Analytical Methods

Boundary-value problems (I)

Second-order boundary-value problems

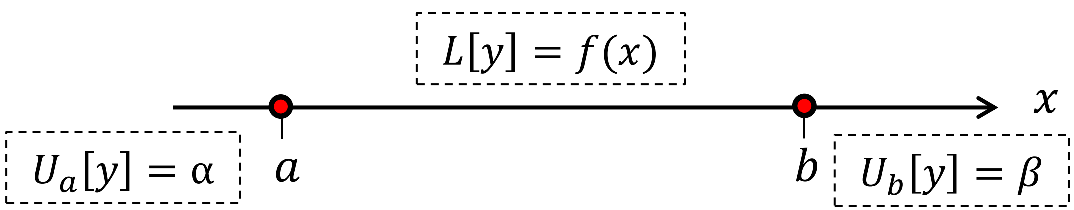
- We ***focus*** on linear boundary-value problems of the differential operator:

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

- The boundary conditions are ***generally*** given by:

$$U_a[y] = a_1 \cdot y(a) + a_2 \cdot y'(a) = \alpha$$

$$U_b[y] = b_1 \cdot y(b) + b_2 \cdot y'(b) = \beta$$



In general

*. The boundary-value problem ***may not*** possess a solution (!)

*. If it does – it may ***not*** be ***unique*** **(!)**

Example 1

- The ***inhomogeneous*** equation: $y'' + y = 1$

$$y(0) = 0 \quad , y(\pi/2) = 0$$

By the variation of parameters method (remember…?), we find a ***unique*** solution:

$$\boxed{\varphi(x) = 1 - cos(x) - sin(x)}$$

- The ***associated homogeneous*** problem: $y'' + y = 0$

$$y(0) = 0 \quad , y(\pi/2) = 0$$

For this case, the solution is ***trivial*** (!)

$$\varphi(x) = c_1 \cdot sin(x) + c_2 \cdot cos(x)$$

$$\begin{aligned} \varphi(0) &= c_2 = 0 \\ \varphi(\pi/2) &= c_1 = 0 \end{aligned} \longrightarrow \boxed{\varphi(x) = 0}$$

Example 2

- The ***inhomogeneous*** equation: $y'' + y = 1$

$$y(0) = 0 \quad , y(\pi) = 0$$

The *general* solution for this case is:

$$\varphi(x) = c_1 \cdot sin(x) + c_2 \cdot cos(x) + 1$$

Applying the boundary conditions:

$$\varphi(0) = c_2 + 1 = 0$$

$$\varphi(\pi) = -c_2 + 1 = 0$$

$$\boxed{\varphi(x) \to Doesn't\ exist\ !}$$

- The ***associated*** ***homogeneous*** problem: $y'' + y = 0$

$$y(0) = 0 \quad , y(\pi) = 0$$

For this case, the solution ***exists*** (!) - but cannot be fully specified by the BC

$$\boxed{\varphi(x) = C \cdot cos(x)}$$

Theorem:

Let $p(x), q(x)$ *and* $f(x)$ *be continues on* $[a, b]$.

Then, ***either*** *the boundary-value problem has a* ***unique*** *solution*

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

$$U_a[y] = a_1 \cdot y(a) + a_2 \cdot y'(a) = \alpha \qquad U_a[y] = b_1 \cdot y(b) + b_2 \cdot y'(b) = \beta$$

$$\boxed{\varphi(x) \rightarrow unique}$$

or *the associated homogeneous boundary-value problem has a* ***non-trivial*** *solution*

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = 0$$

$$U_a[y] = 0 \quad ,U_b[y] = 0$$

$$\boxed{\varphi(x) \neq 0}$$

Summarizing – again….

Possibility (#1):

$$L[y] = y'' + p(x)\cdot y' + q(x)\cdot y = f(x)$$

$$U_a[y] = a_1\cdot y(a) + a_2\cdot y'(a) = \alpha$$

$$U_b[y] = b_1\cdot y(b) + b_2\cdot y'(b) = \beta$$

The associate homogeneous system has **only** a trivial solution

Unique solution

$$\boxed{\varphi(x) = c_1\cdot\varphi_1(x) + c_2\cdot\varphi_2(x) + \psi(x)}$$

c_1 and c_2 are specified explicitly

Possibility (#2):

$$L[y] = y'' + p(x)\cdot y' + q(x)\cdot y = 0$$

$$U_a[y] = a_1\cdot y(a) + a_2\cdot y'(a) = 0$$

$$U_b[y] = b_1\cdot y(b) + b_2\cdot y'(b) = 0$$

Non-trivial solution

$$\boxed{\varphi(x) = C\cdot\varphi_1(x)}$$

$C \neq 0$ is non-specific

Part A *– Green's function method*

(I) Mathematical preliminaries – the Dirac δ function

Paul Dirac

- Introducing the ***Dirac delta function*** $\delta(x-x_0)$ - defined via:

$$\boxed{\delta(x-x_0)=0 \quad ; \quad x\neq x_0}$$

$$\boxed{\int_{-\infty}^{\infty}\delta(x-x_0)=1} \quad \dashrightarrow \quad \lim_{n\to 0}\int_{x_0-n}^{x_0+n}\delta(x-x_0)=1$$

- From that definition, $\delta(x)$ must be ***infinitely high and infinitely narrow*** – i.e. ***like a spike*** – at x_0.

- Such a function doesn't exist in the usual scene… and it is thus considered as a "***generalized function***"

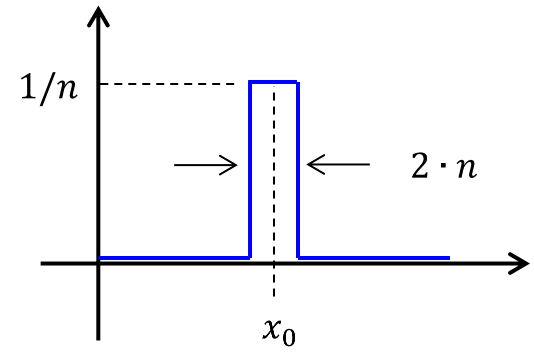
- The $\delta(x-x_0)$ can be *rigorously* defined as the ***limit*** of a ***sequence*** of functions $\delta_n(x-x_0)$:

$$\delta(x-x_0)=\lim_{n\to 0}\delta_n(x-x_0)$$

- This limit however – *does not exist* (!!)

➢ A ***customary example*** – the rectangular function:

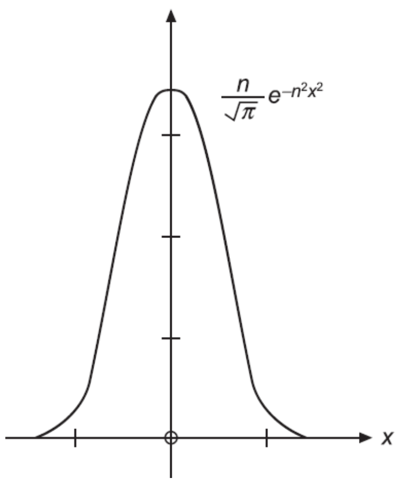
$$\delta_n(x-x_0)=\begin{cases}0 & ; \quad x-x_0<-\dfrac{1}{n}\\ \dfrac{1}{2n} & ; \quad -\dfrac{1}{n}<x-x_0<\dfrac{1}{n}\\ 0 & ; \quad x-x_0>\dfrac{1}{n}\end{cases}$$



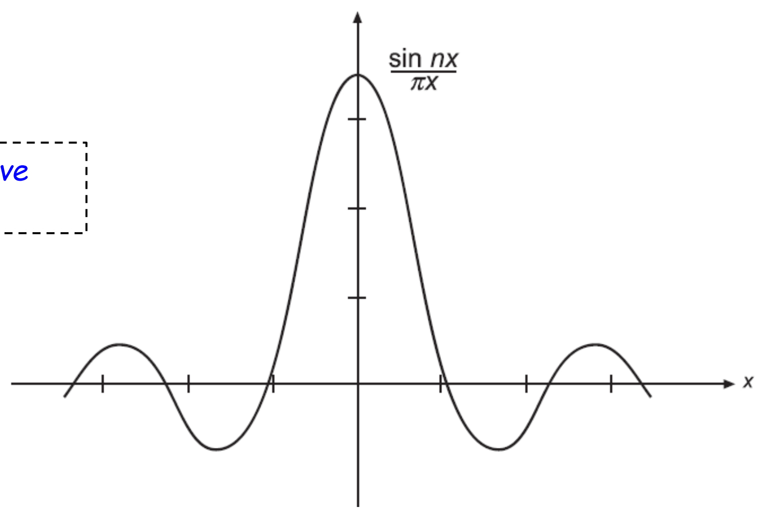
➢ Other common examples:

$$\delta_n(x-x_0)=\frac{n}{\sqrt{\pi}}\cdot\exp(-n^2\cdot(x-x_0)^2)$$

$$\delta_n(x-x_0)=\frac{sin(n\cdot(x-x_0))}{\pi\cdot(x-x_0)}$$



Continuously derivative functions.



Some key-properties of the Dirac δ function

- If $f(x)$ is a "nice" continuously differentiable function.

- A ***fundamental*** property:

$$\boxed{\int_{-\infty}^{\infty} f(x)\cdot\delta(x-x_0)dx = f(x_0)}$$

Sampling / shifting property

$$\int_{-\infty}^{\infty} f(x)\cdot\delta(x-x_0)dx \quad = \lim_{n\to 0}\int_{x_0-n}^{x_0+n} f(x)\cdot\delta(x-x_0)dx \;= f(x_0)\cdot\lim_{n\to 0}\int_{x_0-n}^{x_0+n}\delta(x-x_0)dx \quad = f(x_0)$$

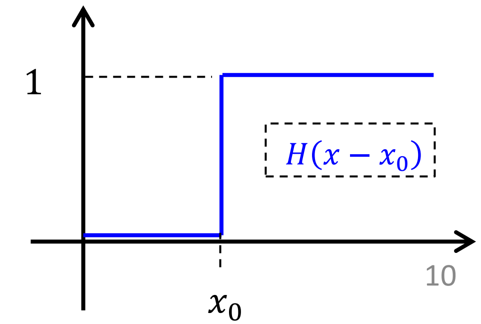
- Integration over the Dirac derivative:

$$\boxed{\int_{-\infty}^{\infty} f(x)\cdot\delta'(x-x_0)dx = -f'(x_0)}$$

Sampling the derivative

- Relation to the ***Heaviside*** "step function":

$$\boxed{\delta(x-x_0) = \frac{d}{dx}H(x-x_0)}$$



(II) Introducing the Green's function method

The aim

To find the ***solution*** of a boundary-value problem with *homogeneous* *B.C.*


George Green

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

$$U_a[y] = 0 \qquad U_b[y] = 0$$

$$\longrightarrow \boxed{\varphi(x) = ?}$$

- We will ***focus*** on cases in which the *associated homogeneous* system has only a ***trivial solution***.

$$y'' + y = 0$$

$$y(0) = 0 \quad , y(\pi/2) = 0$$

$$\downarrow$$

$$\boxed{\varphi(x) = 0}$$

$$y'' + y = 0$$

$$y(0) = 0 \quad , y(\pi) = 0$$

$$\downarrow$$

$$\boxed{\varphi(x) = C \cdot cos(x)}$$

$$L[y] = f(x)$$

$$U_a[y] = 0 \qquad U_b[y] = 0$$

The same L operator

The same homogeneous BC

$f(x) \rightarrow \delta(x - x_0)$

- The ***solution*** of the above boundary-value problem is given by:

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

- Where "***Green's function***" $G(x, x_0)$ is the solution of the boundary-value problem:

$$L[G(x, x_0)] = \delta(x - x_0)$$

$$U_a[G(a, x_0)] = 0 \qquad U_b[G(b, x_0)] = 0$$

Green's function is readily considered as the "***fundamental solution***" of the boundary-value problem

- This can be easily evident via:

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

$$L[\varphi(x)] = L\left[\int_a^b G(x, x_0) \cdot f(x_0) dx_0\right] = \int_a^b L[G(x, x_0)] \cdot f(x_0) dx_0$$

L operates on x only

$$= \int_a^b \delta(x - x_0) \cdot f(x_0) dx_0 = f(x)$$

$L[G(x, x_0)] = \delta(x - x_0)$

Property of the Dirac function

Green's function - Physical interpretation

- Green's function is the "*influence function*" of a "***point source***" located at x_0 on an arbitrary point x in the domain $a \leq x \leq b$

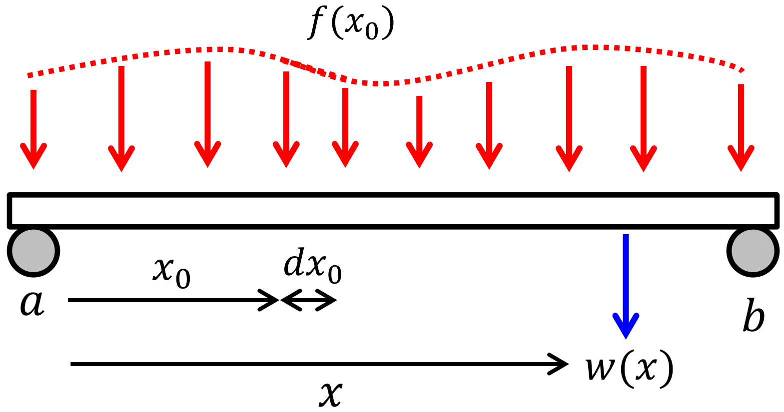
- The ***general*** "***source***" can be considered, for example, as:

Singular point-load (Mechanical)

Heat source (Diffusion)

Point electric charge (Electromagnetism)

An example: beam deflections



- $G(x, x_0)$ represents the deflection at a point x due to a ***singular*** *unit-force* – applied at x_0

$$w(x) = G(x, x_0)$$

- The deflection $w(x)$ due to an ***increment*** of *distributed force* $f(x_0)$ applied to a ***segment*** dx_0 is thus:

$$w(x) = G(x, x_0) \cdot f(x_0) \cdot dx_0$$

- By using ***superposition***, the deflection $w(x)$ due to a *distributed force* $f(x)$ ***along*** $a \le x_0 \le b$ is thus:

$$w(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

Theorem - uniqueness and existence of Green's function

Consider the inhomogeneous boundary-value problem with homogeneous BC:

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

$$U_a[y] = a_1 \cdot y(a) + a_2 \cdot y'(a) = 0 \qquad U_a[y] = b_1 \cdot y(b) + b_2 \cdot y'(b) = 0$$

*If the associated homogeneous boundary-value problem has **only** a **trivial** solution:*

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = 0$$

$$U_a[y] = 0 \quad ,U_b[y] = 0$$

$$\dashrightarrow \boxed{\varphi(x) = 0}$$

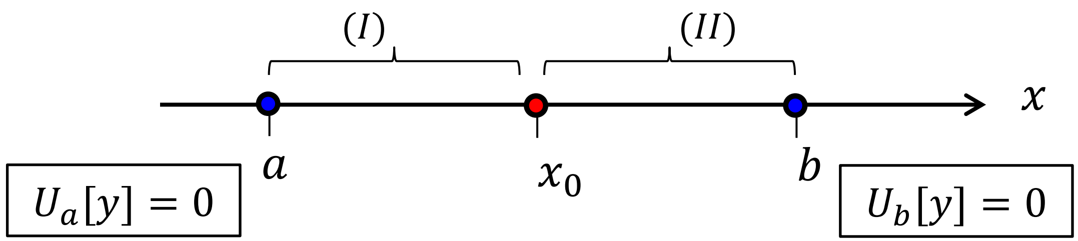
*Then there **exist** an **unique** Green's function (x, x_0) that is associated with the problem.*

Proof: Provided by the ***construction*** of Green's function (in the following)

(III) Construction of Green's function

- Green's function satisfies:

$$\boxed{L[G] = \frac{d^2G(x,x_0)}{dx^2} + p(x)\cdot\frac{dG(x,x_0)}{dx} + q(x)\cdot G(x,x_0) = \delta(x-x_0)}$$

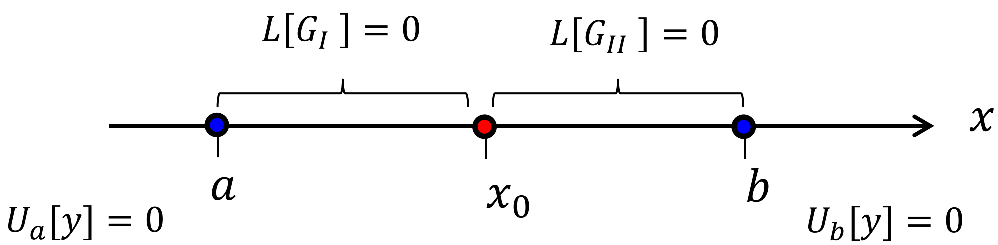


- Recall that $\delta(x-x_0)$ *vanishes* for $x \neq x_0$.

- Thus, for $x \neq x_0$ the equation reduces into:

$$L[G] = \frac{d^2G(x,x_0)}{dx^2} + p(x)\cdot\frac{dG(x,x_0)}{dx} + q(x)\cdot G(x,x_0) = 0$$

The solutions are φ_1 and φ_2 $\dashrightarrow$ $L[\varphi_1(x)] = 0 \qquad L[\varphi_2(x)] = 0$



- Decomposing the solution into the two ***sub-regions*** - in which $L[G] = 0$:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) & (a \le x < x_0) \\ G_{II}(x, x_0) & (x_0 < x \le b) \end{cases}$$

Sub-region (I)

Sub-region (II)

- Where:

$$G_I(x, x_0) = A_1(x_0) \cdot \varphi_1(x) + A_2(x_0) \cdot \varphi_2(x) \qquad (a \le x < x_0)$$

$$G_{II}(x, x_0) = B_1(x_0) \cdot \varphi_1(x) + B_2(x_0) \cdot \varphi_2(x) \qquad (x_0 < x \le b)$$

- We need to construct ***connection conditions*** at $x = x_0$ for the two solutions.

- We seek for $G(x, x_0)$ with the "***weakest singularity***" that satisfies:

$\boxed{G(x,x_0)|_{x=x_0} \rightarrow \textit{Continues}}$

$$\frac{d^2G(x,x_0)}{dx^2} + p(x)\cdot\frac{dG(x,x_0)}{dx} + q(x)\cdot G(x,x_0) = \delta(x-x_0)$$

$$\Big\downarrow \quad \text{Integrating in the vicinity of } x_0\ (n\rightarrow 0)$$

$$\int_{x_0-n}^{x_0+n}\left(\frac{d^2G(x,x_0)}{dx^2} + p(x)\cdot\frac{dG(x,x_0)}{dx} + q(x)\cdot G(x,x_0)\right)dx = 1$$

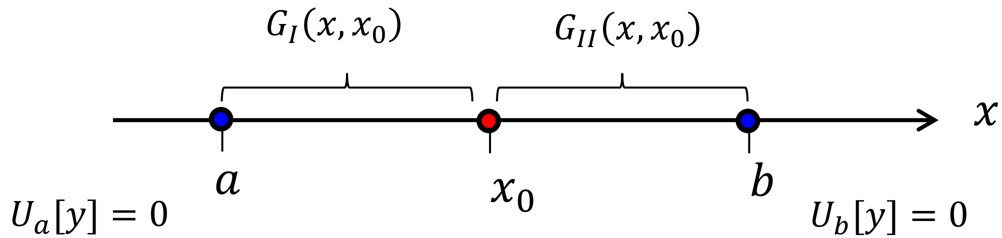
$$\Big\downarrow \quad p(x) \text{ and } q(x) \text{ are continues}$$

$n \rightarrow 0$
$G(x,x_0)$ *is continues*

$$\left[\frac{dG(x_0+n,x_0)}{dx} - \frac{dG(x_0+n,x_0)}{dx}\right] + \quad \cancel{p(x_0)\cdot[G(x_0+n,x_0) - G(x_0-n,x_0)]} +$$

$$+[q(x_0)-p'(x_0)]\cdot\cancel{\int_{x_0-n}^{x_0+n}G(x,x_0)\,dx} = 1$$

$\boxed{\frac{dG(x,x_0)}{dx}\Big|_{x=x_0} \rightarrow \textit{Jumps by unity}}$



- We obtained to ***two connection*** conditions !

$$\boxed{G_I(x, x_0)\Big|_{x=x_0} = G_{II}(x, x_0)\Big|_{x=x_0}}$$

$G(x, x_0) \rightarrow$ *Continues*

$$\boxed{\frac{dG_{II}(x, x_0)}{dx}\Bigg|_{x=x_0} - \frac{dG_I(x, x_0)}{dx}\Bigg|_{x=x_0} = 1}$$

$\frac{dG(x,x_0)}{dx} \rightarrow$ *Jumps by unity*

- Substituting $G_I(x, x_0)$ and $G_{II}(x, x_0)$ yields:

$$(B_1 - A_1) \cdot \varphi_1(x_0) + (B_2 - A_2) \cdot \varphi_2(x_0) = 0$$

$$(B_1 - A_1) \cdot \varphi'_1(x_0) + (B_2 - A_2) \cdot \varphi'_2(x_0) = 1$$

- Using the matrix form:

$$\underbrace{\begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi'_1 & \varphi'_2 \end{pmatrix}}_{\color{blue}{Wronskian}} \cdot \begin{pmatrix} B_1 - A_1 \\ B_2 - A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Since the ***associated homogeneous*** problem has only a ***trivial solution*** (*we focus on this case only*):

$$W(\varphi_1, \varphi_2, x_0) = \varphi_1 \cdot \varphi'_2 - \varphi'_1 \cdot \varphi_2 \neq 0$$

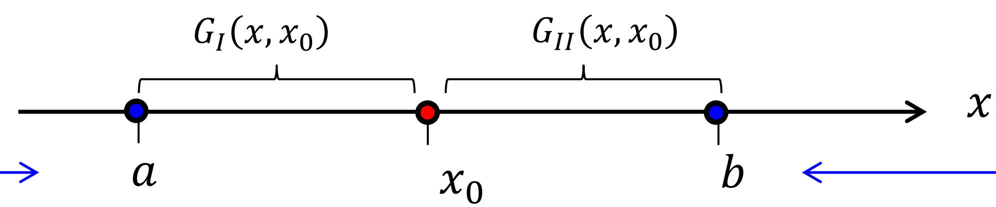
- Thus, the coefficients are ***unequally determined*** via:

$$\boxed{\begin{aligned} B_1 - A_1 &= -\frac{\varphi_2(x_0)}{W(\varphi_1, \varphi_2, x_0)} \\ B_2 - A_2 &= \frac{\varphi_1(x_0)}{W(\varphi_1, \varphi_2, x_0)} \end{aligned}}$$

Note that: $G(x, x_0)$ *is determined by the solutions of the* ***homogenous system***

Summary – conditions for Green's function

$$G(x,x_0) = \begin{cases} G_I(x,x_0) = A_1(x_0)\cdot\varphi_1(x) + A_2(x_0)\cdot\varphi_2(x) & (a \le x < x_0) \\ G_{II}(x,x_0) = B_1(x_0)\cdot\varphi_1(x) + B_2(x_0)\cdot\varphi_2(x) & (x_0 < x \le b) \end{cases}$$



- ***Two conditions*** on $G(x,x_0)$ continuity and unit-jump derivative provide:

$$\boxed{G_I(x,x_0)\Big|_{x=x_0} = G_{II}(x,x_0)\Big|_{x=x_0}} \qquad \boxed{\frac{dG_{II}(x,x_0)}{dx}\Big|_{x=x_0} - \frac{dG_I(x,x_0)}{dx}\Big|_{x=x_0} = 1}$$

- The ***two additional conditions*** are provided by the BC on $G(x,x_0)$:

$$\boxed{U_a[G_I(a,x_0)] = 0} \qquad \boxed{U_b[G_{II}(b,x_0)] = 0}$$

(IV) Methodology: construction of Green's function

$$L[y] = f(x)$$

$$U_a[y] = 0 \qquad U_b[y] = 0$$

$$\downarrow$$

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

Step I: Find the general solutions of the homogeneous system

$$L[\varphi_1(x)] = 0 \qquad L[\varphi_2(x)] = 0$$

Step II: Formulate Green's function via:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = A_1(x_0) \cdot \varphi_1(x) + A_2(x_0) \cdot \varphi_2(x) & (a \le x < x_0) \\ G_{II}(x, x_0) = B_1(x_0) \cdot \varphi_1(x) + B_2(x_0) \cdot \varphi_2(x) & (x_0 < x \le b) \end{cases}$$

Step III: Find the coefficients by the connection and boundary conditions:

$$G_I(x, x_0)\Big|_{x=x_0} = G_{II}(x, x_0)\Big|_{x=x_0}$$

$$U_a[G_I(a, x_0)] = 0$$

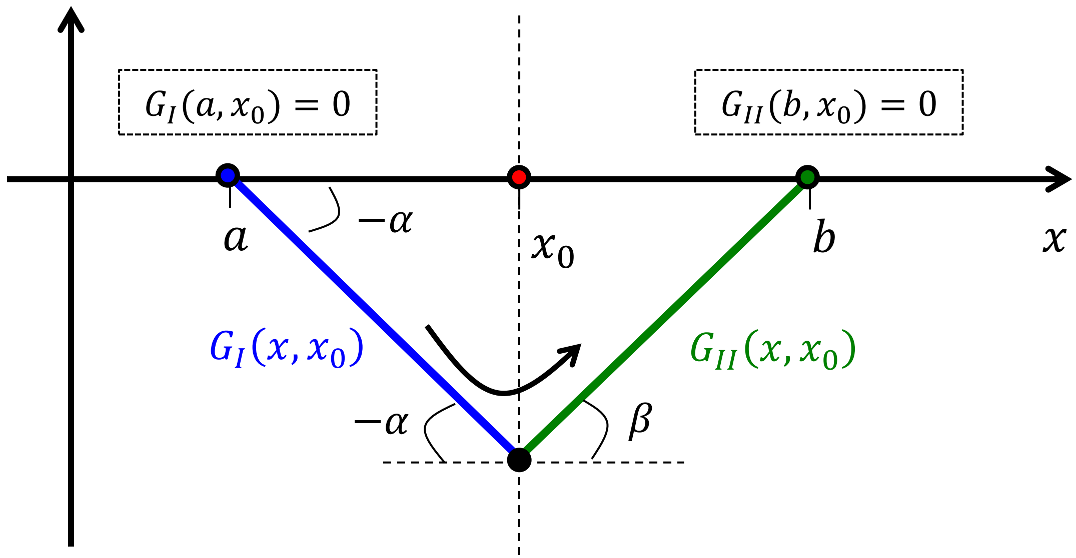
$$\frac{dG_{II}(x, x_0)}{dx}\Big|_{x=x_0} = \frac{dG_I(x, x_0)}{dx}\Big|_{x=x_0} + 1$$

$$U_b[G_{II}(b, x_0)] = 0$$

Qualitative geometrical interpretation of $G(x, x_0)$

A specific example – linear form & zero edge conditions

$$G(x,x_0) = \begin{cases} G_I(x,x_0) & (a \le x \le x_0) \\ G_{II}(x,x_0) & (x_0 \le x \le b) \end{cases}$$



α	β
0^o	45^o
-10^o	$\sim 40^o$
-30^o	$\sim 23^o$
-45^o	0^o

$$G_I(x,x_0)\Big|_{x=x_0} = G_{II}(x,x_0)\Big|_{x=x_0}$$

Non-linear relation between α and β

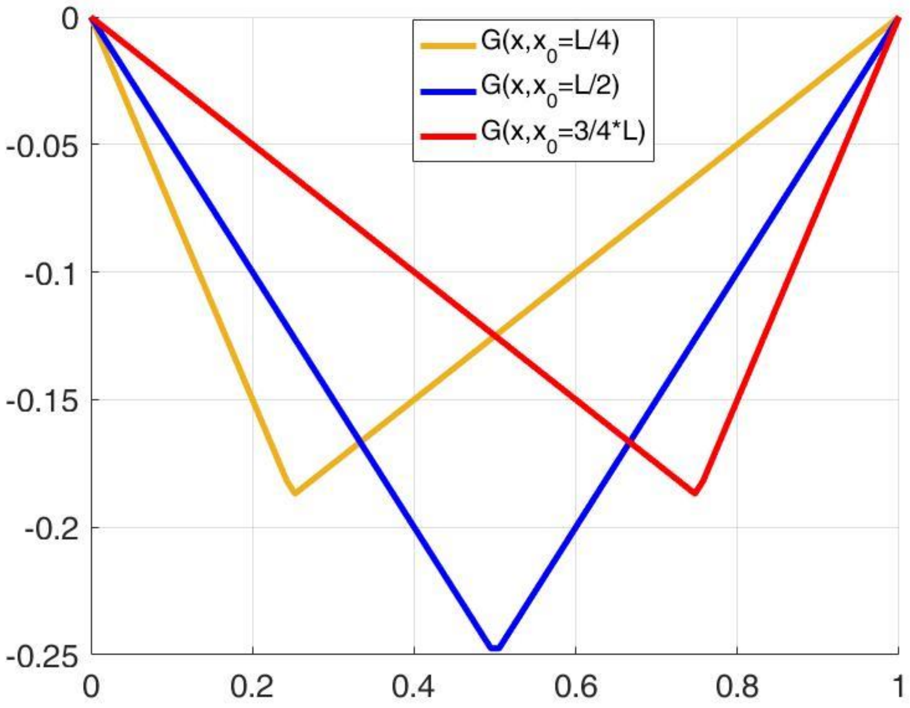
$$\frac{dG_{II}(x,x_0)}{dx}\Big|_{x=x_0} = \frac{dG_I(x,x_0)}{dx}\Big|_{x=x_0} + 1 \longrightarrow tg(\beta) = tg(\alpha) + 1$$

Example 1

$$L[y] = y'' = f(x)$$

$$y(0) = y(L) = 0$$

$$G(x,x_0) = \begin{cases} G_I(x,x_0) = \left(\dfrac{x_0}{L} - 1\right) \cdot x & (0 \le x < x_0) \\ G_{II}(x,x_0) = \left(\dfrac{x}{L} - 1\right) \cdot x_0 & (x_0 < x \le L) \end{cases}$$



Example 1B (At home..)

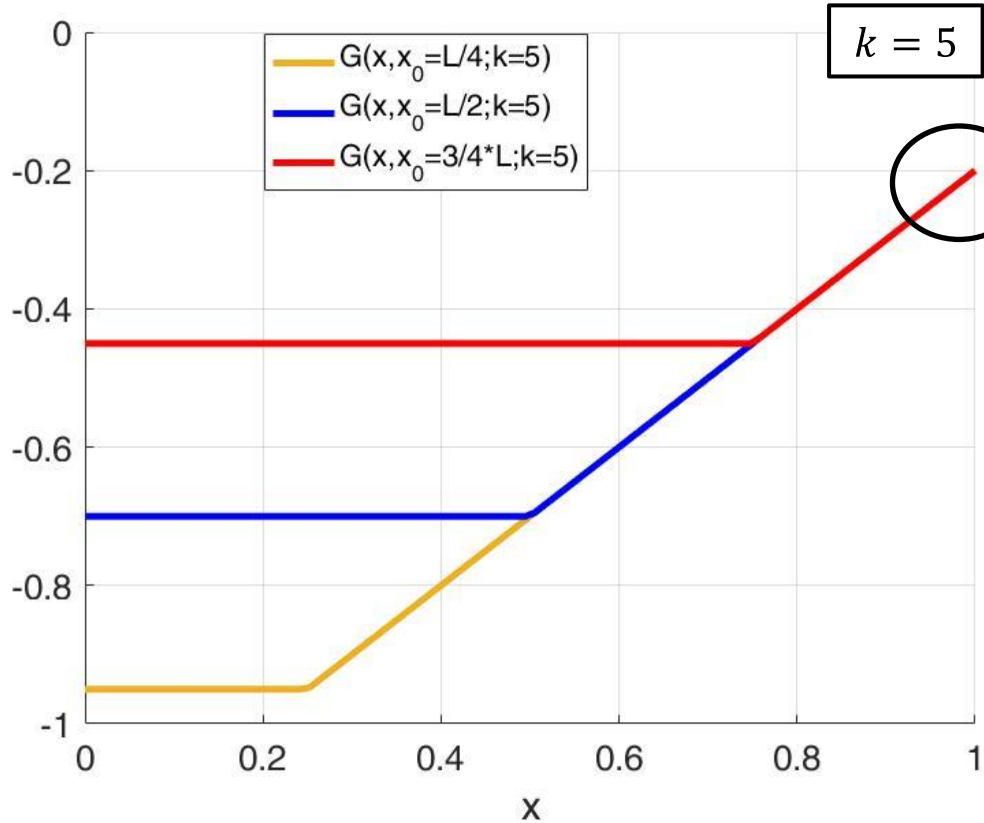
$$\boxed{\begin{array}{c} L[y] = y'' = f(x) \\ \left.\frac{dy}{dx}\right|_{x=0} = 0 \\ \left.\left(k \cdot y + \frac{dy}{dx}\right)\right|_{x=L} = 0 \end{array}}$$

$$\frac{dy}{dx} \propto Force/stress$$

$$k \propto Elastic\ "spring"$$

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = (x_0 - L) - \frac{1}{k} & (0 \leq x < x_0) \\ G_{II}(x, x_0) = (x - L) - \frac{1}{k} & (x_0 < x \leq L) \end{cases}$$

$$\boxed{k = 5}$$



Example 1B (At home..)

$$L[y] = y'' = f(x)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0$$

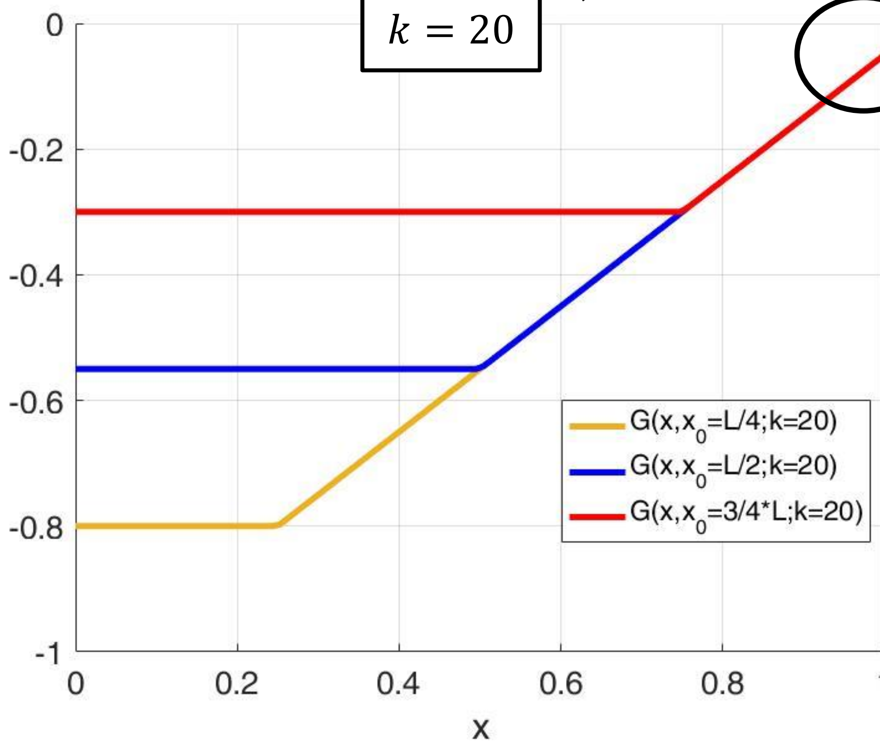
$$\left.\left(k \cdot y + \frac{dy}{dx}\right)\right|_{x=L} = 0$$

$\frac{dy}{dx} \propto Force/stress$

$k \propto Elastic\ "spring"$

$$G(x,x_0) = \begin{cases} G_I(x,x_0) = (x_0 - L) - \frac{1}{k} & (0 \le x < x_0) \\ G_{II}(x,x_0) = (x - L) - \frac{1}{k} & (x_0 < x \le L) \end{cases}$$

$k = 20$



Example 2 – Periodic B.C.

$$L[y] = y'' - k^2 \cdot y = f(x)$$
$$(\, k > 0)$$

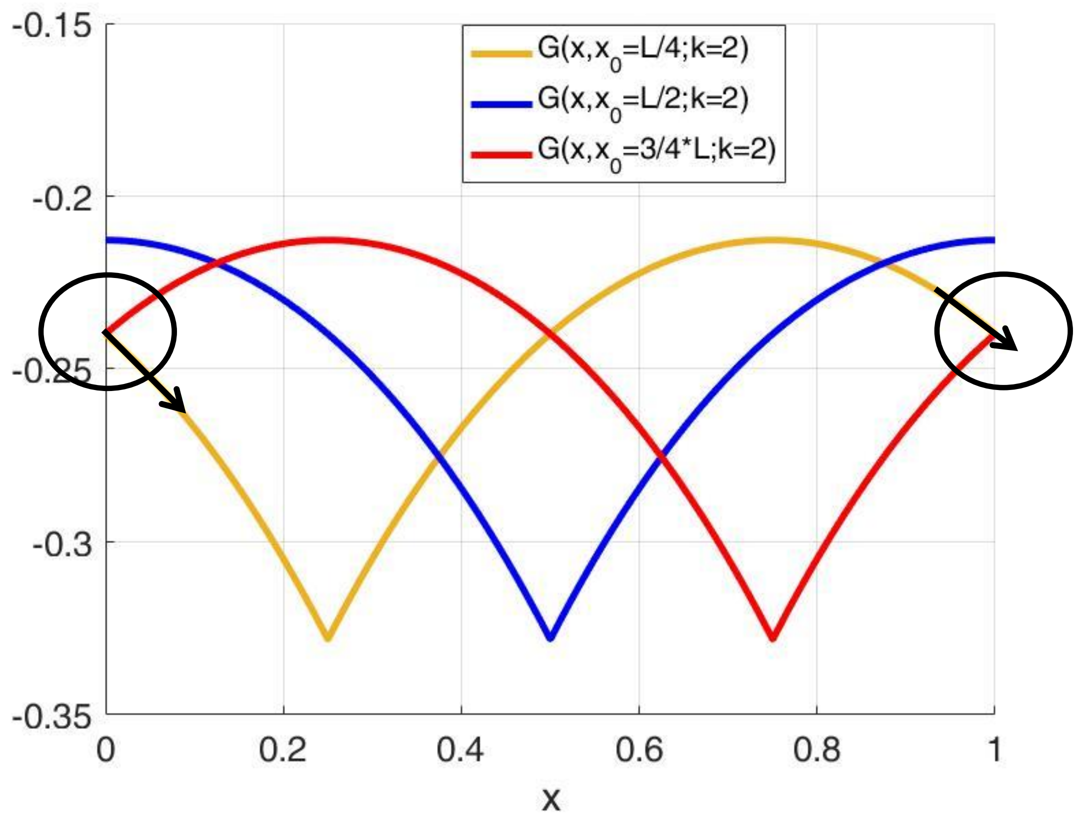
$$y(0) = y(L)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{dy}{dx}\right|_{x=L}$$

$$G_I(x,x_0) = A \cdot \left[\exp\big(k \cdot (L + x - x_0)\big) + \exp\big(k \cdot (x_0 - x)\big)\right]$$

$$G_{II}(x,x_0) = A \cdot \left[\exp\big(k \cdot (L + x_0 - x)\big) + \exp\big(k \cdot (x_0 - x)\big)\right]$$

$$A = \frac{1}{2 \cdot k} \cdot \frac{1}{1 - \exp(k \cdot L)}$$



Periodic B.C. (!)

(V) Finding the solution & Integration over Green's function

$$L[y] = f(x)$$

$$U_a[y] = 0 \qquad U_b[y] = 0$$

- Assuming that we found Green's function for a given boundary-value problem:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) & (a \le x < x_0) \\ G_{II}(x, x_0) & (x_0 < x \le b) \end{cases}$$

x → running coordinate (solution)

x_0 → point-load location (source)

- The solution is given by the integration on x_0 (not on x…):

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

How ??
(Conceptually strange…)

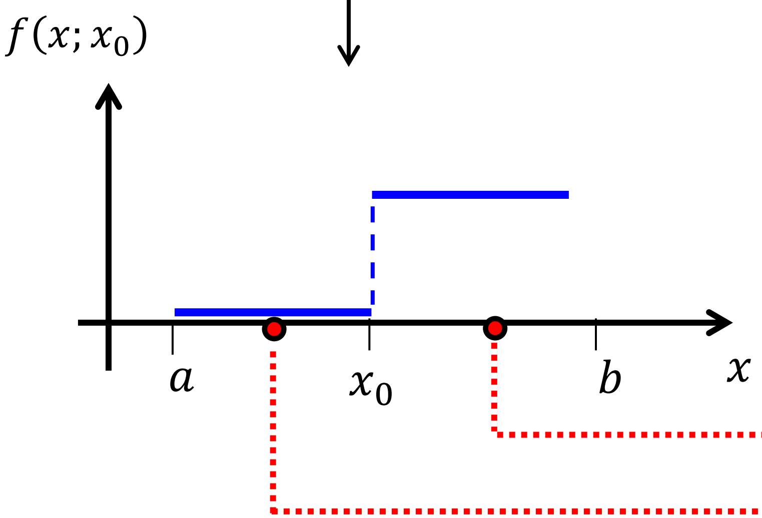
A rapid perspective change is required !

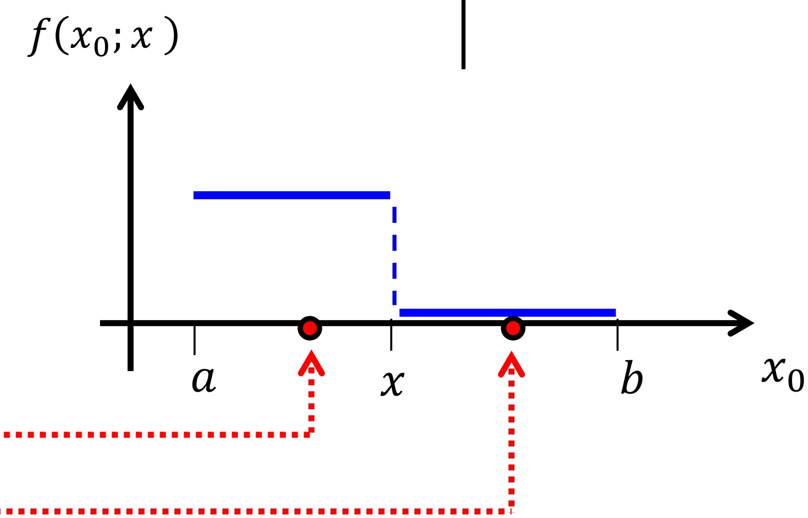
An instructive example:

- Consider the following stepwise function (not a Green's function..):

$$f(x,x_0) = \begin{cases} 0 & (a \le x < x_0) \\ 1 & (x_0 < x \le b) \end{cases}$$

$$f(x,x_0) = \begin{cases} 1 & (a \le x_0 < x) \\ 0 & (x < x_0 \le b) \end{cases}$$





$$\boxed{F(x) = \int_a^b f(x,x_0)dx_0 \;\; = \int_a^x 1 \cdot dx_0 = x - a}$$

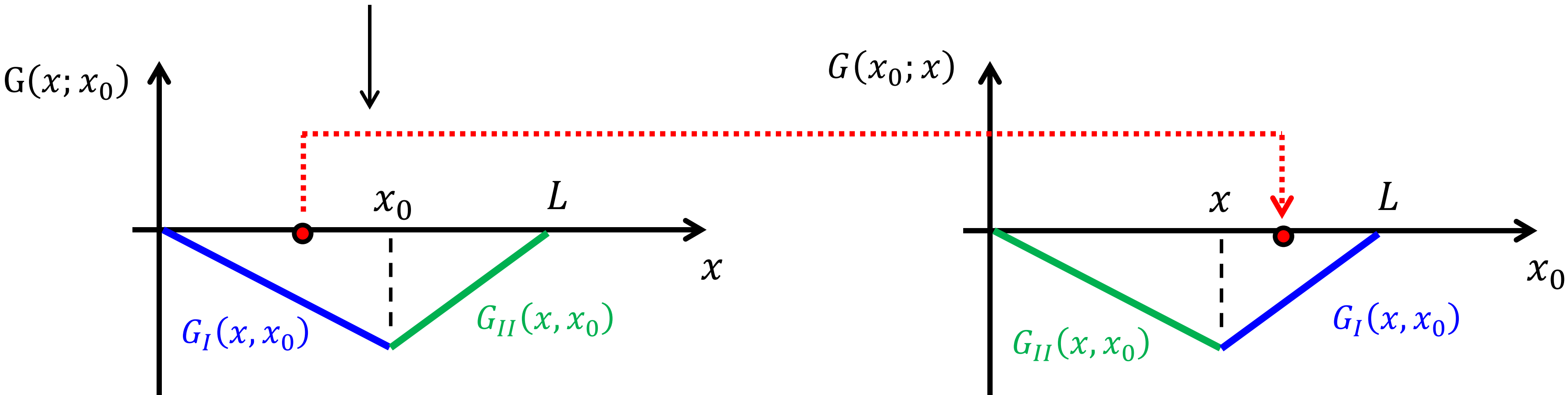
Another example (from exercise 1):

$$L[y] = y'' = f(x)$$

$$y(0) = y(L) = 0$$

➢ Consider the following Green's function (exercise 1):

$$G(x,x_0) = \begin{cases} G_I(x,x_0) = \left(\dfrac{x_0}{L} - 1\right)\cdot x & (0 \le x < x_0) \\[2ex] G_{II}(x,x_0) = \left(\dfrac{x}{L} - 1\right)\cdot x_0 & (x_0 < x \le L) \end{cases}$$



$$G(x,x_0) = \begin{cases} G_{II}(x,x_0) = \left(\dfrac{x}{L} - 1\right)\cdot x_0 & (0 \le x_0 < x) \\[2ex] G_I(x,x_0) = \left(\dfrac{x_0}{L} - 1\right)\cdot x & (x < x_0 \le L) \end{cases}$$

$$G(x,x_0)=\begin{cases} G_{II}(x,x_0)=\left(\frac{x}{L}-1\right)\cdot x_0 & (0\le x_0<x) \\ G_I(x,x_0)=\left(\frac{x_0}{L}-1\right)\cdot x & (x<x_0\le L) \end{cases}$$

$$\boxed{\begin{array}{c} L[y]=y''=f(x) \\ y(0)=y(L)=0 \end{array}}$$

$$\downarrow$$

$$\boxed{\varphi(x)=\int_a^b G(x,x_0)\cdot f(x_0)dx_0}$$

$$\downarrow$$

$$\varphi(x)=\int_0^L G(x,x_0)\cdot f(x_0)dx_0=$$

$$=\int_0^x G_{II}(x,x_0)\cdot f(x_0)dx_0+\int_x^L G_I(x,x_0)\cdot f(x_0)dx_0$$

$$=\left(\frac{x}{L}-1\right)\cdot\int_0^x x_0\cdot f(x_0)dx_0+x\cdot\int_x^L\left(\frac{x_0}{L}-1\right)\cdot f(x_0)dx_0$$

*A **general** solution is obtained for **any** $f(x)$!!*

Example: Uniform unit-load along $0 \leq x \leq L$

$$\boxed{f(x) = 1}$$

$$\varphi(x) = \left(\frac{x}{L} - 1\right) \cdot \int_0^x x_0 \cdot dx_0 + x \cdot \int_x^L \left(\frac{x_0}{L} - 1\right) \cdot dx_0$$

$$\downarrow$$

$$\boxed{\varphi(x) = \frac{1}{2} \cdot x \cdot (x - L)}$$

Example: Linear load along $0 \leq x \leq L$

$$\boxed{f(x) = x}$$

$$\varphi(x) = \left(\frac{x}{L} - 1\right) \cdot \int_0^x x_0^2 \cdot dx_0 + x \cdot \int_x^L \left(\frac{x_0}{L} - 1\right) \cdot x_0 \cdot dx_0$$

$$\downarrow$$

$$\boxed{\varphi(x) = \frac{1}{6} \cdot x \cdot (x^2 - L^2)}$$