COMP 3011
DESIGN AND ANALYSIS OF ALGORITHMS
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Dynamic Programming

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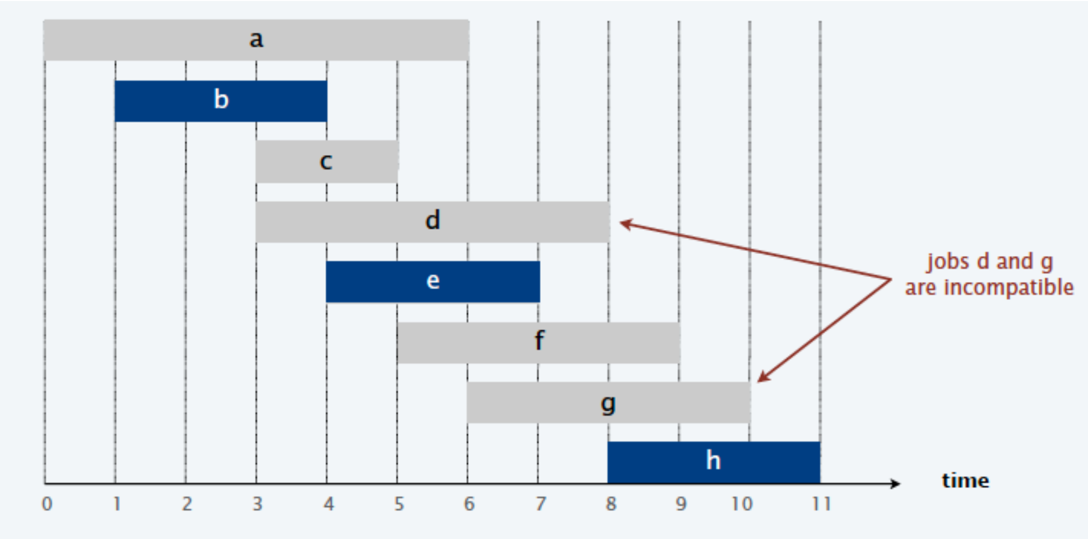
DYNAMIC PROGRAMMING

Algorithmic Paradigms

fancy name for caching intermediate results in a table for later reuse

- ***Greedy***. Process the input in some order, myopically making irrevocable decisions.
- ***Divide-and-conquer***. Break up a problem into independent subproblems; solve each subproblem; combine solutions to subproblems to form solution to original problem.
- ***Dynamic programming***. Break up a problem into a series of overlapping subproblems; combine solutions to smaller subproblems to form solution to large subproblem.



Interval Scheduling Problem

Given a set of jobs $J = \{1, 2, \ldots, n\}$

- Job j starts at s_j and finishes at $f_j \geq s_j$.
- Two jobs are compatible if they don't overlap.

Goal: find maximum subset of mutually compatible jobs.

What if different jobs bring different values?

Weighted Interval Scheduling Problem

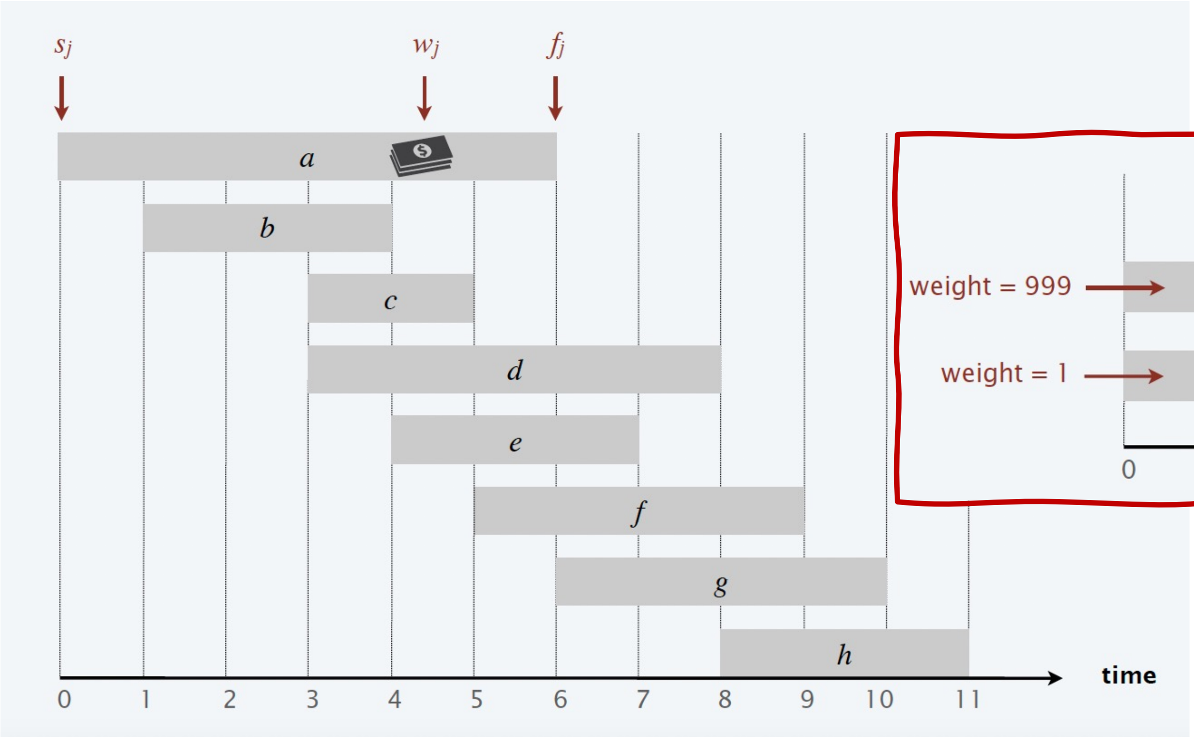
Given a set of jobs $J = \{1, 2, \ldots, n\}$

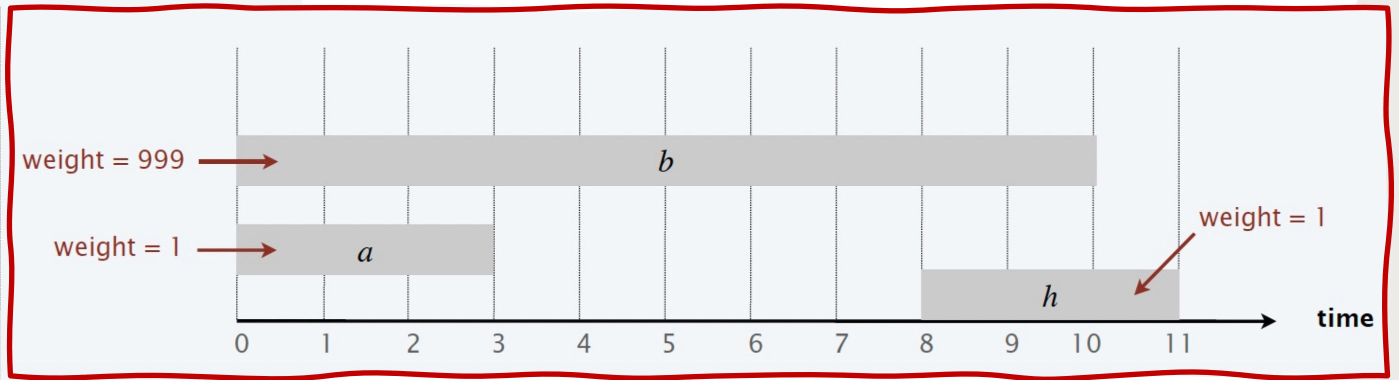
- Job j starts at s_j, finishes at f_j, and has weight $w_j > 0$.
- Two jobs are compatible if they don't overlap.

Goal: find max-weight subset of mutually compatible jobs.

Earliest finish-time first

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.





Weighted Interval Scheduling Problem

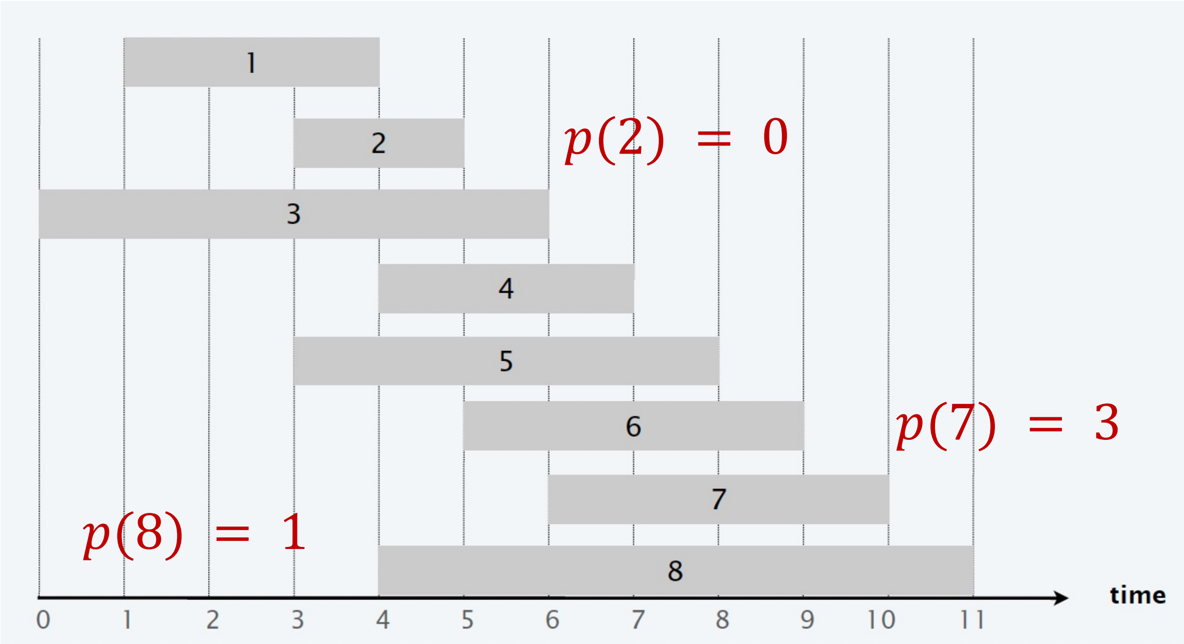
i is rightmost interval that ends before j begins

Jobs are in ascending order of finish time: $f_1 \leq f_2 \leq \ldots \leq f_n$.

***Definition*.**

- $p(j)$ = largest index $i < j$ such that job i is compatible with j.
- $OPT(j)$ = max weight of any subset of mutually compatible jobs for subproblem consisting only of jobs $1, 2, \ldots, j$.

***Goal*.** $OPT(n)$ = max weight of any subset of mutually compatible jobs.



***Case 1*.** $OPT(j)$ does not select job j.

- Must be an optimal solution to problem consisting of remaining jobs $1, 2, \ldots, j-1$.

***Case 2*.** $OPT(j)$ selects job j.

- Collect profit w_j.
- Can't use jobs $\{p(j)+1, p(j)+2, \ldots, j-1\}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \ldots, p(j)$.

Weighted Interval Scheduling Problem

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- Collect profit w_j.
- Can't use jobs $\{p(j) + 1, p(j) + 2, \ldots, j - 1\}$.
- Must include optimal solution to problem consisting of remaining compatible jobs $1, 2, \ldots, p(j)$.

optimal substructure property

Bellman equation

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{OPT(j-1),\ w_j + OPT(p(j))\} & \text{if } j > 0 \end{cases}$$

Exponential running time!

Recursive algorithm
to compute $OPT(j)$:

```
COMPUTE-OPT(j)
IF (j = 0)
    RETURN 0.
ELSE
    RETURN max {COMPUTE-OPT(j – 1), w_j + COMPUTE-OPT(p[j]) }.
```

Case 1

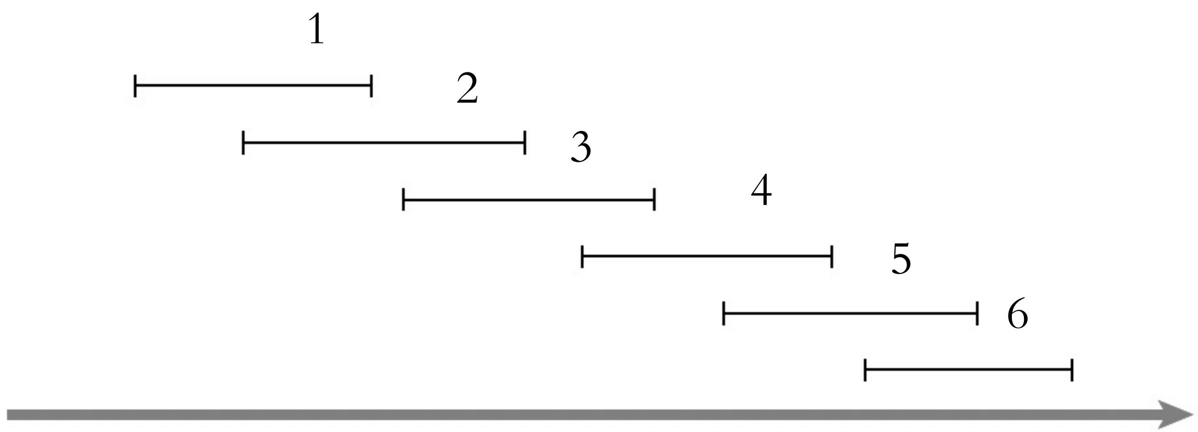
Case 2

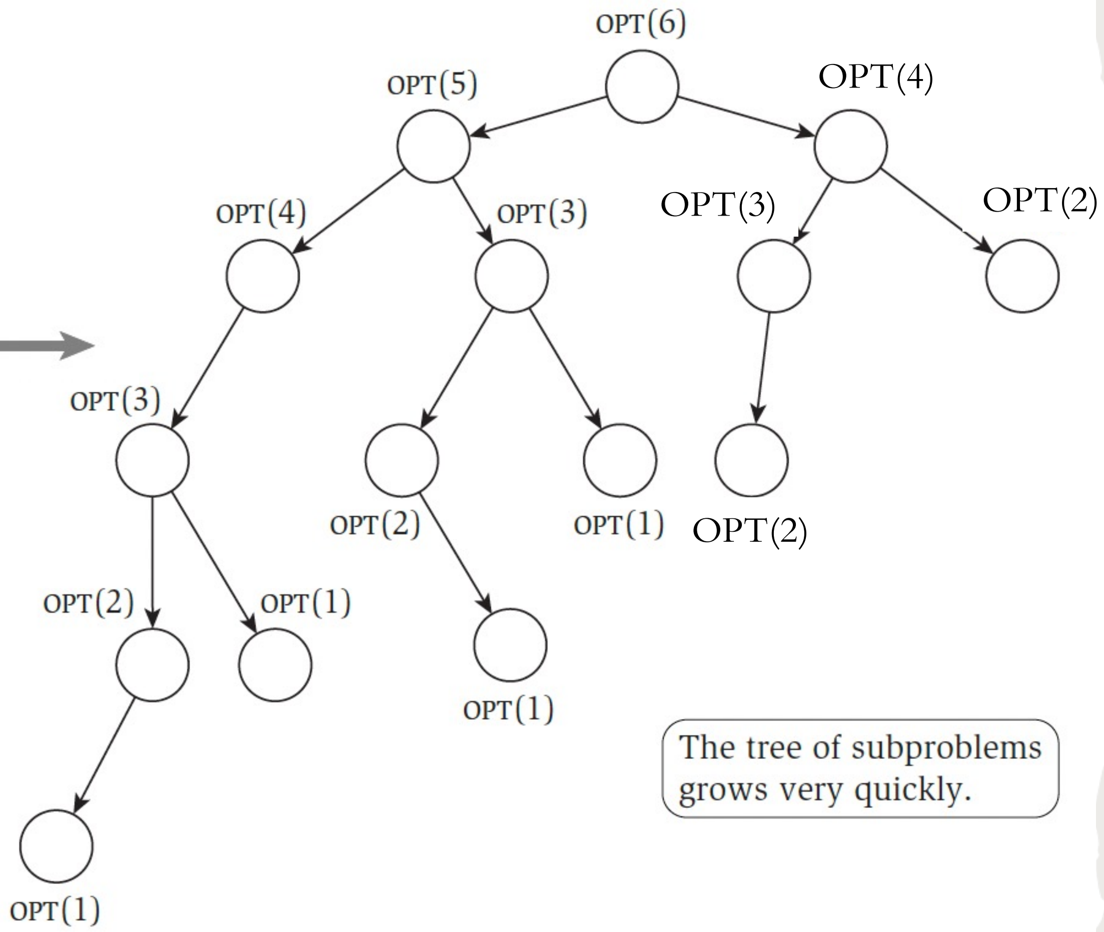
How to compute $p(j)$?

binary search!

Weighted Interval Scheduling Problem

Example:





$$p(1) = p(2) = 0, p(j) = j - 2 \text{ for } j \geq 3$$

Grows like Fibonacci sequence!

Memoizing the Recursion

Observation: the recursive algorithm Compute-Opt is really only solving $n+1$ different subproblems: Compute-Opt(0), Compute-Opt(1), . . . , Compute-Opt(n).

Technique: store the value of Compute-Opt the first time we compute it and then simply use this precomputed value in place of all future recursive calls.

Memoization

TOP-DOWN($n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n$)

Sort jobs by finish time and renumber so that $f_1 \le f_2 \le \ldots \le f_n$.

Compute $p[1], p[2], \ldots, p[n]$ via binary search.

$M[0] \leftarrow 0$. global array

RETURN M-COMPUTE-OPT(n).

M-COMPUTE-OPT(j)

IF ($M[j]$ is uninitialized)

$M[j] \leftarrow \max \{$ M-COMPUTE-OPT $(j-1)$, w_j + M-COMPUTE-OPT($p[j]$) $\}$.

RETURN $M[j]$.

Memoizing the Recursion

Theorem. Memoized version of algorithm takes $O(n \log n)$ time.

Proof.

- Sort by finish time: $O(n \log n)$ via Mergesort.
- Compute $p[j]$ for each j: $O(n \log n)$ via binary search.

- M-COMPUTE-OPT(j): each invocation takes $O(1)$ time and either
  - (1) returns an initialized value $M[j]$.
  - (2) initializes $M[j]$ and makes two recursive calls
- How many M-Compute-Opt calls?
  - M has $n+1$ entries, which are initially empty.
  - Each time the procedure invokes the recurrence, issuing two recursive calls to M-Compute-Opt, it fills in a new entry, and hence increases the number of filled-in entries by 1.
  - There can be at most $O(n)$ calls to M-Compute-Opt.
- Overall running time of M-COMPUTE-OPT(n) is $O(n)$.

Question:
DP algorithm computes optimal value.
How to find optimal solution?

Keep track of an optimal solution
in addition to its value

E.g. maintain an additional array S so that $S[i]$ contains an optimal set of intervals among $\{1, 2, \ldots, i\}$.

Blow up the running time by an additional factor of $O(n)$:

- While a position in the M array can be updated in $O(1)$ time, writing down a set in the S array takes $O(n)$ time.

Memoizing the Recursion

To avoid this $O(n)$ blow-up, we do not explicitly maintain S, but rather recover the optimal solution from values saved in the array M after the optimum value has been computed.

Job j belongs to an optimal solution for the set of intervals $\{1, \ldots, j\}$ if and only if

$$v_j + OPT(p(j)) \geq OPT(j-1).$$

```
Find-Solution(j)
  If j = 0 then
    Output nothing
  Else
    If v_j + M[p(j)] ≥ M[j − 1] then
      Output j together with the result of Find-Solution(p(j))
    Else
      Output the result of Find-Solution(j − 1)
    Endif
  Endif
```

Theorem. Given the array M of the optimal values of the sub-problems, Find-Solution returns an optimal solution in $O(n)$ time.

- Find-Solution calls itself recursively only on strictly smaller values, it makes a total of $O(n)$ recursive calls;
- It spends constant time per call.

Bottom-up Dynamic Programming

Theorem. The bottom-up version takes $O(n \log n)$ time.

Iterative algorithm
to compute $OPT(j)$:

BOTTOM-UP$(n, s_1, \ldots, s_n, f_1, \ldots, f_n, w_1, \ldots, w_n)$

Sort jobs by finish time and renumber so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

Compute $p[1], p[2], \ldots, p[n]$.

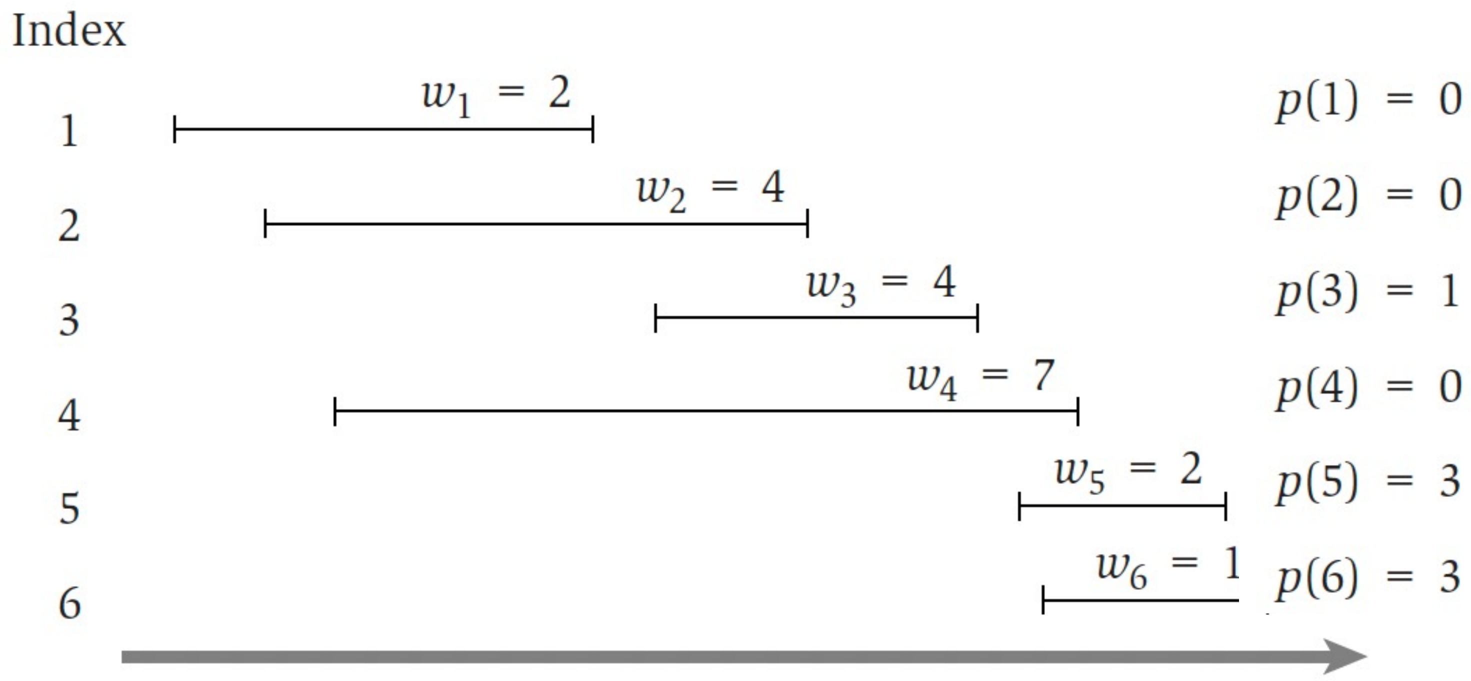
$M[0] \leftarrow 0.$ previously computed values

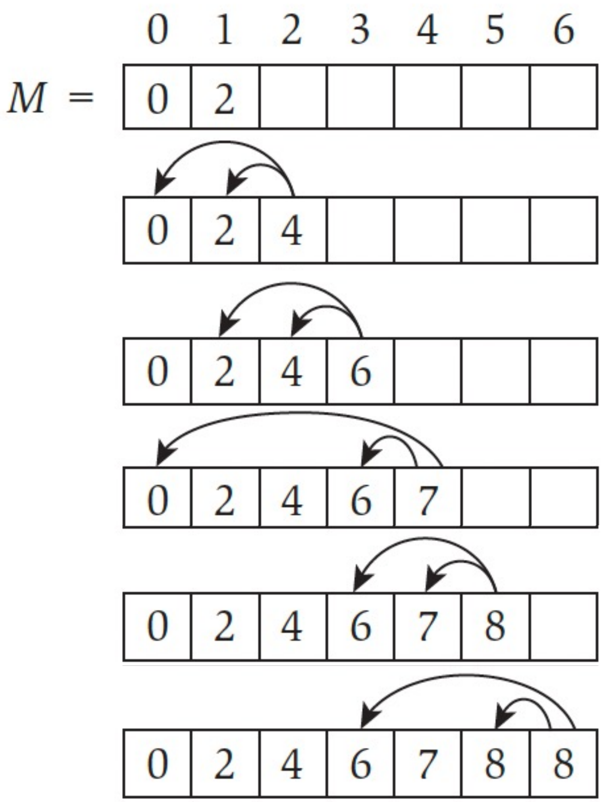
FOR $j = 1$ TO n

$\quad M[j] \leftarrow \max \{ M[j-1],\ w_j + M[p[j]] \}.$

Bottom-up Dynamic Programming

Example:





Algorithmic Paradigms

fancy name for caching intermediate results in a table for later reuse

- ***Greedy***. Process the input in some order, myopically making irrevocable decisions.
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Two More Examples

Max Contiguous Subarray

5	–1	31	–61	59	26	–53	58	97	–93	–23	84

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Given an array x of n integer (positive or negative), find a contiguous subarray whose sum is maximum.

***Approach 1. Brute-force algorithm*.**

- For each i and j: compute
$$a[i] + a[i+1] + \cdots + a[j]$$
- Takes $\Theta(n^3)$ time.

***Approach 2. Apply "cumulative sum" trick*.**

- Precompute cumulative sums:
$$S[i] = a[0] + a[1] + \cdots + a[i].$$
- Now $a[i] + a[i+1] + \cdots + a[j] = S[j] - S[i-1]$.
- Improves running time $\Theta(n^2)$.

Let $OPT(i)$ = max sum of any subarray of x whose rightmost index is i.

Goal: $\max_i OPT(i)$

take only element i

take element i together with best subarray ending at index $i-1$

$$OPT(i) = \begin{cases} x_1, & \text{if } i = 1 \\ \max\{x_i, OPT(i-1) + x_i\}, & \text{if } i > 1 \end{cases}$$

Bellman equation

Running time: $O(n)$.

Monotonically Increasing Subsequence

- Give an efficient algorithm to find the longest monotonically increasing sequence in a sequence of n numbers.
- Example: given (5,2,8,7,3,1,6,4)

  (2,6),

  (1,4),

  (2,3,6)

$(s_1, s_2, \cdots, s_{n-1}, s_n)$

What do we want to know about the first $n-1$ numbers?

What are the subproblems?

Monotonically Increasing Subsequence

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  (2,6), (1,4), (2,3,6)

$(s_1, s_2, \cdots, s_{n-1}, s_n)$

What do we want to know about the first $n-1$ numbers?

Let $OPT(i)$ = the length of the longest sequence ending with the ith character.

Goal: $\max_i OPT(i)$

$$OPT(i) = \begin{cases} 0, & if\ i = 0 \\ \max_{0 \le j < i : s[j] < s[i]} \{OPT(j) + 1\}, & if\ i > 1 \end{cases}$$

Running time: $O(n^2)$.

Subset Sum Problem
and
Knapsack Problem

Subset Sum Problem

Given n items $\{1, \dots, n\}$, and each has a given nonnegative integral weight w_i (for $i = 1, \dots, n$). We are also given an integral bound W. We would like to select a subset S of the items so that $\sum_{i \in S} w_i \leq W$ and, subject to this restriction, $\sum_{i \in S} w_i$ is as large as possible.

For Interval Scheduling, we could simply delete each request that conflicted with request i.

Greedy Approaches:

Start selecting items as long as the total weight remains below W.

- sort the items by decreasing weight

$$\{W/2 + 1, W/2, W/2\}$$

- sort by increasing weight

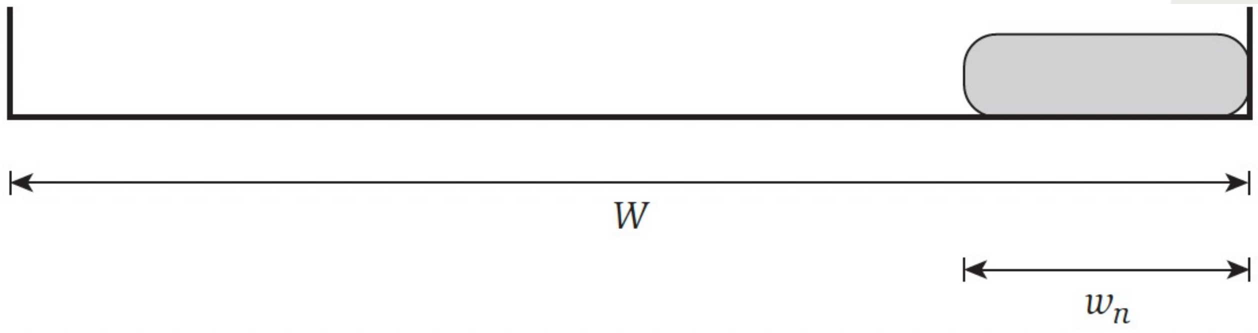
$$\{1, W/2, W/2\}$$

Motivated by ***Weighted Interval Scheduling***:

- consider subproblems involving only the first i items.
- $OPT(i)$ = the best possible solution using a subset of the items $\{1, \dots, i\}$.
- Case 1. $i \notin OPT$, then

$$OPT(i) = OPT(i-1).$$

- Case 2. $i \in OPT$:
  - for the subset of requests $S \subseteq \{1, \dots, i-1\}$ that we will accept, we only have $W - w_i$ weight left.
  - But $OPT(i)$ does not reflect the remaining weight!



Subset Sum Problem: two variables

For for each $i = 0, 1, \ldots, n$ and each integer $0 \le w \le W$:

- $OPT(i, w)$ = the value of the optimal solution using a subset of the items $\{1, \ldots, i\}$ with maximum allowed weight w.

$$OPT(i, w) = \max_{S \subseteq \{1,\cdots,i\}:\ \sum_{l \in S} w_l \le w} \sum_{j \in S} w_j$$

- $OPT(n, W)$ is the quantity we're looking for.

optimal substructure property

***Case 1**. $OPT(i, w)$ does not select item i.*

- $OPT(i, w)$ selects best of $\{1, 2, \ldots, i-1\}$ subject to weight limit w.

***Case 2**. $OPT(i, w)$ selects item i.*

- Collect w_i.
- New weight limit = $w - w_i$.
- $OPT(i, w)$ selects best of $\{1, 2, \ldots, i-1\}$ subject to new weight limit.

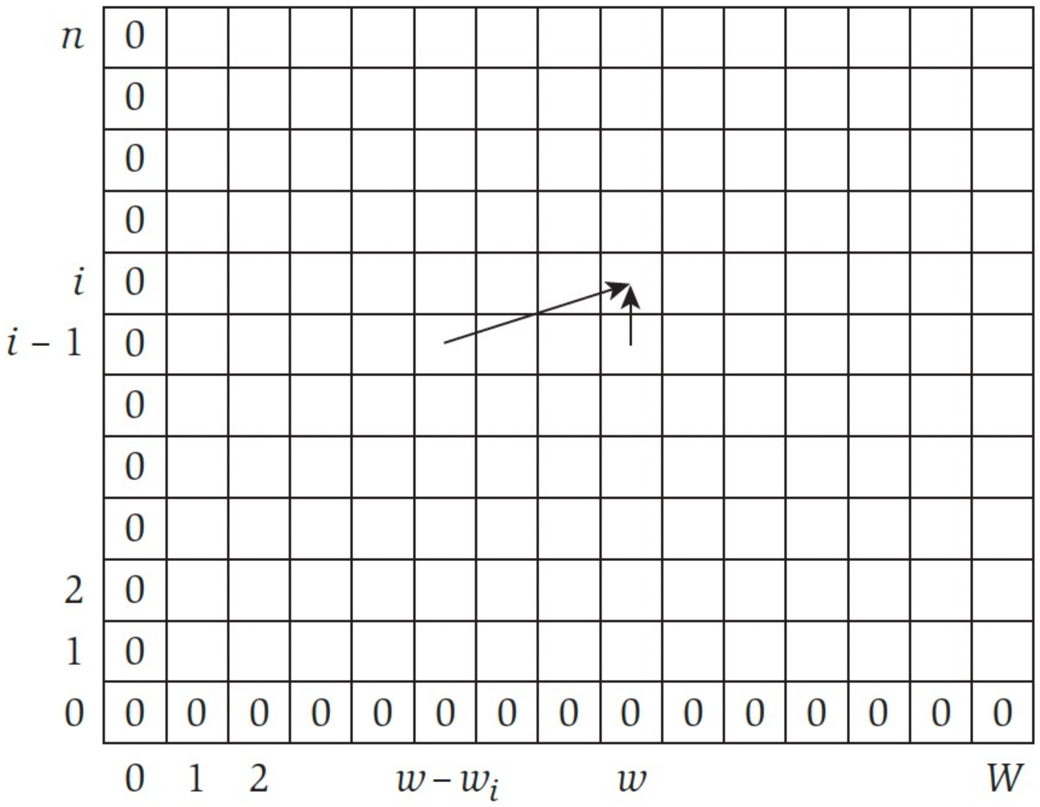
$$OPT(i, w) = \begin{cases} OPT(i-1, w), & \text{if } w < w_i \\ \max\big(OPT(i-1, w), w_i + OPT(i-1, w - w_i)\big), & \textit{otherwise} \end{cases}$$

Subset Sum Problem

$$OPT(i,w) = \begin{cases} OPT(i-1,w), & \text{if } w < w_i \\ \max\big(OPT(i-1,w), w_i + OPT(i-1,w-w_i)\big), & otherwise \end{cases}$$

Subset-Sum(n, W)
  Array $M[0 \ldots n, 0 \ldots W]$
  Initialize $M[0, w] = 0$ for each $w = 0, 1, \ldots, W$
  For $i = 1, 2, \ldots, n$
    For $w = 0, \ldots, W$
      Use the above recurrence to compute $M[i, w]$
    Endfor
  Endfor
  Return $M[n, W]$

- The leftmost column and bottom row are always 0.
- The entry for $OPT(i,w)$ is computed from the two other entries $OPT(i-1,w)$ and $OPT(i-1,w-w_i)$.



The two-dimensional table of OPT values.

Subset Sum Problem

A sample example:

- $W = 6$, items $w_1 = 2, w_2 = 2, w_3 = 3$

3							
2							
1							
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Initial values

3							
2							
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Filling in values for $i = 1$

3							
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Filling in values for $i = 2$

3	0	0	2	3	4	5	5
2	0	0	2	2	4	4	4
1	0	0	2	2	2	2	2
0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6

Filling in values for $i = 3$

trace back through the array

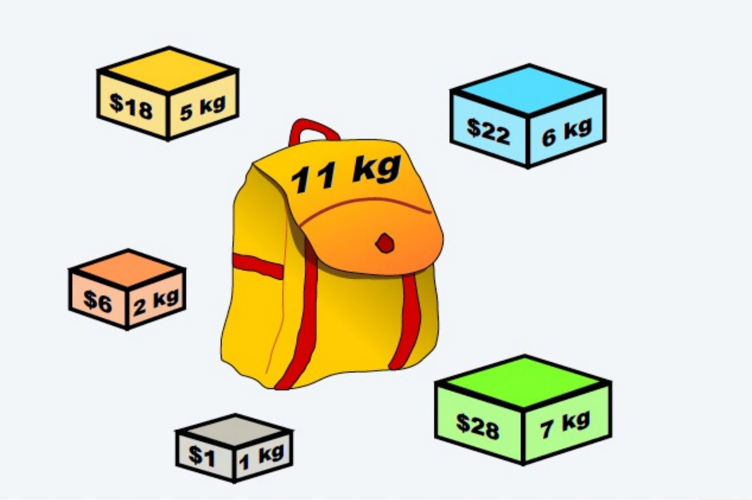
***Theorem*.**

- The DP Algorithm correctly computes the optimal value of the problem, and runs in $O(nW)$ time.
- Given a table M of the optimal values of the subproblems, the optimal set S can be found in $O(n)$ time.

Knapsack Problem

Pack knapsack so as to maximize total value of items taken.

- There are n items: item i provides value $v_i > 0$ and weighs $w_i > 0$.
- Value of a subset of items = sum of values of individual items.
- Knapsack has weight limit of W.
- All values and weights are integral.



Example: $W = 11$

- The subset { 1, 2, 5 } has value $35 (and weight 10).
- The subset { 3, 4 } has value $40 (and weight 11).

i	v_i	w_i
1	$1	1 kg
2	$6	2 kg
3	$18	5 kg
4	$22	6 kg
5	$28	7 kg

Greedy Algorithms:

- Greedy-by-value: repeatedly add item with maximum v_i.
- Greedy-by-weight: repeatedly add item with minimum w_i.
- Greedy-by-ratio: repeatedly add item with maximum ratio v_i / w_i.

$W = 100$
$v_1 = 51, w_1 = 100$
$v_2 = 50, w_2 = 50$
$v_3 = 50, w_3 = 50$

$W = 100$
$v_1 = w_1 = 1$
$v_2 = 99, w_2 = 100$

Dynamic programming: two variables

Let $OPT(i, w)$ = optimal value of knapsack problem with items $1, \dots, i$, subject to weight limit w.

Goal: Computing $OPT(n, W)$.

Case 1. $OPT(i, w)$ does not select item i.
- $OPT(i, w)$ selects best of $\{1, 2, \dots, i-1\}$ subject to weight limit w.

Case 2. $OPT(i, w)$ selects item i.
- Collect value v_i.
- New weight limit = $w - w_i$.
- $OPT(i, w)$ selects best of $\{1, 2, \dots, i-1\}$ subject to new weight limit.

optimal substructure property

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max\{\, OPT(i-1, w),\ \ v_i + OPT(i-1, w - w_i)\,\} & \text{otherwise} \end{cases}$$

Dynamic programming: two variables

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ \ v_i + OPT(i-1, w-w_i)\,\} & \text{otherwise} \end{cases}$$

KNAPSACK($n, W, w_1, \ldots, w_n, v_1, \ldots, v_n$)

FOR $w = 0$ TO W

  $M[0, w] \leftarrow 0.$

previously computed values

FOR $i = 1$ TO n

  FOR $w = 0$ TO W

    IF $(w_i > w)$ $M[i, w] \leftarrow M[i-1, w].$

    ELSE $M[i, w] \leftarrow \max\{\, M[i-1, w],\ v_i + M[i-1, w-w_i]\,\}.$

RETURN $M[n, W].$

Dynamic programming

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{\, OPT(i-1,w),\ v_i + OPT(i-1, w-w_i)\,\} & \text{otherwise} \end{cases}$$

i	v_i	w_i
1	\$1	1 kg
2	\$6	2 kg
3	\$18	5 kg
4	\$22	6 kg
5	\$28	7 kg

$W = 11$

subset of items $1, \ldots, i$

	0	1	2	3	4	5	6	7	8	9	10	11
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	35	40

weight limit w

Theorem.

- The DP algorithm solves the knapsack problem with n items and maximum weight W in $\Theta(nW)$ time and $\Theta(nW)$ space.
- After computing optimal values, can trace back to find solution: $OPT(i,w)$ takes item i iff $M[i,w] > M[i-1,w]$.