Homework 3A

Question 1

Consider $f(x)$ as a continuously differentiable function, and show the following properties of the Dirac δ function.

1.1. $x \cdot \delta(x) = 0$

1.2. $\int_a^b H'(x - x_0) \cdot f(x)dx = f(x_0)$ where $a \le x_0 \le b$

$H(x - x_0)$ is a step (Heaviside) function

(*Hint: use integration by parts*)

1.3. $\int_a^b \delta\big(k \cdot (x - x_0)\big) \cdot f(x)dx = \frac{1}{k} f(x_0)$ where $a \le x_0 \le b$ and $k > 0$

1.4. $\int_a^b \delta'(x - x_0) \cdot f(x)dx = -f'(x_0)$ where $a \le x_0 \le b$

(*Hint: use integration by parts*)

Question 2

Find Green's function of the following boundary-value problems.

2.1. $L[y] = y'' = f(x)$; $\frac{dy}{dx}|_{x=0} = 0$, $\left(y + \frac{dy}{dx}\right)|_{x=1} = 0$

2.2. $L[y] = y'' + y' - 2 \cdot y = f(x)$; $y(0) = 0$, $\lim_{y \to \infty} |y(x)| < \infty$

2.3. $L[y] = y'' - y' = f(x)$; $y(0) = 0$, $\frac{dy}{dx}|_{x=1} = 0$

<u>Solution:</u>

2.1. $G(x, x_0) = \begin{cases} G_I(x, x_0) & 0 \le x < x_0 \\ G_{II}(x, x_0) & x_0 < x < 1 \end{cases}$, $G_I(x, x_0) = x_0 - 2$, $G_{II}(x, x_0) = x - 2$

2.2. $G(x, x_0) = \begin{cases} G_I(x, x_0) & 0 \le x < x_0 \\ G_{II}(x, x_0) & x_0 < x < \infty \end{cases}$,

$$G_I(x, x_0) = \frac{1}{3} \cdot [-\exp(x) + \exp(-2 \cdot x)] \cdot \exp(-x_0) \quad ,$$

$$G_{II}(x, x_0) = \frac{1}{3} \cdot [\exp(-x_0) - \exp(2 \cdot x_0)] \cdot \exp(-2 \cdot x)$$

2.3. $G(x, x_0) = \begin{cases} G_I(x, x_0) & 0 \le x < x_0 \\ G_{II}(x, x_0) & x_0 < x < 1 \end{cases}$,

$G_I(x, x_0) = \exp(-x_0) - \exp(x - x_0)$ $G_{II}(x, x_0) = \exp(-x_0) - 1$

Question 3

Find the solution of the following inhomogeneous boundary-value problem via the Green's function method.

$$y'' - y = f(x)$$

$$y(0) = 0 \quad , \quad y(1) = 0$$

Where:

$$f(x) = \begin{cases} e^x & 0 \le x \le 1/2 \\ 0 & else \end{cases}$$

Identities:

$\int \sinh(x) \cdot e^x \cdot dx = \frac{1}{4} \cdot (e^{2x} - 2x)$ $\qquad$ $\int \cosh(x) \cdot e^x \cdot dx = \frac{1}{4} \cdot (e^{2x} + 2x)$

Solution:

Green's function:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) & 0 \le x < x_0 \\ G_{II}(x, x_0) & x_0 < x < 1 \end{cases}$$

Where:

$G_I(x, x_0) = \left[\sinh(x_0)\frac{\cosh(1)}{\sinh(1)} - \cosh(x_0)\right] \cdot \sinh(x)$ $\qquad$ $G_{II}(x, x_0) = \left[\sinh(x)\frac{\cosh(1)}{\sinh(1)} - \cosh(x)\right] \cdot \sinh(x_0)$

The solution:

$$\varphi(x) = \int_0^L G(x, x_0) \cdot f(x_0) dx_0 = \int_0^x G_{II}(x, x_0) \cdot e^{x_0} dx_0 + \int_x^{1/2} G_I(x, x_0) \cdot e^{x_0} dx_0 = I_1 + I_2$$

Where:

$$I_1 = \int_0^x G_{II}(x, x_0) \cdot e^x dx_0 = \frac{1}{4} \cdot [(e^{2x} - 1) - 2 \cdot x] \cdot \left[\sinh(x)\frac{\cosh(1)}{\sinh(1)} - \cosh(x)\right]$$

$$I_2 = \int_x^{1/2} G_I(x, x_0) \cdot e^{x_0} dx_0 = \sinh(x) \cdot \left[\frac{1}{4} \cdot (e - e^{2x} - 1 + 2x) \cdot \frac{\cosh(1)}{\sinh(1)} - \frac{1}{4} \cdot (e - e^{2x} + 1 - 2x)\right]$$