電磁気学Ｉ　演習 第４回 解答

【VA-64'】以下のベクトルの回転を求めよ。

(1) $3xz^2\hat{x} - y\hat{y} - x^2\hat{z}$

(2) $z\hat{\rho} + z^2\hat{z}$

(3) $\sin\theta\hat{r}$

<u>解答</u>

(1)

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xz^2 & -y & -x^2 \end{vmatrix}$$

$$= \hat{x}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ -y & -x^2 \end{vmatrix} - \hat{y}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 3xz^2 & -x^2 \end{vmatrix} + \hat{z}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ 3xz^2 & -y \end{vmatrix}$$

$$= -\hat{y}\left(-2x - 6xz\right) = 2x(3z+1)\hat{y}$$

(2)

$$\frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial z \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{z} \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial z \\ z & 0 & z^2 \end{vmatrix}$$

$$= \frac{1}{\rho}\left\{\hat{\rho}\begin{vmatrix} \partial/\partial\varphi & \partial/\partial z \\ 0 & z^2 \end{vmatrix} - \rho\hat{\varphi}\begin{vmatrix} \partial/\partial\rho & \partial/\partial z \\ z & z^2 \end{vmatrix} + \hat{z}\begin{vmatrix} \partial/\partial\rho & \partial/\partial\varphi \\ z & 0 \end{vmatrix}\right\}$$

$$= \frac{1}{\rho}\left[\hat{\rho}\left\{\frac{\partial}{\partial\varphi}(z^2) - \frac{\partial}{\partial z}(0)\right\} - \rho\hat{\varphi}\left\{\frac{\partial}{\partial\rho}(z^2) - \frac{\partial}{\partial z}(z)\right\} + \hat{z}\left\{\frac{\partial}{\partial\rho}(0) - \frac{\partial}{\partial\varphi}(z)\right\}\right]$$

$$= \hat{\varphi}$$

(3)

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\varphi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ \sin\theta & 0 & 0 \end{vmatrix} = -\frac{\cos\theta}{r}\hat{\varphi}$$

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【VA-71】ベクトル $\mathbf{F} = -x^3 y\hat{x} + xy^3\hat{y} + z^2\hat{z}$ に関して、Fig.71に示した各辺の長さが1の立方体で xy 面上の曲線 C_1 によって切り取られる S_1 に対してストークスの定理が成り立つことを確認せよ。また、面積を $S_2 \sim S_6$ としたときにも成り立つことを確認せよ。

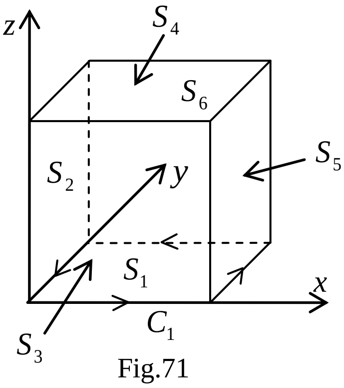

Fig.71

解答

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{l} = \int_{x=0}^{1} \mathbf{F}\cdot(\hat{x}dx)\Big|_{y=z=0} + \int_{y=0}^{1} \mathbf{F}\cdot(\hat{y}dy)\Big|_{\substack{x=1\\z=0}} + \int_{x=1}^{0} \mathbf{F}\cdot(\hat{x}dx)\Big|_{\substack{y=1\\z=0}} + \int_{y=1}^{0} \mathbf{F}\cdot(\hat{y}dy)\Big|_{x=z=0}$$

$$= 0 + \int_{y=0}^{1} y^3 dy - \int_{x=1}^{0} x^3 dx + 0$$

$$= \left[\frac{y^4}{4}\right]_0^1 - \left[\frac{x^4}{4}\right]_1^0 = \frac{1}{2}$$

また、

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \hat{z}\left(x^3 + y^3\right)$$

$$\iint_{S_1} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{y=0}^{1} \nabla\times\mathbf{F}\cdot\hat{z}dxdy\Big|_{z=0}$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left(x^3 + y^3\right)dxdy$$

$$= \frac{1}{2}$$

したがって、

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{l} = \iint_{S_1} \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

であるから、曲線 C_1 によって切り取られる S_1 に対してストークスの定理が成り立つ。

$S_2 \sim S_6$ の法線面積分の和

$$\iint_{S_2} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{y=0}^{1}\int_{z=0}^{1} \nabla\times\mathbf{F}\cdot(-\hat{x}dydz)\Big|_{x=0} = \int_{y=0}^{1}\int_{z=0}^{1} 0dydz = 0$$

$$\iint_{S_3} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{z=0}^{1} \nabla\times\mathbf{F}\cdot(-\hat{y}dxdz)\Big|_{y=0} = \int_{x=0}^{1}\int_{z=0}^{1} 0dxdz = 0$$

$$\iint_{S_4} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{y=0}^{1} \nabla\times\mathbf{F}\cdot(\hat{z}dxdy)\Big|_{z=1} = \int_{x=0}^{1}\int_{y=0}^{1} \left(x^3+y^3\right)dxdy = \frac{1}{2}$$

$$\iint_{S_5} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{y=0}^{1}\int_{z=0}^{1} \nabla\times\mathbf{F}\cdot(\hat{x}dydz)\Big|_{x=1} = -\int_{y=0}^{1}\int_{z=0}^{1} 0dydz = 0$$

$$\iint_{S_6} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \int_{x=0}^{1}\int_{z=0}^{1} \nabla\times\mathbf{F}\cdot(\hat{y}dxdz)\Big|_{y=1} = \int_{x=0}^{1}\int_{z=0}^{1} 0dxdz = 0$$

よって、

$$\iint_{S_2+S_3+S_4+S_5+S_6} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \frac{1}{2}$$

したがって、

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{l} = \iint_{S_2+S_3+S_4+S_5+S_6} \nabla\times\mathbf{F}\cdot d\mathbf{S}$$

であるから面積を $S_2 \sim S_6$ としたときにもストークスの定理が成り立つ

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【VA-87】uをスカラー関数、$\mathbf{A}$ をベクトル関数とするとき、以下の公式が成り立つことを示せ。

(1) $\nabla\times(u\mathbf{A}) = \nabla u\times\mathbf{A} + u\nabla\times\mathbf{A}$

(2) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

<u>解答</u>

(1)

$\nabla\times(u\mathbf{A}) = \nabla u\times\mathbf{A} + u\nabla\times\mathbf{A}$ の証明

$$
\begin{aligned}
\nabla\times(u\mathbf{A}) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ uA_x & uA_y & uA_z \end{vmatrix} \\
&= \hat{x}\left\{\frac{\partial}{\partial y}(uA_z)-\frac{\partial}{\partial z}(uA_y)\right\}-\hat{y}\left\{\frac{\partial}{\partial x}(uA_z)-\frac{\partial}{\partial z}(uA_x)\right\}+\hat{z}\left\{\frac{\partial}{\partial x}(uA_y)-\frac{\partial}{\partial y}(uA_x)\right\} \\
&= \hat{x}\left\{A_z\frac{\partial u}{\partial y}+u\frac{\partial A_z}{\partial y}-A_y\frac{\partial u}{\partial z}-u\frac{\partial A_y}{\partial z}\right\} \\
&\quad -\hat{y}\left\{A_z\frac{\partial u}{\partial x}+u\frac{\partial A_z}{\partial x}-A_x\frac{\partial u}{\partial z}-u\frac{\partial A_x}{\partial z}\right\} \\
&\quad +\hat{z}\left\{A_y\frac{\partial u}{\partial x}+u\frac{\partial A_y}{\partial x}-A_x\frac{\partial u}{\partial y}-u\frac{\partial A_x}{\partial y}\right\} \\
&= u\left[\hat{x}\left\{\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right\}-\hat{y}\left\{\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right\}+\hat{z}\left\{\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right\}\right] \\
&\quad +\hat{x}\left\{A_z\frac{\partial u}{\partial y}-A_y\frac{\partial u}{\partial z}\right\}-\hat{y}\left\{A_z\frac{\partial u}{\partial x}-A_x\frac{\partial u}{\partial z}\right\}+\hat{z}\left\{A_y\frac{\partial u}{\partial x}-A_x\frac{\partial u}{\partial y}\right\} \\
&= u\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}+\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ A_x & A_y & A_z \end{vmatrix} \\
&= \nabla u\times\mathbf{A}+u\nabla\times\mathbf{A}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\nabla\times\mathbf{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \\
&= \hat{x}\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ A_y & A_z \end{vmatrix}-\hat{y}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ A_x & A_z \end{vmatrix}+\hat{z}\begin{vmatrix} \partial/\partial x & \partial/\partial y \\ A_x & A_y \end{vmatrix} \\
&= \hat{x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\hat{y}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\hat{y}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)
\end{aligned}
$$

$$
\nabla\cdot(\nabla\times\mathbf{A})=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)=0
$$

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