Formula sheet – Analytical methods in mechanical engineering (362.2.6091)

1. Ordinary differential equations

1.1. Normal form

The general form of a homogeneous linear second-order differential equation:

$$a_0(x)\cdot\frac{d^2y}{dx^2}+a_1(x)\cdot\frac{dy}{dx}+a_2(x)\cdot y=0$$

The normal form is:

$$\frac{d^2u}{dx^2}+Q(x)\cdot u=0$$

The transformation is:

$$\varphi(x)=u(x)\cdot p(x)\quad\text{where}\quad p(x)=e^{-\frac{1}{2}\int\left(\frac{a_1}{a_0}\right)dx}$$

1.2. Adjoint forms

The general linear second-order differential operator:

$$L=a_0(x)\cdot\frac{d^2}{dx^2}+a_1(x)\cdot\frac{d}{dx}+a_2(x)$$

The adjoint operator produces:

$$\tilde{L}=a_0\cdot\frac{d^2}{dx^2}+(2a_0'-a_1)\cdot\frac{d}{dx}+(a_0''-a_1'+a_2)$$

Lagrange identity:

$$z\cdot L[y]-y\cdot\tilde{L}[z]=\frac{d}{dx}F(x,y,y',z,z')$$

Green’s identity:

$$\int_a^b\{z\cdot L[y]-y\cdot\tilde{L}[z]\}dx=[F(x,y,y',z,z')]_{x=a}^{b}$$

Where:

$$F(x,y,y',z,z')=a_0\cdot(z\cdot y'-z'\cdot y)+(a_1-a_0')\cdot z\cdot y$$

Self-adjoint form:

$$L=\tilde{L}\quad;\quad a_1=a_0'=p(x)\,,\ a_2=q(x)$$

$$F(x,y,y',z,z')=p\cdot(z\cdot y'-z'\cdot y)$$

Transformation of $L[y(x)]$ into a self-adjoint form:

$$L[y(x)]\cdot\frac{p(x)}{a_0}=0\quad;\quad p(x)=e^{\int\left(\frac{a_1}{a_0}\right)dx}$$

2. Series solutions of differential equations

Series solution in a regular point:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$

2.1.Frobeniou method

$$y'' + p(x) \cdot y' + q(x) \cdot y = 0$$

$$x \cdot p(x) = \sum_{k=0}^{\infty} p_k \cdot x^k = p_0 + p_1 \cdot x + \cdots \;\; ; \;\; x^2 \cdot q(x) = \sum_{k=0}^{\infty} q_k \cdot x^k = q_0 + q_1 \cdot x + \cdots$$

The indicial equation:

$$f(r) = r \cdot (r-1) + p_0 \cdot r + q_0 = 0$$

Case I ($r_1 > r_2$):

$$\varphi_1(x) = |x^{r_1}| \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad , \quad \varphi_2(x) = |x^{r_2}| \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

Case II ($r_1 = r_2 = r$):

$$\varphi_1(x) = |x^r| \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad , \quad \varphi_2(x) = \varphi_1(x) \cdot \ln(|x|) + |x^r| \cdot \sum_{n=1}^{\infty} b_n \cdot x^n$$

Case III ($r_1 > r_2$; $r_1 - r_2 = m = 1,2,\ldots$):

$$\varphi_1(x) = |x^{r_1}| \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad , \quad \varphi_2(x) = \mathrm{K} \cdot \varphi_1(x) \cdot \ln(|x|) + |x^{r_2}| \cdot \sum_{n=1}^{\infty} b_n \cdot x^n$$

2.2 The Legendre equation

$$(1-x^2) \cdot y'' - 2 \cdot x \cdot y' + l \cdot (1+l) \cdot y = 0 \qquad (l = const)$$

The Legendre polynomials ($l = 0,1,2,3,4,5$):

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2} \cdot (3x^2 - 1)$, $P_3(x) = \frac{1}{2} \cdot (5x^3 - 3x)$,

$P_4(x) = \frac{1}{8} \cdot (35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8} \cdot \left(63x^5 - 70x^3 + 15x\right)$

Orthogonal relations:

$$\int_{-1}^{1} P_l(x) \cdot P_n(x) dx = \frac{2}{2l+1} \delta_{ln}$$

2.3. The Bessel equation

$$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0 \qquad (\nu = const)$$

The solutions are:

$$\varphi(x) = A \cdot J_\nu(x) + B \cdot J_{-\nu}(x) \quad ; \quad \nu \neq k = 1,2,3 \ldots.$$
$$\varphi(x) = A \cdot J_\nu(x) + B \cdot Y_\nu(x) \quad ; \quad \nu = k = 1,2,3 \ldots.$$

The Bessel functions of the first kind:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot x^{2n+\nu}$$
$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot sin(x) \ , \ J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot cos(x)$$

The Bessel function of the second kind:

$$Y_\nu(x) = \frac{cos(\nu \cdot \pi) \cdot J_\nu(x) - J_{-\nu}(x)}{sin(\nu \cdot \pi)}$$

The Gamma function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad (x > 0)$$
$$\Gamma(x+1) = x \cdot \Gamma(x)$$
$$\Gamma(n+1) = n! \quad (x = n = 1,2,3 \ldots)$$

Asymptotic expressions ($x \ll 1$):

$$J_\nu(x) \approx \frac{1}{2^\nu \cdot \Gamma(\nu+1)} \cdot x^\nu + \cdots$$
$$Y_{\nu=0}(x) \approx \frac{2}{\pi} \cdot \left[\gamma + ln\left(\frac{x}{2}\right) + \cdots\right], \ Y_{\nu\neq 0}(x) \approx -\frac{\Gamma(\nu) \cdot 2^\nu}{\pi} \cdot x^{-\nu} + \cdots$$

Asymptotic expressions ($x \gg 1, \nu$):

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi \cdot x}} \cdot cos\left(x - \frac{\nu \cdot \pi}{2} - \frac{\pi}{4}\right) \quad , \quad Y_\nu(x) \approx \sqrt{\frac{2}{\pi \cdot x}} \cdot sin\left(x - \frac{\nu \cdot \pi}{2} - \frac{\pi}{4}\right)$$

Identities

$$J_{-k}(x) = (-1)^k \cdot J_k(x) \quad ; \quad k = 1,2,3 \ldots.$$
$$\frac{d}{dx}\left(x^\nu \cdot J_\nu(x)\right) = x^\nu \cdot J_{\nu-1}(x) \quad , \quad \frac{d}{dx}\left(x^{-\nu} \cdot J_\nu(x)\right) = -x^{-\nu} \cdot J_{\nu+1}(x) \quad \text{- hold also for } Y_\nu(x)$$
$$\frac{dJ_\nu(x)}{dx} = \frac{1}{2} \cdot [J_{\nu-1}(x) - J_{\nu+1}(x)] \ , \ \left(\frac{\nu}{x}\right) \cdot J_\nu(x) = \frac{1}{2} \cdot [J_{\nu-1}(x) + J_{\nu+1}(x)] \quad \text{- hold also for } Y_\nu(x)$$

Zeroes of Bessel function:

$$J_\nu(\chi_{\nu n}) = 0 \quad (n = 1,2,3 \ldots)$$

Orthogonal relations:

$$\int_0^a x \cdot J_\nu(\chi_{\nu n} \cdot x/a) \cdot J_\nu(\chi_{\nu m} \cdot x/a) = \frac{a^2}{2} \cdot \left(J_{\nu+1}(\chi_{\nu n})\right)^2 \cdot \delta_{mn}$$

3. Boundary value problems - Green's function method

3.1. Homogeneous boundary conditions

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$
$$U_a[y] = 0 \quad , \quad U_b[y] = 0$$

The solution is given by:

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

Where $G(x, x_0)$ is Green's function that satisfies:

$$L[G] = \delta(x - x_0)$$
$$U_a[G] = 0 \quad , \quad U_b[G] = 0$$

Note that for

$$L[y] = a_0(x) \cdot y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

3.2. Non-homogeneous boundary conditions

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$
$$U_a[y] = \alpha \quad , \quad U_b[y] = \beta$$

The solution is given by:

$$\varphi(x_1) = \int_a^b \tilde{G}(x, x_1) \cdot f(x) dx + \left[F\left(x, \varphi, \frac{d\varphi}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx} \right) \right]_{x=a}^{b}$$

Where $\tilde{G}(x, x_1)$ is the adjoint Green's function that satisfies:

$$\tilde{L}[\tilde{G}] = \delta(x - x_1)$$
$$\tilde{U}_a[\tilde{G}] = 0 \quad , \quad \tilde{U}_b[\tilde{G}] = 0$$

where:

$$\tilde{G}(x_0, x_1) = G(x_1, x_0)$$

4. Boundary value problems – Eigenvalue problems

4.1. Strum-Liouvile systems

The Strum-Liouville equation:

$$\frac{d}{dx}\left(p(x)\cdot\frac{dy}{dx}\right)+\left(q(x)+\lambda\cdot s(x)\right)\cdot y=0$$

The self-adjoint operator:

$$L=\tilde{L}=\frac{d}{dx}\left(p(x)\cdot\frac{d}{dx}\right)+q(x)$$

The eigenvalue problem:

$$L[y]=-\lambda\cdot s(x)\cdot y$$

Regular system:

$$a_1\cdot y(a)+a_2\cdot y'(a)=0$$
$$b_1\cdot y(b)+b_2\cdot y'(b)=0$$

Periodic system:

$$y(a)=y(b)$$
$$y'(a)=y'(b)$$
$$p(x=a)=p(x=b)$$

Singular system:

$$lim_{x\to a}\,y(x)<\infty$$
$$p(x=a)=0$$
$$b_1\cdot y(b)+b_2\cdot y'(b)=0$$

Eigenfunction orthogonality of S-L systems $\{\varphi_n\}$:

$$\langle\varphi_m|s|\varphi_n\rangle=\int_a^b\varphi_m(x)\cdot s(x)\cdot\varphi_n(x)dx=\|\varphi_n\|^2\cdot\delta_{mn}$$

With the norm:

$$\|\varphi_n\|=\sqrt{\langle\varphi_n|s|\varphi_n\rangle}=\left[\int_a^b\varphi_n(x)\cdot s(x)\cdot\varphi_n(x)dx\right]^{\frac{1}{2}}$$

4.2. Some eigenfunction series expansions

Sine series

$$\varphi_n(x)=\sin\left(\frac{\pi nx}{L}\right)$$

Inner product and orthogonal relations:

$$\langle\varphi_n|\varphi_m\rangle=\int_0^L\varphi_n(x)\cdot\varphi_m(x)\,dx=\frac{L}{2}\cdot\delta_{nm}$$

The series expansion and coefficients:

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot sin\left(\frac{\pi n x}{L}\right)$$

$$a_n = \frac{\langle f|\varphi_n\rangle}{\langle\varphi_n|\varphi_n\rangle} = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot sin\left(\frac{\pi n x}{L}\right) dx$$

Fourier series (periodic functions)

$$\phi(x) = 1 \ , \quad \varphi_n(x) = \sin\left(\tfrac{\pi n x}{L}\right) \quad , \quad \psi_n(x) = \cos\left(\tfrac{\pi n x}{L}\right)$$

Inner product:

$$\langle g|h\rangle = \int_{-L}^{L} g(x) \cdot h(x)\, dx$$

Orthogonal relations:

$$\langle\phi|\varphi_n\rangle = 0 \quad , \quad \langle\phi|\psi_n\rangle = 0 \quad , \quad \langle\varphi_n|\psi_m\rangle = 0$$

$$\langle\varphi_n|\varphi_m\rangle = L \cdot \delta_{nm} \quad , \quad \langle\psi_n|\psi_m\rangle = L \cdot \delta_{nm} \quad , \quad \langle\phi|\phi\rangle = 2 \cdot L$$

The series expansion and coefficients:

$$f(x) = b_o + \sum_{n=1}^{\infty} \left[a_n \cdot sin\left(\frac{\pi n x}{L}\right) + b_n \cdot cos\left(\frac{\pi n x}{L}\right)\right]$$

$$a_n = \frac{\langle f|\varphi_n\rangle}{\langle\varphi_n|\varphi_n\rangle} = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot sin\left(\frac{\pi n x}{L}\right) dx$$

$$b_n = \frac{\langle f|\psi_n\rangle}{\langle\psi_n|\psi_n\rangle} = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot cos\left(\frac{\pi n x}{L}\right) dx$$

$$b_0 = \frac{\langle f|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{1}{2 \cdot L} \cdot \int_{-L}^{L} f(x) dx$$

Fourier-Legendre series (spherical coordinates)

Inner product and orthogonal relations:

$$\langle P_l|1|P_n\rangle = \langle P_l|P_n\rangle = \int_{-1}^{1} P_l(x) \cdot P_n(x) dx = \frac{2}{2l+1} \delta_{ln}$$

The series expansion and coefficients:

$$f(x) = \textstyle\sum_{l=1}^{\infty} A_l \cdot P_l(x) \ ; \quad (-1 \le x \le 1)$$

$$A_l = \frac{\langle f|P_l\rangle}{\langle P_l|P_l\rangle} = \frac{2 \cdot l + 1}{2} \cdot \int_{-1}^{1} f(x) \cdot P_l(x)\, dx$$

Fourier-Bessel series (cylindrical coordinates)

$$J_{\nu;n} = J_\nu(\chi_{\nu n} \cdot x/a) \ ; \ J_\nu(\chi_{\nu n}) = 0 \quad (n = 1,2,3 \ldots)$$

Inner product and orthogonal relations:

$$\langle J_{\nu;n}|x|J_{\nu;m}\rangle = \int_0^a x \cdot J_\nu(\chi_{\nu n} \cdot x/a) \cdot J_\nu(\chi_{\nu m} \cdot x/a) = \frac{a^2}{2} \cdot \left(J_{\nu+1}(\chi_{\nu n})\right)^2 \cdot \delta_{mn}$$

The series expansion and coefficients:

$$f(x) = \sum_{n=1}^{\infty} A_n \cdot J_\nu(\chi_{\nu n} \cdot x/a) \;\; ; \quad (0 < x < a)$$

$$A_n = \frac{\langle f|x|J_{\nu;n}\rangle}{\langle J_{\nu;n}|x|J_{\nu;n}\rangle} = \frac{2}{a^2 \cdot \left(J_{\nu+1}(\chi_{\nu n})\right)^2} \cdot \int_0^a x \cdot f(x) \cdot J_\nu(\chi_{\nu n} \cdot x/a)\, dx$$