Analytical Methods in Engineering

Homework #1

June 8, 2017

1 Question 1

1.1 $y'' + 2y' - 8y = 0$

Assuming $y = e^{rx}$:

$$r^2 e^{rx} + 2re^{rx} - 8e^{rx} = 0$$

$$r^2 + 2r - 8 = 0 \rightarrow r_{1,2} = \{2, -4\}$$

$$\Rightarrow \boldsymbol{\varphi(x) = c_1 \cdot e^{2x} + c_2 \cdot e^{-4x}}$$

1.2 $y'' + 2y' + 1.25y = 0$

Assuming $y = e^{rx}$:

$$r^2 e^{rx} + 2re^{rx} + 1.25e^{rx} = 0$$

$$r^2 + 2r + 1.25 = 0 \rightarrow r_{1,2} = \{-1 \pm 0.5i\}$$

$$\Rightarrow \boldsymbol{\varphi(x) = c_1 \cdot e^{-x} \left(\cos \frac{x}{2}\right) + c_2 \cdot e^{-x} \left(\sin \frac{x}{2}\right)}$$

1.3 $y'' - 6y' + 9y = 0$

Assuming $y = e^{rx}$:

$$r^2 e^{rx} - 6re^{rx} + 9e^{rx} = 0$$

$$r^2 - 6r + 9 = 0 \rightarrow r_{1,2} = \{3\}$$

$$\Rightarrow \boldsymbol{\varphi(x) = c_1 \cdot e^{3x} + c_2 \cdot x \cdot e^{3x}}$$

1.4 $x^4 y'' + 3x^3 y' + 1.25x^2 y = 0$

Dividing by x^2 ($\because x > 0$): $x^2 y'' + 3xy' + 1.25y = 0$, Assuming $y = x^r$:

$$x^2 \cdot r(r-1) \cdot x^{r-2} + 3x \cdot rx^{r-1} + 1.25x^r = 0$$

$$r^2 + 2r + 1.25 = 0 \to r_{1,2} = \{-1 \pm 0.5i\}$$

$$y = c_1 x^{-1-\frac{i}{2}} + c_2 x^{-1+\frac{i}{2}} = \dots$$

$$= \frac{c_1}{x} e^{\ln(x)^{-i/2}} + \frac{c_2}{x} e^{\ln(x)^{i/2}} = \dots$$

$$= \frac{c_1}{x} e^{-i/2 \ln(x)} + \frac{c_2}{x} e^{i/2 \ln(x)}$$

$$y = \frac{c_1}{x} \left(\cos \frac{\ln x}{2} - i \sin \frac{\ln x}{2} \right) + \dots$$

$$\dots + \frac{c_2}{x} \left(\cos \frac{\ln x}{2} + i \sin \frac{\ln x}{2} \right)$$

$$\boldsymbol{y(x) = \frac{c_3}{x} \cos \frac{\ln x}{2} + \frac{c_4}{x} \sin \frac{\ln x}{2}}$$

2 Question 2

2.1 $y'' + 4y = 0 \quad y(0) = 0, y'(0) = 1$

Assuming $y = e^{rx}$:

$$r^2 e^{rx} + 4e^{rx} = 0$$

$$r^2 + 4 = 0 \rightarrow r_{1,2} = \{\pm 2i\}$$

$$\Rightarrow \varphi = c_1 \cdot \cos 2x + c_2 \cdot \sin 2x$$

$$y(0) = c_1 = 0$$

$$y'(x) = -2c_1 \cdot \sin 2x + 2c_2 \cdot \cos 2x$$

$$y'(0) = 2c_2 = 1 \rightarrow c_2 = \frac{1}{2}$$

$$\boldsymbol{y(x) = \frac{1}{2} \sin 2x}$$

2.2 $y'' + y' + 1.25y = 0$
 $y(0) = 3, y'(0) = 1$

Assuming $y = e^{rx}$:

$$r^2 e^{rx} + re^{rx} + 1.25e^{rx} = 0$$

$$r^2 + r + 1.25 = 0 \rightarrow r_{1,2} = \{-0.5 \pm i\}$$

$$\Rightarrow y = c_1 e^{-x/2} \cos x + c_2 e^{-x/2} \sin x$$

$$y(0) = c_1 = 3$$

$$y' = c_1 \left(-\frac{1}{2} e^{-x/2} \cos x - e^{-x/2} \sin x\right) + \ldots$$

$$+ c_2 \left(-\frac{1}{2} e^{-x/2} \sin x + e^{-x/2} \cos x\right)$$

$$y'(0) = c_1 \left(-\frac{1}{2}\right) + c_2 = 1 \rightarrow c_2 = 2.5$$

$$\boldsymbol{y(x) = 3e^{-\frac{x}{2}} \cos x + 2.5e^{-\frac{x}{2}} \sin x}$$

2.3 $y'' + 4y' + 4y = 0$
$y(-1) = 2,\ y'(-1) = 1$

Assuming $y = e^{rx}$:

$$r^2 + 4r + 4 = 0 \to r_{1,2} = \{-2\}$$

$$\Rightarrow y = c_1 e^{-2x} + c_2 x \cdot e^{-2x}$$

$$y(-1) = \underline{c_1 e^2 - c_2 \cdot e^2 = 2}$$

$$y' = -2c_1 e^{-2x} + c_2 \left(e^{-2x} - 2x \cdot e^{-2x}\right)$$

$$y'(-1) = \underline{-2c_1 e^2 + c_2 \left(e^2 + 2 \cdot e^2\right) = 1}$$

Two equations with two unknowns:

$$\begin{bmatrix} e^2 & -e^2 \\ -2e^2 & e^2 + 2e^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7e^{-2} \\ 5e^{-2} \end{bmatrix}$$

$$\Rightarrow \boldsymbol{y(x) = 7e^{-2}e^{-2x} + 5x \cdot e^{-2}e^{-2x}}$$

$$= y(x) = (7 + 5x)\, e^{-2(1+x)}$$

3 Question 3

$$L[y] = (1 - \ln x)\, y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$$

$$\varphi_1(x) = x$$
$$\varphi_2(x) = \ln x$$

3.1

Substituting presented solutions into the differential equation:

$$L[\varphi_1(x)] = \frac{1}{x} - \frac{1}{x^2} x \equiv 0,$$

hence φ_1 is a solution.

$$L[\varphi_2(x)] = -\frac{(1 - \ln x)}{x^2} + \frac{1}{x^2} - \frac{\ln x}{x^2} \equiv 0,$$

hence φ_2 is a solution.

3.2 Wronskian calculation

$$\varphi_1(x) = x \quad \varphi_2(x) = \ln x$$

$$\varphi_1'(x) = 1 \quad \varphi_2'(x) = \tfrac{1}{x}$$

$$W = \begin{vmatrix} x & \ln x \\ 1 & \frac{1}{x} \end{vmatrix} = 1 - \ln x$$

4 Question 4

4.1 $y'' + 4y' + 4y = x^{-2}e^{-2x} \qquad x > 0$

Repeating steps from 2.3 yields:

$$y = c_1 e^{-2x} + c_2 x \cdot e^{-2x}$$

$$W(\varphi_1, \varphi_2, x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & [1-2x]\, e^{-2x} \end{vmatrix} = e^{-4x}$$

Assuming $C_1(x)$; $C_2(x)$:

$$y = C_1(x)\, e^{-2x} + C_2(x) \cdot x \cdot e^{-2x}$$

Two new equations:

$$C_1'(x)\, e^{-2x} + C_2'(x) \cdot x \cdot e^{-2x} = 0$$
$$C_1'(x) \cdot -2e^{-2x} + C_2'(x)\, [1+2x]\, e^{-2x} = x^{-2}e^{-2x}$$

$$\begin{aligned} C_1(x) = c_1 - \int \frac{f(x)\, \boldsymbol{\varphi_2}(x)}{W(\varphi_1, \varphi_2, x)} dx = \\ = c_1 - \int \frac{x^{-2}e^{-2x} \cdot xe^{-2x}}{e^{-4x}} dx = c_1 - \int \frac{1}{x} dx \end{aligned}$$

$$\boldsymbol{C_1(x) = c_1 - \ln x}$$

$$\begin{aligned} C_2(x) = c_2 + \int \frac{f(x)\, \boldsymbol{\varphi_1}(x)}{W(\varphi_1, \varphi_2, x)} dx = \\ = c_2 + \int \frac{x^{-2}e^{-2x} \cdot e^{-2x}}{e^{-4x}} dx = c_2 + \int \frac{1}{x^2} dx \end{aligned}$$

$$\boldsymbol{C_2(x) = c_2 - \frac{1}{x}}$$

$$y = (c_1 - \ln x)\, e^{-2x} + \left(c_2 - \frac{1}{x}\right) x \cdot e^{-2x}$$

Defining $c_3 = c_1 - 1$

$$\boldsymbol{y = (c_3 - \ln x + c_2 \cdot x)\, e^{-2x}}$$

4.2 $4y'' + y = \frac{2}{\cos(x/2)} \qquad x > 0$

Assuming $y = e^{rx}$:

$$4r^2 + 1 = 0 \rightarrow r_{1,2} = \left\{\pm\frac{i}{2}\right\}$$

$$\varphi_1 = \cos\left(x/2\right) \;\; \varphi_2 = \sin\left(x/2\right)$$

$$\Rightarrow y = c_1 \cos\left(x/2\right) + c_2 \sin\left(x/2\right)$$

$$W\left(\varphi_1, \varphi_2, x\right) = \begin{vmatrix} \cos\left(x/2\right) & \sin\left(x/2\right) \\ \frac{-\sin(x/2)}{2} & \frac{\cos(x/2)}{2} \end{vmatrix} = \frac{1}{2}$$

Assuming $C_1\left(x\right); \quad C_2\left(x\right)$:

$$\begin{aligned} C_1\left(x\right) = c_1 - \int \frac{f\left(x\right) \boldsymbol{\varphi_2}\left(x\right)}{W\left(\varphi_1, \varphi_2, x\right)} dx = \\ = c_1 - \int \frac{\frac{2}{\cos(x/2)} \cdot \sin\left(x/2\right)}{0.5} dx = \\ = c_1 + 8 \int \frac{-1/2 \cdot \sin\left(x/2\right)}{\cos\left(x/2\right)} dx \end{aligned}$$

$$\boldsymbol{C_1\left(x\right) = c_1 + 8\ln\left(\cos\frac{x}{2}\right)}$$

$$\begin{aligned} C_2\left(x\right) = c_2 + \int \frac{f\left(x\right) \boldsymbol{\varphi_1}\left(x\right)}{W\left(\varphi_1, \varphi_2, x\right)} dx = \\ = c_2 + \int \frac{\frac{2}{\cos(x/2)} \cdot \cos\left(x/2\right)}{0.5} dx = \\ = c_2 + 4 \int dx \end{aligned}$$

$$\boldsymbol{C_2\left(x\right) = c_2 + 4x}$$

$$\boldsymbol{y = \left[c_1 + 8\ln\left(\cos\frac{x}{2}\right)\right] \cdot \cos\frac{x}{2} + \left(c_2 + 4x\right) \cdot \sin\frac{x}{2}}$$

5 Question 5

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F_0e^{-\frac{t}{\tau_c}}\sin(\omega t)$$

[1]

5.1 Homogeneous solutions

Assuming underdamped system, and by using solution of form $x=e^{rt}$:

$$mr^2+cr+k=0$$

$$r_{1,2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m}=-\frac{c}{2m}\pm\frac{1}{2}\sqrt{\frac{c^2}{m^2}-4\frac{k}{m}}$$

$$r_{1,2}=-\frac{1}{\tau_c}\pm i\omega_c$$

$$\varphi_1=e^{-\frac{1}{\tau_c}t}\cdot\cos\omega_c t$$

$$\varphi_2=e^{-\frac{1}{\tau_c}t}\cdot\sin\omega_c t$$

5.2 Wronskian calculation

$$\varphi_1'=e^{-\frac{1}{\tau_c}t}\left[-\frac{1}{\tau_c}\cdot\cos\omega_c t-\omega_c\cdot\sin\omega_c t\right]$$

$$\varphi_{\prime 2}=e^{-\frac{1}{\tau_c}t}\left[-\frac{1}{\tau_c}\cdot\sin\omega_c t+\omega_c\cdot\cos\omega_c t\right]$$

$$W(\varphi_1,\varphi_2,t)=\varphi_1\varphi_2'-\varphi_1'\varphi_2=$$

$$=e^{\frac{-2t}{\tau_c}}\left(-\frac{1}{\tau_c}\cdot\sin\omega_c t+\omega_c\cdot\cos\omega_c t\right)\cos\omega_c t-$$

$$-e^{\frac{-2t}{\tau_c}}\left(-\frac{1}{\tau_c}\cdot\cos\omega_c t+\omega_c\cdot\sin\omega_c t\right)\sin\omega_c t$$

$$W(\varphi_1,\varphi_2,t)=e^{\frac{-2t}{\tau_c}}\cdot\omega_c\cdot\left(\cos^2\omega_c t+\sin^2\omega_c t\right)$$

$$\boldsymbol{W(\varphi_1,\varphi_2,t)=\omega_c\cdot e^{\frac{-2t}{\tau_c}}}$$

[1]Supposed to divide by m here?

5.3 Variation of parameters for $\omega \neq \omega_c$

$$
\begin{aligned}
C_1(t) =& c_1 - \int \frac{f(t)\,\boldsymbol{\varphi_2}(t)}{W(\varphi_1, \varphi_2, t)} dt = \\
=& c_1 - \int \frac{F_0 \cdot e^{-\frac{t}{\tau_c}} \cdot \sin(\omega t) \cdot e^{-\frac{1}{\tau_c}t} \cdot \sin \omega_c t}{\omega_c \cdot e^{\frac{-2t}{\tau_c}}} dt = c_1 - \int \frac{F_0 \sin(\omega t) \cdot \sin \omega_c t}{\omega_c} dt \\
=& c_1 - \frac{F_0}{2\omega_c} \int \left[\cos([\omega - \omega_c]\, t) - \cos([\omega + \omega_c]\, t)\right] dt \\
C_1(t) =& c_1 - \frac{F_0}{2\omega_c} \left[\frac{\sin([\omega - \omega_c]\, t)}{\omega - \omega_c} - \frac{\sin([\omega + \omega_c]\, t)}{\omega + \omega_c}\right]
\end{aligned}
$$

$$
\begin{aligned}
C_2(t) =& c_2 + \int \frac{f(t)\,\boldsymbol{\varphi_1}(t)}{W(\varphi_1, \varphi_2, t)} dt = \\
=& c_2 + \int \frac{F_0 \cdot e^{-\frac{t}{\tau_c}} \cdot \sin(\omega t) \cdot e^{-\frac{1}{\tau_c}t} \cdot \cos \omega_c t}{\omega_c \cdot e^{\frac{-2t}{\tau_c}}} dt = c_2 + \int \frac{F_0 \sin(\omega t) \cdot \cos \omega_c t}{\omega_c} dt \\
=& c_2 + \frac{F_0}{2\omega_c} \int \left[\sin([\omega - \omega_c]\, t) + \sin([\omega + \omega_c]\, t)\right] dt \\
C_2(t) =& c_2 - \frac{F_0}{2\omega_c} \left[\frac{\cos([\omega - \omega_c]\, t)}{\omega - \omega_c} + \frac{\cos([\omega + \omega_c]\, t)}{\omega + \omega_c}\right]
\end{aligned}
$$

5.4 Variation of parameters for$\omega = \omega_c$

Substituting $\omega = \omega_c$ in:

$$C_1(t) = c_1 - \frac{F_0}{2\omega_c} \left[\frac{\sin([\omega - \omega_c]\, t)}{\omega - \omega_c} - \frac{\sin([\omega + \omega_c]\, t)}{\omega + \omega_c}\right]$$

$$C_1(t) = c_1 - \frac{F_0}{2\omega_c} \left[t - \frac{\sin 2\omega_c t}{2\omega_c}\right]$$

For C_2 it has to be done before integrating:

$$
\begin{aligned}
C_2(t) =& c_2 + \frac{F_0}{2\omega_c} \int \left[\sin([\omega_c - \omega_c]\, t) + \sin([\omega_c + \omega_c]\, t)\right] dt \\
=& c_2 + \frac{F_0}{2\omega_c} \int \left[\sin(2\omega_c t)\right] dt
\end{aligned}
$$

$$C_2(t) = c_2 - \frac{F_0}{2\omega_c} \frac{\cos(2\omega_c t)}{2\omega_c}$$

5.5 Solution

The solution is given by linear combination of φ_1 and φ_2:

$$
\begin{aligned}
x(t) &= C_1(t)\cdot\varphi_1(t) + C_2(t)\cdot\varphi_2(t) \\
&= \left(c_1 - \frac{F_0}{2\omega_c}\left[t - \frac{\sin 2\omega_C t}{2\omega_c}\right]\right)\cdot e^{-\frac{1}{\tau_c}t}\cdot\cos\omega_c t + \left(c_2 - \frac{F_0}{2\omega_c}\frac{\cos(2\omega_c t)}{2\omega_c}\right)\cdot e^{-\frac{1}{\tau_c}t}\cdot\sin\omega_c t
\end{aligned}
$$

Substituting:
$\tau_c = \frac{3}{2\cdot 2} = 3/4$
$\omega_c = -\left(\frac{c^2}{m^2} - 4\frac{k}{m}\right) = -\frac{3^2}{2^2} + 4\cdot\frac{5/4}{2}{=}1/4$
Continue here after checking about division by m.

6 Question 6

$$e^{2x}y'' + 2y' + y = 0$$

6.1 Normal form

$$\boldsymbol{p\left(x\right)} = e^{-0.5\int\left(\frac{a_1}{a_0}\right)dx} = \boldsymbol{e^{-0.5e^{-2x}}}$$

$$p'\left(x\right) = -e^{0.5e^{-2x}-2x}$$

$$p''\left(x\right) = e^{0.5e^{-2x}-2x}\left(e^{-2x}+2\right)$$

$$Q\left(x\right) = \frac{p''}{p} + \frac{a_1 p'}{a_0 p} + \frac{a_2}{a_0} = \frac{e^{\cancel{0.5e^{-2x}}-2x}\left(e^{-2x}+2\right)}{\cancel{e^{-0.5e^{-2x}}}} + \frac{2\cdot\left(-e^{\cancel{0.5e^{-2x}}-2x}\right)}{e^{2x}\cdot\left(\cancel{e^{-0.5e^{-2x}}}\right)} + \frac{1}{e^{2x}} = 3e^{-2x} - e^{-4x}$$

Normal form:

$$u'' + Q\left(x\right)\cdot u = 0$$

$$\boldsymbol{u'' + \left(3e^{-2x} - e^{-4x}\right)\cdot u = 0}$$

$$y\left(x\right) = e^{-0.5e^{-2x}}\cdot u\left(x\right)$$

6.2 Distance between zeroes

Using Sturm comparison theorem, defining $A, B;\ A > B$:

$$A\left(x\right) = 2 \qquad B\left(x\right) = 3e^{-2x} - e^{-4x}$$

In case of A:

$$u'' + 2u = 0$$

$$\varphi = \cos\sqrt{2}x$$

Zeros are located at intervals of $\frac{\pi}{\sqrt{2}}$. Sturm's theorem states the solution of $u'' + B\left(x\right)u = 0$ will have slower fluctuations than the solution of $u'' + A\left(x\right)u = 0$: Hence the distance between zeroes will be greater than $\frac{\pi}{\sqrt{2}}$. Function $p\left(x\right)$ is positive for all domain, and hence does not affect the zeros locations.

7 Question 7

$$y'' + Q\left(x\right)y = 0 \qquad Q\left(x\right) < 0$$

Using Sturm comparison theorem, we can compare $Q\left(x\right)$ with $A\left(x\right) = 0$, $A > Q$ (given). Solution of the ODE with A instead of Q will be linear: $\alpha\left(x\right) = ax + b$, with at most one zero in whole $\mathbb{R}$ domain. Hence, $\vartheta\left(x\right)$, solution of given ODE, will have at most one zero, or less ("lower frequency" than of $\alpha\left(x\right)$).

8 Question 8

$$L = \frac{d^2}{dx^2}\left(K\frac{d^2}{dx^2}\right) - N\frac{d^2}{dx^2}$$

$$u \cdot L[w] = u\frac{d^2}{dx^2}\left(K\frac{d^2 w}{dx^2}\right) - uN\frac{d^2 w}{dx^2} = u\left[Kw''\right]'' - uNw''$$

Using $\left\{ab'' = (ab' - a'b)' + a''b\right\}$:

$$= \left[u\left(Kw''\right)' - u'Kw''\right]' + u''Kw'' - uNw'' =$$

$$= \left[u\left(Kw''\right)' - u'Kw''\right]' + \left[u''Kw' - \left(u''K\right)' w\right]' + \left(u''K\right)'' w - uNw'' =$$

$$= \left[u\left(Kw''\right)' - u'Kw''\right]' + \left[u''Kw' - \left(u''K\right)' w\right]' + \left(u''K\right)'' w - \left[uNw' - u'Nw\right]' - u''Nw =$$

$$= \left[u\left(Kw''\right)' - u'Kw'' + u''Kw' - \left(u''K\right)' w - uNw' + u'Nw\right]' + \left(u''K\right)'' w - u''Nw =$$

$$= \left[u\left(Kw''\right)' - uNw' + wNu' - w\left(u''K\right)' - u'Kw'' + u''Kw'\right]' + \left(u''K\right)'' w - u''Nw =$$

$$= \underline{\left[u \cdot \left[\left(Kw''\right)' - Nw'\right] + w \cdot \left[Nu' - \left(u''K\right)'\right] + u''Kw' - u'Kw''\right]'} + \underline{\underline{\left(u''K\right)'' w - u''Nw}} =$$

$$= \frac{d}{dx}\underline{F\left(x, u, w, u', w', u'', w'', u''', w'''\right)} + \underline{\underline{w \cdot \tilde{L}\left[u\right]}}$$

$$\tilde{L} = L; \qquad F = u \cdot \left[\left(Kw''\right)' - Nw'\right] + w \cdot \left[Nu' - \left(u''K\right)'\right] + u''Kw' - u'Kw''$$

9 Question 9

$$xy'' + (1-x)\,y' + ny = 0; \qquad n = \text{const}$$

Self adjoint (Sturm-Liouville) operator is given by:

$$L = \tilde{L} = \frac{d}{dx}\left(p\left(x\right) \cdot \frac{d}{dx}\right) + q\left(x\right)$$

$$p\left(x\right) = e^{\int \frac{1-x}{x} dx} = e^{\int \frac{1}{x} - 1 dx} = e^{\ln x - x} = x \cdot e^{-x}$$

Substituting into self-adjoint form:

$$\left[p\left(x\right) \cdot y\right]' + \left[p\left(x\right) \cdot \frac{n}{x}\right] y = 0$$

$$\left[x \cdot e^{-x} \cdot y\right]' + e^{-x} \cdot n \cdot y = 0$$