Part D - The Frobenius method

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General approach – Frobenius method

➢ Consider the following equation with a ***regular-singular*** point at $x = 0$.

$$\boxed{y'' + p(x) \cdot y' + q(x) \cdot y = 0} \qquad /\cdot x^2$$

with the following are ***analytic*** functions:

$$|x| < \rho_p \qquad x \cdot p(x) = \sum_{k=0}^{\infty} p_k \cdot x^k = p_0 + p_1 \cdot x + \cdots$$

$$|x| < \rho_q \qquad x^2 \cdot q(x) = \sum_{k=0}^{\infty} q_k \cdot x^k = q_0 + q_1 \cdot x + \cdots$$

➢ An alternative form:

$$x^2 \cdot y'' + x \cdot [x \cdot p(x)] \cdot y' + x^2 \cdot q(x) \cdot y = 0$$

$$\boxed{x^2 \cdot y'' + x \cdot \left(\sum_{k=0}^{\infty} p_k \cdot x^k\right) \cdot y' + \left(\sum_{k=0}^{\infty} q_k \cdot x^k\right) \cdot y = 0}$$

$$x^2 \cdot y'' + x \cdot \left(\sum_{k=0}^{\infty} p_k \cdot x^k\right) \cdot y' + \left(\sum_{k=0}^{\infty} q_k \cdot x^k\right) \cdot y = 0$$

Note that: for the case of $p_1 = .. p_n = 0$ and $q_1 = .. q_n = 0$ - the equation reduced to Euler equation:

$$x^2 \cdot y'' + x \cdot p_0 \cdot y' + q_0 \cdot y = 0$$

- We seek a power series solution at the form of:

$$\varphi(x) = x^r \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

With the derivatives:

$$\frac{d\varphi}{dx} = \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot x^{r+n-1} \qquad \frac{d^2\varphi}{dx^2} = \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot (r+n-1) \cdot x^{r+n-2}$$

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- Substituting into the equation:

$$x^2 \cdot \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot (r+n-1) \cdot x^{r+n-2} \quad + x \cdot \left(\sum_{k=0}^{\infty} p_k \cdot x^k\right) \cdot \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot x^{r+n-1}$$

$$+ \left(\sum_{k=0}^{\infty} q_k \cdot x^k\right) \cdot \sum_{n=0}^{\infty} a_n \cdot x^{r+n} = 0$$

- Using ***Cauchy multiplication theorem***: $\left(\sum_{k=0}^{\infty} b_k\right) \cdot \left(\sum_{n=0}^{\infty} c_n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} b_{k-n} c_k$

$$\left(\sum_{k=0}^{\infty} p_k \cdot x^k\right) \cdot \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot x^{r+n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_{n-k} \cdot a_k \cdot (r+k) \cdot x^{r+n}$$

$$\left(\sum_{k=0}^{\infty} q_k \cdot x^k\right) \cdot \sum_{n=0}^{\infty} a_n \cdot x^{r+n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} q_{n-k} \cdot a_k \cdot x^{r+n}$$

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- Substituting into the equation and ***combining*** terms yield:

$$\underbrace{\sum_{n=0}^{\infty}\left\{(r+n)\cdot(r+n-1)\cdot a_n+\sum_{k=0}^{n}[p_{n-k}\cdot(r+k)+q_{n-k}]\cdot a_k\right\}}_{=0}\cdot x^{r+n}$$

- All terms ***must vanish*** for ***all*** $n = 0,1,2,\ldots$ values

$$(r+n)\cdot(r+n-1)\cdot a_n+\sum_{k=0}^{n}[p_{n-k}\cdot(r+k)+q_{n-k}]\cdot a_k=0$$

The case of $n = 0$:

$$r\cdot(r-1)\cdot a_0+[p_0\cdot r+q_0]\cdot a_0=0 \qquad (n=0)$$

$$\underbrace{a_0}_{=0}\cdot\underbrace{[r\cdot(r-1)+p_0\cdot r+q_0]}_{=0}=0$$

And/or

Note that $a_0 \neq 0$,
otherwise: $a_1 = a_2 = \cdots = 0$
(See in the folowing)

$$\boxed{f(r)=r\cdot(r-1)+p_0\cdot r+q_0=0}$$

Indicial equation

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$$\boxed{f(r)=r\cdot(r-1)+p_0\cdot r+q_0=0}$$

- By solving the equation, we can now identify the ***unknown*** "r" values for the series solutions!

- There are three options:

$$\boxed{r_1 > r_2 \in \mathcal{R}} \qquad \boxed{r_1 = r_2 \in \mathcal{R}} \qquad \boxed{r_1, r_2 \in \mathbb{C}}$$

$$r_1, r_2 = \lambda \pm i\mu$$

(out of scope)

$$\boxed{f(r)=(r-r_1)\cdot(r-r_2)}$$

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The case of $n \geq 1$:

$$(r+n)\cdot(r+n-1)\cdot a_n + \sum_{k=0}^{n}[p_{n-k}\cdot(r+k)+q_{n-k}]\cdot a_k = 0$$

$k = n$

$$\underbrace{[(r+n)\cdot(r+n-1)+p_0\cdot(r+n)+q_0]}_{f(r+n)}\cdot a_n + \underbrace{\sum_{k=0}^{n-1}[p_{n-k}\cdot(r+k)+q_{n-k}]\cdot a_k}_{g(a_0,a_1,\dots,a_{n-1})} = 0$$

- The coefficients a_n are extracted as a function of $a_0, a_1, \dots, a_{n-1}$ via:

$$\boxed{a_n = -\frac{g(a_0,a_1,\dots,a_{n-1})}{f(r+n)}}$$

Recurrence relation

$(n = 1,2,3,\dots)$

- Thus, two sets of coefficients are obtained for the selection of $r = r_1$ and $r = r_2$.

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- Thus, for $r = r_1$ we obtain a ***first*** series of coefficients:

$$\boxed{\varphi_1(x) = x^{r_1}\cdot\sum_{n=0}^{\infty} a_n\cdot x^n}\qquad \boxed{a_n = -\frac{g(a_1,\dots,a_{n-1})}{f(r_1+n)}}$$

- And, for $r = r_2$ we obtain a ***second*** series solution:

$$\boxed{\varphi_2(x) = x^{r_2}\cdot\sum_{n=0}^{\infty} b_n\cdot x^n}\qquad \boxed{b_n = -\frac{g(b_1,\dots,b_{n-1})}{f(r_2+n)}}$$

- This all looks almost too perfect… BUT….

When this might arise?

Potential problem #1:

$r_1 = r_2 \;\rightarrow\; \varphi_1(x) = \varphi_2(x)$

The two solutions are not independent…

Potential problem #2:

$f(r_1+n) = 0$ *or* $f(r_2+n) = 0$

Coefficients cannot be found…

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- Recall that:

$$f(r) = (r - r_1)\cdot(r - r_2)$$

- Thus:

$$f(r_1 + n) = n\cdot[n + (r_1 - r_2)] \longrightarrow f(r_1 + n) > 0$$

No problems here..

Assuming that $r_1 > r_2$

$$f(r_2 + n) = n\cdot[n + (r_2 - r_1)] \longrightarrow f(r_2 + n) = n\cdot[n - \underbrace{(r_1 - r_2)}_{r_1 - r_2 = m}]$$

$$f(r_2 + n) = n\cdot(n - m) \underbrace{=}_{n=m} 0$$

Potential problem #2:

$r_1 > r_2$ where $r_1 - r_2 = m$

$m = integer\ (e.g.\ 1,2,3 \ldots)$

The Frobenius theorem

$$y'' + p\cdot(x)y' + q(x)\cdot y = 0$$

*Let $x = 0$ be a **singular-regular point** of the differential equations.*

*Let $x\cdot p(x)$ and $x^2\cdot q(x)$ be **analytic functions** at $x = 0$, with their power series **converge** for radius ρ_p and ρ_q respectively:*

$$x\cdot p(x) = p_0 + p_1\cdot x + \cdots = \sum_{n=0}^{m} p_n\cdot x^n \qquad |x| < \rho_p \neq 0$$

$$x^2\cdot q(x) = q_0 + q_1\cdot x + \cdots = \sum_{n=0}^{m} q_n x^n \qquad |x| < \rho_q \neq 0$$

Let r_1 and r_2 be the roots of the indicial equation:

$$f(r) = r\cdot(r-1) + p_0\cdot r + q_0 = 0$$

*Then, there **exists** a unique power series **solution** which **converges** for $\rho \leq min(\rho_p, \rho_q)$ in:*

$0 < x < \rho$ or $-\rho < x < 0$

*We consider **three type cases** of solutions - in regard with real-valued r_1 and r_2 roots.*

Case I:

$r_1 > r_2$ $\quad$ $r_1 - r_2 \neq k$ $\quad$ $(k = 1,2,\dots)$

$$\varphi_1(x) = |x|^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \qquad a_0 = 1$$

$$\varphi_2(x) = |x|^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n \qquad b_0 = 1$$

Example (shown before)

$$4x \cdot y'' + 3 \cdot y' - 3 \cdot y = 0$$

$x \cdot p(x) = 3$

$x^2 \cdot q(x) = -3 \cdot x$

$$f(r) = r \cdot (r-1) + 3 \cdot r = 0$$

$r_1 = 0$

$r_2 = \frac{1}{4}$

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Case II - repeaterd roots:

$r_1 = r_2 = r$

$$\varphi_1(x) = |x|^{r} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \qquad a_0 = 1$$

- *And a second **independent** solution is:*

$$\varphi_2(x) = \varphi_1(x) \cdot ln(|x|) + |x|^{r} \cdot \sum_{n=1}^{\infty} b_n \cdot x^n$$

Example (will be shown soon)

$$x \cdot y'' + y' - y = 0$$

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Case III – roots differing by an integer $\quad r_1 > r_2 \quad r_1 - r_2 = m \quad (m = 1,2,\dots)$

$a_0 = 1$

$$\varphi_1(x) = |x|^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

- As shown before, ***if*** we suggest a second solution in the form:

$$\varphi_2(x) = |x|^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

$$b_n = -\frac{g(b_1,\dots,b_{n-1})}{f(r_2+n)} = -\frac{g(b_1,\dots,b_{n-1})}{n\cdot(n-m)}$$

$b_{n=m}$ - cannot be found

- ***Alternatively***, *a second* ***independent*** *solution is:*

$b_0 = 1$

$$\varphi_2(x) = K\cdot\varphi_1(x)\cdot ln(|x|) + |x|^{r_2}\cdot\sum_{n=0}^{\infty} b_n \cdot x^n$$

K is obtained during finding b_n in the solution

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Note that:

- For ***some cases III*** types $K = 0$ - and the solution will ***reduces*** to the form of ***case I*** (!)

- For example: when both the nominator and the denominator have the term $(n-m)$:

$$b_n = -\frac{g(b_1,\dots,b_{n-1})}{n\cdot(n-m)} = -\frac{\tilde{g}\,(b_1,\dots,b_{n-1})\cdot(n-m)}{n\cdot(n-m)}$$

$$\downarrow$$

$$[b_n\cdot n + \tilde{g}\,(b_1,\dots,b_{n-1})]\cdot(n-m) = 0$$

Problem is eliminated...

- The above relation is ***always satisfied*** for $n = m$. $\quad [b_n\cdot n + \tilde{g}\,(b_1,\dots,b_{n-1})]\cdot 0 = 0$

- ***Any arbitrary*** selection of is ok (!) → even $b_{n=m} = 0$.

Examples (will be shown soon)

$$x\cdot y'' + y' - y = 0 \qquad K \neq 0$$

$$x\cdot y'' + 2\cdot y' + x\cdot y = 0 \qquad K = 0$$

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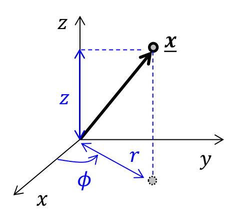
Part E - The Bessel equation

Friedrich Wilhelm Bessel

Motivation – Laplace equation in cylindrical coordinates



- The Laplace equation in cylindrical coordinates (ρ, ϕ, z):

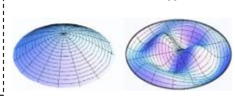
$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \cdot \frac{\partial \Phi}{\partial \rho} + + \frac{1}{\rho^2} \cdot \frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

Φ may represent

Electrostatic potential inside a conducting cylinder

Temperature field within a cylinder (steady-state)

Standing waves of a circular membrane



- Proposing a solution in the form of (separation of variables): $\Phi = R(\rho) \cdot Q(\phi) \cdot Z(z)$
- After substituting (not shown..), the Laplace equation is decomposed into:

z direction

$$\frac{d^2 Z}{dz^2} - k^2 \cdot Z = 0$$

ϕ direction

$$\frac{d^2 Q}{d\phi^2} + \nu^2 \cdot Q = 0$$

R direction

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \cdot \frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right) \cdot R = 0$$

$$x = \rho \cdot k$$

Bessel equation

$$\boxed{\frac{d^2 R}{dx^2} + \frac{1}{x} \cdot \frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) \cdot R = 0}$$

(I) Bessel equation of an order ν

$$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0 \qquad \nu \geq 0$$

- The equation has a ***regular-singular*** point at $x = 0$.

$$p(x) = \frac{1}{x} \longrightarrow x \cdot p(x) = 1 \longrightarrow p_0 = 1$$

$$q(x) = \frac{x^2 - \nu^2}{x^2} \longrightarrow x^2 \cdot q(x) = x^2 - \nu^2 \longrightarrow q_0 = -\nu^2$$

- The indicial equation is: $f(r) = r \cdot (r - 1) + r - \nu^2 = 0$

$$f(r) = r^2 - \nu^2 = 0$$

- The roots of the equation are: $\boxed{r_1 = \nu \quad r_2 - \nu}$

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First solution of Bessel equation ($r_1 = \nu$)

- Proposing first solution in the form of:

$$\varphi_1(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n+\nu}$$

- The derivatives of $\varphi_1(x)$ are:

$$\frac{d\varphi_1}{dx} = \sum_{n=0}^{\infty} a_n \cdot (n + \nu) \cdot x^{n+\nu-1} \qquad \frac{d^2\varphi_1}{dx^2} = \sum_{n=0}^{\infty} a_n \cdot (n + \nu) \cdot (n + \nu - 1) \cdot x^{n+\nu-2}$$

- Substituting into the Bessel equation: $x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0$

$$x^2 \cdot \sum_{n=0}^{\infty} a_n \cdot (n + \nu) \cdot (n + \nu - 1) \cdot x^{n+\nu-2}$$

$$+x \cdot \sum_{n=0}^{\infty} a_n \cdot (n + \nu) \cdot x^{n+\nu-1} + (x^2 - \nu^2) \cdot \sum_{n=0}^{\infty} a_n \cdot x^{n+\nu} = 0$$

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- Multiplying and shifting indices (as we practiced before…):

$$\sum_{n=0}^{\infty} a_n \cdot (n+\nu)\cdot(n+\nu-1)\cdot x^{n+\nu} + \sum_{n=0}^{\infty} a_n \cdot (n+\nu)\cdot x^{n+\nu}$$

$$+\sum_{n=2}^{\infty} a_{n-2}\cdot x^{n+\nu} - \nu^2 \sum_{n=0}^{\infty} a_n \cdot x^{n+\nu} = 0$$

- Combining the a_n terms (and some algebra) yields:

$$\sum_{n=0}^{\infty} a_n \cdot n \cdot (n+2\cdot\nu)\cdot x^{n+\nu} + \sum_{n=2}^{\infty} a_{n-2}\cdot x^{n+\nu} = 0$$

- Combining powers:

$$\underbrace{0\cdot a_0 \cdot x^{\nu}}_{(n=0)} \quad \underbrace{+(1+2\cdot\nu)\cdot a_1 \cdot x^{\nu+1}}_{(n=1)} \quad + \underbrace{\sum_{n=2}^{\infty}[n\cdot(n+2\cdot\nu)\cdot a_n + a_{n-2}]\cdot x^{n+\nu}}_{(n\geq 2)} = 0$$

$$0\cdot a_0 \cdot x^{\nu} + \underbrace{(1+2\cdot\nu)\cdot a_1}_{=0} \cdot x^{\nu+1} + \sum_{n=2}^{\infty}\underbrace{[n\cdot(n+2\cdot\nu)\cdot a_n + a_{n-2}]}_{=0}\cdot x^{n+\nu} = 0$$

- As seen, the a_0 term can be determined to be an ***arbitrary*** value $\boxed{a_0 \rightarrow soon \ldots}$

- The second term must vanish and thus: $\boxed{a_1 = 0}$

- The third term in the brackets must vanish and thus yields:

$$n\cdot(n+2\cdot\nu)\cdot a_n + a_{n-2} = 0$$

$$\downarrow$$

$$\boxed{a_n = -\frac{1}{n\cdot(n+2\cdot\nu)}\cdot a_{n-2}}$$

Recurrence relation

$(n = 2,3,\ldots)$

$$\boxed{a_n = -\frac{1}{n \cdot (n + 2 \cdot \nu)} \cdot a_{n-2}}$$

Recurrence relation

$(n = 2,3, \dots)$

Odd terms:

$$\boxed{a_1 = 0} \longrightarrow \boxed{a_{2n+1} = 0}$$

Even terms:

$$a_{2n} = -\frac{1}{2^2 \cdot n \cdot (n + \nu)} \cdot a_{2n-2}$$

$$\downarrow$$

$$\boxed{a_{2n} = \frac{(-1)^n}{2^{2n} \cdot n! \cdot (n + \nu) \cdot (n + \nu - 1) \cdots (\nu + 1)} \cdot a_0}$$

If ν is an integer (i.e. $\nu = m$)

$$\downarrow$$

$$\boxed{a_{2n} = \frac{(-1)^n \cdot \nu!}{2^{2n} \cdot n! \cdot (n + \nu)!} \cdot a_0}$$

*If ν is **not** an integer ???*

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The Gamma function (Γ):

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \qquad x > 0$$

- The Gamma function is characterized by:

$$\boxed{\Gamma(x + 1) = x \cdot \Gamma(x)}$$

$$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt \overset{\text{Integration by parts}}{=} \int_0^\infty d(-t^x \cdot e^{-t}) - \int_0^\infty (-x \cdot t^{x-1} e^{-t} dt)$$

$$= [-t^x \cdot e^{-t}]_{t=0}^{\infty} + \int_0^\infty x \cdot t^{x-1} e^{-t} dt \quad = x \cdot \int_0^\infty t^{x-1} e^{-t} dt \quad = x \cdot \Gamma(x)$$

- For the case of an ***integer*** $x = n$:

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \qquad \Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1! \qquad \Gamma(3) = 2 \cdot \Gamma(2) = 2!$$

$$\boxed{\Gamma(n + 1) = n!}$$

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The Gamma function (Γ) is a generalization of the factorial (Atzeret "!") for non-integer numbers.

Back to the Bessel function

$$a_{2n} = \frac{(-1)^n}{2^{2n} \cdot n! \cdot (n+\nu)\cdot(n+\nu-1)\cdots(\nu+1)} \cdot a_0$$

$$\downarrow$$

$$a_{2n} = \frac{(-1)^n \cdot \Gamma(\nu+1)}{2^{2n} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot a_0$$

A customary selection of a_0

$$a_0 = \frac{1}{2^\nu \cdot \Gamma(\nu+1)}$$

$$\downarrow$$

$$\boxed{a_{2n} = \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)}}$$

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Summary – The first solution of Bessel equation ($r=\nu$)

$\nu \geq 0$

$$J_\nu(x) \equiv \varphi_1(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n+\nu} \quad = \sum_{n=0}^{\infty} a_{2n} \cdot x^{2n+\nu}$$

$$\downarrow$$

Vanishes at $x \to 0$ for $\nu > 0$

$$\boxed{J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot x^{2n+\nu}}$$

Bessel function of the first kind of order ν

- For the *special case* of $\nu = 0$:

$$\boxed{J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot (n!)^2} \cdot x^{2n}}$$

Bessel function of the first kind of order $\nu = 0$

Approaches unity for $x \to 0$

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n} \cdot (n!)^2} \cdot x^{2n}$$

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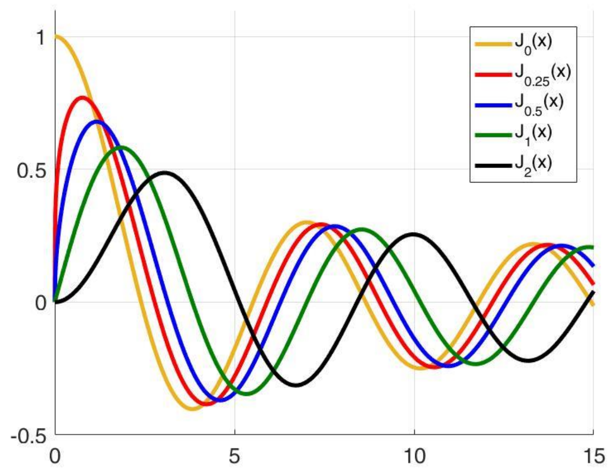
Bessel functions of the first kind ($J_\nu(x)$)



The second solution of Bessel equation ($r_2 = -\nu$)

(I) The case of $\nu \neq k/2$ *(i.e.* $\nu \neq 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$*)*

- The case of $\nu \neq k/2$ corresponds to *Frobenious case I*.

$$r_1 - r_2 = 2 \cdot \nu \neq k$$

- The second solution (for $x > 0$) thus takes the form of :

$$\varphi_2(x) = \sum_{n=0}^{\infty} a_n \cdot x^{n-\nu}$$

- After substituting and ***repeating*** the same procedure ***as before*** it is obtained:

Diverges at $x \to 0$

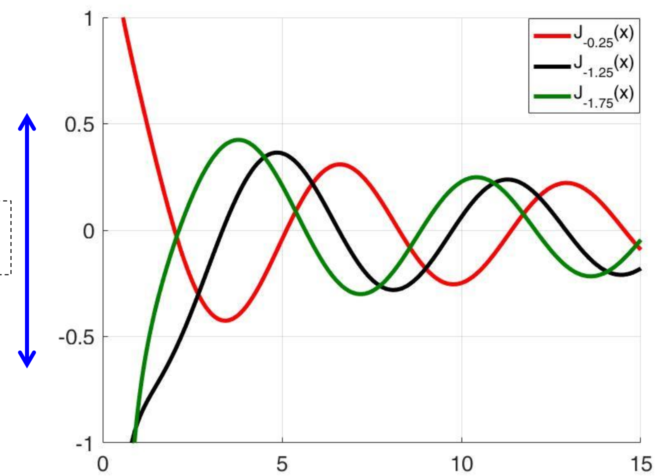
$$\boxed{J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-\nu} \cdot n! \cdot \Gamma(n+1-\nu)} \cdot x^{2n-\nu}}$$

$\nu \neq \frac{k}{2}$

Examples - Bessel functions of the first kind ($J_{-\nu}(x)$)

$\nu \neq \frac{k}{2}$

Diverges at $x \to 0$



(II) Second solution: A half-integer order $\nu = 1/2, 3/2, \ldots$

- The case of an half-integer ν corresponds to *Frobenious case III*.

$$r_1 - r_2 = 2 \cdot \nu = k$$

- Thus, the second solution (for $x > 0$) takes the form of:

$$\varphi_2(x) = K \cdot \varphi_1(x) \cdot ln(x) + x^{-\nu} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

- By using this solution - it is obtained that $K = 0$.

You will experience this in H.W.

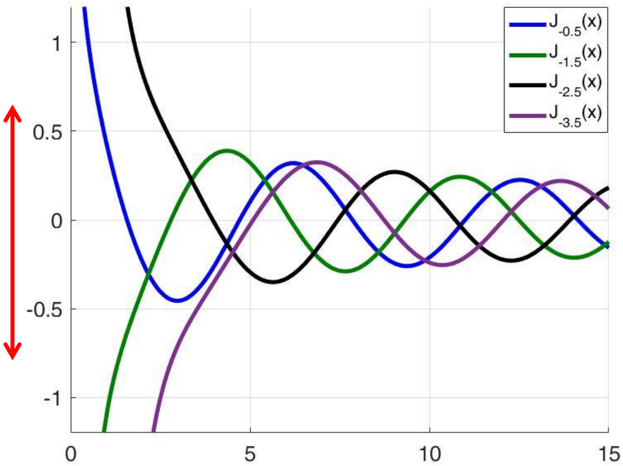
- Finally - the ***second solution*** for Bessel equation of a ***half-integer*** is (as before):

$$\boxed{J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-\nu} \cdot n! \cdot \Gamma(n+1-\nu)} \cdot x^{2n-\nu}}$$

$\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Examples of Bessel functions of the first kind ($J_{-\nu}(x)$) *- Half-integer order*

Diverges at $x \to 0$



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(III) Second solution: An integer order $\nu = 0,1,2,\ldots$

- It ***would be nice*** if we could use $J_{-\nu}(x)$ also for $\nu = k = 0,1,2,\ldots$

But - it is impossible… (!)

- Note that for $\nu = k$ the ***first*** solution yields:

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+k} \cdot n! \cdot \underbrace{(n+k)!}_{=\,\Gamma(n+k+1)}} \cdot x^{2n+k}$$

- Accordingly, the ***second*** solution yields:

$$J_{-k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-k} \cdot n! \cdot (n-k)!} \cdot x^{2n-k}$$

$n < k$ (???)

Since $0! = 1$ $\to (-1)! = \frac{1}{0} = \infty$ $\longrightarrow$ All terms with $n < m$ vanish!

$$= \sum_{n=k}^{\infty} \frac{(-1)^n}{2^{2n-k} \cdot n! \cdot (n-k)!} \cdot x^{2n-k}$$

Shifting indices

$$= (-1)^k \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+k} \cdot n! \cdot (n+k)!} \cdot x^{2n+k} = (-1)^k \cdot J_k(x)$$

!!!!!

The J_k and J_{-k} functions are *not* *independent*!

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How to resolve this problem?

- Introducing ***Weber*** (or ***Neumann***) ***function***:

$$\boxed{Y_\nu(x) = \frac{cos(\nu \cdot \pi) \cdot J_\nu(x) - J_{-\nu}(x)}{sin(\nu \cdot \pi)}}$$

Bessel function of the second kind of order ν

- The function $Y_\nu(x)$ is ***linear independent*** of $J_\nu(x)$.

- The ***second solution*** for Bessel equation of an ***integer order*** is:

$$\boxed{Y_k(x) = \lim_{\nu \to k} Y_\nu(x)}$$

(tedious expressions…)

Note that
$Y_k(x)$ can be obtained explicitly via Frobenius solutions (cases II and III)

Example of an explicit expression ($\nu = 0$):

$$Y_0(x) = \frac{2}{\pi} \cdot \left[\left(\gamma + ln\left(\frac{x}{2}\right) \right) \cdot J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot H_n}{2^{2n} \cdot (n!)^2} \cdot x^{2n} \right]$$

Euler-Mascheroni constant $\gamma \approx 0.5772$

Harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

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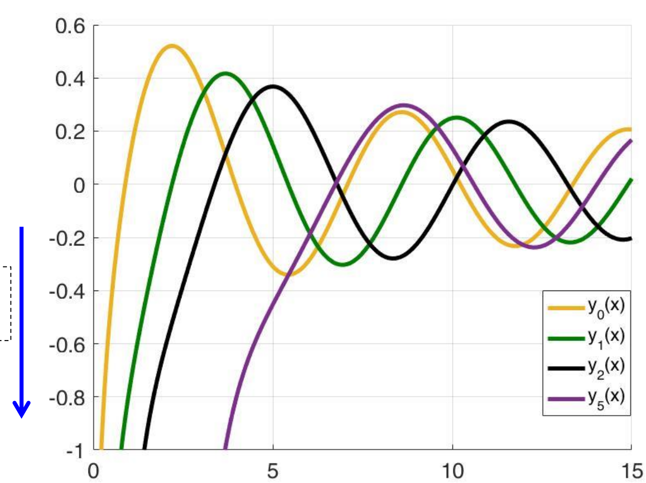
Examples of Bessel functions of the second kind ($Y_\nu(x)$) - an integer order

Diverges at $x \to 0$



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Summary – General solutions of Bessel equation

$$\boxed{x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0}$$

Non-integer orders $(\nu \neq k)$:

$$\boxed{\varphi(x) = A \cdot J_\nu(x) + B \cdot J_{-\nu}(x)} \qquad \nu \neq k$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot x^{2n+\nu} \qquad J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n-\nu} \cdot n! \cdot \Gamma(n+1-\nu)} \cdot x^{2n-\nu}$$

Integer orders $(\nu = k)$:

$$\boxed{\varphi(x) = A \cdot J_\nu(x) + B \cdot Y_\nu(x)} \qquad \nu = k$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \cdot \Gamma(n+\nu+1)} \cdot x^{2n+\nu} \qquad Y_\nu(x) = \frac{cos(\nu \cdot \pi) \cdot J_\nu(x) - J_{-\nu}(x)}{sin(\nu \cdot \pi)}$$

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(II) Asymptotic behavior and reoccurrence relations of Bessel functions

Small x values $(x \ll 1)$:

$$J_\nu(x) \approx \frac{1}{2^\nu \cdot \Gamma(\nu+1)} \cdot x^\nu + \cdots$$

$$Y_{\nu=0}(x) \approx \frac{2}{\pi} \cdot \left[\gamma + ln\left(\frac{x}{2}\right) + \cdots\right]$$

$$Y_{\nu\neq 0}(x) \approx -\frac{\Gamma(\nu) \cdot 2^\nu}{\pi} \cdot x^{-\nu} + \cdots$$

Power-law (or logarithmic) for $x \to 0$

Large x values $(x \gg 1, \nu)$:

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi \cdot x}} \cdot cos\left(x - \frac{\nu \cdot \pi}{2} - \frac{\pi}{4}\right) \qquad Y_\nu(x) \approx \sqrt{\frac{2}{\pi \cdot x}} \cdot sin\left(x - \frac{\nu \cdot \pi}{2} - \frac{\pi}{4}\right)$$

Decaying fluctuations for $x \to \infty$

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Recurrence relations

The same relations are also valid for $Y_\nu(x)$

- Derivatives of Bessel functions:

$$\frac{d}{dx}(x^\nu \cdot J_\nu(x)) = x^\nu \cdot J_{\nu-1}(x)$$

$$\frac{d}{dx}(x^{-\nu} \cdot J_\nu(x)) = -x^{-\nu} \cdot J_{\nu+1}(x)$$

- The ***derivative*** of Bessel function of an order ν can be expressed by the ***adjacent orders***:

$$\frac{dJ_\nu(x)}{dx} = \frac{1}{2} \cdot [J_{\nu-1}(x) - J_{\nu+1}(x)]$$

- Bessel function of an order ν can also be expressed by the ***adjacent orders*** :

$$\left(\frac{\nu}{x}\right) \cdot J_\nu(x) = \frac{1}{2} \cdot [J_{\nu-1}(x) + J_{\nu+1}(x)]$$

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Example – Explicit expressions for Bessel function of half-integer

- On the ***previous lecture slides*** we obtained ***explicit*** expressions for $J_{\pm\frac{1}{2}}(x)$:

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot sin(x) \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot cos(x)$$

Can use the recurrence formula to obtain an explicit expression for $J_{\frac{3}{2}}(x)$?

$$\left(\frac{\nu}{x}\right) \cdot J_\nu(x) = \frac{1}{2} \cdot [J_{\nu-1}(x) + J_{\nu+1}(x)]$$

$$\left(\frac{1/2}{x}\right) \cdot J_{\frac{1}{2}}(x) = \frac{1}{2} \cdot \left[J_{\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)\right] \longrightarrow J_{\frac{3}{2}}(x) = \left(\frac{1}{x}\right) \cdot J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$\boxed{J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot \left[\frac{sin(x)}{x} - cos(x)\right]}$$

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By the same procedure

Explicit expression are obtained for ***all*** *(!) half-integer Bessel function…*

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot sin(x)$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot cos(x)$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot \left[\frac{sin(x)}{x} - cos(x)\right]$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot \left[\frac{cos(x)}{x} + sin(x)\right]$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot \left[\left(\frac{3}{x^2} - 1\right) \cdot sin(x) - \frac{3}{x} \cdot cos(x)\right]$$

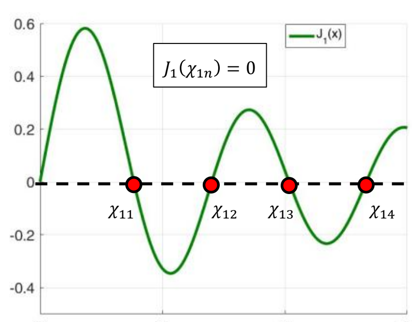
$$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi \cdot x}} \cdot \left[\frac{3}{x} \cdot sin(x) + \left(\frac{3}{x^2} - 1\right) \cdot cos(x)\right]$$

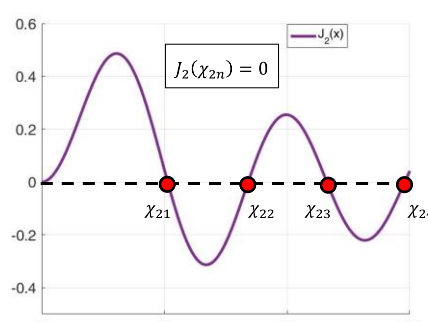
$\vdots$ $\vdots$

(III) Zeroes of Bessel functions

- From the asymptotic forms it is clear that Bessel functions, $J_\nu(x)$ and $Y_\nu(x)$, have ***infinite*** number of ***zeroes***.

- We will be ***chiefly concerned*** with the zeroes of $J_\nu(x)$: $\boxed{J_\nu(\chi_{\nu n}) = 0 \quad (n = 1,2,3\ldots)}$





Zeros of Bessel functions – some numeric data

$J_0(x)$	χ_{01}	χ_{02}	χ_{03}	χ_{04}	χ_{05}
	2.405	5.520	8.654	11.792	14.931

$J_1(x)$	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
	3.882	7.016	10.174	13.324	16.471

$J_2(x)$	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}
	5.136	8.417	11.620	14.796	17.960

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Scaling Bessel function – setting zero conditions

- For a ***given*** ν, the solutions of Bessel equation - Bessel function $J_\nu(x)$ - is ***specified***.

$$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0 \longrightarrow \boxed{\varphi(x) = J_\nu(x)} \quad (*)$$

- Consider now the case where the ***solution must satisfy*** the condition:

$$\boxed{\varphi(x = a) = 0} \quad (**)$$

- Thus, we can ***generate*** a ***set of functions*** that satisfy both (*) and (**):

$$\boxed{\varphi_n(x) = J_\nu(\chi_{\nu n} \cdot x/a)} \longrightarrow \varphi_n(a) = J_\nu(\chi_{\nu n}) = 0$$

- ***The meaning:*** The x coordinate of each $J_\nu(x)$ function is "***stretched***" by $\chi_{\nu n}$, to make the n^th zero of the function appears in $x = a$.

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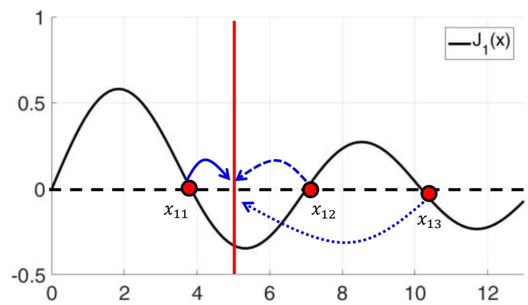
Example $J_1(x)$



- ***If*** the condition of ***zero*** at, say $x = 5$, must be satisfied - the x coordinate can be scaled.

First zero at $x = 5$ $\quad \varphi_1(x) = J_1(\chi_{11} \cdot x/5) \longrightarrow \varphi_1(5) = J_1(\chi_{11}) = 0$

And/or

Second zero at $x = 5$ $\quad \varphi_2(x) = J_1(\chi_{12} \cdot x/5) \longrightarrow \varphi_2(5) = J_1(\chi_{12}) = 0$

And/or

Third zero at $x = 5$ $\quad \varphi_3(x) = J_1(\chi_{13} \cdot x/5) \longrightarrow \varphi_3(5) = J_1(\chi_{13}) = 0$

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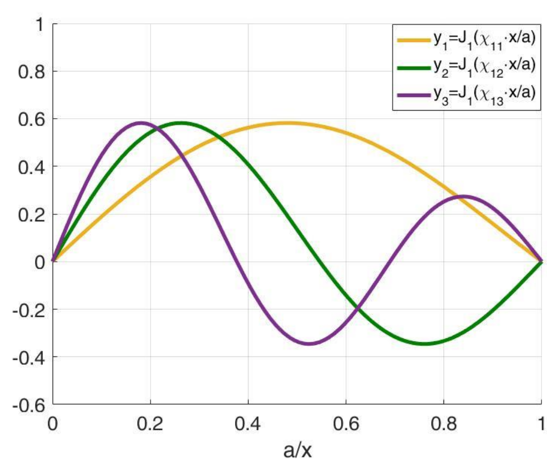
We can ***generate*** *an* ***infinite*** *set of functions that satisfy* $J_1(a) = 0$

$$\boxed{\varphi_n(x) = J_1(x_{1n} \cdot x/a)} \qquad (n = 1,2, \dots)$$



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Orthogonality of the Bessel functions

- For a ***general*** ν-order Bessel equation, an ***infinite set*** of zero-scaled solutions can be obtained.

$$\boxed{\varphi_n(x) = J_\nu(\chi_{\nu n} \cdot x/a)} \qquad (n = 1,2,\dots,\infty)$$

An orthogonal set: Each two functions are orthogonal in the sense of:

(Not so trivial to show…)

$$\int_0^a x \cdot \underbrace{J_\nu(\chi_{\nu n} \cdot x/a)}_{\varphi_n(x)} \cdot \underbrace{J_\nu(\chi_{\nu m} \cdot x/a)}_{\varphi_m(x)}\, dx = \frac{a^2}{2} \cdot (J_{\nu+1}(\chi_{\nu n}))^2 \cdot \delta_{mn}$$

- Note the "weight function" (x) in the integral (*will be discussed later on*).

This set of Bessel functions spans the function space at the range of $0 < x < a$