$. Equation (1) is the special case of equation (3) with $f(t) = 0$ for all t.

Equation (3) admits a simple class of functions as solution:

$$x(t) = e^{\lambda \cdot t} \cdot \left[A + \int_0^t f(z) \cdot e^{-\lambda \cdot z} dz\right] \tag{4}$$

for any constant $A \in \mathbb{R}$. Furthermore, the constant A can be determined by an additional boundary condition because

$$A = x(0) = x(t_0) \cdot e^{-\lambda \cdot t_0} - \int_0^{t_0} f(z) \cdot e^{-\lambda \cdot z} dz$$

for any date $t_0 \in \mathbb{R}$.

It is clear that functions of the type (4) satisfy equation (3). We now show that if a function $x(t)$ solves equation (3), it is necessarily of the type (4).

To be able to solve the FODE, we manipulate a certain function $x(t) \cdot \mu(t)$ instead of manipulating $x(t)$ directly. The auxiliary function $\mu(t)$ is called the **integrating factor**. The integrating factor for this problem is

$$\mu(t) = e^{-\lambda \cdot t}.$$

This integrating factor $\mu(t)$ has the desirable property that $\dot{\mu}(t) = -\lambda \cdot \mu(t)$.

We multiply both sides of the differential equation (3) by the integrating factor to obtain

$$\begin{aligned}
\dot{x}(t) \cdot \mu(t) - \lambda \cdot x(t) \cdot \mu(t) &= f(t) \cdot \mu(t) \\
\dot{x}(t) \cdot \mu(t) + x(t) \cdot \dot{\mu}(t) &= f(t) \cdot \mu(t) \\
\frac{d\left[x(t) \cdot \mu(t)\right]}{dt} &= f(t) \cdot \mu(t).
\end{aligned}$$

Integrating the equation from $t_0 \in \mathbb{R}$ to t we obtain

$$\begin{aligned}
\int_{t_0}^{t} d\left[x(t) \cdot \mu(t)\right] &= \int_{t_0}^{t} f(z) \cdot \mu(z) dz \\
x(t) \cdot \mu(t) - x(t_0) \cdot \mu\left(t_0\right) &= \int_{t_0}^{t} f(z) \cdot \mu(z) dz \\
x(t) &= \frac{x\left(t_0\right) \cdot \mu\left(t_0\right) + \int_{t_0}^{t} f(z) \cdot \mu(z) dz}{\mu(t)} \qquad (5)
\end{aligned}$$

Given the definition of the integrating factor $\mu(t)$,

$$x(t) = e^{\lambda \cdot t} \cdot \left[x\left(t_0\right) \cdot e^{-\lambda \cdot t_0} + \int_{t_0}^{t} f(z) \cdot e^{-\lambda \cdot z} dz \right].$$

Therefore, there exists $A \in \mathbb{R}$ such that

$$x(t) = e^{\lambda \cdot t} \cdot \left[A + \int_{0}^{t} f(z) \cdot e^{-\lambda \cdot z} dz \right].$$

1.3 General Case

We now generalise (3) to allow the coefficient λ to vary with time t. We solve

$$\dot{x}(t) - \lambda(t) \cdot x(t) = f(t), \tag{6}$$

with $\lambda(t) \in \mathbb{R}$ and $f(t) \in \mathbb{R}$.

Equation (6) admits the following class of functions as solution:

$$x(t) = \exp\left(\int_0^t \lambda(s) ds \right) \cdot \left[A + \int_0^t f(z) \cdot \exp\left(- \int_0^z \lambda(s) ds \right) dz \right] \tag{7}$$

for any constant $A \in \mathbb{R}$. Furthermore, the constant A can be determined by an additional boundary condition because

$$A = x(0) = x(t_0) \cdot \exp\left(- \int_0^{t_0} \lambda(s) ds \right) - \int_0^{t_0} f(z) \cdot \exp\left(- \int_0^z \lambda(s) ds \right) dz$$

for any date $t_0 \in \mathbb{R}$.

Some algebra shows that functions of the type (7) satisfy equation (6). We now show that if a function $x(t)$ solves equation (6), it is necessarily of the type (7). As above, we introduce an integrating factor. The integrating factor for this problem is

$$\mu(t) = \exp\left(- \int_0^t \lambda(s) ds \right).$$

This integrating factor $\mu(t)$ has the desirable property that

$$\dot{\mu}(t) = -\lambda(t) \cdot \mu(t).$$

We multiply both sides of equation (6) by the integrating factor to obtain

$$\begin{aligned} \dot{x}(t) \cdot \mu(t) - \lambda(t) \cdot \mu(t) \cdot x(t) &= f(t) \cdot \mu(t) \\ \dot{x}(t) \cdot \mu(t) + x(t) \cdot \dot{\mu}(t) &= f(t) \cdot \mu(t) \\ \frac{d\left[x(t) \cdot \mu(t)\right]}{dt} &= f(t) \cdot \mu(t). \end{aligned}$$

Integrating the equation from $t_0 \in \mathbb{R}$ to t we obtain as earlier equation (5). Therefore the solution to equation (6) is necessarily of the type (7).

1.4 Initial Value Problem

Often, an initial condition for $x(t)$ is given:

$$x\left(t_0\right) = x_0. \tag{8}$$

Equation (6) together with equation (8) form an **initial value problem**. The constant A in (7) must satisfy

$$A = x_0 \cdot \exp\left(-\int_0^{t_0} \lambda(s) ds\right) - \int_0^{t_0} f(z) \cdot \exp\left(-\int_0^{z} \lambda(s) ds\right) dz.$$

Hence the solution to the initial value problem is

$$x(t) = x_0 \cdot \exp\left(\int_{t_0}^{t} \lambda(s) ds\right) + \int_{t_0}^{t} f(z) \cdot \exp\left(\int_{z}^{t} \lambda(s) ds\right) dz. \tag{9}$$

2 Linear Systems of First-Order Differential Equations

We often encounter dynamic systems with several variables that move together over time. For example in the consumption-saving problem that we solved with optimal control methods, in the case of CRRA utility, the solution is characterised by two FODEs:

$$\begin{aligned}\dot{a}(t) &= r \cdot a(t) - c(t),\\ \dot{c}(t) &= \frac{r-\rho}{\gamma} \cdot c(t).\end{aligned}$$

The first FODE is the asset accumulation equation and the second FODE is the Euler equation that characterises optimal consumption over time. To solve explicitly for the optimal consumption path, we need to solve the two FODEs simultaneously. This section presents a method to solve systems of FODEs.

We consider a system of n FODEs with constant coefficients:

$$\begin{aligned}\dot{x}_1(t) &= A_{11} \cdot x_1(t) + A_{12} \cdot x_2(t) + \ldots + A_{1n} \cdot x_n(t) + f_1(t)\\ \dot{x}_2(t) &= A_{21} \cdot x_1(t) + A_{22} \cdot x_2(t) + \ldots + A_{2n} \cdot x_n(t) + f_2(t)\\ &\;\;\vdots\\ \dot{x}_n(t) &= A_{n1} \cdot x_1(t) + A_{n2} \cdot x_2(t) + \ldots + A_{nn} \cdot x_n(t) + f_n(t).\end{aligned}$$

Our goal is to solve for the n functions $x_1(t), x_2(t), \ldots, x_n(t)$.

An alternative way of expressing the system is to write it in matrix form:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(t), \tag{10}$$

where $\dot{\mathbf{x}}(t) \in \mathbb{R}^n$, $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{f}(t) \in \mathbb{R}^n$ are column vectors with n elements. $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a constant $n \times n$ matrix. The system of FODEs is linear because it can be written in matrix form: it involves a linear relationship between the vector

$\dot{\mathbf{x}}(t)$ and the vector $\mathbf{x}(t)$.

If $\mathbf{A}$ is diagonal ($A_{ij} = 0$ for all $i \neq j$), the system would reduce to a collection of n FODEs—one FODE for each $x_i(t)$—that can be solved independently using the techniques from Section 1. If $\mathbf{A}$ is not diagonal, the different entries in $\mathbf{x}(t)$ interact and we must solve the system of FODEs simultaneously.

2.1 General Solution

Assume that $\mathbf{A}$ is diagonalizable. There exists $\mathbf{V} \in \mathbb{R}^{n\times n}$ such that

$$\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}, \tag{11}$$

where $\mathbf{\Lambda} \in \mathbb{R}^{n\times n}$ is a diagonal matrix. The diagonal entries of $\mathbf{\Lambda}$ are the n eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$, and $\mathbf{V}$ is the matrix whose columns are the eigenvectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ of $\mathbf{A}$.

By definition, $\lambda_1, \ldots, \lambda_n$ are the n roots of the polynomial equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

For any $i = 1, \ldots, n$, the eigenvector $\mathbf{z}_i$ associated with the eigenvalue λ_i satisfies

$$(\mathbf{A} - \lambda_i\mathbf{I})\,\mathbf{z}_i = \mathbf{0}.$$

Using the decomposition (11), we rewrite the system (10) as

$$\begin{aligned} \mathbf{V}^{-1}\dot{\mathbf{x}}(t) &= \mathbf{\Lambda}\mathbf{V}^{-1}\mathbf{x}(t) + \mathbf{V}^{-1}\mathbf{f}(t) \\ \dot{\mathbf{y}}(t) &= \mathbf{\Lambda}\mathbf{y}(t) + \mathbf{g}(t), \end{aligned} \tag{12}$$

where we define

$$\begin{aligned} \mathbf{y}(t) &\equiv \mathbf{V}^{-1}\mathbf{x}(t) \\ \mathbf{g}(t) &\equiv \mathbf{V}^{-1}\mathbf{f}(t). \end{aligned}$$

Since the matrix $\mathbf{\Lambda}$ is diagonal, the system is reduced to a collection of n independent FODEs—one for each $y_i(t)$. Once we have solved for $\mathbf{y}(t)$, we can recover $\mathbf{x}(t)$ by

$$\mathbf{x}(t) = \mathbf{V}\mathbf{y}(t).$$

The nature of the eigenvalues and corresponding eigenvectors determines the dynamics of the solution.

2.2 Homogenous Systems

If $\mathbf{f}(t) = \mathbf{0}$, the system (10) is **homogenous**; otherwise it is **nonhomogenous**.

For homogenous systems,

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t). \tag{13}$$

So the transformed system (12) becomes

$$\dot{\mathbf{y}}(t) = \mathbf{\Lambda}\mathbf{y}(t),$$

which leads to n independent FODEs:

$$\dot{y}_i(t) - \lambda_i \cdot y_i(t) = 0$$

for $i = 1, \dots, n$. In other words, each $y_i(t)$ is growing at constant rate λ_i. The analysis of Section 1 shows that the solution to the i^{th} FODE is

$$y_i(t) = A_i \cdot e^{\lambda_i \cdot t}$$

where $A_i \in \mathbb{R}$ is a constant. Finally, $x_1(t), \dots, x_n(t)$ are given by

$$\mathbf{x}(t) = \mathbf{V}\mathbf{y}(t).$$

The columns of $\mathbf{V}$ are the eigenvectors $\mathbf{z}_1, \dots, \mathbf{z}_n$ corresponding to the eigenvalues $\lambda_1, ...\lambda_n$. Hence the solution of the homogenous system (13) is

$$\mathbf{x}(t) = A_1 \cdot \mathbf{z}_1 \cdot e^{\lambda_1 \cdot t} + \dots + A_n \cdot \mathbf{z}_n \cdot e^{\lambda_n \cdot t}. \tag{14}$$

The nature of the eigenvalues and the corresponding eigenvectors determines the dynamics of the solution.

2.3 Example: Two-Variable Homogenous System

Consider a two-variable homogenous system

$$\begin{aligned}\dot{x}_1(t) &= a \cdot x_1(t) + b \cdot x_2(t)\\ \dot{x}_2(t) &= c \cdot x_1(t) + d \cdot x_2(t).\end{aligned}$$

We can write it in matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Assume $\det(\mathbf{A}) = a \cdot d - b \cdot c \neq 0$.

2.3.1 Closed-Form Solution

Equation (14) implies that to determine a closed-form solution of this homogenous system, we need to find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.

The eigenvalues are solutions to

$$\begin{aligned}\det(\mathbf{A} - \lambda\mathbf{I}) &= 0\\ (a-\lambda)\cdot(d-\lambda) - b \cdot c &= 0\\ \lambda^2 - (a+d)\cdot\lambda + (a \cdot d - b \cdot c) &= 0.\end{aligned}$$

Note that the product of the two eigenvalues is equal to the determinant of $\mathbf{A}$:

$$\lambda_1 \cdot \lambda_2 = a \cdot d - b \cdot c = \det(\mathbf{A}). \tag{15}$$

Let $\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$ be the eigenvector correspond to λ_1 and $\begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$ be the eigenvector correspond to λ_2. These vectors are solutions to

$$(\mathbf{A} - \lambda_i \mathbf{I}) \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix} = 0$$

which yields the system

$$\begin{aligned} (a - \lambda_i) \cdot \alpha_i + b \cdot \beta_i &= 0 \\ c \cdot \alpha_i + (d - \lambda_i) \cdot \beta_i &= 0. \end{aligned}$$

Consider the cases where the eigenvalues are real and distinct, the general solution (14) implies

$$\begin{aligned} x_1(t) &= A_1 \cdot \alpha_1 \cdot e^{\lambda_1 \cdot t} + A_2 \cdot \alpha_2 \cdot e^{\lambda_2 \cdot t} \\ x_2(t) &= A_1 \cdot \beta_1 \cdot e^{\lambda_1 \cdot t} + A_2 \cdot \beta_2 \cdot e^{\lambda_2 \cdot t} \end{aligned}$$

where A_1 and A_2 are arbitrary constants.

Note that in the case in which $\lambda_1 = \lambda_2 = \lambda$, the system $[x_1(t), x_2(t)]$ above is still the general solution of the system of FODEs as long as the two eigenvectors $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ are linearly independent.

Also note that any nonhomogenous system with constant terms $[\kappa_1, \kappa_2]$:

$$\begin{aligned} \dot{x}_1(t) &= a \cdot x_1(t) + b \cdot x_2(t) + \kappa_1 \\ \dot{x}_2(t) &= c \cdot x_1(t) + d \cdot x_2(t) + \kappa_2, \end{aligned}$$

can be transformed into an homogenous system.

2.3.2 Stability

Now that we have found a closed-form solution to the system, we can analyse its stability. There are three cases.

1. $\lambda_1 < 0$ and $\lambda_2 < 0$: As shown by (15), since λ_1 and λ_2 have the same sign, $\det(\mathbf{A}) > 0$. As $t \to +\infty$, $x_1(t) \to 0$ and $x_2(t) \to 0$. The system is **stable**. The origin is called a **sink**.

2. $\lambda_1 > 0$ and $\lambda_2 > 0$: As shown by (15), since λ_1 and λ_2 have the same sign, $\det(\mathbf{A}) > 0$. As $t \to +\infty$, $|x_1(t)| \to +\infty$ and $|x_2(t)| \to +\infty$. The system is **unstable**. The origin is called a **source**.

3. λ_1 and λ_2 have opposite sign: As shown by (15), since λ_1 and λ_2 have opposite sign, $\det(\mathbf{A}) < 0$. One part of the solution is stable (it converges to 0 at $t \to +\infty$), the other is unstable (it converges to ∞ at $t \to +\infty$). The origin $(0, 0)$ is called a **saddle point**.

3 Phase Diagrams

Without solving for eigenvalues and eigenvectors explicitly, we can study the properties of a linear system of FODEs by drawing its **phase diagram**.

3.1 Construction of the Phase Diagram

Consider the following linear nonhomogenous system of two FODEs:

$$\dot{x}_1(t) = a \cdot x_1(t) + b \cdot x_2(t) + \kappa_1 \tag{16}$$
$$\dot{x}_2(t) = c \cdot x_1(t) + d \cdot x_2(t) + \kappa_2, \tag{17}$$

with $a < 0$, $b < 0$, $c < 0$, $d > 0$, $\kappa_1 > 0$, and $\kappa_2 > 0$. Since $a \cdot d - b \cdot c < 0$, the eigenvalues of the system are of opposite sign. We are in a **saddle-point equilibrium**.

We construct the phase diagram as follows:

1. Compute the two loci $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. The locus for $\dot{x}_1 = 0$ is given by

$$x_2 = -\frac{a}{b} \cdot x_1 - \frac{\kappa_1}{b}.$$

The locus is a straight line with a negative slope in the (x_1, x_2) plan. The locus for $\dot{x}_2 = 0$ is given by

$$x_2 = -\frac{c}{d} \cdot x_1 - \frac{\kappa_2}{d}.$$

The locus is a straight line with positive slope in the (x_1, x_2) plan.

2. The intersection of the two loci is the **steady-state solution**. Denote the intersection of the two loci as (x_1^*, x_2^*). These two loci divide the (x_1, x_2) plane into four areas.

3. From (16), $\dot{x}_1$ is decreasing in x_2 because $b < 0$. Thus any point above the $\dot{x}_1 = 0$ line must have $\dot{x}_1 < 0$ and any point below the $\dot{x}_1 = 0$ line must have $\dot{x}_1 > 0$. We represent these properties by an horizontal arrow pointing west for any point above the $\dot{x}_1 = 0$ line and an horizontal arrow pointing east for any point below the $\dot{x}_1 = 0$ line.

   Similarly, from (17), $\dot{x}_2$ is increasing in x_2 because $d > 0$. Thus any point above the $\dot{x}_2 = 0$ line must have $\dot{x}_2 > 0$ and any point below the $\dot{x}_2 = 0$ line must have $\dot{x}_2 < 0$. We represent these properties by a vertical arrow pointing north for any point above the $\dot{x}_2 = 0$ line and a vertical arrow pointing south for any point below the $\dot{x}_2 = 0$ line.

   These sets of arrows are called **streamlines**. They determine the trajectories of the system at any point in all the four areas.

4. Last, we derive the saddle path for the system. We know that such a saddle path exist because the eigenvalues of the system have opposite sign.

Drawing the phase diagram of a two-variable system is useful to understand the main features of the dynamic system without solving for $x_1(t)$ and $x_2(t)$ explicitly. The phase diagram is represented at the end of the lecture notes.

3.2 Diagram with a State and a Control Variable

Suppose x_1 is a state variable: information revealed at t does not influence its value at t. Suppose x_2 is a control variable: information revealed at t may influence its value at t. Suppose that we are in the steady state (x_1^*, x_2^*) of the previous phase diagram.

Assume that there is an exogenous, unanticipated increase in κ_2. This increase is revelation of news because it is an unanticipated change to one of the parameters or variables of the system. As κ_2 increases, the $\dot{x}_2 = 0$ locus shifts down, so the new steady state (x_1^{**}, x_2^{**}) is to the south-east of the previous steady state: $x_1^{**} > x_1^*$ and $x_2^{**} < x_2^*$. There is also a new saddle path passing through this new steady state.

Where do we start after the news is revealed at $t = t_r$? That is, what are $x_1(t_r)$ and $x_2(t_r)$? Since x_1 is the state variable, it cannot respond to the news, and $x_1(t_r) = x_1^*$. For the system to converge to the new steady state, it must arrive at the steady-state level x_1^{**} of the state variable along the new saddle path. So $x_2(t_r)$ must be on the new saddle path at x_1^*, and over time both $x_1(t)$ and $x_2(t)$ move along the saddle path until they converge to the new steady state. To sum up, the system jumps from (x_1^*, x_2^*) to $(x_1^*, x_2(t_r))$, and then moves along the saddle path until it reaches (x_1^{**}, x_2^{**}).

The response to the news in the phase diagram is represented at the end of the lecture notes.

4 Nonlinear Systems of First-Order Differential Equations

Unlike the systems of FODEs studied in Sections 2 and 3, which were linear, systems of FODEs in macroeconomics are often nonlinear. For example the typical growth model is characterised by the following nonlinear system of FODEs:

$$\dot{k}(t) = f(k(t)) - c(t) - \delta \cdot k(t), \tag{18}$$

$$\dot{c}(t) = [f'(k(t)) - (\delta + \rho)] \cdot c(t), \tag{19}$$

where $\rho > 0$,and $\delta \in (0,1)$ are parameters, the capital stock $k(t)$ is a state variable with k_0 given, and the production function f satisfies the **Inada conditions**:

$$f(0) = 0,\ f' > 0,\ f'' < 0,\ \lim_{k \to +\infty} f'(k) = 0,\ \lim_{k \to 0} f'(k) = +\infty.$$

It is difficult to solve this system explicitly. But without solving it explicitly, we can characterise its properties by constructing its phase diagram. This is what we do in this section.

4.1 Construction of the Phase Diagram

We draw the phase diagram in a plane with the state variable k on the x-axis and the control variable c on the y-axis. We proceed as follows:

1. We draw the $\dot{k} = 0$ curve defined by

$$c = f(k) - \delta \cdot k$$

and the $\dot{c} = 0$ curve defined by

$$f'(k) = \delta + \rho.$$

The $\dot{k} = 0$ curve is concave and the $\dot{c} = 0$ curve is a vertical line in the (k, c) plane.

2. The intersection of these two loci is the steady state (k^*, c^*) of the system.

3. To determine the directions of the streamlines, we partially differentiate equations (18) and (19):

$$\frac{\partial \dot{k}}{\partial c} = -1 < 0$$
$$\frac{\partial \dot{c}}{\partial k} = c \cdot f''(k) < 0.$$

Therefore as c increases, $\dot{k}$ decreases. So, the horizontal arrows point eastward below the $\dot{k} = 0$ curve and westward above it. Similarly as k increases, $\dot{c}$ decreases. So the vertical arrows point northward to the left of the $\dot{c} = 0$ curve and southward to the right of it.

4. The streamlines drawn yield a saddle-point equilibrium at the steady state (k^*, c^*). The only way for the economy to converge to the steady state is

on the saddle path leading to it. This means that given any initial capital k_0, initial consumption c_0 is such that the pair (k_0, c_0) lies on the saddle path.

The phase diagram is represented at the end of the lecture notes.

4.2 Linearization

The streamlines suggest that we are in a saddle-point equilibrium. We can check the validity of the conclusion by linearizing the nonlinear system (18)–(19) using a first-order Taylor expansion around the steady state:

$$\begin{aligned}\dot{k} &= \dot{k}^* + (k - k^*) \cdot \frac{\partial \dot{k}}{\partial k} + (c - c^*) \cdot \frac{\partial \dot{k}}{\partial c} \\ \dot{c} &= \dot{c}^* + (k - k^*) \cdot \frac{\partial \dot{c}}{\partial k} + (c - c^*) \cdot \frac{\partial \dot{c}}{\partial c}.\end{aligned}$$

Given that $\dot{k}^* = \dot{c}^* = 0$, we have

$$\begin{bmatrix} \dot{k} \\ \dot{c} \end{bmatrix} = \mathbf{J}^* \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix},$$

where $\mathbf{J}^*$ is the Jacobian matrix evaluated at the steady state:

$$\mathbf{J}^* = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k}\Big|_{(k^*,c^*)} & \frac{\partial \dot{k}}{\partial c}\Big|_{(k^*,c^*)} \\ \frac{\partial \dot{c}}{\partial k}\Big|_{(k^*,c^*)} & \frac{\partial \dot{c}}{\partial c}\Big|_{(k^*,c^*)} \end{bmatrix}.$$

This system is a two-variable nonhomogenous system of FODEs for

$$\mathbf{x} = \begin{bmatrix} k \\ c \end{bmatrix}.$$

But it is a two-variable homogenous system for the transformed variable $\mathbf{y}$, where

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^* = \begin{bmatrix} k - k^* \\ c - c^* \end{bmatrix}.$$

The constant matrix A of Section 2 is $\mathbf{J}^*$. The analysis of Section 2 shows that the properties of the steady state depend on the eigenvalues of $\mathbf{J}^*$. The four partial derivatives are

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{(k^*,c^*)} = f'(k^*) - \delta = \rho > 0$$
$$\left.\frac{\partial \dot{k}}{\partial c}\right|_{(k^*,c^*)} = -1 < 0$$
$$\left.\frac{\partial \dot{c}}{\partial k}\right|_{(k^*,c^*)} = c \cdot f''(k^*) < 0$$
$$\left.\frac{\partial \dot{c}}{\partial c}\right|_{(k^*,c^*)} = f'(k^*) - (\delta + \rho) = 0$$

It follows that the Jacobian matrix can be written

$$\mathbf{J}^* = \begin{bmatrix} \rho & -1 \\ c \cdot f''(k) & 0 \end{bmatrix}$$

As shown by (15), the product of the two eigenvalues is the determinant of $\mathbf{J}^*$:

$$\det(\mathbf{J}^*) = c \cdot f''(k) < 0.$$

Therefore, the two eigenvalues have opposite sign. This property establishes that the steady state is a saddle point locally.