Einstein's field equations, Hilbert action, and Energy-Momentum tensor

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1 Introduction

In this work we will present the Einstein's field equations with a quick motivation on how we can obtain it, as originally done by Einstein, where the Newtonian limit was employed as a way to search for a tensor equation that allows the calculations of the metric. Next, we introduce Hilbert action, which is an action dependent of the metric as a rank 2 tensor field and second order derivatives. As we will see, its extremization leads to Einstein's equations and induces a convenient definition for the energy-momentum tensor when matter is coupled in the Lagrangian. For illustration, Klein-Gordon Lagrangian and Maxwell Lagrangian will be considered as examples for the calculations of the energy-momentum tensor of matter fields. Lastly, we show the canonical energy-momentum tensor obtained from Noether's theorem induced by a translational symmetry and compare it with the energy-momentum tensor for Hilbert action, where Belinfante procedure will play a important role to show that both are equivalent by a freedom of adding an anti-symmetric tensor.

2 Einstein's field equation

In this section, we give a insight on the derivation of Einstein's equation from the perspective of a classical limit imposition and then go to the main topic which is the Hilbert action that gives us the same results in a more fundamental way.

2.1 Derivation from Newtonian limit

Before showing Hilbert action, let us give a quick motivation to Einstein's equation. We can obtain it by comparing with the Newtonian case, in the weak field limit. Given a manifold M, the metric on the manifold will be denoted by $g_{\mu\nu}$. In the limit of weak field, the first component is given by

$$g_{00} = -(1 + 2\phi), \tag{1}$$

where ϕ is the scalar potential given by

$$\nabla^2\phi = 4\pi G\rho. \tag{2}$$

Therefore, taking the second derivative of (1),

$$\nabla^2 g_{00} = -8\pi G\rho \tag{3}$$

This is the equation that motivate us to generalize to the general case which is given by

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta}, \tag{4}$$

where $T_{\alpha\beta}$ are the components of the energy momentum tensor and $G_{\alpha\beta}$ are components of the Einstein tensor. Notice that we had one equation that would give the field responsible to the dynamics of a gravitational system, now we have a rank 2 tensor equation with indices varying from 0 to 3. But since the energy-momentum tensor is symmetric, we reduce the number of independent components from 16 to 10.

Now we want to obtain the tensor G which should only depend on the metric and derivatives. First, notice that $T_{\mu\nu}$ should be covariantely conserved, meaning that its covariant derivative is zero. Thus,

$$\nabla^\mu G_{\mu\nu} = 0 \tag{5}$$

Another condition is that it can be only linear on the metric or have second order derivatives because a first order derivative would leave an index free. The only tensor dependent of the metric and is second order in its derivatives is the Riemann tensor given by

$$R^\lambda_{\mu\nu\kappa} = \partial_\nu\Gamma^\lambda_{\kappa\mu} - \partial_\kappa\Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\alpha}\Gamma^\alpha_{\kappa\mu} - \Gamma^\lambda_{\kappa\alpha}\Gamma^\alpha_{\nu\mu} \tag{6}$$

From this tensor, there is only one rank 2 tensor we can build, namely, the Ricci Tensor. It is defined as the contraction of its first and third index (equivalently, the second and fourth) of the covariant Riemann tensor with the metric tensor:

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu} = R^\alpha_{\mu\alpha\nu}. \tag{7}$$

Also, we define the Ricci scalar as $R = g^{\mu\nu}R_{\mu\nu}$. With these quantities, we can build the following tensor depending on the metrics

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 g_{\mu\nu} R \tag{8}$$

Now, our task is to find the two constants. Firstly we use Bianchi's identity that can be stated as

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R, \tag{9}$$

that when substituted in Eq. (5) reads

$$\left(C_1 + \frac{1}{2}C_2\right)\nabla^\mu R_{\mu\nu} = 0 \tag{10}$$

implying that $C_1 = -C_2/2$. The other condition is the Newtonian limit, that is taken when the field is static, meaning that $R_{0000} = 0$ and $R_{ijij} = (\partial^2 g_{00}/\partial_{x^i}\partial_{x^j})/2$. Therefore,

$$G_{00} = 2C_1(R_{i0i0} - R_{0000}) = C_1\nabla^2 g_{00}. \tag{11}$$

This means that $C_1 = 1$. Finally, Einstein's equation is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{12}$$

This simple, but very important, equation allows us to calculate the metric of the space in a manifold given its distribution of mass. So free external interaction system can be totally determined by finding the metric from this equation.

2.2 Hilbert Action

Now we introduce the main subject of this work, the Hilbert action. This action allows us to obtain Einstein's field equation from one of the most important principles of physics, which is the Hamilton principle. To give a little insight on how the action is obtained, we want a Lorentz invariant scalar quantity that is given by only linear terms of the metric and its second derivative. This is obtained by using the Ricci scalar. Thus, the action is given by

$$S = \int_M d^4x\sqrt{-g}R, \tag{13}$$

where g is the determinant of the metric. Now we need to calculate its functional infinitesimal variation and impose $\delta S = 0$. Before, two results are necessary.

Lemma 1. *The infinitesimal variation of the determinant of the metric in respect to coordinates is given by*

$$\delta det(g) = det(g)\delta g_{\mu\nu}g^{\mu\nu} \tag{14}$$

Proof. Using the identity $\log(det(g)) = tr \log g$, we can calculate its variation in both sides,

$$\frac{\delta det(g)}{det(g)} = tr(\delta g g^{-1}) = \delta g_{\mu\nu}g^{\mu\nu} \tag{15}$$

And the result follows as wished. □

A more convenient way of writing this equation is noting that $g_{\mu\rho}g^{\rho\nu} = \delta^\nu_\mu$ implies that $\delta g_{\mu\rho}g^{\rho\nu} + g_{\mu\rho}\delta g^{\rho\nu} = 0$, so that, $\delta g_{\mu\rho}g^{\rho\nu} = -g_{\mu\rho}\delta g^{\rho\nu}$. Therefore

$$\delta det(g) = -det(g)\delta g^{\mu\nu}g_{\mu\nu} \tag{16}$$

The next result is Gauss theorem for manifolds, which we will not prove here.

Theorem 2 (Gauss)**.** *Let M be a manifold, ∂M its border, and T a differentiable function. Then*

$$\int_M \nabla_\mu T^\mu dV = \int_{\partial M} T^\mu d\Sigma_\mu \tag{17}$$

Now we have all the tools to extremize Hilbert action. Since the variation and the integral can commute, and using product rule,

$$\delta S = \int_M d^4x \left(\delta(\sqrt{-g})g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\delta(g^{\mu\nu})R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta(R_{\mu\nu})\right). \tag{18}$$

We have that

$$\delta(\sqrt{-g}) = -\frac{\delta(g)}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{19}$$

where lemma 1 was used. This allows us to rewrite Eq. (18)

$$\delta S = \int_M d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \int_M d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \tag{20}$$

It looks like we already have Einstein's equation form the first integral. Our work now is to show that the second integral is a total derivative that vanishes by supposing that the field and its derivatives vanish on the border of M. In order to do it, we vary the Riemann tensor. From Eq. (6)

$$\delta(R^{\lambda}_{\mu\nu\kappa}) = \partial_\nu \delta(\Gamma^{\lambda}_{\kappa\mu}) - \partial_\kappa \delta(\Gamma^{\lambda}_{\nu\mu}) + \delta(\Gamma^{\lambda}_{\nu\alpha})\Gamma^{\alpha}_{\kappa\mu} + \Gamma^{\lambda}_{\nu\alpha}\delta(\Gamma^{\alpha}_{\kappa\mu}) - \delta(\Gamma^{\lambda}_{\kappa\alpha})\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\lambda}_{\kappa\alpha}\delta(\Gamma^{\alpha}_{\nu\mu}). \tag{21}$$

Adding and subtracting the term $\Gamma^{\alpha}_{\kappa\nu}\Gamma^{\lambda}_{\alpha\mu}$ we get,

$$\begin{aligned} \delta(R^{\lambda}_{\mu\nu\kappa}) &= \partial_\nu \delta(\Gamma^{\lambda}_{\kappa\mu}) + \delta(\Gamma^{\lambda}_{\nu\mu}) + \delta(\Gamma^{\lambda}_{\nu\alpha})\Gamma^{\alpha}_{\kappa\mu} + \Gamma^{\lambda}_{\nu\alpha}\delta(\Gamma^{\alpha}_{\kappa\mu}) + \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\lambda}_{\alpha\mu} + \\ & -\partial_\kappa \delta(\Gamma^{\lambda}_{\nu\mu}) - \delta(\Gamma^{\lambda}_{\kappa\alpha})\Gamma^{\alpha}_{\nu\mu} - \Gamma^{\lambda}_{\kappa\alpha}\delta(\Gamma^{\alpha}_{\nu\mu}) - \Gamma^{\alpha}_{\kappa\nu}\Gamma^{\lambda}_{\alpha\mu}. \end{aligned} \tag{22}$$

The Christoffel symbols transform as $\tilde{\Gamma} = M^{-1}M\Gamma M - M^{-1}\partial M M^{-1}$, but since $\delta(\Gamma)$ is the difference between two Christoffel symbols, the second part vanishes and it transforms as a Lorentz tensor. This allows us to take the covariant derivative of the variation of the Christoffel symbols, and rewrite Eq. (22) as

$$\delta\left(R^{\lambda}_{\mu\nu\kappa}\right) = \nabla_\nu \delta\left(\Gamma^{\lambda}_{\kappa\mu}\right) - \nabla_\kappa \delta\left(\Gamma^{\lambda}_{\nu\mu}\right). \tag{23}$$

The variation of the Ricci tensor is simply the contraction of the index λ and μ

$$\delta\left(R_{\mu\kappa}\right) = \nabla_\alpha \delta\left(\Gamma^{\alpha}_{\kappa\mu}\right) - \nabla_\kappa \delta\left(\Gamma^{\alpha}_{\alpha\mu}\right). \tag{24}$$

Plugging this into the second integral of Eq. (20) leads to

$$\int_M d^4x\sqrt{-g}g^{\mu\nu}\delta(R_{\mu\nu}) = \int_M d^4x\sqrt{-g}g^{\mu\nu}\left(\nabla_\alpha \delta\left(\Gamma^{\alpha}_{\nu\mu}\right) - \nabla_\nu \delta\left(\Gamma^{\alpha}_{\alpha\mu}\right)\right). \tag{25}$$

Compatibility of the covariant derivative with metric gives

$$\int_M d^4x\sqrt{-g}g^{\mu\nu}\delta(R_{\mu\nu}) = \int_M d^4x\sqrt{-g}\left(\nabla_\alpha \left[g^{\mu\nu}\delta\left(\Gamma^{\alpha}_{\nu\mu}\right)\right] - \left[g^{\mu\nu}\nabla_\nu \delta\left(\Gamma^{\alpha}_{\alpha\mu}\right)\right]\right). \tag{26}$$

This is the total derivative we wished. Using Gauss Theorem, this reduces to an integral of the field and its derivatives in the border of M that vanishes by assumption. Thus,

$$\delta S = \int_M d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta(g^{\mu\nu}) \tag{27}$$

and imposing $\delta S = 0$ we obtain Einstein's equations in vacuum.

Next, we can couple the lagrangian with matter. This gives the following action

$$S = \int_M d^4x\left[\sqrt{-g}R + \sqrt{-g}\mathcal{L}_M\right]. \tag{28}$$

The first term is exactly as done before. For the second term, we remember that the definition of functional derivative of a function $F[\rho]$ is given by

$$\int \frac{\delta F}{\delta \rho}\phi\, dx = \frac{d}{d\epsilon}\left(F[\rho + \epsilon\phi]\right)\big|_{\epsilon=0}. \tag{29}$$

In the particular case that F is given by

$$F[\rho] = \int_M f\left(\rho(x)\right) dx, \tag{30}$$

it follows that

$$\int \frac{\delta F}{\delta \rho}\phi \, dx = \int \frac{\partial}{\partial \epsilon} f(\rho + \epsilon\phi)\big|_{\epsilon=0} = \int \phi \frac{\partial\left(f(\rho+\epsilon\phi)\right)}{\partial(\rho+\epsilon\phi)}\big|_{\epsilon=0}\, dx. \tag{31}$$

$$\frac{\delta F}{\delta \rho} = \frac{f(\rho)}{\partial \rho}. \tag{32}$$

So if we define

$$S_M = \int_M d^4x\sqrt{-g}\mathcal{L}_M, \tag{33}$$

we can write the functional derivative of the action of the matter in relation to the metric as

$$\frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}_M}{\partial g^{\mu\nu}} \tag{34}$$

Going back to the variation of the action, we get

$$\begin{aligned}
\delta S &= \int_M d^4x\delta\left(\sqrt{-g}R\right) + \int_M d^4x\delta\left(\sqrt{-g}\mathcal{L}_M\right) && (35)\\
&= \int_M d^4x\delta\left(\sqrt{-g}R\right) + \int_M d^4x\frac{\partial\left(\sqrt{-g}\mathcal{L}_M\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu} && (36)\\
&= \int_M d^4x\delta\left(\sqrt{-g}R\right) + \int_M d^4x\frac{\delta S_M}{\delta g^{\mu\nu}}\delta g^{\mu\nu} && (37)
\end{aligned}$$

Using the result obtained in Eq. (27), the variation of this action reads

$$\delta S = \int_M \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + k\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\right)\delta g^{\mu\nu} \tag{38}$$

Once again, imposing $\delta S = 0$, we obtain Einstein's equation in presence of matter

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{k}{2}T_{\mu\nu} \tag{39}$$

where, by comparison with Eq. (12), $k = 16\pi G$, and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{40}$$

by definition. Note that this tensor is symmetric because the metric tensor is symmetric. Furthermore, it is useful in calculation the energy-momentum tensor for matter field when we want a full symmetric tensor since Noether's theorem does not provide it for every field.

3 Energy-momentum tensor of matter fields

Let us explore more about the energy momentum tensor obtained in Eq. (40) by showing that we can calculate the energy-momentum for fields in flat space by allowing the metric to vary and then employing Minkowski metric. These calculations can be done for general curved space-time, but our purpose is to compare it with the canonical energy momentum from Noether's theorem which is obtained in flat space. From now on, the correct metric signature $diag(1,-1,-1,-1)$ will be used. In this case, the energy momentum tensor changes sign and when reduced to flat space is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}\big|_{g_{\mu\nu}=\eta_{\mu\nu}}. \tag{41}$$

3.1 Klein-Gordon Lagrangian

For the Klein-Gordon Lagrangian, that describes a field that does not change by a Lorentz transformation, the action is given by

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{42}$$

Variation of this action in respect to the metric can be easily calculated:

$$\frac{\delta S}{\delta g^{\alpha\beta}} = \frac{\partial\left(\sqrt{-g}\mathcal{L}\right)}{\partial g^{\alpha\beta}} - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\mathcal{L} + \frac{1}{2}\sqrt{-g}\partial_\alpha\phi\partial_\beta\phi. \tag{43}$$

Thus, the energy-momentum tensor from Eq. (42) is

$$T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \eta_{\alpha\beta}\mathcal{L}. \tag{44}$$

3.2 Maxwell Lagrangian

Let us now consider the Maxwell Lagrangian. Its action describes the electromagnetic field in terms of the 4-vector potential in a spin-1 massless field.

$$S = \int d^4x - \frac{1}{4}\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}F_{\mu\nu} \tag{45}$$

The derivative of the action is given by

$$\begin{aligned}\frac{\delta S}{\delta g^{\alpha\beta}} &= -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\mathcal{L} - \frac{1}{4}\sqrt{-g}\left(\delta^\mu_\alpha\,\delta^\rho_\beta\,g^{\nu\sigma} + \delta^\nu_\alpha\,\delta^\sigma_\beta\,g^{\mu\rho}\right)F_{\rho\sigma}F_{\mu\nu} &(46)\\ &= -\frac{1}{4}\sqrt{-g}\left(F_\beta{}^\nu F_{\alpha\nu} + F^\mu{}_\beta F_{\mu\alpha}\right) - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\mathcal{L}. &(47)\end{aligned}$$

Therefore, the energy-momentum tensor of the electromagnetic field is given by

$$T_{\alpha\beta} = -\frac{1}{2}\left(F_\beta{}^\nu F_{\alpha\nu} + F^\mu{}_\beta F_{\mu\alpha}\right) + \frac{1}{4}\eta_{\alpha\beta}\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{48}$$

Since $F_{0i} = -F_0{}^i = E_i$ (for $diag(1,-1,-1,-1)$ signature) then $F_{0i}F_0{}^i = -E^2$, where E is the electric field. Also, using the the identity $F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2)$, where B is the magnetic field, we find that the T_{00} component of the tensor is

$$T_{00} = \frac{1}{2}\left(E^2 + B^2\right) \tag{49}$$

which is exactly the energy density for the electromagnetic field.

3.3 Canonical energy-momentum tensor

There is another energy-momentum tensor that can be calculated from a Noether's current associated to translational symmetry. Let us derive this current and then compare it to the one we have for the case of electromagnetic fields.

Suppose we have have a small variation of the coordinates x^μ such that $\delta x^\mu = \epsilon^\mu$. This will induce a change in the field by

$$\delta\phi = \frac{\partial\phi}{\partial x^\mu}\epsilon^\mu = \partial_\mu\phi\epsilon^\mu \tag{50}$$

The variation of the Lagrangian is given by

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\epsilon^{\mu} + \frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\phi)}\partial_{\nu}\partial_{\mu}\phi\epsilon^{\mu} && (51)\\
&= \epsilon^{\mu}\partial_{\mu}\mathcal{L} && (52)
\end{aligned}
$$

Since the Lagrangian changes by a total derivative, Noether's theorem implies the following conserved current:

$$\partial_{\mu}T^{\mu\nu} = 0 \tag{53}$$

where

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\partial_{\nu}\phi - \eta_{\mu\nu}\mathcal{L}. \tag{54}$$

This is the canonical energy-momentum tensor obtained as a conserved quantity when equations of motion are used. We can calculate for the Klein-Gordon Lagrangian in Eq. (42) and see that, indeed, it is the same as the one obtained from varying the action. Note that it is also symmetric, but this is not always the case. Let us illustrate an example where the tensor is not symmetric. For the Maxwell Lagrangian, we have

$$\frac{\partial\mathcal{L}}{\partial(\partial_{\alpha}A_{\beta})} = -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial(\partial_{\alpha}A_{\beta})} = -F^{\alpha\beta}. \tag{55}$$

The canonical-energy momentum tensor is given by

$$T_{\mu\nu} = -F_{\mu\beta}\partial_{\nu}A^{\beta} + \frac{1}{4}\eta_{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right) \tag{56}$$

Now we can calculate the T_{00} component of this tensor

$$T_{00} = \frac{1}{2}\left(B^2 + E^2\right) + F_{0i}\partial^{i}A_0. \tag{57}$$

This is not exactly the result we expected. The first term is the energy density and the second term can be dealt with by choosing the coulomb gauge. Note that the tensor $T_{\mu\nu}$ is not symmetric either, and, furthermore, it is not equal to the one calculated in Eq. (48).

3.4 Belinfante symmetrization

Although both definitions of energy-momentum for electromagnetic field is valid, we can change the canonical energy-momentum tensor to symmetrize it and match the one obtained from Hilbert action. We make the following redefinition of the tensor

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_{\rho}\Lambda^{\rho\mu\nu} \tag{58}$$

where $\Lambda_{\rho\mu\nu} = -\Lambda_{\mu\rho\nu}$. Adding this term is allowed because $\partial_{\mu}\partial_{\rho}\Lambda^{\mu\nu\rho} = 0$, so that it is also a conserved quantity. This allows to symmetrize the energy-momentum tensor of Maxwell Lagrangian by adding the term $F_{\rho\mu}\partial^{\rho}A_{\nu}$ leading to

$$
\begin{aligned}
T'_{\mu\nu} &= -F_{\mu\beta}\partial_{\nu}A^{\beta} + \frac{1}{4}\eta_{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right) + F_{\rho\mu}\partial^{\rho}A_{\nu} && (59)\\
&= \frac{1}{4}\eta_{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right) - F_{\mu\rho}\left(\partial_{\nu}A^{\rho} - \partial^{\rho}A_{\nu}\right) && (60)\\
&= \frac{1}{4}\eta_{\mu\nu}\left(F_{\alpha\beta}F^{\alpha\beta}\right) - F_{\mu\rho}F_{\nu}{}^{\rho} && (61)
\end{aligned}
$$

This new tensor agrees with the one obtained in Eq. (48). Note that we also get gauge invariance since Eq. (56) is not gauge invariant.

3.5 Spin is the reason

One might wonder why the canonical stress energy tensor comes out symmetric for scalar fields, while the vector one, and others do not. Surprisingly, spin is the reason. We will not go much into detail but a quick reasoning will be shown.

If one considers a infinitesimal Lorentz transformation on a scalar field, the invariance of the Lagrangian will lead to the following conserved tensor:

$$M^{\mu}{}_{\nu\lambda} = x_\nu T^{\mu}{}_{\lambda} - x_\lambda T^{\mu}{}_{\nu} \tag{62}$$

where $T^{\mu}{}_{\nu}$ are the components of the canonical energy-momentum tensor. This tensor can be interpreted as the orbital angular momentum of the particle. But scalar fields do not change under Lorentz transformations, so its transformation is purely given by geometrical reasons. Instead, if the field transforms according to a non-trivial representation of the Lorentz group, then the tensor in Eq. (62) changes to

$$M^{\mu}{}_{\nu\lambda} = x_\nu T^{\mu}{}_{\lambda} - x_\lambda T^{\mu}{}_{\nu} + S^{\mu}{}_{\nu\lambda} \tag{63}$$

where S is a tensor representing the spin content of the field. This is intrinsic to the field and adds to the orbital angular momentum. Now, if we want this tensor to still be conserved, we find that

$$\begin{aligned} \partial_\mu M^{\mu}{}_{\nu\lambda} &= \partial_\mu \left(x_\nu T^{\mu}{}_{\lambda} - x_\lambda T^{\mu}{}_{\nu} + S^{\mu}{}_{\nu\lambda} \right) && (64) \\ &= \partial_\mu S^{\mu}{}_{\nu\lambda} + T_{\nu\lambda} - T_{\lambda\nu} = 0. && (65) \end{aligned}$$

Thus, conservation of the spin part is given by

$$\partial_\mu S^{\mu}{}_{\nu\lambda} = T_{\lambda\nu} - T_{\nu\lambda}. \tag{66}$$

If spin is not conserved on its own, meaning there is a current of spin, then the canonical energy-momentum tensor is not symmetric. Belinfante actually went further and gave a general expression that symmetrizes the canonical stress energy tensor in term of the spin tensor.