Toric Code: Identifying topological order

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1 Topological order: a quick review

The discovery of Topological order has been a topic of much study in the last few years. It is a very interesting feature of an apparent different type of quantum phase of matter. It is different than the others known state of matter because they are typically described by Landau paradigm, which claims that every phase transition can be characterized by a spontaneous symmetry breaking. A typical example is the Landau model that goes by a $\mathbb{Z}_2$ symmetry break when the temperature crosses a critical point. But topological order is known for not being described by a break of any usual symmetry, instead we need other features to characterize it. Since the main examples of systems that are topologically ordered rises from a quantum many body model, like a set of spins coupled, one of the properties that they have is anyonic excitations and ground state degeneracy robust against perturbations. When the system is put in a compact manifold, we can see a degeneracy that depends on the topology of the manifold, more specific, the number of genus it has. And anyons are particle that has statistics that are neither fermionic nor bosonic, that means that exchange of particles can change the phase of the wavefunction by any number.

1.1 Anyons: How are they possible?

As mentioned above, a very important aspect of topological order is anyonic excitations. These are very instersting particles because we are used to particles being either fermions or bosons. Actually, in quantum field theory, spin-statistics theorem claims that particles in 4 dimensions are either fermions or bosons with spin half spin or interger spin respectively. To prove it, we remember that if we have two identical particles, the exchange of these two particles leads to a change in the wavefunction that should be only a phase to preserve probability distribution (since they are identical)

$$\psi(x_1, x_2) = e^{i\phi}\psi(x_2, x_1) \tag{1}$$

Now the if we exchange both particles once again, we should be back to the situation where the wavefunction is the same, which means

$$\psi(x_1, x_2) = e^{2\pi i\phi}\psi(x_1, x_2). \tag{2}$$

This means that ϕ must be a integer and the wavefuction gains only fermionic or bosonic statistics. But this exchange can also be traced by operators of the Lorentz group acting on two particle states

$$|\psi_1(x_1), \psi_2(x_2) >= a^{x_1\dagger}a^{x_2\dagger}|0> \tag{3}$$

Now all we have to do is act with a translation of distance $|x_2 - x_1|$ and a rotation of π. This will act as an exchange of both particles. For boson there is nothing special, the two-particle states does not change under these operations. But for fermions, rotation of π corresponds to a change of phase by $e^{i\pi/2} = i$. Since there are two particles, the total change of phase in the state is $e^{\pi i}$. Therefore, in 3 space dimensions, particles are either fermions, corresponding to semi integer spin or boson with integer spin.

But we started this section claiming that there exists anyons excitations with any statistics. They indeed exist and do not contradict the theorem above because the spin-statistics theorem is valid for 3 dimensions. In this case, the worldlines of two particles in the second exchange can be untangled and becomes equivalent to no exchange at all. This is what allows us to say that the wavefunction cannot change under a full exchange. But in two dimensions, the worldline cannot be untagled, thus, the wavefunction does not need to be the same under full exchange of particles.

A more mathematically formal way to describe this phenomenom is by noting that in three dimensions, the group that acts as exhange of particles is the permutation group S_n, while in 2 dimensions. it is the braid group B_n. The braid group can be defined by a set of operations acting o N particles. Imagine we line up all these particles and define operators R_i that exchange the i-th particle with $i+1$-th particle. All the elements of B_n are generated by these operator respecting the properties $R_iR_j = R_jR_i$ for $i > j+2$ and $R_iRi+1R_i = R_{i+1}R_iR_{i+1}$.

As we will see, the Toric code is a two dimensional (space) model that allows for anyons excitations and topological ground state.

2 Toric Code

In this section will be introduced the Toric Code that is a $\mathbb{Z}_2$ topological order, and also considered the simplest system to study this phenomenom. It was first introduced by Kitaev, and it allows us to explore many aspects of this new state of matter. Our objective in this text is to describe the anomaly inflow that appears in this system and how to characterize its order from the mean value of some local operators.

2.1 A quick review of the model

Let us start by introducing the model and fix some notations to be used throughout the text. Consider a lattice in two dimensions with a set of spins aligned in the links, each being wither up or down just like our familiar Ising model. It means that each link of this lattice has an associated spin degree of freedom, which is two dimensionsal. Each local degree of freedom has its own Hilbert space, and the total Hilber space of the system is the direct sum of all local Hilbert space, meaning it is 2^N dimensional, where N is the number of links. We want to construct an exactly solvable Hamiltonian interacting these spins. In a lattice described this way, we can define two types of interactions. The first one is called plaquette interactions and is given by $B_p = \prod_{i\in p} Z_i$, where Z_i is the Pauli matrix σ^3 acting only on the spin i. This operator is the product of all Z_i in the plaquette. The other operator is the star operator given by $A_s = \prod_{i\in s} = X_i$, where X_i is the pauli matrix σ^1. This operator acts on a vertex and is the product of all edges intercepting this vertex. Since these two operator commute, we define the Hamiltonian of commuting projector given by

$$H = -J_e \sum A_s - J_m \sum B_p \tag{4}$$

The image below illustrates how each of the commuting projectors act on the lattice.

The states are given by configurations of the spins, and we usually diagonalize in the eigenstates of Z_i, so the spins are either up or down, and B_p returns the product of the eigenvalues of spin in the plaquette, while A_s flip the spins in the vertex it acts.

We can prove that it has a 4-fold ground state degeneracy, and for a manifold with k genus, it is given by 4^k. The excitations appear when acting one of the Pauli operator in any of the spins. Acting With Z_i will create two electric excitations, and X_i will create two magnetic excitations. Note that the total charge is 0 because it is calculated $\mod 2$. We can also define Wilson and 't Hooft operator. Wilson operator will move electric charges and is defined as the product of Z operator on a path in the lattice:

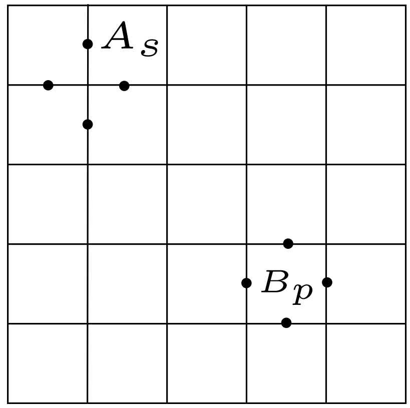

Figure 1: Action of operators A_s and B_p on the spin lattice. Black circles indicate Pauli operators acting on the link.

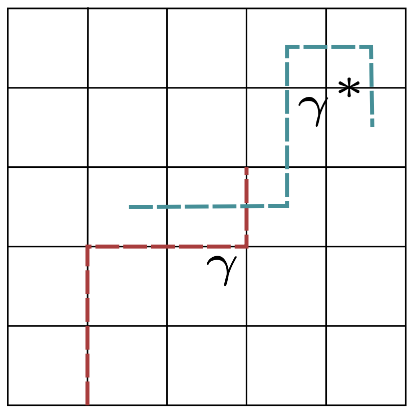

Figure 2: Paths where 't Hooft and Wilson operators are defined. γ is in the original lattice and γ^* is in the dual lattice.

$$W(\gamma) = \prod_{i \in \gamma} Z_i \tag{5}$$

And 't Hooft operators are the product of X operators on a path in the dual lattice:

$$T(\gamma^*) = \prod_{i \in \gamma^*} X_i \tag{6}$$

When the path is a loop (ending point is the same a the starting point), these operators are symmetry operators. And since they are defined on a line, they are one-form symmetries.

2.2 Superselection sectors

As we saw, each commuting projector can be responsible for an excitation. Star operators creates electric particles, and plaquette operators create magnetic vortices. Excitations are denoted by e and m, electric and magnetic respectively. A state can also have both excitations at the same time, which is denoted as $\epsilon = e \times m$, and the trivial, with no excitation, is denoted 1. We can verify that these sector have the following fusion rules:

$$e \times e = m \times m = 1 \tag{7}$$
$$e \times m = \epsilon \tag{8}$$
$$\epsilon \times e = m \tag{9}$$
$$\epsilon \times m = e \tag{10}$$

We can also prove using Wilson and 't Hooft loops that e and m are bosons with nontrivial mutual statistics and ϵ is a fermion.

2.3 Identifying topological order

One might ask if there is some other way to identify topological order, other than searching for anyons.There, indeed, is if the ground state is degenerated. The description is as follows:

Suppose the Hilbert space of the Hamiltonian has a degeneracy in the ground state and denote these states as $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$, $\cdots$. If these states satisfy

$$\langle\psi_i| \, |\psi_j\rangle = \delta_{ij} \tag{11}$$

and for any local operator $\mathcal{O}(x)$

$$\langle\psi_i| \, \mathcal{O}(x) \, |\psi_j\rangle = 0, \quad i \neq j \tag{12}$$

In the case of Toric code, we have 4 ground states given by $|\psi_1\rangle = |0\rangle$, the configurations with only trivial loops, $|\psi_2\rangle = T_x \, |0\rangle$, the one with loops around the x-axis of the torus, $|\psi_3\rangle = T_y \, |0\rangle$, with loops around y-axis, and $|\psi_4\rangle = T_x T_y \, |0\rangle$, with both kinds of loops. The first property is easily checked, since states in different ground state will always have some at least one of the spins in opposed directions. The second property can be verified as follows:

Consider the scalar product $\langle 0| \, \mathcal{O}(x) T_x \, |0\rangle$. Let $W_y(\gamma)$ be a Wilson loop around y. We have that

$$\langle 0| \, \mathcal{O}(x) T_x \, |0\rangle = \langle 0| \, W_y(\gamma) \mathcal{O}(x) T_x W_y(\gamma) \, |0\rangle \tag{13}$$

If the path γ contains x, we can apply plaquette operators to deform this path into one that does not contain x. This allows the path operators to commute with $\mathcal{O}(x)$,

$$\langle 0| \, W_y(\gamma) \mathcal{O}(x) T_x W_y(\gamma) \, |0\rangle = - \langle 0| \, W_y(\gamma) B_p \mathcal{O}(x) W_y(\gamma) B_p T_x \, |0\rangle = - \langle 0| \, \mathcal{O}(x) T_x \, |0\rangle \tag{14}$$

Therefore, $\langle 0| \, \mathcal{O}(x) T_x \, |0\rangle = 0$. Analogously, we can prove that $\langle\psi_i| \, \mathcal{O}(x) \, |\psi_j\rangle = 0$. This let us conclude that there is no local operator that connects distinct ground states in the Toric code. This is the statement that the ground states are locally indistinguishable.

2.4 Border in Toric Code

Now we want to make a little modification in the Toric Code. Let us forget about its name for a moment and put a border in the Torus where it is defined. We can do it by removing the periodic boundary conditions and forcing a borde on either the x-axis or y-axis. Let us work with a border at the line $y = 0$.

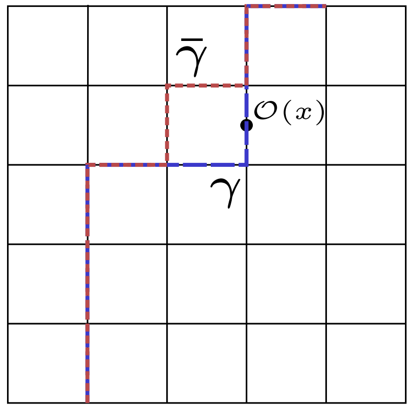

Figure 3: Path on the lattice intersecting local operator. This path can be topologically deformed from γ to $\bar{\gamma}$ by applications of either plaquette or star operators to avoid the point of intersection.

Now the Hamiltonian of this model should be divided by its bulk and edge

$$H = H_b + H_e \tag{15}$$

The bulk is the Toric Code with our familiar commuting projectors

$$H_b = -\sum A_s - \sum B_p \tag{16}$$

But the edge needs to be constructed. We actually have an ambiguity when putting this line defect. It can either go along a line of the lattice, or a line of the dual lattice. In the second case, the star operator become

$$O_i^1 = X_i \tag{17}$$

and the plaquette operator become

$$O_i^2 = Z_{i-1} Z_i Z_{i+1} \tag{18}$$

Note that these operators commute with the Bulk Hamiltonian, but when they intercept each other, they gain a phase $O_i^1 O_j^2 = -O_j^2 O_i^1$. Thus, the edge Hamiltonian is given by

$$H_e = -\lambda_1 \sum O_i^1 - \lambda_2 \sum O_i^2 \tag{19}$$

This phase in the commutation is a 't Hooft anomaly in the border. Now we rescue our commuting projectors to deform these operator. A successive application of plaquette operators takes O_i^2 inside the bulk, and, analogously, star operators take O_i^1 inside the bulk. Both operators "flow" into the bulk and keep their nontrivial commutation relation. That means that we can take an anomaly form the edge of the systems and make it into a bulk anomaly. This is what we call Anomaly Inflow.