Rank-2 Toric Code

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1 Rank-2 Toric code commuting projectors Hamiltonian

Rank-2 Toric Code is a 2 dimensional model of commuting projectors with topological order. It is described by a square lattice with each vertex hosting two independent 2-dimensional degrees of freedom (two independent spins) and a single 2-dimensional degree of freedom on the center of the face in each square. The operators are defined by

$$a_i^x = Z_{1,i}^{-1} Z_{0,i} Z_{0,i-y}^{-1} Z_{1,i+x} \tag{1}$$

$$a_i^y = Z_{2,i}^{-1} Z_{0,i} Z_{0,i-x}^{-1} Z_{2,i+y} \tag{2}$$

$$b_i = X_{1,i}^{-2} X_{2,i}^{-2} X_{0,i}^{-1} X_{0,i-y} X_{0,i-x} X_{0,i-x-y}^{-1} X_{2,i+x} X_{2,i-x} X_{1,i+y} X_{1,i-y} \tag{3}$$

Where the Z and X operators are Pauli operators obeying the algebra $ZX = e^{\frac{2\pi i}{N}} XZ$. The Hamiltonian is the sum of operators over all sites in the lattice

$$H = -\sum_i a_i^x + a_i^y + b_i \tag{4}$$

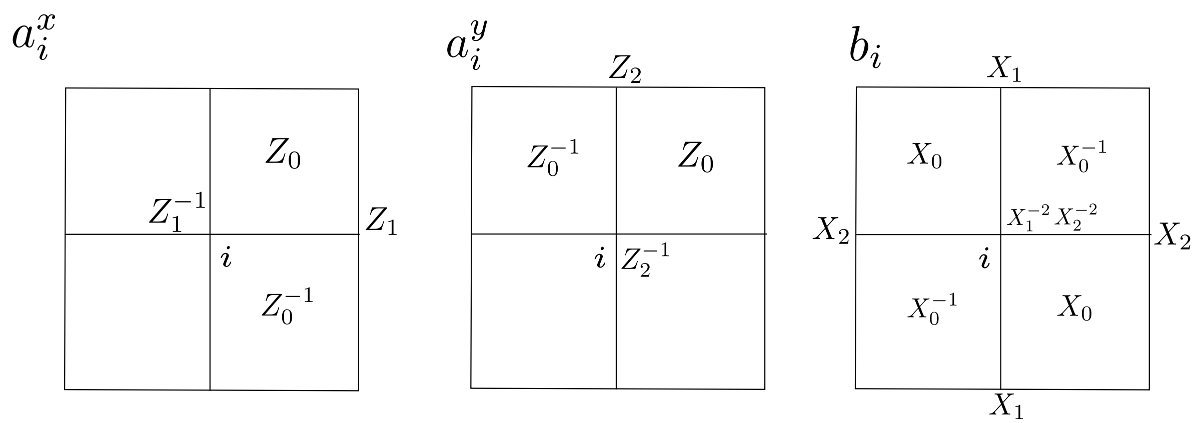

Figure 1: Commuting projectors acting on the lattice in terms of Z_N Pauli matrices.

Note that each of the operators commute with each other and the N-th power of each of them is the identity. This means that this is an exactly solvable model and we can find its ground state degeneracy as we will see.

2 Excitations of the model

Since the operators equal the identity when we take its N-th power, we conclude that its eigenvalues are the N-th root of identity, therefore the ground state is characterized by the configurations of states such that its eigenvalue for a_i^x, a_i^y and b_i has eigenvalue 1. This means that excitations are characterized by states that does not obey this rule. We can create these excitations starting from the ground state and operating with X or Z operators to check for the pattern of excitations created.

2.1 p_x and p_y excitations

When acting with Pauli operators, it might happen that it has nontrivial commutation with one of the commuting projectors creating an excitation located at the point the operator acts. Exciations corresponding to a_i^x will be denoted p_x and a_i^y will be denoted p_y.

If we act with X_1 in one link, it will create a pair of particle antiparticle p_x charge. We will locate them in the horizontal link to the right of the operator being excited. Analogously, the action of X_2 on a vertex will create a pair of p_y particles vertically. that will be located in the vertical link above the operator being excited. Lastly, X_0 being applied in one of the center of the squares will create a pair of each type of particles. The pattern of excitations are illustrated below.

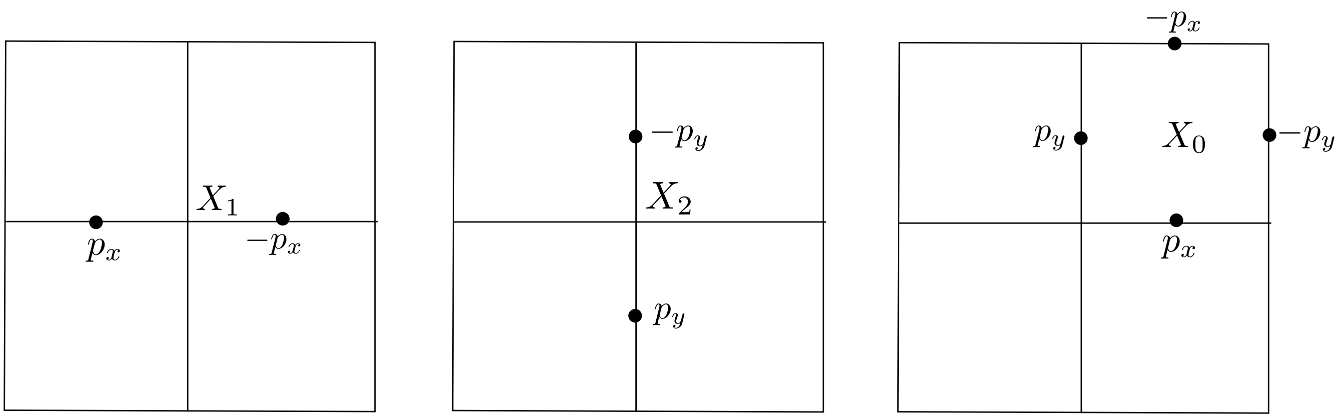

Figure 2: Pattern if excitations when Pauli operators are applied to the ground state.

Now we can see how these charges move in the lattice and if there is any restriction in their mobility. The most simple ones are the horizontal and vertical movement of p_x and p_y respectively. They can freely move applying successive X_1 operators on a horizontal line and X_2 operators on a vertical line.

$$W_1 = \prod_{i=1}^{l} X_{1,x+i,y} \tag{5}$$

$$W_2 = \prod_{i=1}^{l} X_{2,x,y+i} \tag{6}$$

Now the vertical motion of p_x is a little more trick. Note that successive applications of X_0 in a vertical line will move one p_x excitations but will also create p_y's in each along the way. The way to annihilate these extra excitations is by applying X_2 operators together. We find that the operator must be given by

$$W_3 = \prod_{i=1}^{kN} X_{0,x,y+i}(X_{2,x,y+i}^{-1} X_{2,x+1,y+i})^i. \tag{7}$$

And the horizontal motion of p_y is analogous

$$W_4 = \prod_{i=1}^{kN} X_{0,x+i,y}(X_{1,x+i,y}^{-1} X_{1,x+i,y+1})^i. \tag{8}$$

The interesting feature of these operators is that they are limited to intervals multiple of N, meaning that the transverse motion of each excitations is restricted.

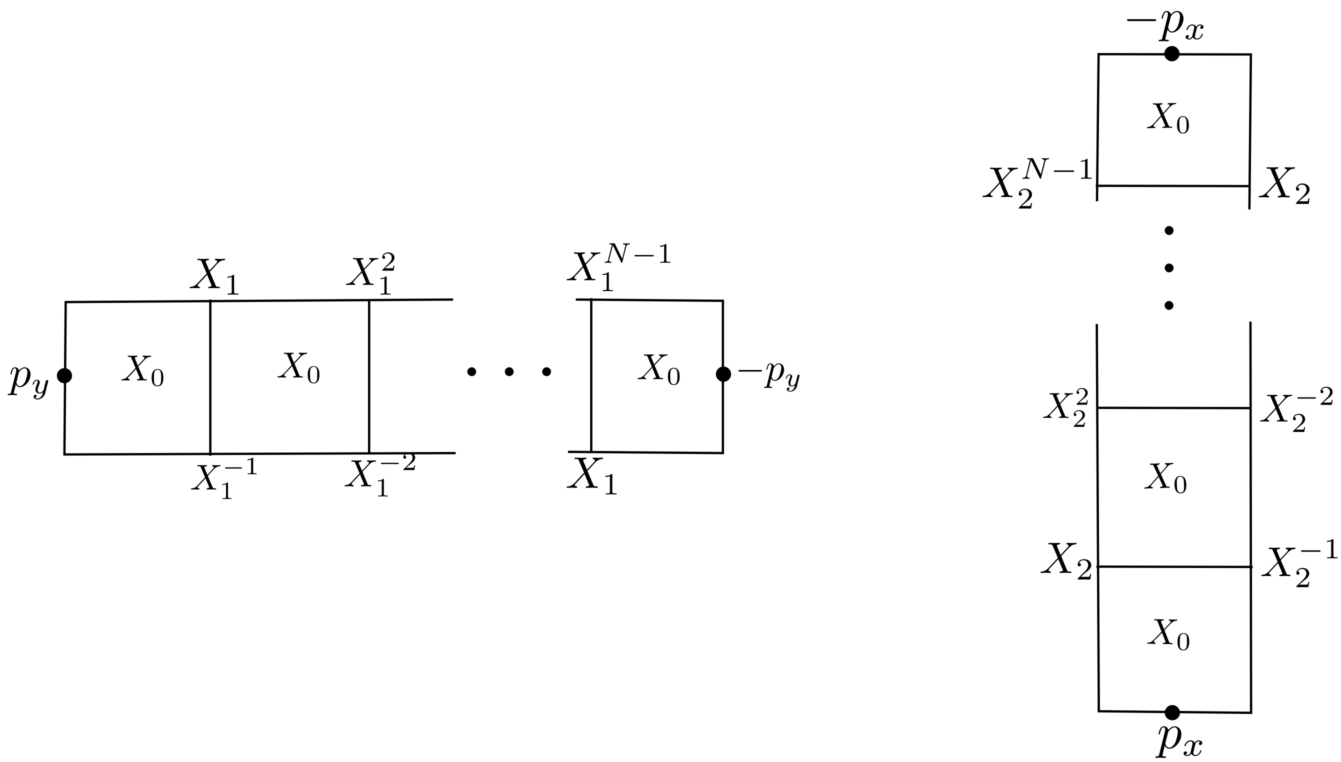

Figure 3: p_x and p_y pair of excitations in their transverse direction.

2.2 m excitations

Now let us see the excitations produced from b_i operators. The $b_i |\psi\rangle = |\psi\rangle$ ground state constraint is violated when any of the Z_i operator is applied and when this happens, an excitation denoted by m on the vertex will be created. Z_1 will create a pair of vertical dipoles, Z_2 will create a pair of horizontal dipoles and Z_0 will create a quadrupole around its square.

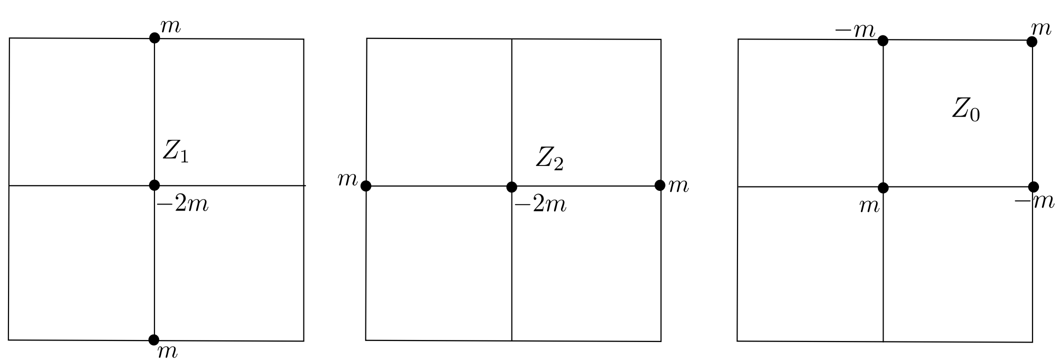

Figure 4: m excitations being created from the application of Z_1, Z_2 and Z_0 operators. All of them create different configurations of dipoles.

We are interested in seeing the motion of a single m particle and the horizontal and vertical m

- m dipole. To move a single m particle, we define line operators that create a pair of particle anti particle in a spacing multiple of N between them. They are defined by

$$T_1 = \prod_{i=1}^{kN} (Z_{1,x,y+i})^i \tag{9}$$

$$T_2 = \prod_{i=1}^{kN} (Z_{2,x+i,y})^i \tag{10}$$

These operators are responsible for the motion of m excitations, and they can only do it by N lattice spacing in both horizontal and vertical directions.

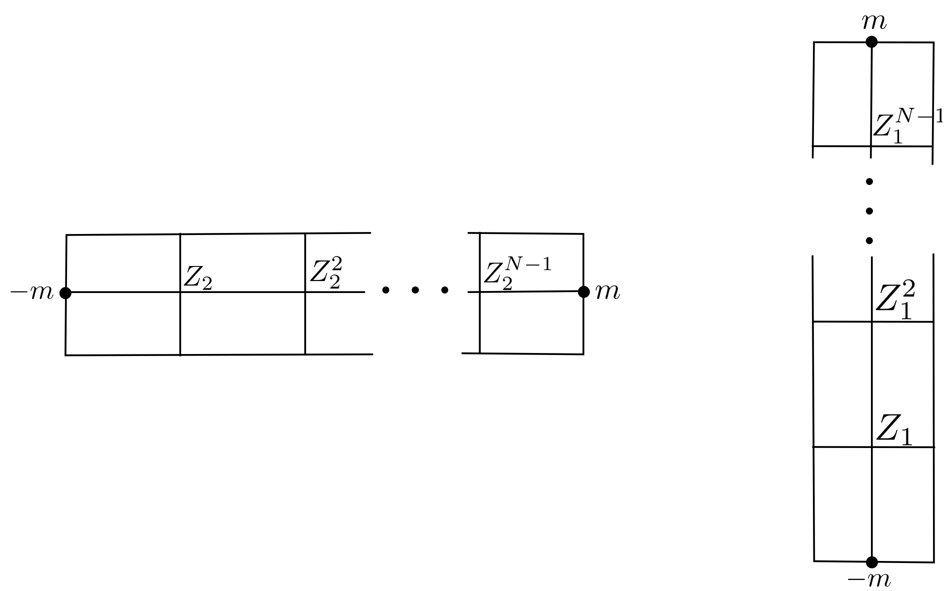

Figure 5: Pair of m excitations being created at the end of N length string operators in the horizontal and vertical directions.

The motion of horizontal and vertical dipoles of m are more simple to describe. They are simply done by successive applications of Z_1 along vertical line, Z_2 in a horizontal line and Z_0 in any direction and can be of any size. Fig. 6 illustrate how each operator act.

$$T_3 = \prod_{i=1}^{L_y} Z_{1,x,y+i}, \quad T_4 = \prod_{i=1}^{L_x} Z_{2,x+i,y} \tag{11}$$

$$T_5 = \prod_{i=1}^{l} Z_{0,x+i,y}, \quad T_6 = \prod_{i=1}^{l} Z_{0,x,y+i} \tag{12}$$

These are all the motions possible in the system. As we can see, there are some restrictions, which we will see to imply in modifications of Wilson and 't Hooft lines. This has implications in the ground state degeneracy that depends on the system sizes.

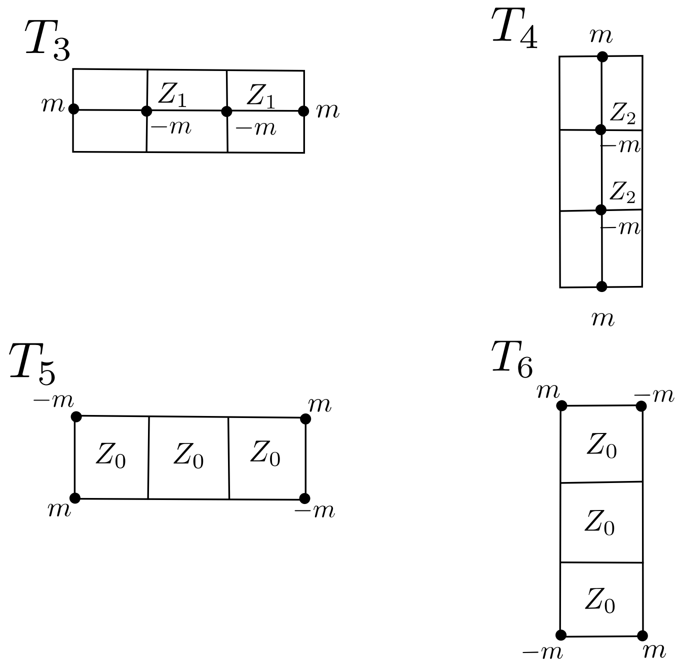

Figure 6: Line operators that move horizontal and vertical dipoles of m particles. Both can move without restrictions.

3 Wilson and 't Hooft lines

In this section we describe the Wilson and 't Hooft lines. As we will see they lead to 't Hooft anomalies that imply in all ground states possible for this model.

We will find the lines in a naive way, but then will be checked that they can be made as an appropriate combinations of the ones we find that lead to simpler nontrivial commutation relations among them.

3.1 First holonomies approach

Z **holonomies:**

The holonomies made up of Z operators will be called 't Hooft lines and four of them are very simple to find. The opertor T_3, T_4, T_5 and T_6 can be applied along the whole circumference around the torus. This is a symmetry of the Hamiltonian because it will overlap both particle and antiparticle they produce and anihillate, leaving no excitations.

$$T_3 = \prod_{i=1}^{L_y} Z_{1,x,y+i}, \quad T_4 = \prod_{i=1}^{L_x} Z_{2,x+i,y} \tag{13}$$

$$T_5 = \prod_{i=1}^{L_x} Z_{0,x+i,y}, \quad T_6 = \prod_{i=1}^{L_y} Z_{0,x,y+i} \tag{14}$$

For the other two holonomies we take T_1 and T_2 to make a loop. But as we saw, T_1 must be a string of size multiple of N, otherwise it excites extra particles, at the same time it must be a multiple of the system size L_x to form a loop. The minimum integer that satisfies both condition is $lcm(L_x, N)$ by definition. For T_2, the operator must be defined in a string of size $lcm(L_y, N)$.

$$T_1 = \prod_{i=1}^{lcm(L_y,N)} (Z_{1,x,y+i})^i, \quad T_2 = \prod_{i=1}^{lcm(L_x,N)} (Z_{2,x+i,y})^i \tag{15}$$

X **Holonomies**

Now, the task is to find loops of X operators. These will be called Wilson loops. We will find them using the translation operators but will be seen that there is a more appropriate way of writing them in a different basis.

Two of them are very easy to write down. Take Eqs. (5) and (6), and consider these lines going around the whole torus. We get

$$W_1 = \prod_{i=1}^{L_x} X_{1,x+i,y}, \quad W_2 = \prod_{i=1}^{L_y} X_{2,x,y+i}. \tag{16}$$

Two more are obtained from Eqs. (7) and (8), but they must go around the torus multiple times, analogous to T_1 and T_2.

$$W_3 = \prod_{i=1}^{lcm(N,L_y)} X_{0,x,y+i}(X^{-1}_{2,x,y+i}X_{2,x+1,y+i})^i, \quad W_4 = \prod_{i=1}^{lcm(N,L_x)} X_{0,x+i,y}(X^{-1}_{1,x+i,y}X_{1,x+i,y+1})^i. \tag{17}$$

The last two Wilson lines are a little bit more tricky to find. But it consists from noting that The lines $W_!$ and W_2 are not equivalent to the ones adjacent to it. Another way to see it, is from the Anyon lattice formalism that explicitly shows that there are six elementary excitations and two of them corresponds to adjacent p_x or p_y particles (this has not been shown here). They are given by

$$W_5 = \prod_{i=1}^{L_x} X_{1,x+i,y}(X_{1,x+i,y+1})^{-1}, \quad W_6 = \prod_{i=1}^{L_y} X_{2,x,y+i}(X_{2,x+1,y+i})^{-1} \tag{18}$$

3.2 ’t Hooft Anomalies

We have found twelve symmetries of the Hamiltonian corresponding to 6 Wilson lines and 6 ’t Hooft lines. We can verify that each Wilson line has a ’t Hooft anomaly with at least one of the T_i operators (non-trivial commutation between two symmetries leading to obstruction in gauging both symmetries at the same time). Unfortunately, this basis of loops generate other non-trivial commutation relations, for instance, W_1 does not commute with T_1 and T_3. It would be very convenient if we could write another basis of Wilson and ’t Hooft loops such that a Wilson loop has a single ’t Hooft pair that does not commute with it. Fortunately there is and we can obtain by taking products between these operators to generate another basis. We find that the convenient operators are given by

$$T_1 = \prod_{i=1}^{lcm(L_y,N)} (Z_{1,x,y+i})^{i-x_0}(Z_{0,x,y+i})^{x-x_0-1}, \quad T_2 = \prod_{i=1}^{lcm(L_x,N)} (Z_{2,x+i,y})^{i-y_0}(Z_{0,x+i,y})^{y-y_0-1} \tag{19}$$

$$T_3 = \prod_{i=1}^{L_y} Z_{1,x,y+i}, \quad T_4 = \prod_{i=1}^{L_x} Z_{2,x+i,y} \tag{20}$$

$$T_5 = \prod_{i=1}^{L_x} Z_{0,x+i,y}, \quad T_6 = \prod_{i=1}^{L_y} Z_{0,x,y+i} \tag{21}$$

$$W_1 = \prod_{i=1}^{L_x} X_{1,x+i,y}(X_{1,x+i,y}(X_{1,x+i,y+1})^{-1})^{y-y_0'}, \quad W_2 = \prod_{i=1}^{L_y} X_{2,x,y+i}(X_{2,x,y+i}(X_{2,x+1,y+i})^{-1})^{x-x_0'}. \tag{22}$$

$$W_3 = \prod_{i=1}^{lcm(N,L_y)} X_{0,x,y+i}(X^{-1}_{2,x,y+i}X_{2,x+1,y+i})^{i-y_0'}, \quad W_4 = \prod_{i=1}^{lcm(N,L_x)} X_{0,x+i,y}(X^{-1}_{1,x+i,y}X_{1,x+i,y+1})^{i-x_0'}. \tag{23}$$

$$W_5 = \prod_{i=1}^{L_x} X_{1,x+i,y}(X_{1,x+i,y+1})^{-1}, \quad W_6 = \prod_{i=1}^{L_y} X_{2,x,y+i}(X_{2,x+1,y+i})^{-1} \tag{24}$$

The constant x_0 and y_0 are chosen so that the commutators $[W_1, T_1]$, $[W_2, T_2]$, $[W_3, T_2]$ and $[W_4, T_1]$ are null. They must be given by $y_0' = y_0 + 1/2(N - gcd(L_y, N))$ and $x_0' = x_0 + 1/2(N - gcd(L_x, N))$.

If we verify the non-trivial commutation relation between these operators

$$W_1T_3 = \omega T_3W_1, \quad W_2T_4 = \omega T_4W_3 \tag{25}$$

$$W_3T_5 = \omega^{n_y} T_5W_3, \quad W_4T_3 = \omega^{n_y} T_3W_4 \tag{26}$$

$$W_5T_1 = \omega^{n_y} T_1W_5 \quad W_6T_2 = \omega^{n_x} T_2W_6, \tag{27}$$

we find that the ground state must be degenerate by $N^2[gcd(L_x, N)gcd(L_y, N)]^2$ states. For instance, we can take $W_3T_5 = \omega^{n_y}T_5W_3$, where $\omega = e^{(2\pi i/N)}$ and $n_y = lcm(L_y, N)/L_y$ is the number of times the number of revolutions the loop makes around the torus. This relation leads to $gcd(L_y, N)$ number of degenerate ground states since $\omega^{n_y} = 1$. Doing the same procedure for each nontrivial commutation relation will reveal the total GSD. But this is not quite correct, we actually overestimated the degeneracy. The correct GSD comes from something very tricky, that is the fact that W_4 is not the most minimal choice of holonomy that generates all others. The most minimal one is

$$W_4 = \prod_{i=1}^{m_xL_x} X_{0,x+i,y}(X^{-1}_{1,x+i,y}X_{1,x+i,y+1})^{i-x_0} \prod_{i=1}^{m_yL_y} X_{0,x,y+i}(X^{-1}_{2,x,y+i}X_{2,x+1,y+i})^{i-y_0} \tag{28}$$

where $m_x = gcd(L_x, N)/gcd(L_x, L_y, N)$ and $m_y = [lcm(L_x, gcd(L_y, N)) + kN]/L_y$ with k being the least positive integer such that m_y is a integer. With this correct holonomy, we get that the GSD is $N^3 gcd(L_x, N)gcd(L_y, N)gcd(L_x, L_y, N)$.

Although these are the most convenient holonomies to use, the ones obtained in Eqs. (13-18) have more concrete interpretations. They are all creation of a pair of some elementary excitation particle and antiparticle that travel around the torus and annihilate each other.

3.3 Braiding statistics

A interesting feature of this model is the abelian statistics between particles. It is trivial to see that they are bosons if we braid them with themselves, but are anyons if we consider braiding operations between different particles.

For instance, we can take a p_x excitation braiding around a m particle. We can do this by creating a pair of m particles and sending one of them to infinity through T_1. This isolates one of the excitations. Next, take a single p_x excitation (that can be created in the same manner) and make it translate around m until it is back to its initial position. The operation just described is analogous to a double exchange. If m is located at (x_0, y_0) and p_x is at (x, y) to the right and below m, we can take a sequence of $W_3^\dagger$, W_1, W_3 and $W_1^\dagger$. The last operator will meet T_1 at position $(y - y_0)$, so that the many-body wave function would get a phase factor of ω^{y-y_0}, where $\omega = e^{2\pi i/N}$. If the excitations have charge $r, s \leq N - 1$, the phase would change to $\omega^{rs(y-y_0)}$. This is the position dependent braiding statistics. Note that if we translate y or y_0 by N, the phase does not change.

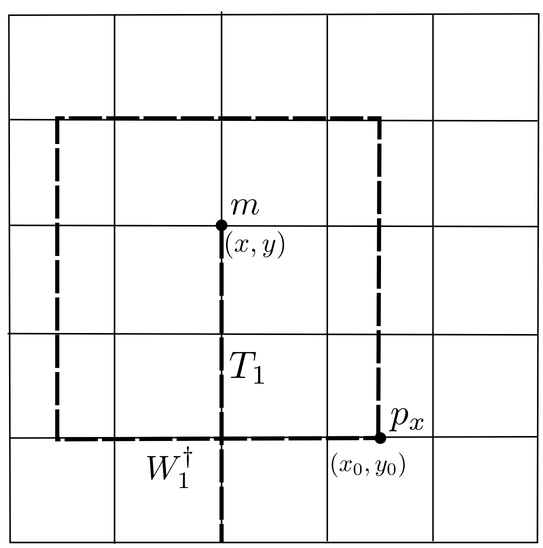

Figure 7: Set up of braiding a p_x excitation around m. The isolated m is created by sending another $-m$ to far away by applying T_1 and the sequence of W operators meets T_1 in the intersection of both paths.

4 Deformation of loops

Given the 12 loop operators we have found, we could ask what type of deformations we can make, inspired in the well known topological deformations of string operators in the usual Toric Code.

Unfortunately, a direct application of commuting projectors on the string operators will deform them creating many others operators in the neighborhood, but we can prove that we are free to make translations without changing ther operational form. Let us start this argument for 't Hooft operators. Note that all the loops being deformed are the ones described in Sec. 3.1, but this will also lead to the deformation of loops described in Sec. 3.2.

Deformation of 't Hooft loops:

First define four operators that are the product of a_i^x and a_i^y around each direction of the torus

$$A_x^{(x)} = \prod_{i=1}^{L_y} a^x_{x,y+i} = \prod_{i=1}^{L_y} (Z_{1,x,y+i})^{-1} Z_{1,x+1,y+i}, \quad A_y^{(x)} = \prod_{i=1}^{L_x} a^x_{x+i,y} = \prod_{i=1}^{L_x} (Z_{0,x+i,y-1})^{-1} Z_{0,x+i,y} \tag{29}$$

$$A_y^{(y)} = \prod_{i=1}^{L_x} a^y_{x+i,y} = \prod_{i=1}^{L_x} (Z_{2,x+i,y})^{-1} Z_{2,x+i,y+1}, \quad A_x^{(y)} = \prod_{i=1}^{L_y} a^y_{x,y+i} = \prod_{i=1}^{L_y} (Z_{0,x-1,y+i})^{-1} Z_{0,x,y+i} \tag{30}$$

Now it is very easy to translate T_3, T_4, T_5 and T_6, just apply the one of the four operators. Note that they can be translated to adjacent lines. For T_2, we need to define a new operator:

$$\mathcal{A}_y^{(y)} = \prod_{i=1}^{n_x L_x} (a^y_{x+i,y})^i = \prod_{i=1}^{n_x L_x} (Z_{2,x+i,y})^{-i} (Z_{2,x+i,y+1})^i \prod_{i=1}^{L_x} (Z_{0,x,y})^{-n_x} \tag{31}$$

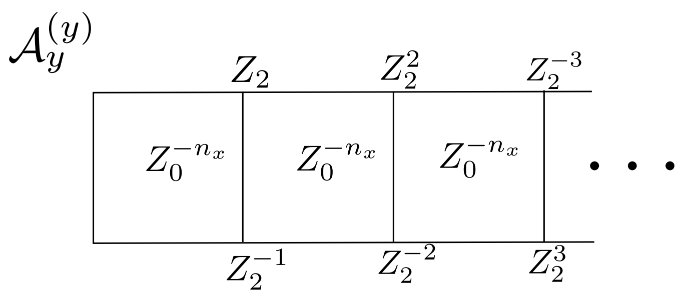

Figure 8: Action of $\mathcal{A}_y^{(y)}$ along a horizontal line of the torus.

Applying Eq. (31) to a T_2 operator will translate the whole string of Z_1's to vertically but will also create a string of $Z_0^{n_x}$'s. These can be translate to the line above by using $A_y^{(x)}$. Next we use $\mathcal{A}_y^{(y)}$ to translate the Z_1's once again, but it will also change the power of each Z_0 to $2n_y$. We can repeat this procedure $gcd(N, L_y)$ times, so that $(Z_0)^{n_y gcd(N,L_y)} = (Z_0)^N = 1$, and we finally translate the operator T_1, but this time by $gcd(N, L_y)$ lattice spacing. The first procedure of this translation is described in Fig. 9.

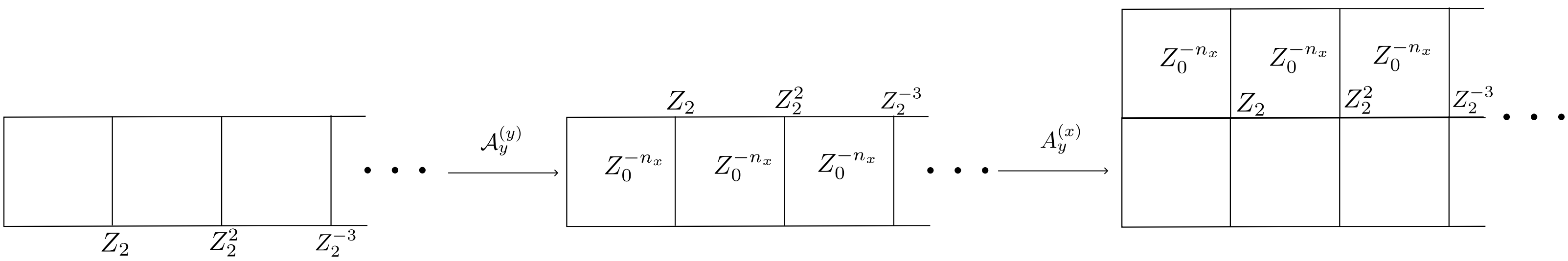

Figure 9: First step in the translation of a T_2 't Hooft loop.

For T_2 the procedure is very analogous, except we define the operator

$$\mathcal{A}_x^{(y)} = \prod_{i=1}^{n_x L_x} (a_{x+i,y}^y)^i. \tag{32}$$

Deformation of Wilson loops:

There are still 6 W loops to deform. Once again we define a product of commuting projectors using b_i.

$$B_y = \prod_{=1}^{L_x} b_{x+i,y} = \prod_{=1}^{L_x} X_{1,x+i,y-1}(X_{1,x+i,y})^{-2} X_{1,x+i,y+1} \tag{33}$$

$$B_x = \prod_{=1}^{L_y} b_{x,y+i} = \prod_{=1}^{L_y} X_{2,x-1,y+i}(X_{2,x,y+i})^{-2} X_{2,x+1,y+i} \tag{34}$$

Consider a W_1 operator located at position $y = y_0$. We can deform it by using the sequence of operators $B_{y+1}^\dagger (B_{y+2}^\dagger)^2 \cdots (B_{y+N-1}^\dagger)^{N-1}$. This will translate W_1 by N lattice spacing. For W_2 we can do the same using B_x.

For W_3 and W_4 we define two other operators. Let us describe the procedure to translate W_4 and then W_3 can be done analogously. Define the operator

$$\mathcal{B}_y^{(x)} = \prod_{i=1}^{lcm(L_x,N)} b_{x+i,y} = \prod_{i=1}^{lcm(L_x,N)} \left[X_{1,x+i,y-1}(X_{1,x+i,y})^{-2} X_{1,x+i,y+1}\right]^i \prod_{i=1}^{L_x} (X_{0,x+i,y-1})^{-n_x} (X_{0,x+i,y})^{-n_x} \tag{35}$$

Acting $\mathcal{B}_{y+1}^{(x)}$ on a W_4 loop located at position y will translate the loop by one lattice spacing. Similarly, we can define $\mathcal{B}_x^{(y)} = \prod_{i=1}^{lcm(L_y,N)} b_{x,y+i}$ to translate any loop W_3 horizontally.

The last two loops left are W_5 and W_6 that can be translated in the same manner we did for W_1 and W_2. This completes the proof that any loop can be translated in the lattice.

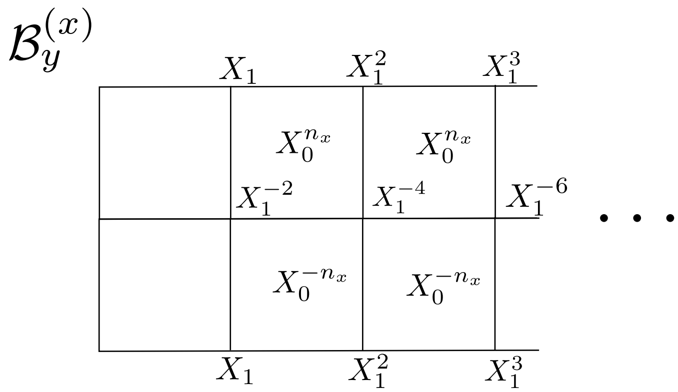

Figure 10: Action of $\mathcal{B}_y^{(x)}$ in a horizontal line.

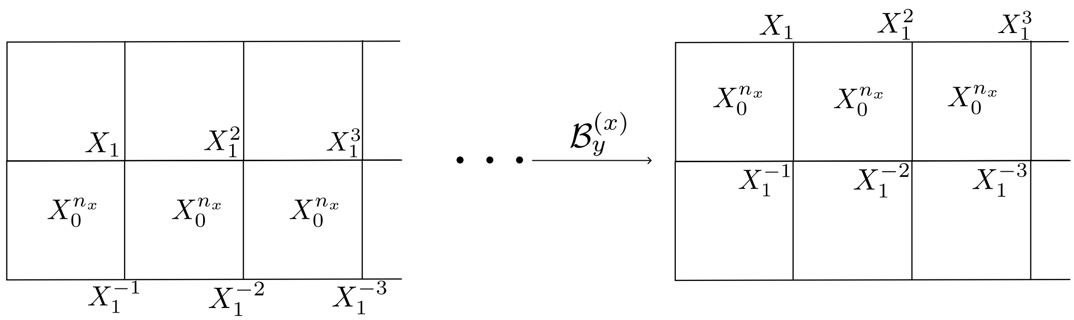

Figure 11: Translation of loop W_4 by one lattice spacing applying operator $\mathcal{B}_y^{(x)}$.

Note on loops with multiple winding

There is one point to think about when defining loops that wind around the torus more than one time. Let us take T_2 as an example. After traveling around the torus once, the product will act with X_1 in the initial position. There are two ways to define this operation, we either take the $L_x + 1$ power of X_1, that is the one described during this whole work, or instead we take the power of the position it is inserted in, which in this case would be simply 1. Both of them work as a symmetry of the Hamiltonian, but we have to understand why.

The most simple one to understand is the first one, as it has a clear interpretation of creating and annihilating pair of excitation. In a given position j, the operator has the following action

$$Z_2^{j+L_x+j+2L_x+j+\cdots+(n_x-1)L_x+j} = Z_2^{n_x j+L_x\frac{(n_x(n_x-1))}{2}} \tag{36}$$

so that

$$T_2 = \prod_{j=1}^{L_x} Z_2^{n_x j+L_x\frac{(n_x(n_x-1))}{2}} \tag{37}$$

In particular, if L_x and N are co-primes, $n_x = N$, and the operator is either the identity (if L_x is even or both L_x and N are odd) or a Wilson loop that already exists. But the second way to define the product, simply leads to

$$\tilde{T}_2 = \prod_{j=1}^{L_x} Z_2^{n_x j} \tag{38}$$

It shows that T_2 is actually the product of $\tilde{T}_2$ and some power of T_4. It is actually usually easier to work with $\tilde{T}_2$ as it is not the combination of any operator. There is one particularly nice property of $\tilde{T}_2$ which is

$$\tilde{T}_2^{gcd(N,L_x)} = I. \tag{39}$$