Aalborg University
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Computational and Applied Topology Exercises

A rigorous approach to AI/Machine Learning, with applications in Topological Data Analysis and Computational Algebraic Topology November 2023

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For those interested in further exercises, you can look at [1, 2].

1. (a) How many different simplicial complexes can you build on up to 4 vertices?

(b) What is the (simplicial) homology of each of these simplicial complexes?

2. Can you think of some applications where simplicial complexes are not appropriate? Can you come up with alternative types of complex that we could use in these situations?

3. Let $P \subset \mathbb{R}^d$ be a finite set of points. Recall that the Vietoris-Rips complex $\mathcal{VR}_\varepsilon(P)$ at scale ε on P is defined as follows: for every subset of points $\{p_i\}_{i=0}^n \subset P$, if for all $0 \leq i, j \leq n$, $\|p_i - p_j\| \leq \varepsilon$, then the simplex $[p_0, \ldots, p_n]$ is in $\mathcal{VR}_\varepsilon(P)$. Furthermore, recall that the Čech complex $\mathcal{C}_\varepsilon(P)$ at scale ε on P is defined as follows: for every subset of points $\{p_i\}_{i=0}^n \subset P$, if $\cap_i B_\varepsilon(p_i) \neq \emptyset$, then the simplex $[p_0, \ldots, p_n]$ is in $\mathcal{C}_\varepsilon(P)$. Where $B_\varepsilon(p_i)$ is the closed ball of radius ε centred at p_i.

(a) Show that

$$\mathcal{C}_\varepsilon(P) \subseteq \mathcal{VR}_{2\varepsilon}(P).$$

(b) Show that

$$\mathcal{VR}_\varepsilon(P) \subseteq \mathcal{C}_{\sqrt{2}\varepsilon}(P).$$

4. Show that for the boundary operator ∂, $\mathrm{im}(\partial_{k+1}) \subseteq \ker(\partial_k)$.

5. Consider the set of digital digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ (in this exact font). Can you find a way to define a function on them whose functional persistence distinguishes them?

6. Consider the simplicial complex X with the following top-dimensional simplicies:

$$\{abcd, bef, ae, df, acg, adg, cdg\}.$$

(a) Draw X.

(b) What are the homology groups of X over ${}^{\mathbb{Z}}/_{2\mathbb{Z}}$?

(c) Put a filtration on X and calculate the persistence diagram.

7. Complete the Python notebook.

References

[1] Tamal Krishna Dey and Yusu Wang. *Computational Topology for Data Analysis.* Cambridge University Press, 2022.

[2] Rob Ghrist. *Elementary Applied Topology.* 1st ed. Createspace, 2014.