Homework 1A

Question 1

Find the general solutions of the following homogeneous equations.

1.1. $y'' + 2y' - 8y = 0$

1.2. $y'' + 2y' + 1.25y = 0$

1.3. $y'' - 6y' + 9y = 0$

1.4. $x^4 y'' + 3x^3 y' + 1.25x^2 y = 0 \quad ; \; x > 0$

Solutions:
1.1. $\varphi = c_1 exp(2x) + c_2 exp(-4x)$
1.2. $\varphi = c_1 exp(-x)cos(x/2) + c_2 exp(-x)sin(x/2)$
1.3. $\varphi = c_1 exp(3x) + c_2 \, x \, exp(3x)$
1.4. $\varphi = c_1 x^{-1} cos(ln(x)/2) + c_2 \, x^{-1} sin(ln(x)/2)$

Question 2

Find the solutions of the following initial-value problems.

2.1. $y'' + 4y = 0 \quad ; \quad y(0) = 0, \; y'(0) = 0$

2.2. $y'' + y' + 1.25y = 0 \quad ; \quad y(0) = 3, \; y'(0) = 1$

2.3. $y'' + 4y' + 4y = 0 \quad ; \quad y(-1) = 2, \; y'(-1) = 1$

Solutions:
2.1. $\varphi = \frac{1}{2} sin(2x)$
2.2. $\varphi = 3exp(-x/2)cos(x) + \frac{5}{2} exp(-x/2)sin(x)$
2.3. $\varphi = 7exp\big(-2(x+1)\big) + 5 \, x \, exp\big(-2(x+1)\big)$

Question 3

Consider the following differential equation:

$$L[y] = \big(1 - ln(x)\big)y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

3.1. Show that $\varphi_1(x) = x$ and $\varphi_2(x) = ln(x)$ are the solutions of the equation.

3.2. Calculate the Wronskian $W(\varphi_1, \varphi_2; x)$ and investigate whether the functions are linear independent solutions of $L[y]$ or not? Justify.

Question 4

Solve the following non-homogeneous equations by the method of variation of parameters.

4.1. $y'' + 4y' + 4y = x^{-2}e^{-2x} \quad ; \quad x > 0$

4.2. $4y'' + y = \frac{2}{cos(x/2)} \quad ; \quad x > 0$

Solutions:
4.1. $\varphi = c_1e^{-2x} + c_2\, xe^{-2x} - e^{-2x}ln(x)$
4.2. $\varphi = c_1cos(x/2) + c_2sin(x/2) + xsin(x/2) + 2ln\big(cos(x/2)\big)cos(x/2)$

Question 5

A forced mass-spring-despot system is defined by: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$.

Consider first the homogeneous system:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

5.1. Find the general solutions of the homogeneous equation $\varphi_1(t),\ \varphi_2(t)$.

5.2. Calculate the Wronsian $W(\varphi_1, \varphi_2; t) = \varphi_1\varphi_2{}' - \varphi_1{}'\varphi_2$.

Consider now the forced system with the following excitation force:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F_0\, e^{-t/\tau_c} sin(\omega t)$$

Using variation of parameters, the solution is given by $x(t) = C_1(t)\varphi_1 + C_1(t)\varphi_2$.

5.3. Find $C_1(t)$ and $C_2(t)$ for and $\omega \neq \omega_c$.

5.4. Find $C_1(t)$ and $C_2(t)$ for and $\omega = \omega_c$.

5.5. Find the particular solution for $\omega = \omega_c$ $x(t) = C_1\varphi_1 + C_2\varphi_2$.

Consider the parameters: $m = 2$, $c = 3$, $k = 5/4$ and $F_0 = 2$, and the initial conditions: $x(0) = 1$ and $v(0) = 0.25$.

Useful trigonometric identities:

$$cos(\alpha) \cdot cos(\beta) = \frac{1}{2}[cos(\alpha - \beta) + cos(\alpha + \beta)]$$

$$sin(\alpha) \cdot sin(\beta) = \frac{1}{2}[cos(\alpha - \beta) - cos(\alpha + \beta)]$$

$$sin(\alpha) \cdot cos(\beta) = \frac{1}{2}[sin(\alpha + \beta) + sin(\alpha - \beta)]$$

Solutions:

5.1. $\varphi_1(t) = cos(\omega_c \cdot t)e^{-t/\tau_c}$, $\varphi_2(t) = sin(\omega_c \cdot t)e^{-t/\tau_c}$

$\tau_c = \frac{1}{2}\frac{c}{m}$ and $\omega_c = \frac{1}{2}\left|\sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}\right|$

5.2. $W(\varphi_1, \varphi_2; t) = \omega_c e^{-2 \cdot t/\tau_c}$

5.3. $C_1(t) = c_1 - \frac{1}{2}\frac{F_0}{m \cdot \omega_c}\left[\frac{1}{(\omega - \omega_c)} sin\big((\omega - \omega_c)t\big) + \frac{1}{(\omega + \omega_c)} sin\big((\omega - \omega_c)t\big)\right]$

$C_2(t) = c_2 - \frac{1}{2}\frac{F_0}{m \cdot \omega_c}\left[\frac{1}{(\omega + \omega_c)} cos\big((\omega + \omega_c)t\big) + \frac{1}{(\omega - \omega_c)} cos\big((\omega - \omega_c)t\big)\right]$

5.4. $C_1(t) = c_1 - \frac{1}{2}\frac{F_0}{m \cdot \omega_c}\left[t - \frac{1}{2\omega_c} sin(2\omega_c \cdot t)\right]$, $C_2(t) = c_2 - \frac{1}{4}\frac{F_0}{m \cdot \omega_c^2} cos(2\omega_c \cdot t)$

5.5. $x(t) = [1 - 2t + 4sin(t/2)]cos(t/4)e^{-3t/4} + [8 - 4cos(t/2)]sin\left(\frac{t}{4}\right)e^{-\frac{3t}{4}}$