Appendices

I. Appendix A

Given a $n_1 \times n_2$ assignment problem, the assignment matrix $Z \in \{0,1\}^{n_1 \times n_2}$ encodes the mapping of $\mathbf{x}_i$ to $\mathbf{y}_{i'}$ if $\mathbf{z}_{i,i'} = 1$. $\mathbf{z} \in \{0,1\}^{n_1 n_2}$ is the row-vectorized replica of Z such that

$$\mathbf{z} = \left[\mathbf{z_{1,1}} ... \mathbf{z_{1,n_2}} \; ... \; \mathbf{z_{n_1,1}} ... \mathbf{z_{n_1,n_2}} \;\right]^T. \tag{1}$$

The pairwise affinity matrix A is given by

$$\mathbf{A} = \begin{bmatrix} \Omega_2\left(c_{1,1}, c_{1,1}\right) = 0 & \cdots & \Omega_2\left(c_{1,1}, c_{1,n_2}\right) \\ \vdots & & \vdots \\ \Omega_2\left(c_{1,n_2}, c_{1,1}\right) = 0 & \cdots & \Omega_2\left(c_{1,n_2}, c_{1,n_2}\right) \\ & \vdots & \\ \Omega_2\left(c_{n_1,1}, c_{1,1}\right) & \cdots & \Omega_2\left(c_{n_1,1}, c_{1,n_2}\right) = 0 \\ \vdots & & \vdots \\ \Omega_2\left(c_{n_1,n_2}, c_{1,1}\right) & \cdots & \Omega_2\left(c_{n_1,n_2}, c_{1,n_2}\right) = 0 \end{bmatrix}. \tag{2}$$

and

$$a_{\widehat{i,j}} = a_{(i-1)n_2+i',(j-1)n_2+j'} = \Omega_2\left(c_{ii'}, c_{jj'}\right).$$

In the probabilistic framework, the assignment vector has the same indexing as in Eq. 1

$$\mathbf{p} = \left[\; P(c_{1,1}) \quad \dots \quad P(c_{1,n_2}) \quad ... \quad P(c_{n_1,1}) \quad \dots \quad P(c_{n_1,n_2}) \;\right]^T. \tag{3}$$

and the joint and conditional probabilities, $P\left(c_{ii'}, c_{jj'}\right)$, and $P\left(c_{ii'}|c_{jj'}\right)$, respectively, follow the same indexing as in Eq. 2.

II. Appendix B

In this section we show that the two-step iterative scheme is monotonically decreasing the objective function. The proof has two parts, the first (ending in Eq. 5) derives a result that is used in the second part. The first step of the PM is a single iteration of the Power Iteration scheme that converges in the Frobenius norm, and thus decreases the objective function for each entry of $P_t\left(c_{ii'}\right)$

$$\begin{aligned} &\left(\left(\sum_{jj'} P_t\left(c_{ii'}|c_{jj'}\right) P_{t+1}\left(c_{jj'}\right)\right) - P_{t+1}\left(c_{ii'}\right)\right)^2 \\ &\quad\le \left(\left(\sum_{jj'} P_t\left(c_{ii'}|c_{jj'}\right) P_t\left(c_{jj'}\right)\right) - P_t\left(c_{ii'}\right)\right)^2 \\ &\quad= \left(P_{t+1}\left(c_{ii'}\right) - P_t\left(c_{ii'}\right)\right)^2 \end{aligned} \tag{4}$$

Denote by $S_t = \sum_{jj'} P_t\left(c_{ii'}|c_{jj'}\right) P_{t+1}\left(c_{jj'}\right)$, hence

$$P_t^2\left(c_{ii'}\right) - 2P_{t+1}\left(c_{ii'}\right) P_t\left(c_{ii'}\right) \ge S_t^2 - 2P_{t+1}\left(c_{ii'}\right) S_t$$

Assume WLOG $P_{t+1}\left(c_{ii'}\right) \ge P_t\left(c_{ii'}\right)$, (the same proof mutatis mutandis holds for $P_{t+1}\left(c_{ii'}\right) \le P_t\left(c_{ii'}\right)$), then

$$P_t^2\left(c_{ii'}\right) - P_{t+1}\left(c_{ii'}\right) P_t\left(c_{ii'}\right) \le 0$$

and

$$\begin{aligned} 0 &\ge S_t^2 - 2P_{t+1}\left(c_{ii'}\right) S_t + P_{t+1}\left(c_{ii'}\right) P_t\left(c_{ii'}\right) \\ &\ge S_t^2 - P_{t+1}\left(c_{ii'}\right) S_t - P_t\left(c_{ii'}\right) S_t + P_{t+1}\left(c_{ii'}\right) P_t\left(c_{ii'}\right) \\ &= \left(S_t - P_{t+1}\left(c_{ii'}\right)\right)\left(S_t - P_t\left(c_{ii'}\right)\right). \end{aligned}$$

As $P_{t+1}\left(c_{ii'}\right) > P_t\left(c_{ii'}\right) \ge 0$ and $S_t \ge 0$ then

$$S_t - P_{t+1}\left(c_{ii'}\right) < S_t - P_t\left(c_{ii'}\right)$$

and

$$S_t - P_{t+1}\left(c_{ii'}\right) < 0 \tag{5}$$

The result in Eq. 5 will be used in the second part of the convergence proof, where we show that the second step of the PM, decrease the objective function. Namely, we aim to show that

$$\begin{aligned} &\left[\sum_{jj'} P_t\left(c_{ii'}|c_{jj'}\right) P_{t+1}\left(c_{jj'}\right) - P_{t+1}\left(c_{ii'}\right)\right]^2 \ge \\ &\quad\left[\sum_{jj'} P_{t+1}\left(c_{ii'}|c_{jj'}\right) P_{t+1}\left(c_{jj'}\right) - P_{t+1}\left(c_{ii'}\right)\right]^2 \\ &= \left[\left(\sum_{jj'} P_t\left(c_{ii'}|c_{jj'}\right) \frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)} P_{t+1}\left(c_{jj'}\right)\right) - P_{t+1}\left(c_{ii'}\right)\right]^2. \end{aligned} \tag{6}$$

Simplifying the above expression we get

$$S_t\left(\left(\frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)}\right)^2 - 1\right) - 2P_{t+1}\left(c_{ii'}\right)\left(\frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)} - 1\right) \le 0$$

Assuming $P_{t+1}\left(c_{ii'}\right) > P_t\left(c_{ii'}\right)$ as before, we have that $\frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)} - 1 > 0$, and $\frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)} + 1 > 2$. Thus,

$$0 \ge S_t\left(\frac{P_{t+1}\left(c_{ii'}\right)}{P_t\left(c_{ii'}\right)} + 1\right) - 2P_{t+1}\left(c_{ii'}\right) \ge 2\left(S_t - P_{t+1}\left(c_{ii'}\right)\right) \tag{7}$$

Equation 7 is validated by the first part of the proof (Eq. 5), and this implies the reduction of the objective function in Eq. 6. The proof of the complementary case, $P_{t+1}\left(c_{ii'}\right) < P_t\left(c_{ii'}\right)$, can be derived mutatis mutandis.