Analytical Methods in Engineering

Homework #2

June 8, 2017

1 Question 1

1.1 $\left(2+x^2\right)y''-xy'+4y=0$

Defining y and it's derivatives:

$$y\left(x\right)=\sum_{m=0}^{\infty}a_m\cdot x^m$$

$$y'\left(x\right)=\sum_{m=1}^{\infty}a_m\cdot m\cdot x^{m-1}$$

$$y''\left(x\right)=\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}$$

Substituting into given equation:

$$\left(2+x^2\right)\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}-x\sum_{m=1}^{\infty}a_m\cdot m\cdot x^{m-1}+4\sum_{m=0}^{\infty}a_m\cdot x^m=0$$

Separating first term:

$$2\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}+x^2\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}-\sum_{m=1}^{\infty}a_m\cdot m\cdot x^{m}+4\sum_{m=0}^{\infty}a_m\cdot x^m=0$$

Shifting indices:

$$2\sum_{s=0}^{\infty}a_{s+2}\cdot\left(s+2\right)\left(s+1\right)\cdot x^s+\sum_{s=2}^{\infty}a_s\cdot s\left(s-1\right)\cdot x^s-\sum_{s=1}^{\infty}a_s\cdot s\cdot x^s+4\sum_{s=0}^{\infty}a_s\cdot x^s=0$$

Separation for s:

$$\begin{cases} s=0: & 4a_2+4a_0=0 & \rightarrow \boldsymbol{a_2=-a_0} \\ s=1: & 12a_3+3a_1=0 & \rightarrow \boldsymbol{a_3=-a_1/4} \\ s>1: & \sum_{s=2}^{\infty}\left[2\left(s+2\right)\left(s+1\right)a_{s+2}+\left(s^2-2s+4\right)a_s\right]x^s=0 & \rightarrow \boldsymbol{a_{s+2}=\frac{-\left(s^2-2s+4\right)}{2\left(s+2\right)\left(s+1\right)}\cdot a_s} \end{cases}$$

$$\underline{y\left(x\right)=a_0\left[1-x^2+x^4/6-x^6/30+\ldots\right]+a_1\left[x-x^3/4+7x^5/160-19x^7/1920\right]}$$

1.2 $\left(1+x^2\right)y''-4xy'+6y=0$

Substituting y and it's derivatives into given equation:

$$\left(1+x^2\right)\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}-4x\sum_{m=1}^{\infty}a_m\cdot m\cdot x^{m-1}+6\sum_{m=0}^{\infty}a_m\cdot x^m=0$$

Separating first term:

$$\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}+x^2\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}-\sum_{m=1}^{\infty}4a_m\cdot m\cdot x^m+6\sum_{m=0}^{\infty}a_m\cdot x^m=0$$

Shifting indices:

$$\sum_{s=0}^{\infty}a_{s+2}\cdot\left(s+2\right)\left(s+1\right)\cdot x^s+\sum_{s=2}^{\infty}a_s\cdot s\left(s-1\right)\cdot x^s-\sum_{s=1}^{\infty}4a_s\cdot s\cdot x^s+\sum_{s=0}^{\infty}6a_s\cdot x^s=0$$

Separation for s:

$$\begin{cases} s=0: & 2a_2+6a_0=0 & \rightarrow \boldsymbol{a_2=-3a_0} \\ s=1: & 6a_3+2a_1=0 & \rightarrow \boldsymbol{a_3=-\frac{a_1}{3}} \\ s>1: & \sum_{s=2}^{\infty}\left[\left(s+2\right)\left(s+1\right)a_{s+2}+\left(s^2-5s+6\right)a_s\right]x^s=0 & \rightarrow \boldsymbol{a_{s+2}=\frac{-\left(s^2-5s+6\right)}{\left(s+2\right)\left(s+1\right)}\cdot a_s} \end{cases}$$

$$\underline{y\left(x\right)=a_0\left[1-3x^2\right]+a_1\left[x-\frac{x^3}{3}\right]}$$

1.3 $y''-2xy'+\lambda y=0$

Substituting y and it's derivatives into given equation:

$$\sum_{m=2}^{\infty}a_m\cdot m\left(m-1\right)\cdot x^{m-2}-2x\sum_{m=1}^{\infty}a_m\cdot m\cdot x^{m-1}+\lambda\sum_{m=0}^{\infty}a_m\cdot x^m=0$$

Shifting indices:

$$\sum_{s=0}^{\infty}a_{s+2}\cdot\left(s+2\right)\left(s+1\right)\cdot x^s-\sum_{s=1}^{\infty}2a_s\cdot s\cdot x^s+\sum_{s=0}^{\infty}\lambda a_s\cdot x^s=0$$

Separation for s:

$$\begin{cases} s=0: & 2a_2+\lambda a_0=0 & \rightarrow \boldsymbol{a_2=-\frac{\lambda}{2}a_0} \\ s>0: & \sum_{s=1}^{\infty}\left[\left(s+2\right)\left(s+1\right)a_{s+2}+\left(\lambda-2s\right)a_s\right]x^s=0 & \rightarrow \boldsymbol{a_{s+2}=\frac{\left(2s-\lambda\right)}{\left(s+2\right)\left(s+1\right)}\cdot a_s} \end{cases}$$

$$\underline{y\left(x\right)=a_0\left[1-\frac{\lambda}{2}x^2+\frac{\lambda}{2}\cdot\frac{\lambda-4}{4\cdot3}x^4+\frac{\lambda}{2}\cdot\frac{\lambda-4}{4\cdot3}\cdot\frac{8-\lambda}{6\cdot5}x^6+\dots\right]+}$$

$$\underline{+a_1\left[x+\frac{2-\lambda}{3\cdot2}x^3+\frac{2-\lambda}{3\cdot2}\cdot\frac{6-\lambda}{5\cdot4}x^5\overset{?}{\pm}\frac{2-\lambda}{3\cdot2}\cdot\frac{6-\lambda}{5\cdot4}\cdot\frac{10-\lambda}{7\cdot6}x^7+\dots\right]}$$

2 Question 2

Repeating the same pattern as in Question 1 yields:

$$\sum_{s=0}^{\infty} a_{s+2} \cdot (s+2)(s+1) \cdot x^s + \sum_{s=1}^{\infty} 5a_s \cdot x^s = 2 + 6x^2$$

$$2a_2 + \sum_{s=1}^{\infty} \left[a_{s+2} \cdot (s+2)(s+1) + 5a_s\right] \cdot x^s = 2 + 6x^2$$

Separation for s:

$$\begin{cases} s=0: & 2a_2 = 2 & \rightarrow a_2 = 1 \\ s=1: & 6a_3 + 5a_0 = 0 & \rightarrow a_3 = -\frac{5}{6}a_0 \\ s=2: & 12a_4 + 5a_1 = 6 & \rightarrow a_4 = \frac{6-5a_1}{12} = \frac{1}{2} - \frac{5a_1}{12} \\ s=3: & 20a_5 + 5a_2 = 0 & \rightarrow a_5 = -\frac{5}{20} = -\frac{1}{4} \\ s=4: & 30a_6 + 5a_3 = 0 & \rightarrow a_6 = -\frac{5}{30}a_3 = \frac{5}{36}a_0 \\ s=5: & 42a_7 + 5a_4 = 0 & \rightarrow a_7 = -\frac{5}{42}a_4 = \frac{-5}{84} + \frac{25}{484}a_1 \end{cases}$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$y(x) = a_0 + a_1 x + x^2 - \frac{5}{6}a_0 x^3 + \left(\frac{1}{2} - \frac{5a_1}{12}\right)x^4 - \frac{1}{4}x^5 + \frac{5}{36}a_0 x^6 + \left(\frac{-5}{84} + \frac{25}{484}a_1\right)x^7$$

$$\underline{y(x) = a_0\left[1 - \frac{5}{6}x^3 + \frac{5}{36}x^6 + \ldots\right] + a_1\left[x - \frac{5}{12}x^4 + \overset{???}{\boldsymbol{\frac{25}{484}}}x^7 + \ldots\right] + 1 \cdot \left[x^2 + \frac{1}{2}x^4 - \frac{1}{4}x^5 - \frac{5}{84}x^7 + \ldots\right]}$$

3 Question 3

3.1 $x(x-1) \cdot y'' + 6x^2 \cdot y' + 3y = 0 \qquad x > 0$

Def:

$$\varphi(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$\varphi'(x) = \sum_{n=0}^{\infty} a_n (r+n) \cdot x^{r+n-1}$$

$$\varphi''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) \cdot x^{r+n-2}$$

Sub:

$$x(x-1) \cdot \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) \cdot x^{r+n-2} + 6x^2 \cdot \sum_{n=0}^{\infty} a_n (r+n) \cdot x^{r+n-1} + 3\sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\left(x^2-x\right)\cdot\sum_{n=0}^{\infty}a_n\left(r+n\right)\left(r+n-1\right)\cdot x^{r+n-2}+6x^2\cdot\sum_{n=0}^{\infty}a_n\left(r+n\right)\cdot x^{r+n-1}+3\sum_{n=0}^{\infty}a_nx^{r+n}=0$$

$$\sum_{n=0}^{\infty}a_n\left(r+n\right)\left(r+n-1\right)\cdot x^{r+n}-\sum_{n=0}^{\infty}a_n\left(r+n\right)\left(r+n-1\right)\cdot x^{r+n-1}+6\cdot\sum_{n=0}^{\infty}a_n\left(r+n\right)\cdot x^{r+n+1}+3\sum_{n=0}^{\infty}a_nx^{r+n}=0$$

Shifting:

$$\sum_{s=1}^{\infty}a_{s-1}\left(r+s-1\right)\left(r+s-2\right)\cdot x^{r+s-1}-\sum_{n=0}^{\infty}a_n\left(r+n\right)\left(r+n-1\right)\cdot x^{r+n-1}+$$
$$+6\cdot\sum_{s=2}^{\infty}a_{s-2}\left(r+s-2\right)\cdot x^{r+s-1}+3\sum_{s=1}^{\infty}a_{s-1}x^{r+s-1}=0$$

Rearranging:

$$-a_0r\left(r-1\right)\cdot x^{r-1}+\left[a_0r\left(r-1\right)-a_1\left(r+1\right)r+3a_0\right]\cdot x^r+$$
$$+\sum_{s=2}^{\infty}\left[a_{s-1}\left(r+s-1\right)\left(r+s-2\right)-a_s\left(r+s\right)\left(r+s-1\right)+6a_{s-2}\left(r+s-2\right)+3a_{s-1}\right]\cdot x^{r+s-1}=0$$

All terms must vanish:

$$a_0r\left(r-1\right)=0$$
$$\left[a_0r\left(r-1\right)-a_1\left(r+1\right)r+3a_0\right]=0\qquad\Rightarrow\quad a_1=a_0\frac{3+r\left(r-1\right)}{r+1}$$
$$\left[a_{s-1}\left(r+s-1\right)\left(r+s-2\right)-a_s\left(r+s\right)\left(r+s-1\right)+6a_{s-2}\left(r+s-2\right)+3a_{s-1}\right]=0$$

Since $a_0\neq0,$:

$$r\left(r-1\right)=0\quad\rightarrow\quad r_1=1;r_2=0$$

$$\varphi_1\left(x\right)=\sum_{n=0}^{\infty}a_nx^{n+1}$$

$$\varphi_2\left(x\right)=\sum_{n=0}^{\infty}b_nx^{n}$$

3.2 $4x^2\cdot y''+\left(3x+1\right)\cdot y=0\qquad x>0$

$$\varphi\left(x\right)=x^r\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{r+n}$$

$$\varphi'\left(x\right)=\sum_{n=0}^{\infty}a_n\left(r+n\right)\cdot x^{r+n-1}$$

$$\varphi''\left(x\right)=\sum_{n=0}^{\infty}a_n\left(r+n\right)\left(r+n-1\right)\cdot x^{r+n-2}$$

Sub:

$$4x^2 \cdot \sum_{n=0}^{\infty} a_n \left(r+n\right)\left(r+n-1\right) \cdot x^{r+n-2} + \left(3x+1\right) \cdot \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$4 \cdot \sum_{n=0}^{\infty} a_n \left(r+n\right)\left(r+n-1\right) \cdot x^{r+n} + 3\sum_{n=0}^{\infty} a_n x^{r+n+1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

Shifting

$$4 \cdot \sum_{n=0}^{\infty} a_n \left(r+n\right)\left(r+n-1\right) \cdot x^{r+n} + 3\sum_{n=1}^{\infty} a_{n-1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

$$\left[4 \cdot a_0 r\left(r-1\right) + a_0\right] \cdot x^r + \sum_{n=1}^{\infty} \left[4a_n \left(r+n\right)\left(r+n-1\right) + 3a_{n-1} + a_n\right] \cdot x^{r+n} = 0$$

$$\left[4 \cdot a_0 r\left(r-1\right) + a_0\right] = 0 \quad \rightarrow \quad 4r\left(r-1\right) + 1 = 0 \quad \rightarrow \quad \boldsymbol{r_{1,2} = 1/2}$$

$$\left[4a_n \left(r+n\right)\left(r+n-1\right) + 3a_{n-1} + a_n\right] = 0 \quad \rightarrow \quad \boldsymbol{a_n = a_{n-1} \frac{3}{\left(r+n\right)\left(r+n-1\right)+1}}$$

Solutions:

$$\varphi_1\left(x\right) = x^{0.5} \sum_{n=0}^{\infty} a_n x^n$$

$$\varphi_2\left(x\right) = x^{0.5} \sum_{n=0}^{\infty} b_n x^n$$