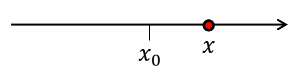
Analytical Methods

Solutions in power series of differential equations

Power series - a review

- Consider a power series:

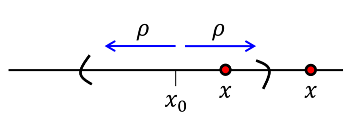
$$\sum_{n=0}^{m} a_n (x - x_0)^n$$



- The power series ***converges*** at a ***fixed*** x - if the following limit exists:

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n$$

*Theorem: A power series **converges** at a **symmetric** open interval of a **radious** ρ around x_0.*



An ***absolute*** ***convergence*** is obtained for : $|x - x_0| < \rho$

Non-convergence (***divergence***) is obtained for : $|x - x_0| > \rho$

What is the convergence radius?

$$\sum_{n=0}^{m} a_n (x - x_0)^n = \sum_{n=0}^{m} C_n$$

- Following the ***ratio test***:

$$\frac{|C_{n+1}|}{|C_n|} \xrightarrow[n\to\infty]{} K$$

An ***absolute convergence*** is obtained if: $K < 1$

A ***divergence*** is obtained for : $K > 1$

- Substituting the power series terms:

$$\boxed{|x - x_0| < \frac{1}{L} \equiv \rho}$$

Convergence radius (ρ)

$$\frac{|C_{n+1}|}{|C_n|} = \left|\frac{a_{n+1}}{a_n}\right| \cdot |x - x_0| \xrightarrow[n\to\infty]{} L \cdot |x - x_0| < 1$$

Convergence

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Example

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n+1} \cdot n \cdot (x-2)^n$$

- Following the ***ratio test***:

$$\frac{|C_{n+1}|}{|C_n|} = \frac{|(-1/2)^{n+2} \cdot (n+1) \cdot (x-2)^{n+1}|}{|(-1/2)^{n+1} \cdot n \cdot (x-2)^n|} = \frac{1}{2} \cdot \frac{n+1}{n} \cdot |x-2|$$

$$\xrightarrow[n\to\infty]{} \frac{1}{2} \cdot \underbrace{|x-2|}_{\rho} < 1$$

Convergence condition

- The ***radius of converges*** is thus: $\rho = 2$:

- The power series ***converges*** for ***any*** x at the range of $0 < x < 4$:

- The power series ***diverges*** for $x < 0$ and $x > 4$:

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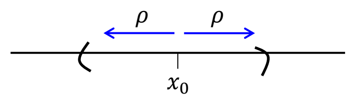
Analytic function

$$\sum_{n=0}^{m} a_n (x - x_0)^n \xrightarrow[m\to\infty]{} f(x)$$

- The function $f(x)$ is termed ***analytic function*** if it can be represented as a power series which ***converges*** for an open interval radius $\rho > 0$ around x_0 :

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$



- Thus, ***analytic function*** is smooth which all its derivatives are defined at x_0 - ***AND*** its power series ***converges*** for a certain $\rho > 0$ to $f(x)$.

- A specific case of a converged power series is ***Taylor series***:

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \xrightarrow[\boxed{\text{Taylor series}}]{} a_n = \frac{1}{n!}\left[\frac{d^n}{dx^n} f(x)\right]_{x=x_0}$$

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Example 1:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

- Following the ***ratio test***:

$$\frac{|C_{n+1}|}{|C_n|} = \frac{\left|\frac{1}{(n+1)!}x^{n+1}\right|}{\left|\frac{1}{n!}x^n\right|} = \frac{n!}{(n+1)!}\cdot|x| \xrightarrow[n\to\infty]{} \frac{1}{n+1}|x| < 1$$

$\rho = \infty$
Convergence for ***all*** x

Example 2:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

- This function ***obviously diverges*** at $x = 1$.

- Following the ***ratio test***:

$$\frac{|C_{n+1}|}{|C_n|} = \frac{|x^{n+1}|}{|x^n|} = |x| < 1$$

Radius of convergence
$\rho = 1$

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Example 3:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \dots + (-1)^n \cdot x^{2n} + \dots = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$$

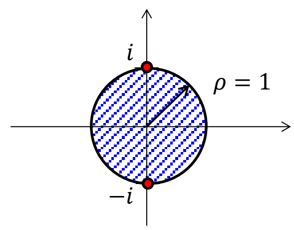
- The denominator ***never*** ***vanishes*** for ***any*** x.
- The series however ***obviously*** ***diverges*** for ***any*** $x > 1$.
- Are we missing something?
- Using the ***ratio*** ***test***:

$$\frac{|C_{n+1}|}{|C_n|} = |x^2| < 1$$

Radius of convergence
$\rho = 1$

- The power series diverges when the denominator vanishes:

$$1 + x^2 = 0 \longrightarrow x = \pm i$$



Radius of convergence → including the ***complex plane****!*

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Part A - Series solutions in an ordinary point

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$$\boxed{a_0(x)\cdot y'' + a_1(x)\cdot y' + a_2(x)\cdot y = 0}$$

- An ***ordinary point*** of the differential equation $x = x_0$ is defined for: $\boxed{a_0(x_0) \neq 0}$

- Thus, the equation can be formulated as:

$$\boxed{y'' + p(x)\cdot y' + q(x)\cdot y = f(x)}$$

with the following ***analytic*** functions:

$$p(x) = \frac{a_1(x)}{a_0(x)} \quad , \qquad q(x) = \frac{a_2(x)}{a_0(x)},$$

- A ***singular point*** of the differential equation $x = x_0$ is defined for: $\boxed{a_0(x_0) = 0}$

- If $a_1(x_0) \neq 0$ and/or $a_2(x_0) \neq 0$, the functions $p(x)$ and/or $q(x)$ are not-bounded and thus ***not analytic***.

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Theorem 2:

$$y'' + p(x)\cdot y' + q(x)\cdot y = 0$$

Let $x = x_0$ be an ***ordinary point*** *of the differential equations, and $p(x)$ and $q(x)$* ***analytic functions*** *at $x = x_0$, with their power series* ***converge*** *for radius $\rho_p$and ρ_q respectively:*

$$p(x) = \sum_{n=0}^{m} p_n(x-x_0)^n \qquad |x-x_0| < \rho_p \neq 0$$

$$q(x) = \sum_{n=0}^{m} q_n(x-x_0)^n \qquad |x-x_0| < \rho_q \neq 0$$

Then, there ***exists*** *a unique* ***solution*** *in the form of a power series which* ***converges*** *for a radius ρ.*

$$\varphi(x) = \sum_{n=0}^{m} a_n(x-x_0)^n = a_0\cdot\varphi_1(x) + a_1\cdot\varphi_2(x) \qquad |x-x_0| < \rho \qquad \rho \leq min\{\rho_p, \rho_q\}$$

The coefficients a_0 and a_1 are ***arbitrary****, and determined by the* ***initial conditions****.*

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Example:

$$y'' + y = 0$$

- We wish to find the power series solution, expended in $x_0 = 0$ - an ***ordinary point***.
- $p(x) = 0$ *and* $q(x) = 1$ $\rightarrow$ *analytical functions* at $x_0 = 0$ with $\rho_p, \rho_q = \infty$
- We seek a power series solution at the form of:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n \quad = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots$$

- The derivatives of $\varphi(x)$ are the derivatives of the power series:

$$\frac{d\varphi}{dx} = a_1 + a_2 \cdot 2 \cdot x + \cdots \quad = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$\frac{d^2\varphi}{dx^2} = 2 \cdot a_2 + 6 \cdot a_3 \cdot x + \cdots \quad = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2}$$

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- Substituting into the equation:

$$y'' + y = 0 \quad \longrightarrow \quad \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} + \sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

- ***Shifting indices*** for the first series, to combine the power terms:

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} \longrightarrow \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^n$$

- The equation can be written as:

$$\sum_{n=0}^{\infty} \underbrace{[a_{n+2} \cdot (n+2) \cdot (n+1) + a_n]}_{=0} \cdot x^n = 0$$

Must vanish for *any* selection of x and n

- The term in the ***brackets*** must ***vanish***!

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- Thus, coefficients of the power series must fulfil the relation:

$$a_{n+2} \cdot (n+2) \cdot (n+1) + a_n = 0$$

$$\downarrow$$

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)\cdot(n+1)}}$$

Recurrence relation

- The *even* and *odd* coefficients of the power series are evaluated *separately*:

Even sequence (n=0,2,4,6...)

$$a_2 = -\frac{a_0}{2\cdot 1} \qquad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!} \qquad a_{2n} = (-1)^n \frac{a_0}{(2n)!}$$

$$(n = 0,1,2 \ldots)$$

Odd sequence (n=1,3,5,..)

$$a_3 = -\frac{a_1}{3\cdot 2} \qquad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!} \qquad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

- Substituting the coefficients into the power series solution:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = \sum_{n=0}^{\infty} a_{2n} \cdot x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} \cdot x^{2n+1}$$

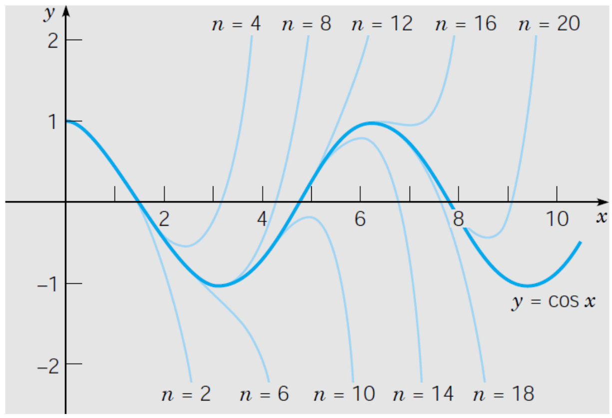
$$= a_0 \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n}}_{\varphi_1(x)} + a_1 \cdot \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}}_{\varphi_2(x)}$$

- The sharp-eye student - which was outstanding in calculus (Hedva) - may notice that:

$$\varphi_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = cos(x) \qquad \varphi_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = sin(x)$$

$$\boxed{\varphi(x) = a_0 \cdot cos(x) + a_1 \cdot sin(x)}$$

$$\varphi_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} = cos(x)$$



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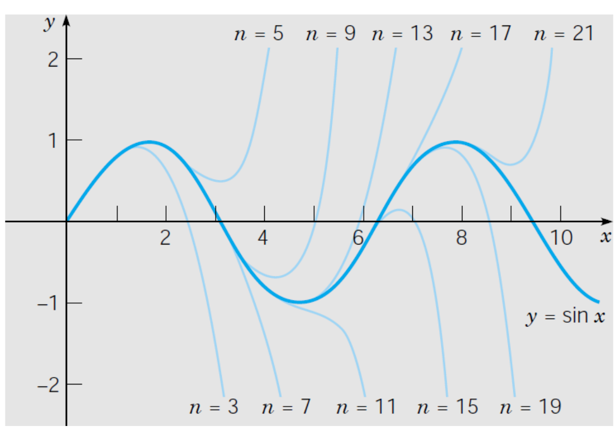
$$\varphi_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1} = sin(x)$$



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Example – Airy equation:

$$y'' - x \cdot y = 0$$

- Also here, we wish to find the power series solution, expended in $x_0 = 0$ - an ***ordinary point***.
- $p(x) = 0$ *and* $q(x) = -x$ → *analytical functions* at $x_0 = 0$ with $\rho_p, \rho_q = \infty$
- We again seek a power series solution at the form of:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$\varphi' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$\varphi'' = \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2}$$

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- Substituting into the equation:

$$y'' - xy = 0 \longrightarrow \sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} - \sum_{n=0}^{\infty} a_n \cdot x^{n+1} = 0$$

- ***Shifting indices*** for the ***first*** series:

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} \longrightarrow \sum_{n=0}^{\infty} a_{n+2} \cdot (n+2) \cdot (n+1) \cdot x^n$$

- ***Shifting indices*** for the ***second*** series:

$$\sum_{n=0}^{\infty} a_n \cdot x^{n+1} \longrightarrow \sum_{n=1}^{\infty} a_{n-1} \cdot x^n$$

- The equation can be written as:

$$\underset{(n=0)}{\boxed{a_2 \cdot 2}} + \underset{(n=1,2\dots)}{\sum_{n=1}^{\infty} [a_{n+2} \cdot (n+2) \cdot (n+1) - a_{n-1}] \cdot x^n} = 0$$

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$$\underbrace{a_2 \cdot 2}_{=0} + \sum_{n=1}^{\infty} \underbrace{[a_{n+2} \cdot (n+2) \cdot (n+1) - a_{n-1}] \cdot x^n}_{=0} = 0$$

- Both terms ***must*** ***vanish*** !!

$$\boxed{a_2 = 0} \quad (n = 0)$$

$$\boxed{a_{n+2} = \frac{a_{n-1}}{(n+2) \cdot (n+1)}} \quad (n = 1,2 \dots)$$

Recurrence relation

- As seen, ***every*** ***third*** ***term*** in the power series is determined by the reoccurrence relation.

- Since $a_2 = 0$, using the reoccurrence formula for the sequence $\boldsymbol{n = 3, 6, 9} \dots$ we obtain:

$$a_5 = \frac{a_2}{5 \cdot 4} = 0 \quad (n = 3) \xrightarrow[\text{Similarly...}]{} a_8 = a_{11} = a_{14} = \cdots = 0 \quad (n = 6,9,11, \dots)$$

$$\boxed{a_{3n-1} = 0} \quad (n = 1,2,3 \dots)$$

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- Using the reoccurrence formula for the sequence $\boldsymbol{n = 1, 4, 7} \dots$ we obtain :

$$a_3 = \frac{a_0}{3 \cdot 2} \quad (n = 1) \qquad a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6} \quad (n = 4) \qquad a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} \quad (n = 7)$$

$$\boxed{a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-4) \cdot (3n-3) \cdot (3n-1) \cdot 3n}} \quad (n = 1,2,3 \dots)$$

- Using the reoccurrence formula for the sequence $\boldsymbol{n = 2, 5, 8} \dots$ we obtain :

$$a_4 = \frac{a_1}{4 \cdot 3} \quad (n = 2) \qquad a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7} \quad (n = 5) \qquad a_{10} = \frac{a_7}{10 \cdot 9} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} \quad (n = 8)$$

$$\boxed{a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n-3) \cdot (3n-2) \cdot 3n \cdot (3n+1)}} \quad (n = 1,2,3 \dots)$$

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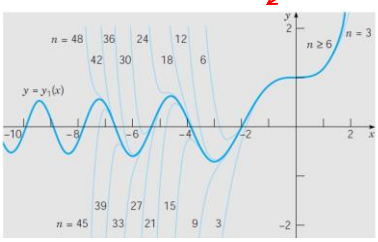
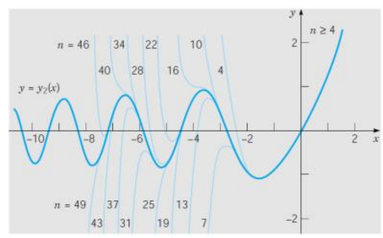
- Substituting the coefficients into the power series solution:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n = \cdots =$$

$$= a_0 \underbrace{\sum_{n=0}^{\infty}\left[1 + \frac{x^{3n}}{2\cdot 3 \cdots (3n-1)\cdot 3n}\right]}_{\varphi_1(x)} + a_1 \underbrace{\sum_{n=0}^{\infty}\left[x + \frac{x^{3n+1}}{3\cdot 4 \cdots 3n\cdot (3n+1)}\right]}_{\varphi_2(x)}$$





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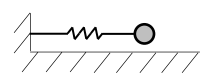
Example – The Hermit equation:

(Harmonic oscillator – quantum physics)

$$y'' - 2xy' + \lambda y = 0 \quad ; \quad \lambda = const$$



- The power series solution $\varphi_\lambda(x)$, expended in $x_0 = 0$, is:

See H.W 2

$$\varphi_\lambda(x) = a_0 \underbrace{\left[1 - \frac{\lambda}{2}x^2 + \frac{(\lambda-4)\cdot\lambda}{4\cdot 3\cdot 2}x^4 + \frac{(\lambda-8)\cdot(\lambda-4)\cdot\lambda}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^6 + \cdots\right]}_{\text{Hermit function type 1}}$$

$$+a_1 \underbrace{\left[x - \frac{\lambda-2}{3\cdot 2}x^3 + \frac{(\lambda-6)\cdot(\lambda-2)}{5\cdot 4\cdot 3\cdot 2}x^5 + \frac{(\lambda-10)\cdot(\lambda-6)\cdot(\lambda-2)}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}x^7 + \cdots\right]}_{\text{Hermit function type 2}}$$

- As seen, for $\lambda = 2\cdot k$ $(k = 1,2,3,\ldots)$ - one of the power series is ***truncated***!

- The power series becomes a finite polynom → termed as ***Hermit polynomial*** $H_\lambda(x)$

A finite polynom always converges !

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$$\varphi_\lambda(x) = a_0\left[1 - \frac{\lambda}{2}x^2 + \frac{(\lambda-4)\cdot\lambda}{4\cdot3\cdot2}x^4 + \frac{(\lambda-8)\cdot(\lambda-4)\cdot\lambda}{6\cdot5\cdot4\cdot3\cdot2}x^6 + \cdots\right]$$

$$+a_1\left[x - \frac{\lambda-2}{3\cdot2}x^3 + \frac{(\lambda-6)\cdot(\lambda-2)}{5\cdot4\cdot3\cdot2}x^5 + \frac{(\lambda-10)\cdot(\lambda-6)\cdot(\lambda-2)}{7\cdot6\cdot5\cdot4\cdot3\cdot2}x^7 + \cdots\right]$$

The Hermit polynomials:

$\lambda = 0 \longrightarrow H_0(x) = 1$

$\lambda = 2\cdot1 \longrightarrow H_1(x) = 2\cdot x$

Scaling factor (orthonormal basis)

$\lambda = 2\cdot2 = 4 \longrightarrow H_2(x) = -2\cdot(1-2x^2)$

$\lambda = 2\cdot3 = 6 \longrightarrow H_3(x) = -12\cdot\left(x - \frac{2}{3}x^3\right)$

Olinde Rodrigues

$$\boxed{H_n(x) = (-1)^n\cdot e^{x^2}\cdot\frac{d^n}{dx^n}\left[e^{-x^2}\right]}$$

The Rodrigues formula

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Part B - The Legendre equation

Adrien-Marie Legendre

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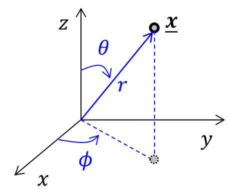
Motivation – Laplace equation in spherical coordinates



➢ The Laplace equation in spherical coordinates (r, ϕ, θ):

$$\frac{1}{r}\cdot\frac{\partial^2}{\partial r^2}(r\cdot\Phi)+\frac{1}{r^2\cdot sin(\theta)}\cdot\frac{\partial}{\partial\theta}\left(sin(\theta)\cdot\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\cdot sin(\theta)}\cdot\frac{\partial^2\Phi}{\partial\phi^2}=0$$

Φ may represent
Electrostatic potential
Temperature field (steady-state)
Standing spherical waves

➢ Proposing a solution in the form of (separation of variables): $\Phi=\frac{U(r)}{r}\cdot P(\theta)\cdot Q(\phi)$

➢ After substituting (not shown..), the Laplace equation is decomposed into:

m and l are "eigenvalues"
(we will get there later on..)

r direction

$$\frac{d^2U}{dr^2}-\frac{l\cdot(l+1)}{r^2}\cdot U=0$$

ϕ direction

$$\frac{d^2Q}{d\phi^2}+m^2\cdot Q=0$$

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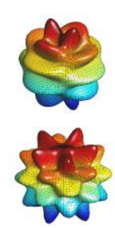
θ direction

$$\frac{1}{sin(\theta)}\cdot\frac{d}{d\theta}\left(sin(\theta)\cdot\frac{dP}{d\theta}\right)+\left[l\cdot(1+l)-\frac{m^2}{sin^2(\theta)}\right]\cdot P=0$$

$$\Big\downarrow\ x=cos(\theta)$$

*The **Generalized Legendre** equation*

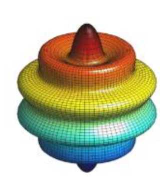
$$\boxed{\frac{d}{dx}\left((1-x^2)\cdot\frac{dP}{dx}\right)+\left[l\cdot(1+l)-\frac{m^2}{1-x^2}\right]\cdot P=0}$$



➢ For the case of ***azimuthal symmetry*** $m=0$ and the equation reduces into:

*The **Legendre** equation*

$$\boxed{\frac{d}{dx}\left((1-x^2)\cdot\frac{dP}{dx}\right)+l\cdot(1+l)\cdot P=0}$$



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Legendre equation

$$(1-x^2)\cdot y'' - 2\cdot x\cdot y' + l\cdot(1+l)\cdot y = 0 \qquad (l = const)$$

- The equation has a ***singular point*** at $x = 1$.
- We wish to find the power series solution, expended in $x_0 = 0$ - an ***ordinary point***.

$$p(x) = -\frac{2\cdot x}{(1-x^2)} \qquad\qquad q(x) = \frac{l\cdot(1+l)}{(1-x^2)}$$

- $p(x)$ and $q(x)$ → *analytical functions* at $x_0 = 0$.
- Their radius of convergence is $\rho_p, \rho_q = 1$.
- Thus, a power series solution for Legendre equation will converges at:

$$\boxed{-1 < x < 1}$$

- We again seek for a power series solution:

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$\varphi' = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} \qquad\qquad \varphi'' = \sum_{n=2}^{\infty} a_n \cdot n\cdot(n-1)\cdot x^{n-2}$$

- Substituting into the equation: $(1-x^2)\cdot y'' - 2x\cdot y' + l\cdot(1+l)\cdot y = 0$

$$(1-x^2)\cdot\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)\cdot x^{n-2}$$

$$-2x\cdot\sum_{n=1}^{\infty} a_n\cdot n\cdot x^{n-1} \quad + l\cdot(1+l)\cdot\sum_{n=0}^{\infty} a_n\cdot x^n = 0$$

$$(1-x^2)\cdot\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)\cdot x^{n-2} \quad -2x\cdot\sum_{n=1}^{\infty} a_n\cdot n\cdot x^{n-1} \quad +l\cdot(1+l)\cdot\sum_{n=0}^{\infty} a_n\cdot x^n=0$$

- Expanding:

$$\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)\cdot x^{n-2} \; -\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)\cdot x^n$$

$$-2\cdot\sum_{n=1}^{\infty} a_n\cdot n\cdot x^n \quad +l\cdot(1+l)\cdot\sum_{n=0}^{\infty} a_n\cdot x^n=0$$

Shifting indices

$$\sum_{n=0}^{\infty} a_{n+2}\cdot(n+2)\cdot(n+1)\cdot x^n$$

$$\sum_{n=0}^{\infty} a_{n+2}\cdot(n+2)\cdot(n+1)\cdot x^n \; -\sum_{n=2}^{\infty} a_n\cdot n\cdot(n-1)\cdot x^n$$

$$-2\cdot\sum_{n=1}^{\infty} a_n\cdot n\cdot x^n \quad +l\cdot(1+l)\sum_{n=0}^{\infty} a_n\cdot x^n=0$$

- Combining power terms:

$$\overset{(n=0)}{\underbrace{[2\cdot a_2+l\cdot(1+l)\cdot a_0]}_{=0}}+\overset{(n=1)}{\underbrace{[3\cdot 2\cdot a_3-2a_1+l\cdot(1+l)\cdot a_1]}_{=0}}\cdot x+$$

$$\sum_{n=2}^{\infty}\overset{(n\geq 2)}{\underbrace{[a_{n+2}\cdot(n+2)\cdot(n+1)-a_n\cdot n\cdot(n-1)-2a_n\cdot n+l\cdot(1+l)a_n]}_{=0}}\cdot x^n=0$$

- All terms ***must vanish*** *!!*

➢ The first brackets yields:

$$a_2 = -\frac{l \cdot (1+l)}{2} \cdot a_0 \qquad (n=0)$$

➢ The second brackets yields:

$$a_3 = -\frac{2 - l \cdot (1+l)}{3 \cdot 2} \cdot a_1 = -\frac{(1-l) \cdot (2+l)}{3 \cdot 2} \cdot a_1 \qquad (n=1)$$

➢ The third brackets ($n \geq 2$) yields:

$$a_{n+2} = \frac{n \cdot (n+1) - l \cdot (1+l)}{(n+2) \cdot (n+1)} \cdot a_n \qquad (n \geq 2)$$

$$\downarrow$$

$$\boxed{a_{n+2} = \frac{(n-l) \cdot (n+l+1)}{(n+2) \cdot (n+1)} \cdot a_n}$$

Recurrence relation

$(n = 2,3, \ldots)$

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➢ The *even* and *odd* coefficients of the power series are evaluated *separately !*

Even sequence (n=0,2,4,6...)

$$a_{n+2} = \frac{(n-l) \cdot (n+l+1)}{(n+2) \cdot (n+1)} \cdot a_n$$

$$a_2 = -\frac{l \cdot (1+l)}{2} \cdot a_0$$

$$a_4 = \frac{(2-l) \cdot (2+l+1)}{4 \cdot 3} a_2 = -\frac{l \cdot (1+l) \cdot (2-l) \cdot (3+l)}{4!} a_0$$

$$a_6 = \frac{(4-l) \cdot (5+l)}{6 \cdot 5} a_4 = -\frac{l \cdot (1+l) \cdot (2-l) \cdot (3+l) \cdot (4-l) \cdot (5+l)}{6!} a_0$$

$$a_{2n} = -\frac{l \cdot (1+l) \cdot (2-l) \cdots (2n-2-l) \cdot (2n-1+l)}{(2n)!} a_0$$

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Odd sequence (n=1,3,5 …)

$$a_3 = \frac{(1-l)\cdot(2+l)}{3\cdot 2}\cdot a_1$$

$$\boxed{a_{n+2} = \frac{(n-l)\cdot(n+l+1)}{(n+2)\cdot(n+1)}\cdot a_n}$$

$$a_5 = \frac{(3-l)\cdot(3+l+1)}{5\cdot 4}a_3 \qquad = \frac{(1-l)\cdot(2+l)\cdot(3-l)\cdot(4+l)}{5!}a_1$$

$$a_7 = \frac{(5-l)\cdot(6+l)}{7\cdot 6}a_5 \qquad = \frac{(1-l)\cdot(2+l)\cdot(3-l)\cdot(4+l)\cdot(5-l)\cdot(6+l)}{7!}a_1$$

$$\boxed{a_{2n+1} = \frac{(1-l)\cdot(2+l)\cdots(2n-1-l)\cdot(2n+l)}{(2n+1)!}a_1}$$

- The power series solution $\varphi_l(x)$, expended in $x_0 = 0$, is:

$$\varphi_l(x) = a_0\left[\underbrace{1 - \frac{l\cdot(1+l)}{2}x^2 - \frac{l\cdot(1+l)\cdot(2-l)\cdot(3+l)}{4!}x^4 - \cdots}_{\text{Legendre function type 1}}\right]$$

$$+a_1\left[\underbrace{x + \frac{(1-l)\cdot(2+l)}{3\cdot 2}x^3 + \frac{(1-l)\cdot(2+l)\cdot(3-l)\cdot(4+l)}{5!}x^5 + \cdots}_{\text{Legendre function type 2}}\right]$$

- As can be seen, for $l = 0,1,2,3,\ldots$ one of the power series is ***truncated***!
- The power series that is ***not truncated*** → ***Legendre function*** $Q_l(x)$

$Q_l(x)$ *converges for:* $-1 < x < 1$

- The ***truncated*** series → a finite polynom → ***Legendre polynomials*** $P_l(x)$

$P_l(x)$ *converges for:* $-1 \leq x \leq 1$

The Legendre polynomials:

$l = 0 \longrightarrow P_0(x) = 1$

$l = 1 \longrightarrow P_1(x) = x$

$l = 2 \longrightarrow P_2(x) = \frac{1}{2} \cdot (3x^2 - 1)$

Scaling factor to achieve:
$P_l(x = 1) = 1$

$l = 3 \longrightarrow P_3(x) = -\frac{3}{2} \cdot \left(x - \frac{5}{3}x^3\right) = \frac{1}{2} \cdot (5x^3 - 3x)$

$$\boxed{P_l(x) = \sum_{k=0}^{l} \frac{(-1)^k \cdot (2n-2k)!}{2^n \cdot k! \cdot (n-2k)!} x^{n-2k}}$$

Olinde Rodrigues

$$\boxed{P_l(x) = \frac{1}{2^l \cdot l!} \cdot \frac{d^l}{dx^l}[(x^2-1)^l]}$$

The Rodrigues formula

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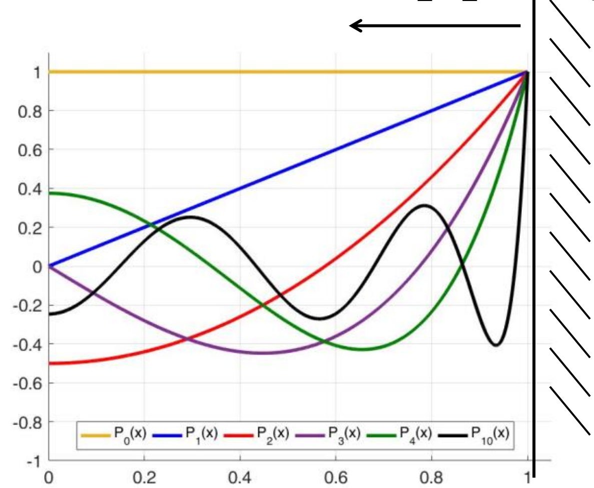
The Legendre polynomials:

$P_l(x)$ exsist for:
$-1 \leq x \leq 1$

$-1 \leq x \leq 1$



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Orthogonality of Legendre polynomials

$$\boxed{\int_{-1}^{1} P_n(x) \cdot P_m(x) dx = \frac{2}{2n+1} \delta_{mn}}$$

Spanning the function space at the range of $-1 \leq x \leq 1$

$m = 0, n = 0 \longrightarrow \int_{-1}^{1} P_0(x) \cdot P_0(x) dx = \int_{-1}^{1} 1 \cdot 1 \cdot dx = 2$

$m = 0, n = 1 \longrightarrow \int_{-1}^{1} P_0(x) \cdot P_1(x) dx = \int_{-1}^{1} 1 \cdot x \cdot dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$

$m = 1, n = 1 \longrightarrow \int_{-1}^{1} P_1(x) \cdot P_1(x) dx = \int_{-1}^{1} x \cdot x \cdot dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

Part C - Singular points

Singular points of a differential equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

- A ***singular point*** of the differential equation $x = x_0$ is defined for: $\boxed{a_0(x_0) = 0}$

- And for the formulation:

$$y'' + p(x)y' + q(x)y = 0$$

with the following ***not analytic*** functions:

$$p(x) = \frac{a_1(x)}{a_0(x)} \;, \qquad q(x) = \frac{a_2(x)}{a_0(x)},$$

Note that: For singular points, uniqueness and existence theorem is generally not applicable !!!

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Examples:

$$\boxed{x^2 \cdot y'' + x \cdot y + (x^2 - \nu^2) \cdot y = 0}$$ *(Bessel equation)*

- The point $x = 0$ is a *singular point*

$$p(x) = \frac{1}{x} \qquad\qquad q(x) = \frac{x^2 - \nu^2}{x^2}$$

$$\boxed{(1 - x^2) \cdot y'' - 2x \cdot y + l \cdot (1 + l) \cdot y = 0}$$ *(Legendre equation)*

- The point $x = \pm 1$ are *singular points*

$$p(x) = \frac{-2x}{1 - x^2} \qquad\qquad q(x) = \frac{l \cdot (1 + l)}{1 - x^2}$$

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$$\boxed{x^2 \cdot (x-1) \cdot (x+2)^3 \cdot y'' + 7x^2 \cdot y' + 5x^2 \cdot y = 0}$$

(Kishkashta's equation)

Prof. Kishkashta (1976-1981)

- The points $x = 1$ and $x = -2$ are *singular points*
- The point $x = 0$ is a ***not*** a *singular point*

$$p(x) = \frac{7\cancel{x^2}}{\cancel{x^2}(x-1)\cdot(x+2)^3} = \frac{7}{(x-1)\cdot(x+2)^3}$$

$$q(x) = \frac{5x^2}{x^2(x-1)\cdot(x+2)^3} = \frac{5}{(x-1)\cdot(x+2)^3}$$

- Another view of the equation:

$$x^2 \cdot [\underbrace{(x-1)\cdot(x+2)^3 \cdot y'' + 7 \cdot y' + 5 \cdot y}_{=0}] = 0$$

Must vanish for *any* selection of x

$$\boxed{(x-1)\cdot(x+2)^3 \cdot y'' + 7 \cdot y' + 5 \cdot y = 0}$$

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Examples for problems in singular points

$$\boxed{x^2 \cdot y'' + \alpha \cdot x \cdot y' + \beta \cdot y = 0}$$

(Euler equation)

- The point $x = 0$ is a *singular point*

Specific examples of Euler equation

$$\boxed{x^2 y'' - 2y = 0}$$

- The solutions are: $\varphi_1(x) = x^2$ $\quad \varphi_2(x) = \dfrac{1}{x}$

Solution diverges for $x \to 0$

$$\boxed{x^2 y'' - 2xy' + 2y = 0}$$

- The solutions are: $\varphi_1(x) = x^2$ $\quad \varphi_2(x) = x$

Both solution seems ok… but…

- The solutions ***cannot satisfy*** non-zero initial conditions $y(x=0) \neq 0$:

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"Weak" singularities - regular-singular points

$$y'' + p(x) \cdot y' + q(x) \cdot y = 0$$

- In a singular point $x = x_0$ - the following functions ***not analytic***:

$$p(x) = \frac{a_1(x)}{a_0(x)} \,, \qquad q(x) = \frac{a_2(x)}{a_0(x)} ,$$

- A special case of "***weak***" a singularity emerges when the following functions ***are analytic***:

$$(x - x_0) \cdot p(x) \quad = p_0 + p_1 \cdot x + p_2 \cdot x^2 + \cdots = \sum_{n=0}^{\infty} p_n \cdot x^n$$

Analytic at $x = x_0$:
Regular-singular point

$$(x - x_0)^2 \cdot q(x) \quad = q_0 + q_1 \cdot x + q_2 \cdot x^2 + \cdots = \sum_{n=0}^{\infty} q_n \cdot x^n$$

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Examples:

$$\boxed{x^2 \cdot y'' + \alpha \cdot x \cdot y' + \beta \cdot y = 0}$$

(Euler equation)

- The point $x = 0$ is a ***regular-singular point***

$$p(x) = \frac{\alpha}{x} \longrightarrow x \cdot p(x) = \alpha$$

$$q(x) = \frac{\beta}{x^2} \longrightarrow x^2 \cdot q(x) = \beta$$

$$\boxed{x^2 \cdot y'' + x \cdot y' + (x^2 - \nu^2) \cdot y = 0}$$

(Bessel equation)

- The point $x = 0$ is a ***regular-singular point***

$$p(x) = \frac{1}{x} \longrightarrow x \cdot p(x) = 1$$

$$q(x) = \frac{x^2 - \nu^2}{x^2} \longrightarrow x^2 \cdot q(x) = x^2 - \nu^2$$

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$$\boxed{(1-x^2)\cdot y'' - 2x\cdot y + l\cdot(1+l)\cdot y = 0}$$

(Legendre equation)

- The point $x = \pm 1$ are ***regular-singular points***

$$p(x) = \frac{-2x}{1-x^2} \qquad q(x) = \frac{l\cdot(1+l)}{1-x^2}$$

- For the point $x = 1$, for example:

$$(1-x)\cdot p(x) = \frac{-2x}{1+x} \quad = (-2x)\cdot(1-x+x^2+\cdots) = \sum_{n=0}^{\infty} p_n \cdot x^n$$

$$(1-x)^2\cdot q(x) = \frac{l\cdot(1+l)}{1+x}\cdot(1-x)$$

$$= l\cdot(1+l)\cdot(1-x+x^2+\cdots)\cdot(1-x) = \sum_{n=0}^{\infty} p_n \cdot x^n$$

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$$\boxed{x^2\cdot(x-1)\cdot(x+2)^3\cdot y'' + 7x^2\cdot y' + 5x^2\cdot y = 0}$$

Prof. Kishkashta (1976-1981)

- The point $x = 0$ is an ***ordinary point***.
- The point $x = 1$ is a ***regular-singular point***
- The points $x = -2$ is an "***irregular***" *singular points*
- Let's take a look at $p(x)$, for example:

Analytic at $x = 1$

$$(1-x)\cdot p(x) = \frac{7}{(x+2)^3}$$

$$p(x) = \frac{7}{(x-1)\cdot(x+2)^3}$$

$$(x+2)\cdot p(x) = \frac{7}{(1-x)\cdot(x+2)^2}$$

Not analytic at $x = -2$

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Series solutions in a (regular) singular point

Example:

$$4x \cdot y'' + 3 \cdot y' - 3 \cdot y = 0$$

- We wish to find the power series solution, expended at $x = 0$ - a ***regular-singular point***

$$p(x) = \frac{3}{x} \longrightarrow x \cdot p(x) = 3$$

$$q(x) = \frac{-3}{x} \longrightarrow x^2 \cdot q(x) = -3 \cdot x$$

- We seek a power series solution at the form of:

$r \in \mathcal{R}$
yet unknown…

$$\varphi(x) = x^r \cdot \sum_{n=0}^{\infty} a_n \cdot x^n \quad = \sum_{n=0}^{\infty} a_n \cdot x^{r+n} \quad = a_0 \cdot x^r + a_1 \cdot x^{1+r} + a_2 \cdot x^{2+r} + \cdots$$

- The derivatives of $\varphi(x)$ are the derivatives of the power series:

$$\frac{d\varphi}{dx} = r \cdot a_0 \cdot x^{r-1} + (r+1) \cdot a_1 \cdot x^r + \cdots \quad = \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot x^{r+n-1}$$

$$\frac{d^2\varphi}{dx^2} = \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot (r+n-1) \cdot x^{r+n-2}$$

➢ Substituting into the equation: $4x \cdot y'' + 3 \cdot y' - 3 \cdot y = 0$

$$\longrightarrow \quad 4x \cdot \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot (r+n-1) \cdot x^{r+n-2}$$

Shifting index

$$+3 \cdot \sum_{n=0}^{\infty} a_n \cdot (r+n) \cdot x^{r+n-1} \quad -3 \cdot \sum_{n=0}^{\infty} a_n \cdot x^{r+n} = 0$$

➢ Combining power terms:

$$\underbrace{[4 \cdot a_0 \cdot r \cdot (r-1) + 3 \cdot a_0 \cdot r]}_{=0}^{(n=0)} \cdot x^{r-1} +$$

$$+\sum_{n=1}^{\infty} \underbrace{[4 \cdot a_n \cdot (r+n) \cdot (r+n-1) + 3 \cdot a_n \cdot (r+n) - 3 \cdot a_{n-1}]}_{=0}^{(n \geq 1)} \cdot x^{r+n-1} = 0$$

➢ All terms ***must vanish*** !!

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➢ The ***first brackets*** yields:

$$a_0 \cdot [4 \cdot r \cdot (r-1) + 3 \cdot r] = 0 \qquad (n=0)$$

➢ The ***second brackets*** yields:

$$4 \cdot a_n \cdot (r+n) \cdot (r+n-1) + 3 \cdot a_n \cdot (r+n) - 3 \cdot a_{n-1} = 0 \qquad (n \geq 1)$$

$$a_n \cdot [4 \cdot (r+n) \cdot (r+n-1) + 3 \cdot (r+n)] - 3 \cdot a_{n-1} = 0$$

$$\downarrow$$

$$\boxed{a_n = \frac{3}{4 \cdot (r+n) \cdot (r+n-1) + 3 \cdot (r+n)} \cdot a_{n-1}}$$

Recurrence relation

$(n = 1,2,3, \ldots)$

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Let's take a look at the ***first brackets***:

$$\underbrace{a_0}_{=0} \cdot \underbrace{[4 \cdot r \cdot (r-1) + 3 \cdot r]}_{=0} = 0 \qquad (n=0)$$

And/or

➢ The case of $a_0 = 0$ is ***not applicable***:

$$a_0 = 0 \xrightarrow[\text{Recurrence relation}]{} a_1 = 0 \dashrightarrow a_n = 0 \longrightarrow \boxed{\varphi \equiv 0}$$

Recurrence relation

➢ The only applicable case is thus:

$$\boxed{f(r) = \quad 4 \cdot r \cdot (r-1) + 3 \cdot r = 0}$$

Indicial equation

$$\boxed{f(r) = 4 \cdot r \cdot (r-1) + 3 \cdot r = 0} \qquad (n=0)$$

➢ The roots of the equation are: $\boxed{r_1 = 0}$ $\boxed{r_2 = \frac{1}{4}}$

➢ We identified the "unknown r" values that produce the power series solutions:

$$\varphi_1(x; r_1) = x^{r_1} \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$\varphi_2(x; r_2) = x^{r_2} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

➢ For ***each case***, we now can identify the series coefficients $(a_0, \dots, a_n)$ and $(b_0, \dots, b_n)$ by using the ***reoccurrence relation*** (!)

Case I - $r = 0$

$$a_n = \frac{3}{n \cdot (4n-1)} \cdot a_{n-1}$$

$$\varphi_1(x) = x^0 \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$a_1 = \frac{3}{1 \cdot (4 \cdot 1 - 1)} \cdot a_0 = \frac{3}{1 \cdot 3} a_0$$

$$a_2 = \frac{3}{2 \cdot (4 \cdot 2 - 1)} \cdot a_1 = \frac{3}{2 \cdot 7} a_1 = \frac{3^2}{(1 \cdot 2)(3 \cdot 7)} a_0$$

$$a_3 = \frac{3}{3 \cdot (4 \cdot 3 - 1)} \cdot a_2 = \frac{3}{3 \cdot 11} a_2 = \frac{3^3}{(1 \cdot 2 \cdot 3)(3 \cdot 7 \cdot 11)} a_0$$

$$\boxed{a_n = \frac{3^n}{n! \cdot [3 \cdot 7 \cdot 11 \cdots (4 \cdot n - 1)]} \cdot a_0}$$

Free selection of a_0

$$\varphi_1(x) = x^0 \cdot \sum_{n=0}^{\infty} a_n \cdot x^n$$

$$\boxed{\varphi_1(x; r_1) = \sum_{n=0}^{\infty} \frac{3^n}{n! \cdot [3 \cdot 7 \cdot 11 \cdots (4 \cdot n - 1)]} \cdot x^n}$$

We can choose $a_0 = 1$ (for simplicity)

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Case II - $r = 1/4$

$$b_n = \frac{3}{n \cdot (4n+1)} \cdot b_{n-1}$$

$$\varphi_2(x) = x^{\frac{1}{4}} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

$$b_1 = \frac{3}{1 \cdot (4 \cdot 1 + 1)} \cdot b_0 = \frac{3}{1 \cdot 5} b_0$$

$$b_2 = \frac{3}{2 \cdot (4 \cdot 2 + 1)} \cdot b_1 = \frac{3}{2 \cdot 9} b_1 = \frac{3^2}{(1 \cdot 2)(5 \cdot 9)} b_0$$

$$\boxed{b_n = \frac{3^n}{n! \cdot [5 \cdot 9 \cdots (4 \cdot n + 1)]} \cdot b_0}$$

Free selection of b_0

$$\varphi_2(x) = x^{\frac{1}{4}} \cdot \sum_{n=0}^{\infty} b_n \cdot x^n$$

$$\boxed{\varphi_2(x; r_1) = x^{\frac{1}{4}} \cdot \sum_{n=0}^{\infty} \frac{3^n}{n! \cdot [5 \cdot 9 \cdots (4 \cdot n + 1)]} \cdot x^n}$$

We can choose $b_0 = 1$ (for simplicity)

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Summary:

$$4x \cdot y'' + 3 \cdot y' - 3 \cdot y = 0$$

Cannot be solved analytically by the conventional methods…

- The ***general solution*** of the equation is:

$$\varphi(x) = c_1 \cdot \varphi_1(x) + c_2 \cdot \varphi_2(x)$$

- Where $\varphi_1(x)$ and $\varphi_2(x)$ are:

$$\varphi_1(x) = \sum_{n=0}^{\infty} \frac{3^n}{n! \cdot [3 \cdot 7 \cdot 11 \cdots (4 \cdot n - 1)]} \cdot x^n$$

$$\varphi_2(x) = x^{\frac{1}{4}} \cdot \sum_{n=0}^{\infty} \frac{3^n}{n! \cdot [5 \cdot 9 \cdots (4 \cdot n + 1)]} \cdot x^n$$

*Show at home: the solutions are **linearly independent** and **converge** for $0 < x < \infty$*

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