Homework 2B

Question 3

Find the first three terms of the power series solutions of the following.

3.1. $x \cdot (x-1) \cdot y'' + 6x^2 \cdot y' + 3 \cdot y \quad ; \quad x > 0$

3.2. $4 \cdot x^2 \cdot y'' + (3x+1)y = 0 \quad ; \quad x > 0$

Solutions:

2.1. $r_1 = 1 \ , \ r_2 = 0$

$$\varphi_1 = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \cdots \quad , \quad \varphi_2 = 3 \cdot \varphi_1 \cdot ln(x) + 1 - \frac{21}{4} \cdot x^2 - \frac{19}{4}x^3 + \cdots$$

Hint: *K is arbitrary and can be chosen to set $b_0 = 1$.*

2.2. $r_1 = 1/2 \ , \ r_2 = 1/2$

$$\varphi_1(x) = x^{1/2} + \sum_{n=1}^{\infty}(-1)^n \cdot \left(\frac{3}{4}\right)^n \cdot \frac{1}{(n!)^2} \cdot x^{n+\frac{1}{2}} = x^{1/2} - \frac{3}{4} \cdot x^{\frac{3}{2}} + \frac{9}{64} \cdot x^{\frac{5}{2}} + \cdots,$$

$$\varphi_2(x) = ln(x) \cdot \left[x^{\frac{1}{2}} - \frac{3}{4} \cdot x^{\frac{3}{2}} + \frac{9}{62} \cdot x^{\frac{5}{2}} + \cdots\right] + \frac{3}{2} \cdot x^{\frac{3}{2}} + \frac{27}{64} \cdot x^{\frac{5}{2}} + \cdots$$

Question 4

Using Frobenius method, find the three first terms of the power series solutions of the following Bessel equation of an order of $\nu = 3/2$:

$$x^2 \cdot y'' + x \cdot y' + \left(x^2 - \frac{9}{4}\right) \cdot y = 0 \quad ; \quad x > 0$$

Solutions:

$r_1 = 3/2 \ , \ r_2 = -3/2$

$$\varphi_1 = x^{\frac{3}{2}} - \frac{1}{12}x^{\frac{7}{2}} + \frac{1}{280}x^{\frac{11}{2}} + \cdots , \quad \varphi_2 = x^{-\frac{3}{2}} + \frac{1}{2} \cdot x^{\frac{1}{2}} - \frac{1}{8} \cdot x^{\frac{5}{2}} + \cdots$$

Question 5

Obtain the following recurrence relations of Bessel functions:

$$\frac{d}{dx}\left(x^\nu \cdot J_\nu(x)\right) = x^\nu \cdot J_{\nu-1}(x) \quad , \quad \frac{d}{dx}\left(x^{-\nu} \cdot J_\nu(x)\right) = -x^{-\nu} \cdot J_{\nu+1}(x) \qquad \text{(a-b)}$$

$$\frac{dJ_\nu(x)}{dx} = \frac{1}{2} \cdot [J_{\nu-1}(x) - J_{\nu+1}(x)] \quad , \quad \left(\frac{\nu}{x}\right) \cdot J_\nu(x) = \frac{1}{2} \cdot [J_{\nu-1}(x) + J_{\nu+1}(x)] \qquad \text{(c-d)}$$

Hint: *First obtain (a) and (b) by using $J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot x^{2n+\nu}$. Then, use (a-b) to obtain (c-d).*