Analytical Methods

Second-order linear differential equations

<u>Part A</u> – A rapid review on elementary solutions of differential equations

Linear differntial equations

- A second-order linear differential equation may be written in the form:

$$\underbrace{a_0(x)\cdot y'' + a_1(x)\cdot y' + a_2(x)\cdot y}_{\text{Physical system}} = \underbrace{g(x)}_{\text{Generalized load}} \qquad \boxed{a \le x \le b \equiv \Delta}$$

$a_i(x)$ and $g(x)$ are ***real-valued*** ***continues*** functions in an interval Δ

- If $g(x) = 0 \rightarrow$ ***homogeneous*** equation

- If $g(x) \neq 0 \rightarrow$ ***nonhomogeneous*** equation

Initial-value problem:

$$y(x_0) = y_0 \quad , y'(x_0) = y_1 \quad ; \quad x_0 \in \Delta$$

Boundary-value problem:

$$U_a(y) = \alpha \quad , U_b(y) = \beta$$

$$(e.g. \quad y(x_a) = y_a \quad , y(x_b) = y_b)$$

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- In case where $a_0(x) \neq 0$, we can obtain:

$$\boxed{y'' + p(x)y' + q(x)y = f(x)}$$

with the following ***continues*** functions in Δ :

$$p = \frac{a_1}{a_0} \; , \qquad q = \frac{a_2}{a_0}, \qquad f = \frac{g}{a_0}$$

- Introducing the ***linear differential operator***:

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

Note that: For any twice differential functions $y_1(x)$ and $y_2(x)$, and for any constants c_1 and c_2 , the operator has the property:

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$$

Show this…

- The differential equations can thus be written in the form:

$$\boxed{L[y] = f(x)}$$

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- A **solution** of the differential equation $y = \varphi(x)$ is a twice differential function which satisfies:

$$L[\varphi(x)] = f(x)$$

Theorem 1 : Existence-uniqueness Theorem:

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

$$y(x_0) = y_0 \quad , y'(x_0) = y_1$$

Initial-value problem

*Let the functions $p(x), q(x), f(x)$ be continues on Δ. For any $x_0 \in \Delta$ and constant y_0 and y_1, there **exists** a **unique** solution $\varphi(x)$ for the **initial-value problem**.*

- Does the "existence" and/or "uniqueness" also appear for boundary-value problems ?

No simple answer here…

Solutions of homogeneous equations

$$L[y] = y'' + p(x)y' + q(x)y = 0$$

Theorem 2:

If $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of the homogeneous differential equation $L[y] = 0$, the combination $c_1\varphi_1(x) + c_2\varphi_2(x)$ with c_1 and c_2 as constants is also a solution of $L[y] = 0$.

Proof:

Recall the linearity of $L[y]$:

$$L[c_1\varphi_1 + c_2\varphi_2] = c_1 L[\varphi_1] + c_2 L[\varphi_2]$$

Since however $\varphi_1(\mathrm{x})$ and $\varphi_2(\mathrm{x})$ are solutions of $L[y] = 0$ - $L[\varphi_1] = 0$ and $L[\varphi_2] = 0$:

$$L[c_1\varphi_1 + c_2\varphi_2] = 0$$

Thus,$c_1\varphi_1(x) + c_2\varphi_2(x)$ is also a solution of $L[y] = 0$.

How does this agree with the existence-uniqueness theorem ??

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Linear dependence/independence

The functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are said to be linearly dependent on an interval Δ if there exist constants $c_1, c_2, \dots, c_n$,not all zero, such that:

$$c_1\varphi_1(x) + c_2\varphi_2(x) + \cdots + c_n\varphi_n(x) = 0$$

The functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$are said to be linearly independent ***if and only if*** *$c_1 = c_2 = \cdots = c_n = 0$ for all $x \in \Delta$*

Example 1:

$$\varphi_1 = x\,, \qquad \varphi_2 = -2x \quad (-\infty < x < \infty)$$

$$c_1\varphi_1 + c_2\varphi_2 = c_1 x - 2c_2 x \quad = x(c_1 - 2c_2) \underbrace{=}_{?} 0$$

Linearly dependent

For the selection of $c_1 = 2,\ c_2 = 1$, for example $\rightarrow$ zero

Example 2:

$$\varphi_1 = 1\,, \qquad \varphi_2 = x \quad (-\infty < x < \infty)$$

Linearly independent

$$c_1\varphi_1 + c_2\varphi_2 = c_1 + c_2 x \underbrace{=}_{?} 0$$

Only $c_1 = c_2 = 0$ can eliminate the expression for all $x \in \Delta$

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The Wronskian

Józef Hoene-Wroński

The ***Wronskian*** of two differential functions $\varphi_1(x)$ *and* $\varphi_2(x)$ on an interval Δ is defined by the determinant:

$$W(\varphi_1, \varphi_2; x) = det \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi'_1 & \varphi'_2 \end{pmatrix} = \varphi_1 \varphi'_2 - \varphi'_1 \varphi_2$$

Theorem 2:

$$L[y] = y'' + p(x)y' + q(x)y = 0 \qquad a \le x \le b \equiv \Delta$$

*Let $\varphi_1(x)$ and $\varphi_2(x)$ be the solutions of $L[y] = 0$ on an interval Δ. Then, $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent **if and only if** $W(\varphi_1, \varphi_2; x) \neq 0$ for all $x \in \Delta$.*

- To prove the theorem, we need to show that *for all* $x \in \Delta$:

Path A: $W(\varphi_1, \varphi_2; x) \neq 0 \rightarrow \varphi_1(x)$ *and* $\varphi_2(x)$ are linearly independent.

Path B: $\varphi_1(x)$ *and* $\varphi_2(x)$ are linear independent solutions of $L[y] = 0 \rightarrow W(\varphi_1, \varphi_2; x) \neq 0$.

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Proof - Path A

- Let $\varphi_1(x)$ and $\varphi_2(x)$ solutions of $L[y] = 0$.
- Linear independence emerges where the following vanishes only for $c_1 = c_2 = 0$ – for all $x \in \Delta$.

$$c_1\varphi_1(x) + c_2\varphi_2(x) = 0$$

- By deriving the above we obtain:

$$c_1\varphi'_1(x) + c_2\varphi'_2(x) = 0$$

- For ***any*** selection of $x_0 \in \Delta$ - $\varphi_1(x_0)$ and $\varphi_2(x_0)$ are set → producing two linear equations:

$$\begin{cases} c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = 0 \\ c_1\varphi'_1(x_0) + c_2\varphi'_2(x_0) = 0 \end{cases}$$

C_1 and C_2 are unknowns…

- This system of equations can be written as:

$$\underbrace{\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix}}_{\textit{Coefficient matrix}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}_{\textit{Unknowns}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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Proof (contd. 1) - Path A

$$\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Assuming that $W(\varphi_1, \varphi_2; x) \neq 0$:

$$det \begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix} \neq 0$$

- Since the determinant of the homogenous system doesn't vanish, there exists ***only*** *trivial solution* for the system - $c_1 = c_2 = 0$.

- Since $W(\varphi_1, \varphi_2; x) \neq 0$: occurs for ***any*** $x_0 \in \Delta \;\rightarrow\; c_1 = c_2 = 0$ for *any* $x \in \Delta$.

- Therefore, $\varphi_1(x)$ and $\varphi_2(x)$ are the ***linear independence***.

Path A: $W(\varphi_1, \varphi_2; x) \neq 0 \;\rightarrow\; \varphi_1(x)$ and $\varphi_2(x)$ are linearly independent

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Proof (contd. 2) - Path B

- Let now ***assume*** that that exist $x_0 \in \Delta$, for which $W(\varphi_1, \varphi_2; x) = 0$:

$$W(\varphi_1, \varphi_2; x) = 0 \;\rightarrow\; det \begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix} = 0$$

- Thus, exist c_1 and c_2, where $(C_1)^2 + (C_2)^2 \neq 0$ such that:

$$\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- This leads to:

$$c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = 0$$

$$c_1\varphi'_1(x_0) + c_2\varphi'_2(x_0) = 0$$

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Proof (contd. 2) - Path B

- Now, let $\varphi_1(x)$ and $\varphi_2(x)$ be (I) ***solutions*** of $L[y]=0$ and (II) ***linear independent*** functions
  → This should lead to a ***contradiction***.

(I) $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of $L[y]=0$

- The linear combination $\varphi(x)=c_1\varphi_1(x)+c_2\varphi_2(x)$ is also a solution of $L[y]=0$:

$$L[\varphi(x)] = L[c_1\varphi_1(x)+c_2\varphi_2(x)] = 0$$

- Recall from before that:

$$c_1\varphi_1(x_0)+c_2\varphi_2(x_0)=0 \longrightarrow \varphi(x_0)=0$$

$$c_1\varphi'_1(x_0)+c_2\varphi'_2(x_0)=0 \longrightarrow \varphi'(x_0)=0$$

$$\boxed{\varphi(x)=c_1\varphi_1(x)+c_2\varphi_2(x)\equiv 0}$$

For ***any*** C_1 and C_2
(specifically, $(C_1)^2+(C_2)^2\neq 0$)

- Thus, following the Uniqueness theorem → $\varphi(x)$ must be the ***trivial solution*** (!)

Proof (contd. 3) - Path B

(I) $\varphi_1(x)$ and $\varphi_2(x)$ are solutions of $L[y]=0$

$$\boxed{c_1\varphi_1(x)+c_2\varphi_2(x)=0 \leftrightarrow c_1,c_2\neq 0}$$

Contradiction (!!!)

(II) $\varphi_1(x)$ and $\varphi_2(x)$ are linear independent functions

$$\boxed{c_1\varphi_1(x)+c_2\varphi_2(x)=0 \leftrightarrow c_1=c_2=0}$$

- This ***contradiction*** originates from the ***assumption*** that $W(\varphi_1,\varphi_2;x)=0$.

Path B: $\varphi_1(x)$ and $\varphi_2(x)$ are linear independent solutions of $L[y]=0$ → $W(\varphi_1,\varphi_2;x)\neq 0$.

Example 1:

$$\varphi_1 = sin(x)\,, \qquad \varphi_2 = cos(x) \quad (-\infty < x < \infty)$$

Linearly independent

$$W = \varphi_1\varphi'_2 - \varphi'_1\varphi_2 = -sin^2(x) - cos^2(x) = -1$$

Example 2:

$$\varphi_1 = x\,, \qquad \varphi_2 = -2x \quad (-\infty < x < \infty)$$

Linearly dependent

$$W = \varphi_1\varphi'_2 - \varphi'_1\varphi_2 = -2x - (-2x) = 0$$

Example 3:

$$\varphi_1 = 1\,, \qquad \varphi_2 = x \quad (-\infty < x < \infty)$$

Linearly independent

$$W = \varphi_1\varphi'_2 - \varphi'_1\varphi_2 = 1 - 0 = 1$$

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Example 4:

$$\varphi_1 = 1\,, \qquad \varphi_2 = x^2 \quad (-\infty < x < \infty)$$

$$W = \varphi_1\varphi'_2 - \varphi'_1\varphi_2 = 2 \cdot x - 0 = 2 \cdot x$$

?!?!?!

- For $x = 0 \rightarrow W(\varphi_1, \varphi_2) = 0$.
- For $x \neq 0 \rightarrow W(\varphi_1, \varphi_2) \neq 0$.
- The functions $\varphi_1(x) = 1$ and $\varphi_2(x) = x^2$ are clearly ***linear independent*** (remember Taylor's series).
- Something isn't right (!)

Any ideas what is wrong ????

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Theorem 3:

$$L[y] = y'' + p(x)y' + q(x)y = 0 \qquad a \le x \le b \equiv \Delta$$

$$y(x_0) = y_0 \;\; , y'(x_0) = y_1$$

Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent solutions of $L[y] = 0$ on an interval Δ. Then, ***every solution*** *of $L[y] = 0$ can be expressed* ***uniquely*** *as $\varphi(x) = c_1\varphi_1(x) + c_2\varphi_2(x)$, where the constants c_1 and c_1 are determined by the* ***initial condition****.*

Proof:

Suppose that $\varphi(x)$ is a solution of $L[y] = 0$, we can calculate its value and derivative at x_0:

$$y(x_0) = c_1\varphi_1(x_0) + c_2\varphi_2(x_0) = y_0$$

$$y'(x_0) = c_1\varphi'_1(x_0) + c_2\varphi'_2(x_0) = y_1$$

$$\downarrow$$

$$\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

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$$\underbrace{\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix}}_{W(\varphi_1, \varphi_2; x_0)} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$$

Since $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent solutions of $L[y] = 0$, following theorem 2 $W(\varphi_1, \varphi_2; x_0) \neq 0$.

Thus, the coefficients c_1 and c_2 are determined ***uniquely***.

The coefficients c_1 and c_2 can be found ***explicitly*** by ***Cramer's rule***:

$$c_1 = \frac{\det\begin{pmatrix} y_0 & \varphi_2(x_0) \\ y_1 & \varphi'_2(x_0) \end{pmatrix}}{\det\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix}} = \frac{y_0\varphi'_2(x_0) - y_1\varphi_2(x_0)}{W(\varphi_1, \varphi_2; x_0)}$$

$$c_2 = \frac{\det\begin{pmatrix} \varphi_1(x_0) & y_0 \\ \varphi'_1(x_0) & y_1 \end{pmatrix}}{\det\begin{pmatrix} \varphi_1(x_0) & \varphi_2(x_0) \\ \varphi'_1(x_0) & \varphi'_2(x_0) \end{pmatrix}} = \frac{y_1\varphi_1(x_0) - y_0\varphi'_1(x_0)}{W(\varphi_1, \varphi_2; x_0)}$$

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The Wronskian and the differential equation

Example

$$L[y] = y'' - 2y' + \frac{3}{4}y = 0 \quad (-\infty < x < \infty)$$

- The solutions for the equation are $\varphi_1 = exp(x/2)$ and $\varphi_2 = exp(3x/2)$ *(show by substituting…)*

- The Wronskian of the solutions is:

$$W = \varphi_1 \varphi'_2 - \varphi'_1 \varphi_2 = \frac{1}{2} exp\left(\frac{x}{2}\right) exp\left(\frac{3x}{2}\right) - \frac{3}{2} exp\left(\frac{x}{2}\right) exp\left(\frac{3x}{2}\right)$$

$$W = -\frac{1}{2} exp(2x)$$

$W(\varphi_1, \varphi_2; x_0) \neq 0 \rightarrow$ *φ_1 and φ_2 independent*

- *A more general form ?*

$$W = c \cdot exp\left(-\int p(x)dx\right)$$

A coincidence ??

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Theorem 4 - Abel's theorem:

$$L[y] = y'' + p(x)y' + q(x)y = 0 \qquad a \leq x \leq b \equiv \Delta$$

Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent solutions of $L[y] = 0$ on an interval Δ. Then, the Wronskian of the solutions $W(\varphi_1, \varphi_2; x)$ is given by:

$$\boxed{W = c \cdot exp\left(-\int p(x)dx\right)}$$

Proof:

Let $\varphi_1(x)$ and $\varphi_2(x)$ be the solutions of $L[y] = 0$:

$$\varphi_1'' + p(x)\varphi_1' + q(x)\varphi_1 = 0 \qquad /\cdot \varphi_2$$

$$\varphi_2'' + p(x)\varphi_2' + q(x)\varphi_2 = 0 \qquad /\cdot \varphi_1$$

$$(II) - (I) \downarrow$$

$$[\varphi_2''\varphi_1 - \varphi_1''\varphi_2] + p(x) \cdot [\varphi_2'\varphi_1 - \varphi_1'\varphi_2] + q(x) \cdot [\varphi_2\varphi_1 - \varphi_1\varphi_2] = 0$$

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$$\underbrace{[\varphi_2''\varphi_1 - \varphi_1''\varphi_2]}_{W'(\varphi_1,\varphi_2;x)} + p(x)\cdot \underbrace{[\varphi_2'\varphi_1 - \varphi_1'\varphi_2]}_{W(\varphi_1,\varphi_2;x)} = 0$$

Also note that:

$$\frac{dW}{dx} = \frac{d}{dx}(\varphi_2'\varphi_1 - \varphi_1'\varphi_2) = \varphi_2''\varphi_1 - \varphi_1''\varphi_2$$

Thus:

$$W' + p(x)W = 0$$

The general solution of the equation is:

$$\frac{W'}{W} = -p(x) \longrightarrow (ln(W))' = -p(x) \longrightarrow ln(W) = C - \int p(x)dx$$

$$\boxed{W = c\cdot exp\left(-\int p(x)dx\right)}$$

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A few insights

$$\boxed{W = c\cdot exp\left(-\int p(x)dx\right)}$$

Abel's formula

- The exponent is always **_non-zero_**.
- **_Only_** the **_coefficient_** **_determines_** whether $W = 0$ or $W \neq 0$.
- By specifying the integration range $x_0 \to x$, the coefficient is specified to be $c = W(\varphi_1, \varphi_2; x_0)$:

$$\boxed{W(\varphi_1,\varphi_2;x) = W(\varphi_1,\varphi_2;x_0)\cdot exp\left(-\int p(x)dx\right)}$$

Abel's formula

- Thus, if $W(\varphi_1,\varphi_2;x_0) \neq 0$ for a **_specific_** $x_0 \in \Delta$ $\rightarrow$ $W(\varphi_1,\varphi_2;x) \neq 0$ for **_any_** arbitrary $x \in \Delta$
- It is thus needed to calculate $W(\varphi_1,\varphi_2;x_0)$ **_only_** for a specific $x_0 \in \Delta$ – to find if $W = 0$ or $W \neq 0$.

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Required knowledge from previous studies

- Finding solutions of homogeneous differential equations with *constant coefficients* of n[th] order.

$$y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_1y'+a_0=0$$

Strategy: *Proposing solutions in the form of:* $\varphi=exp(r\cdot x)$

- Finding solutions of homogeneous ***Euler's*** differential equation:

$$x^2y''+\alpha\cdot x\cdot y'+\beta\cdot y=0$$

Strategy: Change variable $x=exp(z)$, find solutions in the form of $\varphi(z)=exp(r\cdot z)$ and change back to x.

i.e. proposing solutions in the form of: $\varphi(x)=x^r$

Solutions of Euler equation (a reminder)

$$\boxed{x^2y''+\alpha\cdot x\cdot y'+\beta\cdot y=0} \qquad (x>0)$$

- The solutions are obtained at the form of: $\varphi(x)=x^r$

- Substituting into the equation:

$$x^2\cdot r\cdot(r-1)\cdot x^{r-2}+\alpha\cdot x\cdot r\cdot x^{r-1}+\beta\cdot x^r=0$$

$$\downarrow$$

$$\boxed{r^2+(\alpha-1)\cdot r+\beta=0}$$

- The roots of the equation are:

$$r_{1,2}=\frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\cdot\beta}}{2}$$

<u>*Case 1: Real distinct roots:*</u> $(\alpha-1)^2-4\cdot\beta>0 \longrightarrow r_1 \neq r_2 \in \mathcal{R}$

$$\boxed{\varphi_1(x)=x^{r_1} \qquad \varphi_2(x)=x^{r_2}}$$

<u>*Case 2: Real equal roots:*</u> $(\alpha-1)^2-4\cdot\beta=0 \longrightarrow r_1=r_2=r \in \mathcal{R}$

[e.g. using reduction of order method - $\varphi_2(x)=\varphi_1(x)\cdot\phi(x)$]

$$\boxed{\varphi_1(x)=x^{r} \qquad \varphi_2(x)=x^{r}\cdot ln(x)}$$

<u>*Case 3: Complex conjugate roots:*</u> $(\alpha-1)^2-4\cdot\beta<0 \longrightarrow r_1,r_2=\lambda\pm i\mu$

$$\boxed{\varphi_1(x)=x^{\lambda+i\mu} \qquad \varphi_2(x)=x^{\lambda-i\mu}}$$

$$\boxed{\varphi_1(x)=x^{\lambda}cos\big(\mu\cdot ln(x)\big) \qquad \varphi_2(x)=x^{\lambda}sin\big(\mu\cdot ln(x)\big)}$$

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Solutions of nonhomogeneous equations

(variation of parameters)

$$L[y]=y''+p(x)y'+q(x)y=f(x)$$

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- We wish to find the solution of the non-homogeneous equation:

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

- The general solution of the ***homogeneous*** equation is given by:

$$\varphi_h = c_1\varphi_1(x) + c_2\varphi_2(x) \quad \longrightarrow \quad L[\varphi_h] = 0$$

- The c_1 and c_2 are ***constants***, or ***"parameters"***.

- ***An idea:*** to ***vary*** these ***parameters*** in order to find a solution to the *non-homogeneous* equation, i.e. assuming a solution in the form of:

$$\varphi = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x) \quad \longrightarrow \quad L[\varphi] = f(x)$$

Unknown functions (varied parameters)

Homogeneous solutions

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Variation of parameters method

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

- We assume a solution for $L[y] = f$ in the form of:

$$\varphi = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x)$$

- We need to ***formulate*** ***two equations*** that will lead us to determine $C_1(x)$ and $C_2(x)$.

- The first derivative of φ yields:

$$\varphi' = C_1\varphi'_1 + C_2\varphi'_2 + C'_1\varphi_1 + C'_2\varphi_2$$

- Let's propose that:

$$\boxed{C'_1\varphi_1 + C'_2\varphi_2 = 0} \qquad \text{Eq. \#1}$$

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- The ***first*** derivative of φ thus yields:

$$\varphi' = C_1\varphi'_1 + C_2\varphi'_2$$

- The ***second*** derivative of φ yields:

$$\varphi'' = C_1\varphi''_1 + C_2\varphi''_2 + C'_1\varphi'_1 + C'_2\varphi'_2$$

- ***Substituting*** φ and its derivatives into $L[y] = f$ yields:

$$L[\varphi] = (C_1\varphi''_1 + C_2\varphi''_2 + C'_1\varphi'_1 + C'_2\varphi'_2) +$$

$$+p(x)\cdot(C_1\varphi'_1 + C_2\varphi'_2) + q(x)(C_1\varphi_1 + C_2\varphi_2) = f(x)$$

- Rearranging…

$$C'_1\varphi'_1 + C'_2\varphi'_2 + C_1\cdot\overbrace{[\varphi''_1 + p(x)\varphi'_1 + q(x)\varphi_1]}^{L[\varphi_1]=0} + C_2\cdot\overbrace{[\varphi''_2 + p(x)\varphi'_2 + q(x)\varphi_2]}^{L[\varphi_2]=0} = f(x)$$

- Thus, the second equation for determining $C_1(x)$ and $C_2(x)$ is:

$$\boxed{C'_1\varphi'_1 + C'_2\varphi'_2 = f(x)}$$ Eq. #2

- Thus, we have now two equations for $C_1{}'(x)$ and $C_2{}'(x)$:

$$\begin{cases} C'_1\varphi_1 + C'_2\varphi_2 = 0 \\ C'_1\varphi'_1 + C'_2\varphi'_2 = f(x) \end{cases}$$

- In a matrix form:

$$\begin{pmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi'_1(x) & \varphi'_2(x) \end{pmatrix}\begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Ringing any bells?

- By using ***Cramer's rule***:

$$C'_1 = \frac{\det\begin{pmatrix} 0 & \varphi_2(x) \\ f(x) & \varphi'_2(x) \end{pmatrix}}{W(\varphi_1,\varphi_2;x)} = -\frac{f(x)\varphi_2(x)}{W(\varphi_1,\varphi_2;x)}$$

$$C'_2 = \frac{\det\begin{pmatrix} \varphi_1(x) & 0 \\ \varphi'_1(x) & f(x) \end{pmatrix}}{W(\varphi_1,\varphi_2;x)} = \frac{f(x)\varphi_1(x)}{W(\varphi_1,\varphi_2;x)}$$

- After integration we obtain:

$$C_1(x) = c_1 - \int \frac{f(x)\varphi_2(x)}{W(\varphi_1,\varphi_2;x)}dx \qquad C_2(x) = c_2 + \int \frac{f(x)\varphi_1(x)}{W(\varphi_1,\varphi_2;x)}dx$$

- Substituting these functions into the solution:

$$\varphi = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x)$$

$$= \underbrace{c_1\varphi_1(x) + c_2\varphi_2(x)}_{\varphi_h} \underbrace{-\varphi_1(x)\cdot\int\frac{f(x)\varphi_2(x)}{W(\varphi_1,\varphi_2;x)}dx + \varphi_2(x)\cdot\int\frac{f(x)\varphi_1(x)}{W(\varphi_1,\varphi_2;x)}dx}_{\varphi_p}$$

Homogeneous solution *Particular solution*

- Thus, the particular solution of $L[y] = f$ is given by:

$$\boxed{\varphi_p(x) = \int^x f(s)\frac{[\varphi_1(s)\varphi_2(x) - \varphi_1(x)\varphi_2(s)]}{W(\varphi_1,\varphi_2;s)}ds}$$

<u>Methodology: Solution for non-homogeneous equations by variation of parameters</u>

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

<u>Step 1:</u> find the general solutions of the homogeneous equation - $\varphi_1(x)$ and $\varphi_2(x)$:

$$\varphi_h = c_1\cdot\varphi_1(x) + c_2\cdot\varphi_2(x) \longrightarrow \boxed{L[\varphi_h] = 0}$$

<u>Step 2:</u> find the coefficients:

$$C_1(x) = c_1 - \int \frac{f(x)\varphi_2(x)}{W(\varphi_1,\varphi_2;x)}dx \qquad C_2(x) = c_2 + \int \frac{f(x)\varphi_1(x)}{W(\varphi_1,\varphi_2;x)}dx$$

<u>Step 3:</u> find the solution of the non-homogeneous equation:

$$\varphi = C_1(x)\cdot\varphi_1(x) + C_2(x)\cdot\varphi_2(x) \longrightarrow \boxed{L[\varphi] = f(x)}$$

<u>Step 4:</u> Find the constants c_1 and c_2 - using the ***<u>initial conditions</u>***

Example – Non-homogeneous Euler's equation

Find the ***particular solution*** $\varphi_p(x)$ of the following Euler's equation:

$$x^2y'' - 2xy' + 2y = 6x^4 \qquad (x > 0)$$

The solutions of the homogeneous equation $L[y] = 0$ are:

$$\varphi_1 = x\,, \qquad \varphi_2 = x^2$$

(Show by substituting..)

The Wronskian of the solutions is:

$$W(\varphi_1, \varphi_2; x) = \varphi_2'\varphi_1 - \varphi_1'\varphi_2 = x^2$$

The coefficients of the solution for the non-homogeneous equation are:

$$C_1(x) = c_1 - \int \frac{f(x)\varphi_2(x)}{W(\varphi_1, \varphi_2; x)}dx \quad = c_1 - \int \frac{6x^4x^2}{x^2}dx = c_1 - \frac{6}{5}x^5$$

$$C_2(x) = c_2 + \int \frac{f(x)\varphi_1(x)}{W(\varphi_1, \varphi_2; x)}dx \quad = c_2 + \int \frac{6x^4x}{x^2}dx = c_2 + \frac{3}{2}x^4$$

The solution of the non-homogeneous equation is:

$$\varphi = C_1(x) \cdot \varphi_1(x) + C_2(x) \cdot \varphi_2(x)$$

$$= \left(c_1 - \frac{6}{5}x^5\right) \cdot x + \left(c_2 + \frac{3}{2}x^4\right) \cdot x^2$$

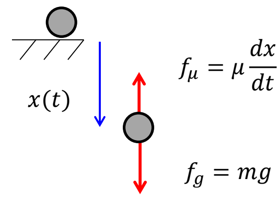
$$= \underbrace{c_1x + c_2x^2}_{\varphi_h} + \underbrace{\frac{3}{10}x^6}_{\varphi_p}$$

$$\longrightarrow \quad \boxed{\varphi_p = \frac{3}{10}x^6}$$

Show by substituting that $\varphi_p(x)$ solves the non-homogeneous equation:

$$L[\varphi_p] = f(x)$$

Example – Falling particle



- Consider a particle of a mass "m" falls trough air.
- Let $x(t)$ be the distance as a function of time.
- Two forces are acting on the particle:

Gravity force $\quad f_g = mg$

"Friction" force by air $\quad f_\mu = -\mu \frac{dx}{dt}$

- The equation of motion is thus:

$$m\frac{d^2x}{dt^2} = mg - \mu\frac{dx}{dt}$$

Non-homogeneous equation

Rearranging:

$$\boxed{\frac{d^2x}{dt^2} + \frac{\mu}{m}\frac{dx}{dt} = g} \longrightarrow L[x] = g$$

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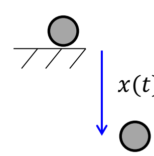
- The general solution of the ***homogeneous*** equation is:



$$x_h(t) = c_1 + c_2 \cdot e^{-\left(\frac{\mu}{m}\right)\cdot t} = c_1 + c_2 e^{-\left(\frac{t}{\tau}\right)}$$

$\tau \equiv m/\mu$ (characteristic time)

- The particular solution of the ***non-homogeneous*** equation is:

$$x_p(t) = \frac{mg}{\mu}t$$

(show by variation of parameters..)

- Thus, the location of the particle:

$$x(t) = x_h(t) + x_p(t) = c_1 + c_2 \cdot e^{-\left(\frac{\mu}{m}\right)\cdot t} + \frac{mg}{\mu}t$$

- By applying the ***initial conditions*** (in rest), the specific solution is:

$$x\Big|_{t=0} = \frac{dx}{dt}\Big|_{t=0} = 0 \longrightarrow \boxed{x(t) = \frac{m^2 g}{\mu}\left[e^{-\left(\frac{t}{\tau}\right)} - 1\right] + \frac{mg}{\mu}t}$$

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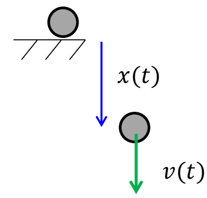
The ***displacement*** of the particle is:



(note that $x(0) = 0$)

$$x(t) = \frac{m^2 g}{\mu}\left[e^{-\left(\frac{t}{\tau}\right)} - 1\right] + \frac{mg}{\mu}t$$

The ***velocity*** of the particle is:

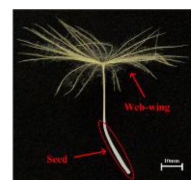
(note that $v(0) = 0$)

$$v(t) = \frac{dx}{dt} = \frac{m^2 g}{\mu}\left[1 - e^{-\left(\frac{t}{\tau}\right)}\right]$$

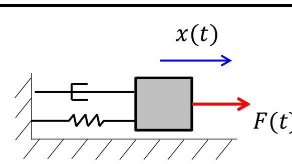
The "***limiting velocity***" (or "***terminal velocity***") for a falling particle in steady state is:

$$v(t \to \infty; t \gg \tau) = \frac{mg}{\mu}$$

This is the ***maximal*** velocity of a falling particle in ***specific*** air-friction conditions.



Example – Mass-spring-dashpot



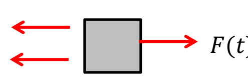
- Consider a mass-spring-dashpot model.
- Let $x(t)$ be the distance from the point of static equilibrium.
- Three forces are acting on the mass:

Excitation force $\quad F(t)$

Spring force $\quad f_k = -kx$

Dissipation force $\quad f_c = -c\dfrac{dx}{dt}$

$f_k = kx$



$f_c = c\dfrac{dx}{dt}$

- The equation of motion is thus:

$$m\frac{d^2x}{dt^2} = F(t) - kx - c\frac{dx}{dt}$$

Non-homogeneous equation

Rearranging:

(See H.W. 1…)

$$\boxed{\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}} \longrightarrow L[x] = F(t)/m$$