Part B – *Qualitative analysis by separation and comparison theorems*

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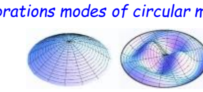
Qualitative behavior of solutions

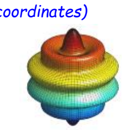
- ***Exact analytical*** solutions of second-order differential equations can be obtained ***only*** for ***specific*** cases.
- Many ***important*** equations, however, cannot be solved explicitly…
- A few examples:

Airy equation $$y'' - x \cdot y = 0$$ *(Light diffraction through a circular aperture)*

Bessel equation $$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0$$ *(vibrations modes of circular membrane)*



Legendre equation $$(1 - x^2) \cdot y'' - 2x \cdot y' + l \cdot (l + 1) \cdot y = 0$$ *(Heat conduction and electrostatics in spherical coordinates)*



- ***Instead*** - we seek for ***qualitative characteristics*** that are of importance from the point of view of physical applications

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(I) Qualitative analysis via the separation theorem

Theorem 5 – Sturm separation theorem

$$L[y] = y'' + p(x)y' + q(x)y = 0 \qquad a \le x \le b \equiv \Delta$$

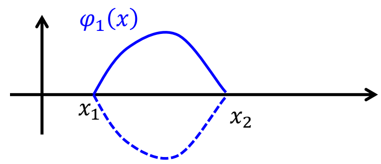
Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent solutions of $L[y] = 0$ on an interval Δ .

*Then, between two **successive zeroes** of $\varphi_1(x)$ there exists **exactly one** zero of $\varphi_2(x)$.*

- ***Meaning:*** *the zeroes of $\varphi_1(x)$ and $\varphi_2(x)$ occur **alternatively**.*

Proof:

Let x_1 and x_2 be **succesive zeroes** of $\varphi_1(x)$, i.e. $\varphi_1(x_1) = \varphi_1(x_2) = 0$:



Note that $\varphi_1(x)$,can be either positive or negative between these zeroes ($x_1 < x < x_2$).

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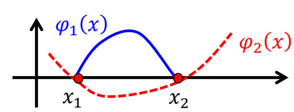
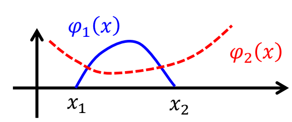
Four possibilities for $\varphi_2(x)$

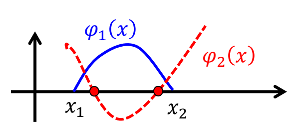
Case I: $\varphi_2(x)$ **vanishes** at x_1 or/and x_2.



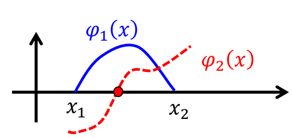
Case II: $\varphi_2(x)$ **doesn't vanish** at $x_1 \le x \le x_2$.



Case III: $\varphi_2(x)$ vanishes **twice or more** at $x_1 < x < x_2$.



Case IV: $\varphi_2(x)$ vanishes **exactly once** at $x_1 < x < x_2$.



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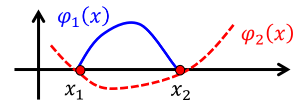
Case I:

Assuming that $\varphi_2(x)$ ***vanishes*** at x_1 or/and x_2.



Since however $\varphi_1(x)$ *and* $\varphi_2(x)$ *are linearly independent solutions of* $L[y]=0$:

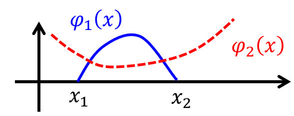
$$W(\varphi_1,\varphi_2,x)=\varphi_1\varphi'_2-\varphi'_1\varphi_2\neq 0$$

Contradiction!

Thus, $\varphi_2(x_1)\neq 0$, $\varphi_2(x_2)\neq 0$ - otherwise $W(\varphi_1,\varphi_2;x_1\ or\ x_2)=0$.

Case II:

Assuming that $\varphi_2(x)$ ***doesn't vanish*** at $x_1\leq x\leq x_2$.



We can define a twice differential function:

$$y=\frac{\varphi_1}{\varphi_2}\quad\longrightarrow\quad y(x_1)=0\qquad y(x_2)=0$$

Following ***Rolle's theorem*** (HEDVA..) - the derivative $y'(x)$ *must vanish* at least once at $x_1<x<x_2$.

The derivative $y'(x)$ is:

$$y'=\frac{\varphi'_1\varphi_2-\varphi_1\varphi'_2}{(\varphi_2)^2}\quad=-\frac{W(\varphi_1,\varphi_2,x)}{(\varphi_2)^2}\neq 0$$

Contradiction!

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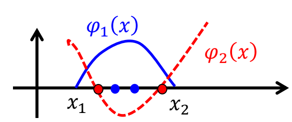
- So, following cases (I-II), $\varphi_2(x)$ must vanishes ***once or more*** at $x_1<x<x_2$.

Case III:

Assuming that $\varphi_2(x)$ vanishes ***twice or more*** at $x_1<x<x_2$.



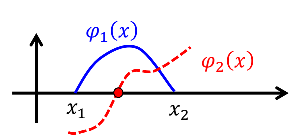
Thus, following the above, $\varphi_1(x)$ *must* also vanish - *at least once* - between the zeroes of $\varphi_2(x)$.

Contradiction!

However, it is *initially assumed* that x_1 and x_2 are *successive zeroes* of $\varphi_1(x)$.

Case IV – the only possibility left…

$\varphi_2(x)$ ***must*** vanishes ***exactly once*** at $x_1<x<x_2$.



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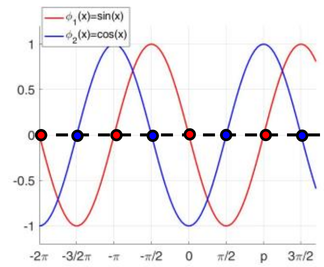
Simple example 1:

$$y'' + y = 0$$

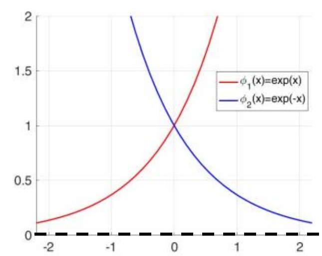
- The solutions are $\varphi_1(x) = sin(x)$ *and* $\varphi_2(x) = cos(x)$.
- The zeroes of $\varphi_1(x) = sin(x)$ *occur at* $x = n\pi$.
- The zeroes of $\varphi_2(x) = cos(x)$ *occur at* $x = \frac{\pi}{2} + n\pi$.



Simple example 2:

$$y'' - y = 0$$

- The solutions are $\varphi_1(x) = e^x$ *and* $\varphi_2(x) = e^{-x}$.
- Neither $\varphi_1(x)$ *nor* $\varphi_2(x)$ *vanishes in* $-\infty < x < \infty$.



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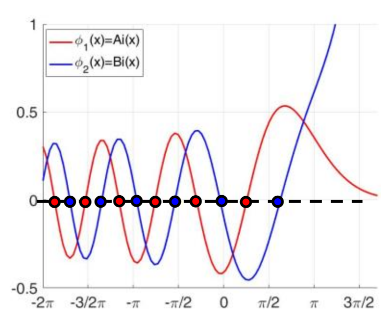
Less trivial example:

$$y'' - xy = 0$$

Airy equation

- This equation doesn't have explicit analytical solutions.
- Local solutions are obtained by a series expansion (we will see later on..).
- The solutions of the equation are "Airy functions": $\varphi_1(x) = Ai(x)$ *and* $\varphi_2(x) = Bi(x)$
- The zeros of $\varphi_1(x)$ *and* $\varphi_2(x)$ are ***alternating***.



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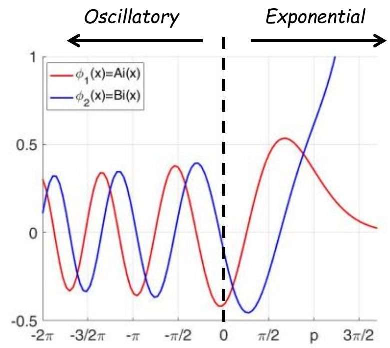
$$y'' - xy = 0$$

Airy equation

- The equation is characterized by a ***turning point*** at $x = 0$.
- For $x > 0$ the solution is "***increasingly exponential***".
- For $x < 0$ the solution is "***increasingly oscillatory***".

→ the zeros of $\varphi_1(x)$ *and* $\varphi_2(x)$ are getting ***closer and closer*** as $x < 0$.



(II) Normal form and the comparison theorem

$$a_0(x) \cdot y'' + a_1(x) \cdot y' + a_2(x) \cdot y = 0 \quad \xrightarrow{\quad} \quad u'' + Q(x) \cdot u = 0$$

General form | *(??)* | *Normal form*

- To proceed we wish to transfer the equation into a "*normal form*".
- We ***propose*** a solution for the equation in the form of:

$$\varphi(x) = u(x) \cdot p(x) \longrightarrow \varphi' = u'p + up' \longrightarrow \varphi'' = u''p + 2u'p' + up''$$

- Substituting the φ into the equation:

$$a_0(x) \cdot (u''p + 2u'p' + up'') + a_1(x) \cdot (u'p + up') + a_2(x) \cdot (up) = 0$$

- Rearranging:

$$a_0 \cdot u'' \cdot p + u' \cdot (2a_0p' + a_1p) + u \cdot (a_0p'' + a_1p' + a_2p) = 0$$

$$u''\cdot p + u'\cdot\left(\cancel{2p' + \frac{a_1}{a_0}p}\right) + u\cdot\underbrace{\left(p'' + \frac{a_1}{a_0}p' + \frac{a_2}{a_0}p\right)}_{Q(x)\cdot p} = 0$$

- We select $p(x)$ to make the second term vanish :

$$2\cdot p' + \frac{a_1}{a_0}\cdot p = 0 \longrightarrow p(x) = e^{-\frac{1}{2}\int\left(\frac{a_1}{a_0}\right)dx}$$

- Finally, dividing by $p(x)$ we obtain the *normal form*:

$$\boxed{u'' + Q(x)\cdot u = 0}$$

Normal form

- Where:

$$Q(x) = \frac{p''}{p} + \frac{a_1}{a_0}\frac{p'}{p} + \frac{a_2}{a_0} \qquad p(x) = e^{-\frac{1}{2}\int\left(\frac{a_1}{a_0}\right)dx}$$

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Example - Bessel equation:

Bessel equation $\quad x^2\cdot y'' + x\cdot y' + (x^2 - \nu^2)\cdot y = 0 \qquad \nu = const\ , \qquad x > 0$

- To transfer into the *normal form* we identify:

$$a_0 = x^2\ , a_1 = x\ , a_2 = x^2 - \nu^2$$

- Thus, $p(x)$ is:

$$p(x) = e^{-\frac{1}{2}\int\left(\frac{a_1}{a_0}\right)dx} = e^{-\frac{1}{2}\int\left(\frac{x}{x^2}\right)dx} = e^{-\frac{1}{2}ln(x)}$$

$$\longrightarrow \boxed{p(x) = x^{-\frac{1}{2}}} \qquad p'(x) = -\frac{1}{2}x^{-\frac{3}{2}} \qquad p''(x) = -\frac{3}{4}x^{-\frac{5}{2}}$$

- Calculating $Q(x)$:

$$Q(x) = \frac{p''}{p} + \frac{a_1}{a_0}\frac{p'}{p} + \frac{a_2}{a_0} = -\frac{3}{4}x^{-2} + \left(\frac{x}{x^2}\right)\cdot\left(-\frac{1}{2}x^{-1}\right) + \left(\frac{x^2 - \nu^2}{x^2}\right)$$

$$\longrightarrow \boxed{Q(x) = 1 + \frac{1 - 4\nu^2}{4x^2}}$$

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$$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0 \longrightarrow u'' + \left(1 + \frac{1 - 4\nu^2}{4x^2}\right) u = 0$$

Bessel equation — *Normal form*

- For the ***special case*** of $\nu = 1/2$, the normal form reduces into:

$$u'' + u = 0$$

- The *solutions* of the ***normal form*** (with $\nu = 1/2$) are:

$$u_1 = sin(x) \qquad u_2 = cos(x)$$

- The *solutions* of ***Bessel equation*** on an order $\nu = 1/2$, are thus :

$$\varphi(x) = u(x)p(x) \ = u(x)x^{-\frac{1}{2}} \longrightarrow \boxed{\varphi_1(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{sin(x)}{\sqrt{x}}} \qquad \boxed{\varphi_2(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{cos(x)}{\sqrt{x}}}$$

$$J_{\nu=1/2}(x) \qquad J_{\nu=-1/2}(x)$$

- These ***solutions*** are denoted as "***Bessel functions of the <u>first kind</u> of an order*** $\nu = 1/2$"

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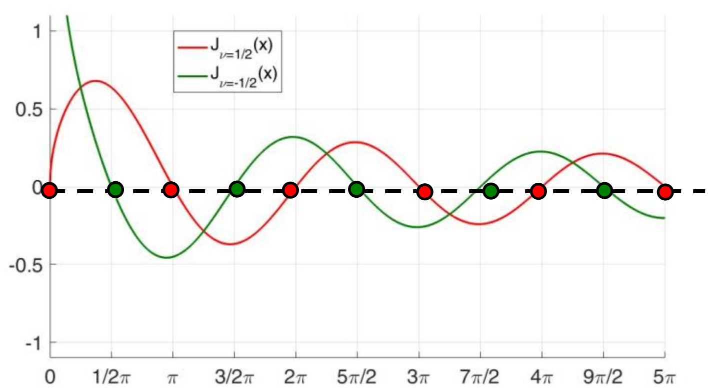
$$J_{\nu=1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{sin(x)}{\sqrt{x}} \longrightarrow J_{\nu=1/2}(x) = 0 \rightarrow x = n\pi$$

$$J_{\nu=-1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{cos(x)}{\sqrt{x}} \longrightarrow J_{\nu=-1/2}(x) = 0 \rightarrow x = n\pi + \frac{\pi}{2}$$

*The zeros of $J_{\nu=1/2}(x)$ and $J_{\nu=-1/2}(x)$ are **alternating***
(Sturm separation theorem..)



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Theorem 6 – Sturm comparison theorem

Let $\varphi(x)$ *and* $\psi(x)$ *be nontrivial solutions of the following* ($\varphi \neq 0, \psi \neq 0$)*:*

$$y'' + A(x) \cdot y = 0 \qquad y'' + B(x) \cdot y = 0 \qquad a \leq x \leq b \equiv \Delta$$

Where $A(x) > B(x)$ *for all* $x \in \Delta$ *.*

Then, between any two zeros of $\psi(x)$ *(if existing..) there is* ***at least*** *one zero of* $\varphi(x)$ *.*

Simple examples:

$\boxed{A(x)=4}$ $\quad y'' + 4 \cdot y = 0 \;\rightarrow\; \varphi(x) = sin(2x) \;\rightarrow\; \varphi(x) = 0 \rightarrow x = \frac{n\pi}{2}$

$\boxed{B(x)=1}$ $\quad y'' + 1 \cdot y = 0 \;\rightarrow\; \psi(x) = sin(x) \;\rightarrow\; \psi(x) = 0 \rightarrow x = n\pi$

$\boxed{A(x) > B(x) \text{ - Faster fluctuations}}$

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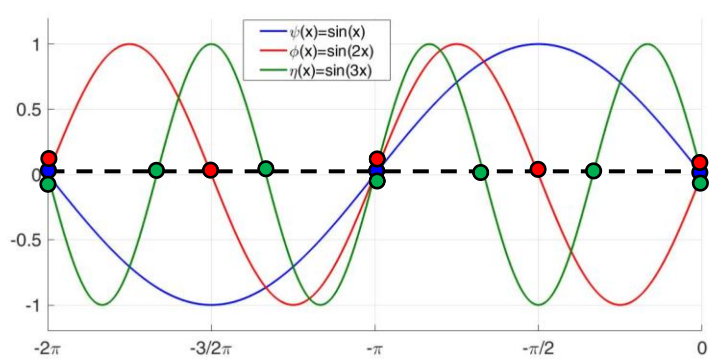
$$y'' + 1y = 0 \longrightarrow \psi(x) = sin(x) \longrightarrow \psi(x) = 0 \rightarrow x = n\pi$$

$$y'' + 4y = 0 \longrightarrow \varphi(x) = sin(2x) \longrightarrow \varphi(x) = 0 \rightarrow x = \frac{n\pi}{2}$$

$$y'' + 16y = 0 \longrightarrow \eta(x) = sin(4x) \longrightarrow \eta(x) = 0 \rightarrow x = \frac{n\pi}{4}$$



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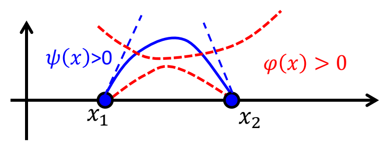
Proof - *Sturm comparison theorem:*

Let $\varphi(x)$ and $\psi(x)$ be nontrivial solutions of the following equations with $A(x) > B(x)$:

$$\varphi'' + A(x)\cdot\varphi = 0 \qquad\qquad \psi'' + B(x)\cdot\psi = 0$$

Let x_1 and x_2 be succesive zeroes of $\psi(x)$, i.e. $\psi(x_1) = \psi(x_2) = 0$:



Considering, for example, that: $\psi(x) > 0 \longrightarrow \psi'(x_1) \geq 0 \qquad \psi'(x_2) \leq 0$

Assuming - ***in contradiction*** - that $\varphi(x)$ has ***no zeros*** for for all $x_1 < x < x_2$.

→ For example: $\varphi(x) > 0$

➢ ***From one side:*** The Wronskian of $\psi(x)$ and $\varphi(x)$ is:

$$W(\varphi,\psi;x) = \varphi(x)\psi'(x) - \varphi'(x)\psi(x)$$

The $W(\varphi,\psi;x)$ at $x = x_1$ and $x = x_2$ yields:

$$W(\varphi,\psi;x_1) = \varphi(x_1)\psi'(x_1) - \varphi'(x_1)\cancel{\psi(x_1)} \qquad = \underbrace{\varphi(x_1)}_{\geq 0}\underbrace{\psi'(x_1)}_{\geq 0} \geq 0$$

$$W(\varphi,\psi;x_2) = \varphi(x_2)\psi'(x_2) - \varphi'(x_2)\cancel{\psi(x_2)} \qquad = \underbrace{\varphi(x_2)}_{\geq 0}\underbrace{\psi'(x_2)}_{\leq 0} \leq 0$$

Thus, ***exists*** (at least) one x in the range $x_1 < x < x_2$ - for which: $\boxed{\dfrac{dW(\varphi,\psi;x)}{dx} \leq 0}$

Contradiction!

➢ From the ***other side***, however:

$$\frac{dW(\varphi,\psi;x)}{dx} = \varphi(x)\psi''(x) - \varphi''(x)\psi(x) \qquad \boxed{\frac{dW(\varphi,\psi;x)}{dx} > 0}$$

$\varphi'' + A(x)\cdot\varphi = 0$
$\psi'' + B(x)\cdot\psi = 0$

$$= -\varphi(x)\cdot B(x)\psi(x) + A(x)\varphi(x)\cdot\psi(x) \quad = \underbrace{[A(x) - B(x)]}_{>0}\cdot\underbrace{\varphi(x)\psi(x)}_{>0}$$

Example:

$\boxed{\text{Airy equation}}$ $\qquad \underbrace{y'' - x}_{A(x)} \cdot y = 0 \qquad (-\infty < x < -1)$

- The equation ***cannot*** be solved explicitly by the methods that we currently have in hand..
- Yet, we can say ***something*** about its solution in the above range.
- We can ***propose*** another equation that we ***do know*** its solution:

$$\underbrace{y'' + 1}_{B(x)} \cdot y = 0 \longrightarrow \varphi = sin(x)$$

$$sin(x) = 0 \ ; \ x = n\pi$$

- Noting that $A(x) > B(x)$, we can now use the ***comparison theorem*** to investigate the zeroes of Airy equation solution.
- The solutions of the equation, "***Airy functions***" $Ai(x)$ and $Bi(x)$, thus:
  - Possess ***infinite zeroes*** at the range $-\infty < x < -1$.
  - The distance between successive zeroes of $Ai(x)$ or $Bi(x)$ is ***less than*** $\Delta x < \pi$.

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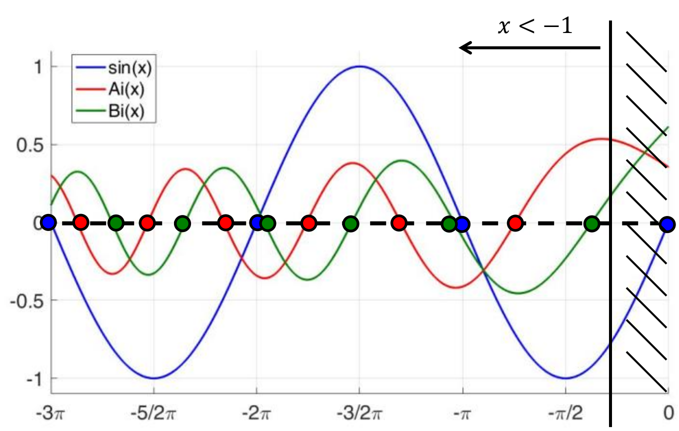
$$y'' + 1y = 0 \longrightarrow \psi(x) = sin(x) \longrightarrow \psi(x) = 0 \rightarrow x = n\pi$$

$$y'' - xy = 0 \longrightarrow \begin{matrix} \varphi_1(x) = A_i(x) \\ \varphi_2(x) = B_i(x) \end{matrix} \longrightarrow \begin{matrix} \varphi_1(x) = 0 \\ \varphi_2(x) = 0 \end{matrix}$$



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Example:

Bessel equation

$$x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0 \qquad (0 < x)$$

- Also here, the equation ***cannot*** be solved explicitly by the methods that we have in hand..
- Changing into its ***normal form***:

Normal form

$$u'' + \underbrace{\left(1 + \frac{1 - 4\nu^2}{4x^2}\right)}_{B(x;\nu)} u = 0 \qquad y = u(x)/\sqrt{x}$$

Note that: The zeroes of ***Bessel equation*** solutions are also the zeroes of the ***normal form***

- We should distinguish between ***three cases***:

Case (I) - $B(x) > 1$, obtained for $0 \leq \nu < 1/2$.

Case (II) - $B(x) < 1$, obtained for $\nu > 1/2$.

Case (III) - $B(x) = 1$, obtained for $\nu = 1/2$.

Case (I) - $0 \leq \nu < 1/2$

$$u'' + \underbrace{\left(1 + \frac{1 - 4\nu^2}{4x^2}\right)}_{B(x;\nu)>1} u = 0$$

- We can ***propose*** (again..) the following equation that we ***do know*** its solution:

$$y'' + \underbrace{1}_{A(x)=1} \cdot y = 0 \longrightarrow \varphi = sin(x)$$

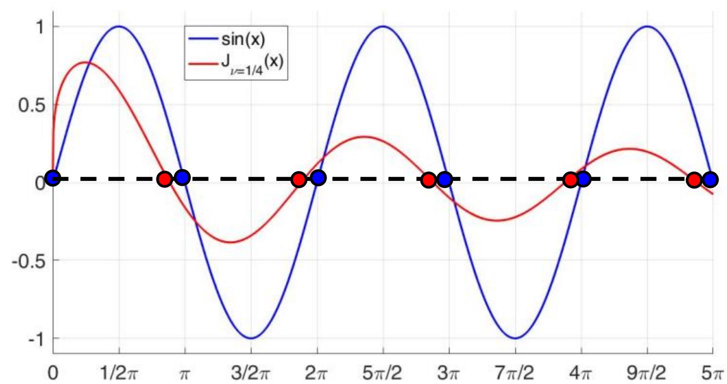
$$sin(x) = 0 \ ; \ x = n\pi$$

- Noting $B(x;\nu) > A(x)$, we can use the ***comparison theorem*** to investigate the zeroes of Bessel equation solutions $y(x) = u(x)/\sqrt{x}$.
- The solutions $y(x)$ with $0 \leq \nu < 1/2$, are thus characterized by:
  - ***Infinite zeroes*** at the range $0 < x < \infty$.
  - The distance between successive zeroes of the solution is ***less than*** $\Delta x < \pi$.

$$y'' + 1y = 0 \longrightarrow \psi(x) = sin(x) \longrightarrow \psi(x) = 0 \rightarrow x = n\pi$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \xrightarrow{\text{Example}} \varphi(x) = J_{\nu=1/4}(x) \longrightarrow J_{\nu=1/4}(x) = 0$$

Bessel function of the first kind of an order $\nu = 1/4$



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Case (II) - $\nu > 1/2$

$$u'' + \underbrace{\left(1 + \frac{1 - 4\nu^2}{4x^2}\right)}_{B(x;\nu)<1} u = 0$$

- We can ***propose*** (again..) the following equation that we ***do know*** its solution:

$$\underbrace{y'' + 1 \cdot y = 0}_{A(x)=1} \longrightarrow \varphi = sin(x)$$

$$sin(x) = 0 \; ; \; x = n\pi$$

- Noting $B(x;\nu) < A(x)$, we can use the ***comparison theorem*** to investigate the zeroes of Bessel equation solutions $y(x) = u(x)/\sqrt{x}$ for $\nu > 1/2$.

- The solutions $y(x)$ with $\nu > 1/2$, are thus characterized by:
  - ***Infinite zeroes*** at the range $0 < x < \infty$.
  - The distance between successive zeroes of the solution is ***greater than*** $\Delta x > \pi$.

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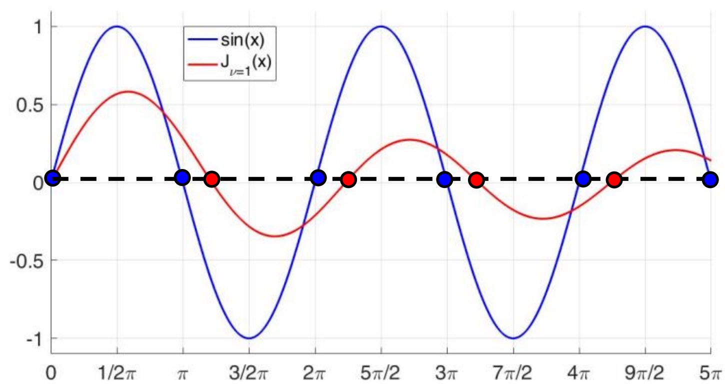
$$y'' + 1y = 0 \quad \longrightarrow \quad \psi(x) = sin(x) \quad \longrightarrow \quad \psi(x) = 0 \;\rightarrow x = n\pi$$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \xrightarrow{\textit{Example}} \quad \varphi(x) = J_{\nu=1}(x) \quad \longrightarrow \quad J_{\nu=1}(x) = 0$$

Bessel function of the ***first kind*** *of an order* $\nu = 1$



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Case (III) - $\nu = 1/2$

$$u'' + \left(1 + \cancel{\frac{1 - 4\nu^2}{4x^2}}\right)u = 0$$

$$\boxed{y = u(x)/\sqrt{x}}$$

- The solutions for this case were already found:

$$\boxed{J_{\nu=1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{sin(x)}{\sqrt{x}}} \qquad \boxed{J_{\nu=-1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{cos(x)}{\sqrt{x}}}$$

- The solutions $y(x)$ with $\nu = 1/2$, are thus characterized by:
  - ***Infinite zeroes*** at the range $0 < x < \infty$.
  - The distance between successive zeroes of the solution is ***exactly*** $\Delta x = \pi$.

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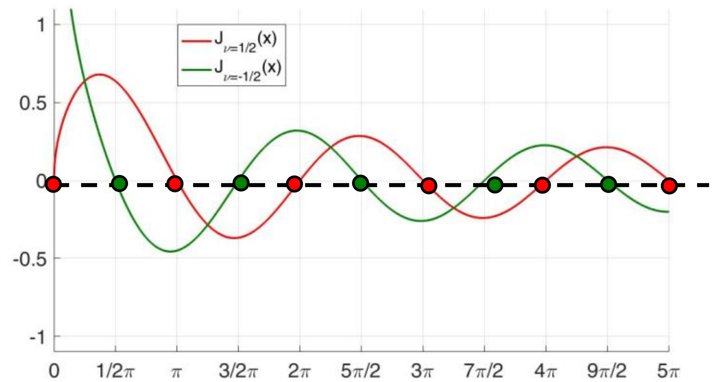
$$J_{v=1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{sin(x)}{\sqrt{x}} \qquad \longrightarrow \qquad J_{v=1/2}(x) = 0 \rightarrow x = n\pi$$

$$J_{v=-1/2}(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{cos(x)}{\sqrt{x}} \qquad \longrightarrow \qquad J_{v=-1/2}(x) = 0 \rightarrow x = n\pi + \frac{\pi}{2}$$



Theorem 7 – A specific variation of comparison theorem

Let $\varphi(x)$ *nontrivial solution of the following:*

$$y'' + B(x) \cdot y = 0 \qquad\qquad a \le x \le b \equiv \Delta$$

Where $0 > B(x)$ *for all* $x \in \Delta$ *.*

Then, the solution $\varphi(x)$ *has* ***at the most*** *one zero at the range.*

Proof:

You can, and will, do it by yourself…
(at H.W.)

Part C – Adjoint forms

(I) The adjoint differential operator

- A second-order linear differential operator:

$$L = a_0(x) \cdot \frac{d^2}{dx^2} + a_1(x) \cdot \frac{d}{dx} + a_2(x)$$

$a_0(x)$, $a_1(x)$ and $a_2(x)$are ***real-valued*** ***continues*** functions in an interval $\Delta \equiv a \le x \le b$

- Consider now two function $y(x)$ and $z(x)$

- We now focus on the product:

$$z \cdot L[y] \;=\; z \cdot [a_0(x) \cdot y'' + a_1(x) \cdot y' + a_2(x) \cdot y]$$

- We will see that this product can ***uniquely*** be written as:

$$z \cdot L[y] \;=\; y \cdot \underbrace{\tilde{L}[z]}_{\text{An adjoint operator}} + \underbrace{\frac{d}{dx} F(x, y, z, y', z')}_{\text{Full differential}}$$

Proof by construction

- And we wish to show that:

$$\underbrace{z \cdot L[y]} = y \cdot \tilde{L}[z] + \frac{d}{dx} F(x, y, z, y', z')$$

- Writing the right side:

$$\boxed{(*)} \quad \overbrace{z \cdot L[y]} = z \cdot [a_0(x) \cdot y'' + a_1(x) \cdot y' + a_2(x) \cdot y]$$

$$= \underbrace{[z \cdot a_0(x) \cdot y'']}_{(I)} + \underbrace{[z \cdot a_1(x) \cdot y']}_{(II)} + y \cdot [a_2(x) \cdot z] \quad \surd$$

Let's expand these terms a bit…

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- Recall the product rule: $\frac{d(z \cdot y)}{dx} = z \cdot \frac{d(y)}{dx} + y \cdot \frac{d(z)}{dx}$

- The ***first term*** yields:

$$\underbrace{z \cdot a_0 \cdot y''}_{(I)} = (z \cdot a_0 \cdot y')' - \underbrace{(z \cdot a_0)' \cdot y'}_{\text{Additional product..}}$$

$$= \underbrace{(z \cdot a_0 \cdot y')'}_{} - \underbrace{\left((z \cdot a_0)' \cdot y\right)'}_{} + (z \cdot a_0)'' \cdot y$$

Full differentials

$$= \frac{d}{dx}[z \cdot a_0 \cdot y' - (z \cdot a_0)' \cdot y] + y \cdot [(z \cdot a_0)'']$$

- The ***second term*** yields:

$$\underbrace{z \cdot a_1 \cdot y'}_{(II)} = \underbrace{(z \cdot a_1 \cdot y)'}_{\text{Full differential}} - (z \cdot a_0)' \cdot y = \frac{d}{dx}[z \cdot a_1 \cdot y] - y \cdot [(z \cdot a_0)']$$

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- Substituting $(I)\&(II)$ back to (*) and combining terms…

$$\boxed{(*)} \quad z\cdot L[y] = z\cdot[a_0(x)\cdot y'' + a_1(x)\cdot y' + a_2(x)\cdot y]$$

$$= \underbrace{\frac{d}{dx}[a_0\cdot(z\cdot y' - z'\cdot y) + (a_1 - a_0')\cdot z\cdot y]}_{F(x,y,z,y',z')} + \underbrace{y\cdot[z\cdot a_2 - (z\cdot a_1)' + (z\cdot a_0)'']}_{y\cdot\tilde{L}[z]}$$

- Identifying the ***differential function***:

$$\boxed{F(x,y,z,y',z') = a_0\cdot(z\cdot y' - z'\cdot y) + (a_1 - a_0')\cdot z\cdot y}$$

- Identifying the ***adjoint operator***:

Adjoint operator

$$\boxed{\tilde{L} \equiv a_0\cdot\frac{d^2}{dx^2} + (2a_0' - a_1)\cdot\frac{d}{dx} + (a_0'' - a_1' + a_2)}$$

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Theorem

*For **any** linear differential operator L - always **exists** a **unique** adjoint operator $\tilde{L}$, such that:*

$$\boxed{z\cdot L[y] - y\cdot\tilde{L}[z] = \frac{d}{dx}F(x,y,z,y',z')}$$

*Specifically, for **second-order** operators:*

$$L = a_0(x)\cdot\frac{d^2}{dx^2} + a_1(x)\cdot\frac{d}{dx} + a_2(x)$$

The adjoint operator is:

$$\tilde{L} \equiv a_0\cdot\frac{d^2}{dx^2} + (2a_0' - a_1)\cdot\frac{d}{dx} + (a_0'' - a_1' + a_2)$$

And the differential function is:

$$F(x,y,z,y',z') = z\cdot a_0\cdot y' + z\cdot(a_1 - a_0)\cdot y$$

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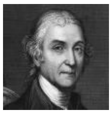


Joseph-Louis Lagrange

$$z \cdot L[y] - y \cdot \tilde{L}[z] = \frac{d}{dx} F(x, y, z, y', z')$$

Lagrange identity

- By ***integrating*** between $a \le x \le b$, we obtain "*Green's (second) Identity*":

George Green

$$\int_a^b \{z \cdot L[y] - y \cdot \tilde{L}[z]\} dx = [F(x, y, z, y', z')]_a^b$$

Green's identity

(II) Self-adjoint operator

- The differential operator (L) and adjoint operator ($\tilde{L}$) of a second-order equation are:

$$L = a_0 \cdot \frac{d^2}{dx^2} + a_1 \cdot \frac{d}{dx} + a_2 \qquad \tilde{L} = a_0 \cdot \frac{d^2}{dx^2} + (2a_0' - a_1) \cdot \frac{d}{dx} + (a_0'' - a_1' + a_2)$$

- The operator L is said to be ***self-adjoint*** if: $\boxed{L = \tilde{L}}$

- This is obtained if the following relations are thus satisfied:

$$a_1 = 2a_0' - a_1$$

$$a_2 = a_0'' - a_1' + a_2$$

$$\rightarrow \quad a_1 = a_0' \equiv p \qquad a_2 \equiv q$$

Self-adjoint differential operator (second-order)

$$L = \tilde{L} = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)$$

Strum-Liouville operator

Examples

- Bessel equation of an order ν :

$$\underbrace{x^2 \cdot y'' + x \cdot y'} + (x^2 - \nu^2) \cdot y = 0$$

$$\overbrace{x \cdot (x \cdot y')'} + (x^2 - \nu^2) \cdot y = 0$$

$$L = \frac{d}{dx}\left(x \cdot \frac{d}{dx}\right) + \left(x - \frac{\nu^2}{x}\right)$$

- Legendre equation of an order l :

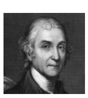
$$\underbrace{(1 - x^2) \cdot y'' - 2x \cdot y'} + l \cdot (l + 1) \cdot y = 0$$

$$\overbrace{\left((1 - x^2) \cdot y'\right)'} + l \cdot (l + 1) \cdot y = 0$$

$$L = \frac{d}{dx}\left((1 - x^2) \cdot \frac{d}{dx}\right) + l \cdot (l + 1)$$

Lagrange and Green's identities - self adjoint operators

$$\boxed{L = \tilde{L} = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)}$$



$$z \cdot L[y] - y \cdot L[z] = \frac{d}{dx} F(x, y, z, y', z')$$

Lagrange identity

$$\int_a^b \{z \cdot L[y] - y \cdot L[z]\} dx = [F(x, y, z, y', z')]_a^b$$

Green's identity

- Where the differential function for a *self-adjoint* ***second-order*** operator is:

$$\boxed{F(x, y, z, y', z') = p(x) \cdot (z \cdot y' - z' \cdot y)}$$

(III) Transformation into a self-adjoint form

- In general, a second order differential operator is ***not*** *(!)* ***self-adjoint***.

- However, ***every*** *homogeneous* equation ***can*** be ***transformed*** into a self-adjoint form.

$$a_0(x)\cdot y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0 \xrightarrow[\text{(how ?)}]{\text{Self-adjoint form}} [p(x)\cdot y']' + q(x)\cdot y = 0$$

$$y'' + \frac{a_1}{a_0}\cdot y' + \frac{a_2}{a_0}\cdot y = 0 \qquad /\cdot p(x) \qquad \textit{(yet undetermined)}$$

$$\underbrace{p(x)}_{A_0(x)}\cdot y'' + \underbrace{p(x)\cdot\frac{a_1}{a_0}}_{A_1(x)}\cdot y' + \underbrace{p(x)\cdot\frac{a_2}{a_0}}_{A_2(x)}\cdot y = 0$$

$$\xrightarrow[\text{Self-adjoint condition}]{} \quad \frac{dA_0(x)}{dx} = A_1(x) \quad \longrightarrow \quad \frac{dp(x)}{dx} = p(x)\cdot\frac{a_1}{a_0}$$

- ***Self-adjoint*** form will thus be obtained for a ***selection*** of $p(x)$ - such that:

$$\frac{dp(x)}{dx} = p(x)\cdot\frac{a_1}{a_0}$$

$$\longrightarrow \quad \boxed{p(x) = \underbrace{C}_{C\,=\,1}\cdot e^{\int\left(\frac{a_1}{a_0}\right)dx}} \qquad \textit{(Selection of } p(x) \textit{ that will produce a self-adjoint form)}$$

- After the transformation, the self-adjoint form of the equation is:

$$[p(x)\cdot y']' + \underbrace{\left[p(x)\cdot\frac{a_2}{a_0}\right]}_{\equiv\, q(x)}\cdot y = 0$$

$$\boxed{[p(x)\cdot y']' + q(x)\cdot y = 0}$$

Example – Legendre equation (the long way..)

- Legendre equation of an order n :

$$\underbrace{(1-x^2)}_{a_0(x)}\cdot y'' - \underbrace{2x}_{a_1(x)}\cdot y' + \underbrace{l\cdot(l+1)}_{a_2(x)}\cdot y = 0$$

- The self-adjoint form of the equation is obtained by:

$$[p(x)\cdot y']' + \left[p(x)\cdot\frac{a_2}{a_0}\right]\cdot y = 0 \qquad\qquad p(x) = e^{\int\left(\frac{a_1}{a_0}\right)dx}$$

- The calculation of $p(x)$ yields:

$$p(x) = e^{\int\left(\frac{a_1}{a_0}\right)dx} = e^{\int\left(\frac{-2x}{1-x^2}\right)dx} = e^{ln(1-x^2)} = 1-x^2$$

- Substituting into the self-adjoint form yields:

$$[(1-x^2)\cdot y']' + \cancel{(1-x^2)}\cdot\frac{l\cdot(l+1)}{\cancel{(1-x^2)}}\cdot y = 0$$

$$\boxed{[(1-x^2)\cdot y']' + l\cdot(l+1)\cdot y = 0}$$

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