1 Example 1

Given equation:

$$y'' + \lambda \cdot y = 0 \qquad 0 < x < 1$$

$$y(0) = 0 \qquad y(1) + ky(1) = 0 \quad k > 0$$

(This is regular S-L (Sturm–Liouville), because BC are regular and the operator is self adjoint)
Find the values of λ: $\lambda_n = \{\lambda_1, \lambda_2 \dots\}$ so $\varphi_n(x; \lambda_n) \neq 0$

Solution:

$$y'' + \lambda y = 0$$

3 options exist: $\longrightarrow \lambda = 0$, $\lambda < 0$, $\lambda > 0$
Starting with $\boxed{\lambda = 0}$

$$y'' = 0$$

$$\varphi(x) = A + B \cdot x$$

$$y(0) = 0 \qquad \Rightarrow \qquad A = 0$$

$$y(1) + y'(1) \cdot k = 0$$

$$\Rightarrow B + Bk = 0 \qquad \Rightarrow \qquad B = 0 \qquad \because k > 0$$

Hence the solution is trivial: $\lambda = 0 \qquad \Rightarrow \qquad \varphi(x) = 0$ and thus $\lambda = 0$ is <u>not an eigenvalue</u>.
Checking for $\boxed{\lambda < 0}$:
Defining $\lambda \equiv -\mu^2$

$$y'' - \mu^2 y = 0$$

$$\rightarrow \left\{e^{\mu x}; e^{-\mu x}\right\}$$

More convenient form:

$$\varphi(x) = A \cdot \sinh(\mu x) + B \cdot \cosh(\mu x)$$

Boundary conditions:

$$y(0) = B \cdot \cosh(0) = 0 \qquad \Rightarrow \qquad B = 0 \qquad \Rightarrow \qquad \varphi(x) = A \cdot \sinh(\mu x)$$

$$y(1) + y'(1) \cdot k = 0$$

$$A \cdot [\sinh(\mu) + \mu \cdot k \cdot \cosh(\mu)] = 0$$

Reminder: $\mu \neq 0$

Because of hyperbolic sine and cosine (essentially, $\mathbf{tanh}\,(\boldsymbol{\mu}) = \boldsymbol{\mu k}$ only for $\boldsymbol{\mu} = \mathbf{0}$) the expression in braces is not equal to 0, hence: $A = 0$. Thus $\lambda < 0$ is not an eigenvalue.

Finally, $\boxed{\lambda > 0}$:

Defining $\lambda \equiv \mu^2$

$$y'' + \mu^2 y = 0$$

$$\varphi(x) = A \cdot \sin(\mu x) + B \cdot \cos(\mu x)$$

Boundary conditions:

$$y(0) = B \cdot \cos(0) = 0 \qquad \Rightarrow \qquad B = 0 \qquad \Rightarrow \qquad \varphi(x) = A \cdot \sin(\mu x)$$

$$y(1) + y'(1) \cdot k = 0$$

$$A \cdot [\sin(\mu) + \mu \cdot k \cdot \cos(\mu)] = 0$$

Non-trivial solution achieved by demanding:

$$\sin(\mu) + \mu \cdot k \cdot \cos(\mu) = 0$$

$$\tan(\mu) = -\mu \cdot k$$

Summary:

For $\lambda = \mu^2 > 0$ a set of solutions exist so:

$$\varphi(x) = A \cdot \sin\left(\sqrt{\lambda_n} x\right) \neq 0$$

Where λ_n are eigenvalues and $\varphi(x)$ are eigenfunctions.

2 Example 2

Given equation:

$$y'' + \lambda \cdot y = 0 \qquad 0 < x < 1$$

$$y(-\pi) = y(\pi) \qquad y'(-\pi) = y\left('\pi\right)$$

Periodic BC.

Easily seen: $P(x) \equiv 1 \Rightarrow P(-\pi) = P(\pi)$

This is a periodic S-L problem.

Find the values of λ: $\lambda_n = \{\lambda_1, \lambda_2 \ldots\}$ so $\varphi_n(x; \lambda_n) \neq 0$

Solution:

Again, 3 options: $\lambda = 0$, $\lambda < 0$, $\lambda > 0$

Going straight for $\boxed{\lambda > 0}$ because the equation is the same and given periodic BC rule out other options:

$\lambda = 0 \qquad \Rightarrow \qquad \varphi(x) = A + Bx \qquad \Rightarrow \qquad A = B = 0$
$\lambda < 0$ doesn't work too (exponentials and stuff)
Hence, $\lambda \equiv \mu^2 > 0$:

$$y'' + \mu^2 y = 0$$

$$\varphi(x) = A \cdot \sin(\mu x) + B \cdot \cos(\mu x)$$

Boundary conditions:
$\varphi(-\pi) = \varphi(\pi)$

$$A \cdot \sin(-\mu\pi) + B \cdot \cos(-\mu\pi) = A \cdot \sin(\mu\pi) + B \cdot \cos(\mu\pi)$$

$$\mapsto 2 \cdot A \cdot \sin(\mu\pi) = 0$$

$\varphi'(-\pi) = \varphi'(\pi)$

$$A \cdot \mu \cdot \cos(-\mu\pi) + B \cdot \mu \cdot \sin(-\mu\pi) = A \cdot \mu \cdot \cos(\mu\pi) + B \cdot \mu \cdot \sin(\mu\pi)$$

$$\mapsto 2 \cdot B \cdot \sin(\mu\pi) = 0$$

There are 2 options now:

$$\sin(\mu\pi) \neq 0 \qquad \Rightarrow \qquad A = B = 0$$

Or:

$$\sin(\mu\pi) = 0 \qquad \Rightarrow \qquad \varphi(x) \neq 0$$

$$\Rightarrow \mu_n = 1, 2, \ldots$$

Summary:
$\mu_1 = 1 \to \begin{cases} A\sin(1 \cdot x) \\ B\cos(1 \cdot x) \end{cases}$
We can see that for single eigenvalue we have **two** eigenfunctions. Same for different eigenvalues.
$\mu_1 = 2 \to \begin{cases} A\sin(2 \cdot x) \\ B\cos(2 \cdot x) \end{cases}$.

Note: for some problems, certain eigenvalues can have more than one eigenfunction while others might have one only.