## 1.4 Measure and Probability

**Def. 1.5 (Measure)**

$\mu$: $\mathcal{F} \to \mathbb{R}$ is a **measure** on $(\Omega, \mathcal{F}) \Leftrightarrow$

1. ${}^{\forall}A \in \mathcal{F}$, $\mu(A) \geq 0$
2. ${}^{\exists}A \in \mathcal{F}$ s.t. $\mu(A) < \infty$
3. $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_i \cap A_j = \emptyset$ $(i \neq j)$ (mutually disjoint)
   $\Rightarrow \quad \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mu(A_i)$ **($\sigma$-additivity)**

$(\Omega, \mathcal{F}, \mu)$: **Measure space**

- $\mu(\Omega) < \infty \Leftrightarrow$ **finite measure**
- ${}^{\exists}\{A_1, A_2, \ldots\}$, $A_i \in \mathcal{F}$, s.t. $\bigcup_i A_i = \Omega$ & $\mu(A_i) < \infty$, $i = 1, 2, \ldots$
  $\Leftrightarrow$ **$\sigma$-finite measure**

**Remark 1.1** $\mu(\emptyset) = 0$

$\because$ By 2, ${}^{\exists}A \in \mathcal{F}$ s.t. $\mu(A) < \infty$.
In 3, $A_1 = A$, $A_2 = A_3 = \cdots = \emptyset$ $(A_i \cap A_j = \emptyset$, $i \neq j)$
$\Rightarrow \mu(A) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots \Rightarrow \mu(\emptyset) = 0$

**Examples of measures**

**Lebesgue measure $\lambda$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$:**

1. $A = \prod_{i=1}^{n}(a_i, b_i)$, $a_i < b_i$ (open interval) $\Rightarrow \lambda(A) = \prod_{i=1}^{n}(b_i - a_i)$
2. $A \in \mathcal{B}(\mathbb{R}) \Rightarrow \lambda(A) = \inf_{\mathcal{C}_A} \sum_{B \in \mathcal{C}_A} \lambda(B)$,
   where $\mathcal{C}_A = \{$Open intervals covering $A\}$

**Counting measure $\nu$:** Measure s.t. $\nu(A) \in \overline{\mathbb{Z}}_+ = \{0, 1, 2, \ldots, +\infty\}$, $A \in \mathcal{F}$

**Probability measure $\mathsf{P}$:** Measure s.t. $\mathsf{P}(\Omega) = 1$

**Properties of measures**

i) $A, B \in \mathcal{F}$ s.t. $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

ii) $A_1, A_2, \ldots \in \mathcal{F}$ (not necessarily disjoint) $\Rightarrow \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mu(A_i)$

iii) $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_1 \subset A_2 \subset \cdots \Rightarrow \mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i)$

iv) $A_1, A_2, \ldots \in \mathcal{F}$ s.t. $A_1 \supset A_2 \supset \cdots$ & ${}^{\exists}n \in \mathbb{N}$ s.t. $\mu(A_n) < \infty$
$\Rightarrow \mu\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i)$

**Remark 1.2** In iv), "${}^{\exists}n \in \mathbb{N}$ s.t. $\mu(A_n) < \infty$" is necessary.

**Example:** ${}^{\forall}a, b \in \mathbb{R}$ $(a < b)$, $\mu((a, b)) = b - a$ (Lebesgue measure)

$$A_i = [i, \infty) \Rightarrow \bigcap_{i=1}^{\infty} A_i = \emptyset \text{ while } {}^{\forall}i \in \mathbb{N},\ \mu(A_i) = +\infty$$

## 1.5 Measurable Functions

**Def. 1.6 (Measurable Function)**

$(E, \mathcal{E})$**,** $(F, \mathcal{F})$**:** Measurable spaces

$f$: $E \to F$ is a **measurable function** from $(E, \mathcal{E})$ to $(F, \mathcal{F})$
(or $\mathcal{E}/\mathcal{F}$-measurable) $\Leftrightarrow$

$${}^{\forall}B \in \mathcal{F}, \quad f^{-1}(B) = \{x \in E : f(x) \in B\} \in \mathcal{E}$$

Especially, $(E, \mathcal{E}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $(F, \mathcal{F}) = (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \Leftrightarrow$ **Borel function**

**Examples: Random variables, stochastic processes:** $(\Omega, \mathcal{F}, \mathsf{P})$: Probability space

- $X$: $\Omega \to \mathbb{R}$ is a random variable $\Leftrightarrow {}^{\forall}B \in \mathcal{B}(\mathbb{R}),\ \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$
$\Leftrightarrow X$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable

- $X$: $\Omega \to D(\mathbb{R})$ is a stochastic process s.t. the sample paths are right-continuous with left limits on $\mathbb{R}$ ($^\forall\omega \in \Omega$, $(X(\omega, t))_{t\in\mathbb{R}}$ is right-continuous with left limits) $\Leftrightarrow$ $X$ is $\mathcal{F}/\mathcal{B}(D(\mathbb{R}))$-measurable

**Properties of measurable functions**

i) Continuous and piecewise continuous functions on $\mathbb{R}^n$ are Borel functions

ii) $f$, $g$: measurable functions $\Rightarrow$
$f + g$: $x \mapsto f(x) + g(x)$, $f \cdot g$: $x \mapsto f(x) \times g(x)$ are measurable functions

iii) $f_1, f_2, \ldots$: measurable functions $\Rightarrow$
$\lim_n f_n$: $x \mapsto \lim_n f(x)$ (pointwise limit) is a measurable function (whenever it exists)

iv) $(E_i, \mathcal{E}_i)$, $i = 1, 2, 3$: Measurable spaces
$f_i$: $E_i \to E_{i+1}$, $i = 1, 2$: Measurable functions
$\Rightarrow$ $f_2 \circ f_1$: $E_1 \to E_3$ ($x \mapsto f_2(f_1(x))$) is measurable

**Example 1.8** $X_1, X_2, \ldots, X_n$**:** Random variables

$f$**:** $\mathbb{R}^n \to \mathbb{R}$: Borel function

$\Rightarrow$ $f(X_1, X_2, \ldots, X_n)$ is a random variable

**Example 1.9 (Unmeasurable function)** $A \notin \mathcal{B}(\mathbb{R})$**:** An unmeasurable set

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A \end{cases} \qquad \Rightarrow \quad \mathbf{1}_A^{-1}(\{1\}) = A \notin \mathcal{B}(\mathbb{R})$$

# 2 Integration and Expectation

**Expectation at Elementary Level:**

$X$**:** A random variable

$X$ **is discrete** $\Leftrightarrow$ $X$ takes only a countable # of different values $x_1, x_2, \dots$

$X$ **is (absolutely) continuous** $\Leftrightarrow$ $^\exists f\colon \mathbb{R} \to \mathbb{R}_+$ (probability density function) s.t.

$$\mathsf{P}(X \leq x) = \int_{-\infty}^{x} f(s)\,\mathrm{d}s, \quad \forall x \in \mathbb{R}$$

**Def. 2.1 (Elementary Definition of Expectation)**

The expectation $\mathsf{E}(X)$ of a random variable $X$ is given by

$$\mathsf{E}(X) = \begin{cases} \displaystyle\sum_{i=1}^{\infty} x_i \,\mathsf{P}(X = x_i) & X \text{ is discrete,} \\ \displaystyle\int x\, f(x)\,\mathrm{d}x & X \text{ is continuous with density } f \end{cases}$$

**Question:** How about more general cases?

- There exist random variables which are neither discrete nor continuous.
- Not all continuous random variables are absolutely continuous (with probability density functions).

**Example 2.1** For $U \sim U[0,1)$, $X = U\,\mathbf{1}_{U>1/2}$ is neither discrete nor continuous