Part D – Eigenfunction expansions

<u>**Series expansion**</u>

- A ***square integrable*** function $f(x)$ in $a \leq x \leq b$ with a ***weight function*** $s(x)$ satisfies:

$$\int_a^b f(x) \cdot s(x) \cdot f(x) dx < \infty$$

Generally $[a, b]$ can be $(-\infty, \infty)$

- Meaning - the function $f(x)$ has a ***finite norm***:

$$\|f(x)\| = \sqrt{\langle f|s|f \rangle} < \infty$$

- Any arbitrary such $f(x)$ can be ***expanded*** in a set of ***orthonormal*** functions $\varphi_n(x)$:

$$\boxed{f(x) \sim \sum_{n=1}^{N} a_n \cdot \varphi_n(x)}$$

An approximation…

*What selection of a_n will provide the "**best**" approximation for $f(x)$?*

- The "***best***" selection ***can*** be defined by ***minimizing*** the mean-square error:

$$E(a_1, a_2, \dots, a_N) = \int_a^b \left[f(x) - \sum_{n=1}^{N} a_n \cdot \varphi_n(x) \right]^2 \cdot s(x)dx$$

$$\frac{\partial E}{\partial a_1} = 0 \qquad \frac{\partial E}{\partial a_2} = 0 \quad \cdots\cdots\cdots \quad \frac{\partial E}{\partial a_N} = 0 \qquad \longrightarrow \qquad \boxed{\frac{\partial E}{\partial a_k} = 0} \qquad (k = 1,2, \dots . N)$$

- An example for the derivation:

$$\frac{\partial E}{\partial a_1} = \frac{\partial}{\partial a_1} \int_a^b \left[f(x) - \sum_{n=1}^{N} a_n \cdot \varphi_n(x) \right]^2 \cdot s(x)dx \quad = \int_a^b \left\{ \frac{\partial}{\partial a_1} \left[f - \sum_{n=1}^{N} a_n \cdot \varphi_n \right]^2 \right\} \cdot s(x)dx$$

$$= \int_a^b \left\{ \frac{\partial}{\partial a_1} \left[\cancel{f^2} - 2 \cdot f \cdot \sum_{n=1}^{N} a_n \cdot \varphi_n + \left(\sum_{n=1}^{N} a_n \cdot \varphi_n \right)^2 \right] \right\} \cdot s(x)dx$$

$$= -2 \cdot \int_a^b [f \cdot s \cdot \varphi_1]dx \quad +2 \cdot a_1 \cdot \int_a^b [\varphi_1 \cdot s \cdot \varphi_1]dx$$

- The "***best***" approximation is obtained if the derivative for ***all*** a_k coefficients satisfies:

$$\frac{\partial E}{\partial a_k} = \frac{\partial}{\partial a_k} \int_a^b \left[f(x) - \sum_{n=1}^{N} a_n \cdot \varphi_n(x) \right]^2 \cdot s(x) dx$$

$$= -2 \cdot \int_a^b [f \cdot s \cdot \varphi_k] dx \quad +2 \cdot a_k \cdot \int_a^b [\varphi_k \cdot s \cdot \varphi_k] dx \quad = 0$$

$$\longrightarrow \quad a_k = \frac{\int_a^b f(x) \cdot s(x) \cdot \varphi_k(x) dx}{\int_a^b \varphi_k(x) \cdot s(x) \cdot \varphi_k(x) dx}$$

The "best" selection for the a_k coefficient


Josef Fourier

Fourier coefficients

$$\boxed{a_k = \frac{\langle f | s | \varphi_k \rangle}{\langle \varphi_k | s | \varphi_k \rangle}}$$

The "best" approximation

Fourier series

$$\boxed{f(x) \sim \sum_{n=1}^{N} a_n \cdot \varphi_n(x)}$$

Some well-known (and useful) series expansions


Josef Fourier

(1) The "customary" Fourier series (periodic functions)

- The function set:

$$\phi(x) = 1 \qquad \varphi_n(x) = \sin\left(\frac{\pi n x}{L}\right) \qquad \psi_n(x) = \cos\left(\frac{\pi n x}{L}\right)$$

- The inner product:

$$\langle g|1|h\rangle = \langle g|h\rangle = \int_{-L}^{L} g(x) \cdot h(x)\, dx$$

- The orthogonal relations:

$$\langle \phi|\varphi_n\rangle = 0 \qquad \langle \phi|\psi_n\rangle = 0 \qquad \langle \varphi_n|\psi_m\rangle = 0$$

$$\langle \varphi_n|\varphi_m\rangle = L \cdot \delta_{nm} \qquad \langle \psi_n|\psi_m\rangle = L \cdot \delta_{nm} \qquad \langle \phi|\phi\rangle = 2 \cdot L$$

- The ***Fourier*** series expansion:

$$\boxed{f(x) \sim b_o + \sum_{n=1}^{N} \left[a_n \cdot \sin\left(\frac{\pi n x}{L}\right) + b_n \cdot \cos\left(\frac{\pi n x}{L}\right) \right]}$$

- The coefficients:

$$\boxed{a_n = \frac{\langle f | \varphi_n \rangle}{\langle \varphi_n | \varphi_n \rangle} = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot \sin\left(\frac{\pi n x}{L}\right) dx}$$

$$\boxed{b_n = \frac{\langle f | \psi_n \rangle}{\langle \psi_n | \psi_n \rangle} = \frac{1}{L} \cdot \int_{-L}^{L} f(x) \cdot \cos\left(\frac{\pi n x}{L}\right) dx}$$

$$\boxed{b_0 = \frac{\langle f | \phi \rangle}{\langle \phi | \phi \rangle} = \frac{1}{2 \cdot L} \cdot \int_{-L}^{L} f(x) dx}$$

(2) The Fourier-Legendre series (spherical problems)

Josef Fourier

Adrien-Marie Legendre

- The function set:

$$\varphi_l(x) = P_l(x)$$

The Legendre polynomials (first six):

$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2} \cdot (3x^2 - 1) \qquad P_3(x) = \frac{1}{2} \cdot (5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8} \cdot (35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8} \cdot (63x^5 - 70x^3 + 15x\ )$$

- The inner product and orthogonal relations:

$$\langle P_l|1|P_n\rangle = \langle P_l|P_n\rangle \quad = \int\limits_{-1}^{1} P_l(x)\cdot P_n(x)dx \quad = \frac{2}{2l+1}\delta_{ln}$$

- The **Fourier-Legendre** series expansion:

$$\boxed{f(x) \sim \sum_{l=1}^{N} A_l \cdot P_l(x)}$$

Fourier-Legendre series

- The coefficients:

$$\boxed{A_l = \frac{\langle f|P_l\rangle}{\langle P_l|P_l\rangle} = \frac{2\cdot l+1}{2}\cdot \int_{-1}^{1} f(x)\cdot P_l(x)\, dx}$$

(3) The Fourier-Bessel series (cylindrical problems)

Josef Fourier

Friedrich Wilhelm Bessel

- The function set:

$$\varphi_n(x) = J_\nu(\chi_{\nu n} \cdot x/a)$$

$$\boxed{\varphi_n(x = a) = J_\nu(\chi_{\nu n}) = 0}$$

- The inner product:

$$\langle J_{\nu;n} | x | J_{\nu;m} \rangle = \int_0^a x \cdot J_\nu(\chi_{\nu n} \cdot x/a) \cdot J_\nu(\chi_{\nu m} \cdot x/a)$$

- The orthogonal relations:

$$\langle J_{\nu;n} | x | J_{\nu;m} \rangle = \frac{a^2}{2} \cdot (J_{\nu+1}(\chi_{\nu n}))^2 \cdot \delta_{mn}$$

- The ***Fourier-Bessel*** series expansion:

$$\boxed{f(x) \sim \sum_{n=1}^{N} A_n \cdot J_\nu(\chi_{\nu n} \cdot x/a)}$$

Fourier-Bessel series

- The coefficients:

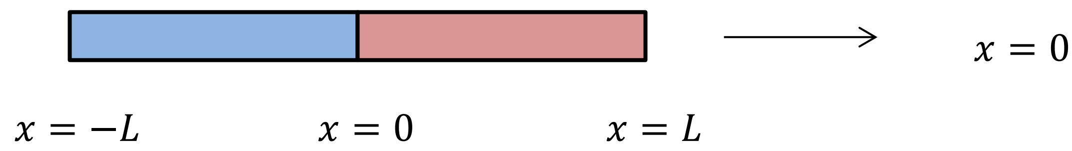
$$A_n = \frac{\langle f | x | J_{\nu;n} \rangle}{\langle J_{\nu;n} | x | J_{\nu;n} \rangle}$$

$$\boxed{A_n = \frac{2}{a^2 \cdot (J_{\nu+1}(\chi_{\nu n}))^2} \cdot \int_0^a x \cdot f(x) \cdot J_\nu(\chi_{\nu n} \cdot x/a)\, dx}$$

Example 1 – Temperature along a rod

- Consider a rod of length $2 \cdot L$ $(L = 1)$, subjected to a temperature field:

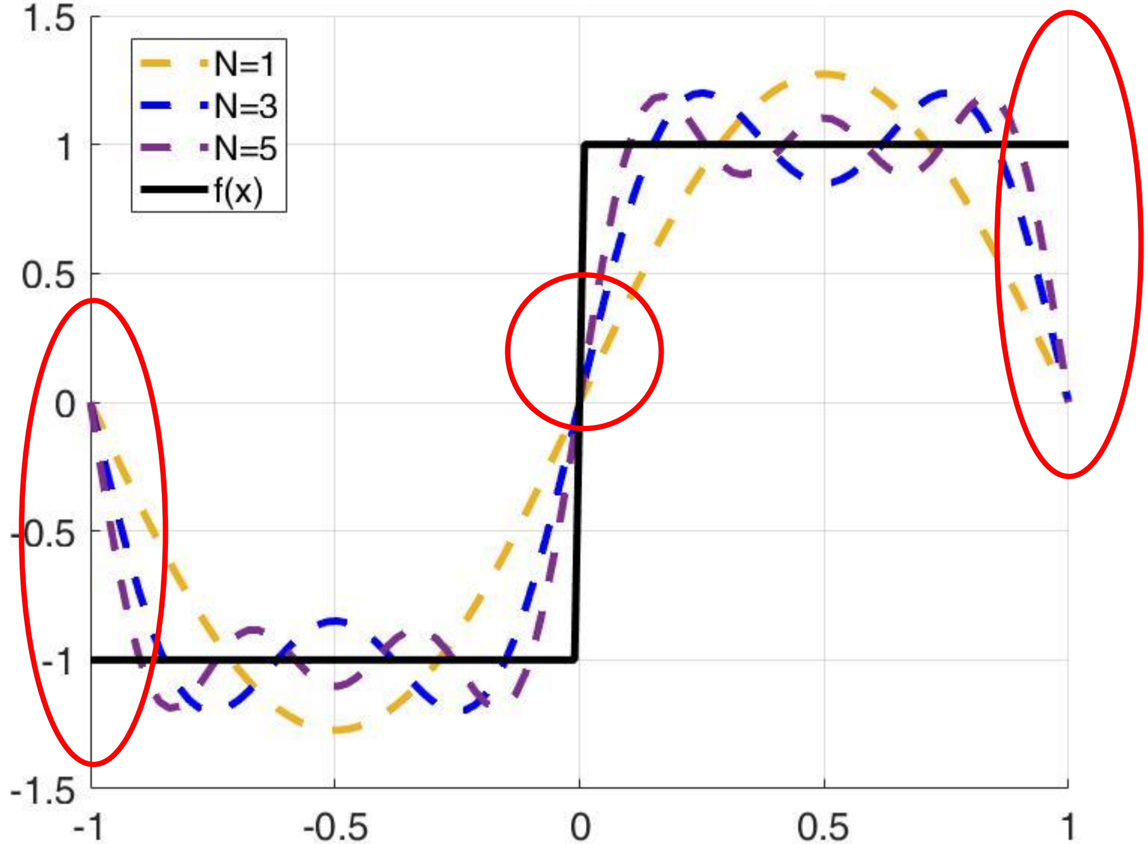
$$f(x) = \begin{cases} -1 & (-L < x < 0) \\ 1 & (0 < x < L) \end{cases}$$



Find the ***Fourier series*** approximation (5 first terms) for $f(x)$ using the eigenfunction set:

$$\left\{1, \sin\left(\frac{\pi n x}{L}\right), \cos\left(\frac{\pi n x}{L}\right)\right\}$$

$$f(x) \sim \frac{4}{\pi} \cdot \sin\left(\frac{\pi x}{L}\right) + \frac{4}{3\pi} \cdot \sin\left(\frac{3\pi x}{L}\right) + \frac{4}{5\pi} \cdot \sin\left(\frac{5\pi x}{L}\right)$$



?

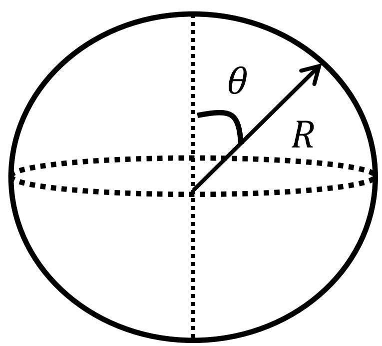
Example 2 – Temperature on a spherical shell

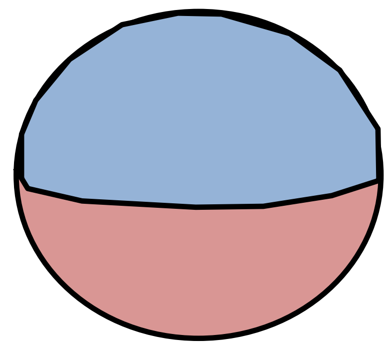
Fourier-Legendre series

- Consider spherical shell of radius R, subjected to a temperature field:

$$f(\theta) = \begin{cases} -1 & (0 < \theta < 90^0) \\ 1 & (90^0 < x < 180^0) \end{cases} \xrightarrow{\quad\quad} f(x) = \begin{cases} -1 & (-1 \leq x < 0) \\ 1 & (0 < x \leq 1) \end{cases}$$

$$\boxed{x = \cos(\theta)}$$





Find the ***Fourier-Legendre series*** approximation (5 first terms) for $f(x)$ using the eigenfunction set of *Legendre polynomials*:

$$\{P_l(x)\}$$

The Legendre polynomials:

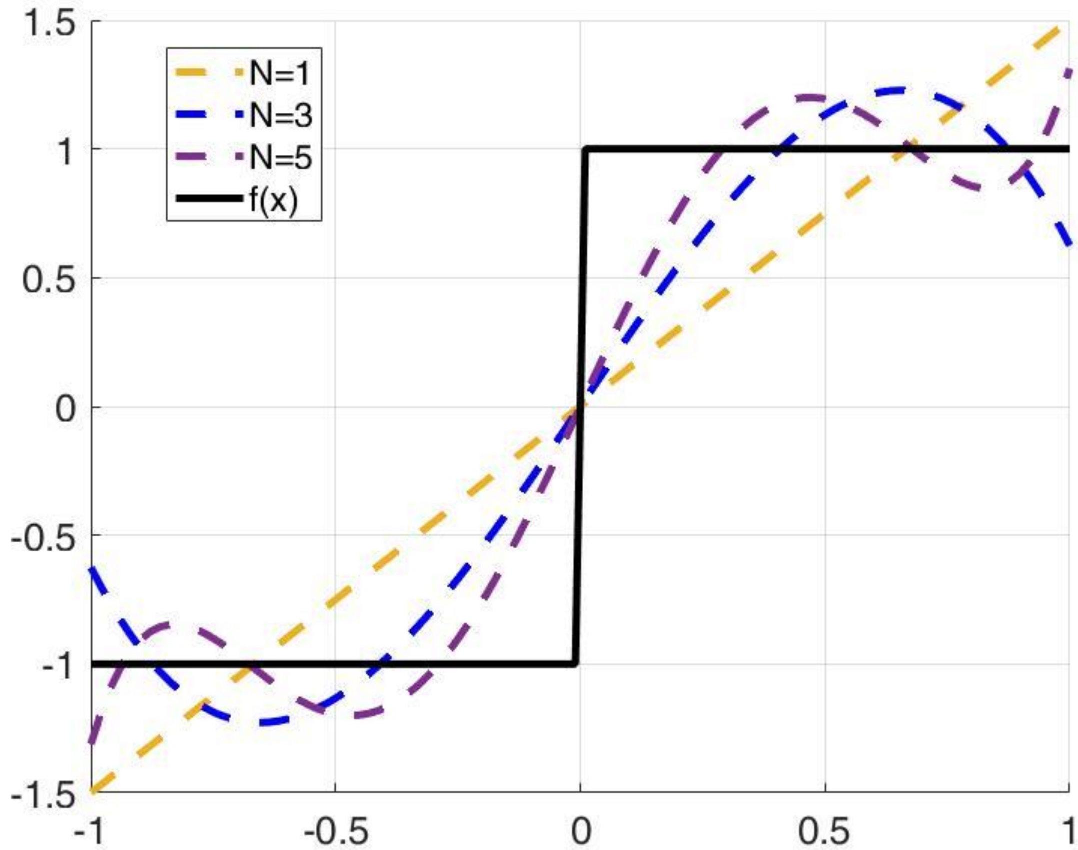
$$P_0(x) = 1 \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2} \cdot (3x^2 - 1) \qquad P_3(x) = \frac{1}{2} \cdot (5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8} \cdot (35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8} \cdot (63x^5 - 70x^3 + 15x\ )$$

$$\boxed{\langle P_n | P_m \rangle \int_{-1}^{1} P_n(x) \cdot P_m(x) dx = \frac{2}{2n+1} \delta_{mn}}$$

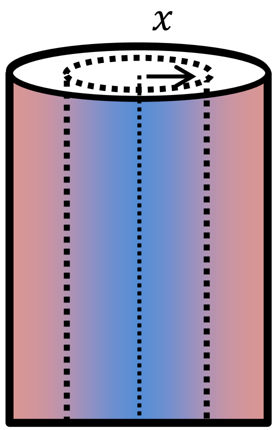
$$\boxed{f(x) \sim \frac{3}{2} \cdot p_0(x) - \frac{7}{8} \cdot p_3(x) + \frac{11}{16} \cdot p_5(x)}$$



Example 3 – Temperature within a cylinder

Fourier-Bessel series

- Consider a cylinder of radius $a = 1$, subjected to a ***radial*** ***linear*** temperature field:



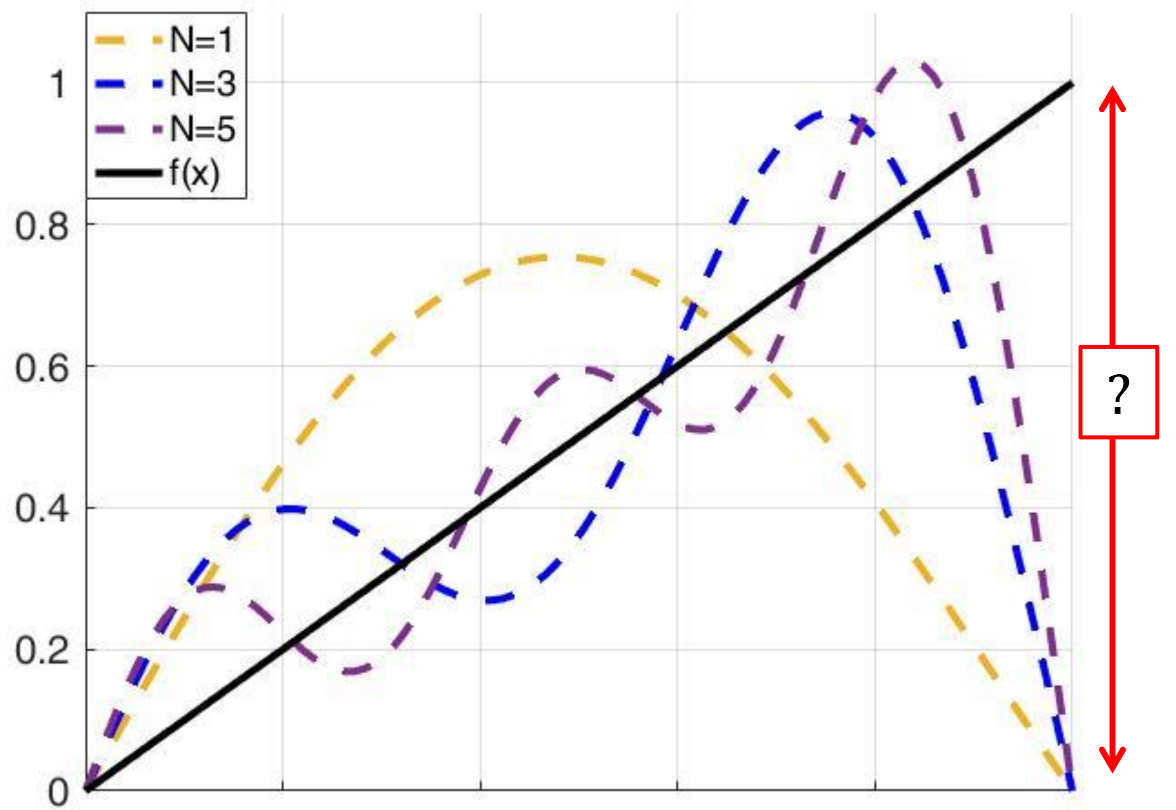
$$f(x) = x$$

Find the ***Fourier-Bessel series*** approximation (5 first terms) for $f(x)$ using the eigenfunction set of *Bessel-zeroes functions of an order* $\nu = 1$:

$$\{J_1(\chi_{1n} \cdot x)\}$$

$$f(x) \sim \sum_{n=1}^{N} A_n \cdot J_1(\chi_{1n} \cdot x) \qquad A_n = \frac{2}{J_2(\chi_{1n}) \cdot \chi_{1n}}$$

$$\boxed{\begin{gathered} f(x) \sim 1.3 \cdot J_1(\chi_{11} \cdot x) - 0.95 \cdot J_1(\chi_{12} \cdot x) + 0.79 \cdot J_1(\chi_{13} \cdot x) \\ -0.69 \cdot J_1(\chi_{14} \cdot x) + 0.62 \cdot J_1(\chi_{15} \cdot x) \end{gathered}}$$



<u>Part E</u> – *Eigenfunction expansions*

(some deeper insights)

(I) Convergence and completeness

- The ***approximation*** of $f(x)$ via a ***finite*** (N-terms) expansion of ***orthonormal*** set:

$$f(x) \sim \sum_{n=1}^{N} a_n \cdot \varphi_n(x) \qquad\qquad a_n = \langle f|s|\varphi_n\rangle$$

> ***Intuitively:*** if the number of terms (N) is taken *larger* and *larger*, we expect to get a "*better*" and "*better*" approximation to $f(x)$

- Our ***intuition*** is correct if the set of functions $\varphi_n(x)$ is ***complete***.

> *Which orthonormal function sets are complete?*

- *Endless* efforts of *generations* of mathematicians – let into a ***precious conclusion***...

"***All*** orthonormal sets of functions normally occurring in mathematical physics have been ***proven*** to be ***complete***"

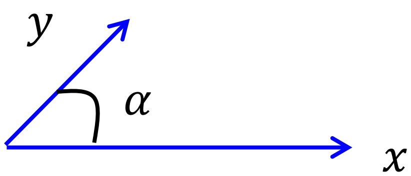
(K. D. Jackson, Classical electrodynamics, 3rd edition, p. 68)

- This course ***focuses*** on solutions of ***physical problems***.

*All the eigenfunction sets that we will handle in this course will be **orthogonal** (**orthonormal**) and **complete** !*

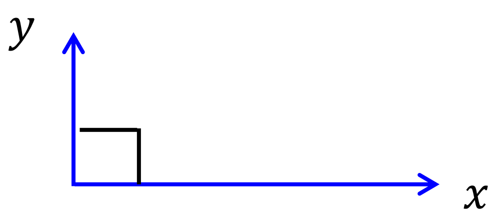
2D vector space - simplified examples

A complete set



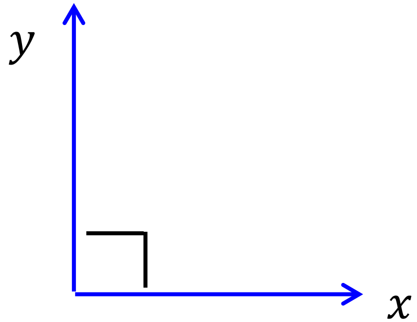
$$|\underline{x}| \neq |\underline{y}|$$

Orthogonal-complete set



$$|\underline{x}| \neq |\underline{y}|$$

Orthonormal-complete set



$$|\underline{x}| = |\underline{y}|$$

Using a complete orthonormal function set:

$$f(x) \sim \sum_{n=1}^{N} a_n \cdot \varphi_n(x)$$

The approximation: a finite number of functions (N) from a complete set (∞ terms)

$$E_N = \int_a^b \left[f(x) - \sum_{n=1}^{N} a_n \cdot \varphi_n(x) \right]^2 \cdot s(x) dx$$

The error: decreases as the number of functions used (N) increases

- The error vanishes for $N \to \infty$:

$$\lim_{N\to\infty} E_N = 0$$

Convergent in the mean to $f(x)$
("Strong" convergence)

- Thus, ***any function*** can be ***accurately*** represented via an ***infinite*** series:

$$\boxed{f(x) = \sum_{n=1}^{\infty} a_n \cdot \varphi_n(x)}$$

Such a function space must have a name…

$$a_n = \langle f|s|\varphi_n \rangle$$

Expansion coefficients

$$\{\varphi_1(x), \varphi_2(x), \dots\}$$

Complete orthonormal function set

(II) Hilbert space

- A ***generalization*** of the 3D vector space – using nD ***vector functions***.


David Hilbert

*Characteristics (**similar** to the vector space):*

- The space includes a ***complete set*** of base functions that can span ***any*** function in the space:

$$\{\varphi_1(x), \varphi_2(x), \ldots\} \longrightarrow f(x)$$

- The space is ***linear***:

$$f(x), g(x) \longrightarrow a_1 \cdot f(x) + a_2 \cdot g(x)$$

Hilbert space functions — *Also a Hilbert space function*

- There is an ***inner product***, defined by:

$$\langle f|s|g\rangle = \int_a^b f(x) \cdot s(x) \cdot g(x) dx$$

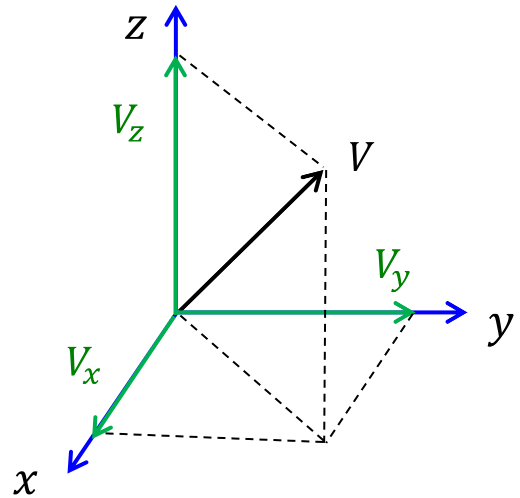
- Any element in the space has a ***norm***, defined by the inner product:

$$\|f(x)\| = \sqrt{\langle f|s|f\rangle}$$

Geometrical representation (schematic)

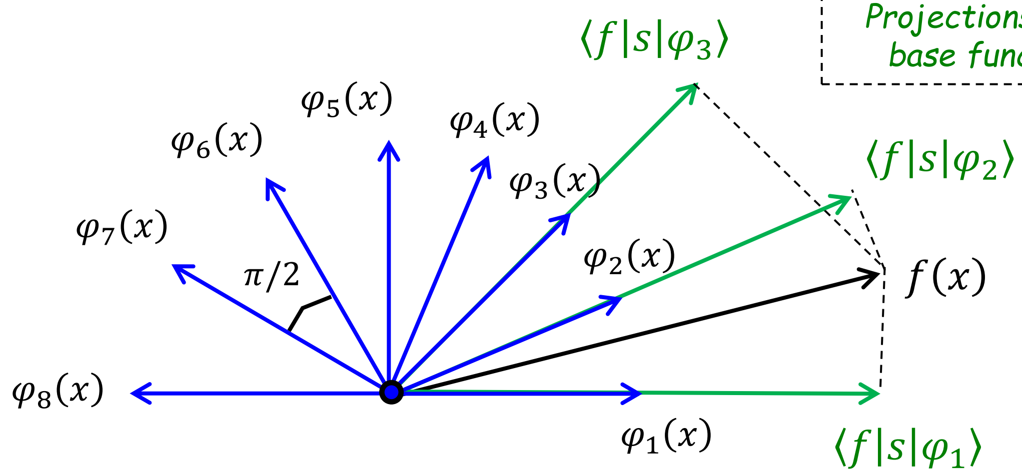
(Adapted from Liboff - Introductory quantum mechanics, 4rd edition)

3D vector space



8D Hilbert space

Projections on the base functions



Example 1

- An example of an Hilbert space is the set of ***square integrable*** functions (finite norm) with weight function $s(x) = 1$:

$$\|f(x)\|^2 = \langle f|1|f\rangle = \int_0^L f(x)\cdot f(x)dx < \infty$$

$$\boxed{\mathcal{H}_1}$$

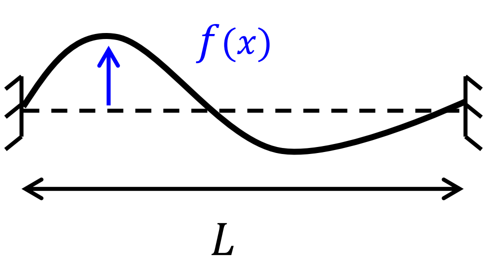
- The function space $\mathcal{H}_1$ is spanned by the orthonormal set of trigonometric functions:

$$\varphi_n(x) = \sin(\kappa_n \cdot x)$$

$$\psi_n(x) = \cos(\kappa_n \cdot x)$$

$$\kappa_n = n \cdot \frac{\pi}{L}$$

- These are in fact the eigenfunctions (normal modes) of a ***finite*** string.



Fourier series

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \varphi_n(x) + \sum_{m=1}^{\infty} b_m \cdot \psi_m(x)$$

$$a_n = \langle f|\varphi_n\rangle \qquad b_m = \langle f|\psi_m\rangle$$

Example 2

- Another example of Hilbert spaces is the same set of functions – over the ***entire*** x interval

$$\|f(x)\|^2 = \langle f|1|f\rangle = \int_{-\infty}^{\infty} f(x) \cdot f(x)dx < \infty \qquad \boxed{\mathcal{H}_2}$$

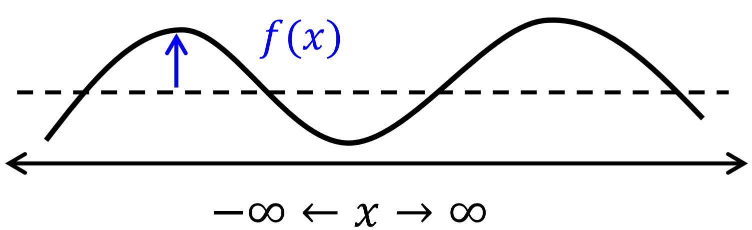
- The function space $\mathcal{H}_2$ is spanned by the orthonormal set of trigonometric functions:

$$\varphi_k(x) = \varphi(x, \kappa) = \sin(\kappa \cdot x)$$

$0 < k$ - Continues parameter

$$\psi_k(x) = \psi(x, \kappa) = \cos(\kappa \cdot x)$$

- These are in fact the eigenfunctions (normal modes) of an ***infinite*** string



Fourier integral

$$f(x) = \int_0^{\infty} a_k \cdot \varphi(x, k)dk + \int_0^{\infty} b_k \cdot \psi(x, k)dk$$

$$a_k = \langle f|\varphi_k\rangle \qquad b_k = \langle f|\psi_k\rangle$$

(III) Completeness (closure) relations

(1) Discrete eigenfunction set

$\varphi_n(x) \qquad n = 1,2 \ldots.$

- For a ***complete*** ***othonormal*** set φ_n, the series expansion of $f(x)$ is:

$$f(x) = \sum_{n=1}^{\infty} a_n \cdot \varphi_n(x) \qquad\qquad a_n = \langle f|s|\varphi_n \rangle$$

$$f(x) = \sum_{n=1}^{\infty} \left[\int_a^b f(\xi) \cdot s(\xi) \cdot \varphi_n(\xi) d\xi \right] \cdot \varphi_n(x)$$

$$f(x) = \int_a^b \underbrace{\left\{ \sum_{n=1}^{\infty} \varphi_n(\xi) \cdot s(\xi) \cdot \varphi_n(x) \right\}}_{\boxed{??}} \cdot f(\xi) d\xi$$

$$f(x) = \int_a^b \{ \quad \delta(x-\xi) \quad \} \cdot f(\xi)\, d\xi$$


Paul Dirac

$$\downarrow$$

Closure relations

Try at home…

$$\sum_{n=1}^{\infty} \varphi_n(\xi) \cdot s(\xi) \cdot \varphi_n(x) = \delta(x-\xi)$$

Discrete set

Seems analogues…

(?)

Orthonormal relations

$$\int_a^b \varphi_n(x) \cdot s(x) \cdot \varphi_m(x) dx = \delta_{nm}$$

Discrete set

(2) Continues eigenfunction set

$\varphi_k(x) = \varphi(x,k) \quad k \in \mathcal{R}$

- For a ***continues*** ***othonormal*** set the series expansion of $f(x)$ is:

$$f(x) = \int_0^\infty a_k \cdot \varphi(x,k)dk \qquad a_k = \langle f|s|\varphi_k\rangle$$

$$f(x) = \int_0^\infty \left[\int_a^b f(\xi)\cdot s(\xi)\cdot \varphi(\xi,k)d\xi\right]\cdot \varphi(x,k)dk$$

$$f(x) = \int_a^b \underbrace{\left\{\int_0^\infty \varphi(\xi,k)\cdot s(\xi)\cdot \varphi(x,k)dk\right\}}_{\delta(x-\xi)}\cdot f(\xi)d\xi$$

Closure relations

Try at home...

$$\int_0^\infty \varphi(\xi,k)\cdot s(\xi)\cdot \varphi(x,k)dk = \delta(x-\xi)$$

Continues set

Analogues !

Orthonormal relations

$$\int_a^b \varphi(x,k)\cdot s(x)\cdot \varphi(x,p)dx = \delta(k-p)$$

Continues set