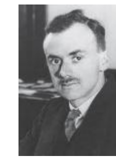
Part C – Eigenfunctions orthogonality

(S-L systems)

Mathematical preliminaries: Inner product and orthogonal functions

Inner product – function space

- Consider two ***real-value*** functions $\varphi_1(x)$ and $\varphi_2(x)$ in the interval $a \le x \le b$



Paul Dirac

- The ***inner-product*** of these two functions is defined as:

Bra-Ket notation

$$\boxed{\langle \varphi_1 | \varphi_2 \rangle \equiv \int_a^b \varphi_1(x) \cdot \varphi_2(x) dx}$$

- The *weighted* ***inner-product*** – with a *weight function* $s(x)$ is:

$$\langle \varphi_1 | s | \varphi_2 \rangle \equiv \int_a^b \varphi_1(x) \cdot s(x) \cdot \varphi_2(x) dx$$

Note that:

$\langle \varphi_1 | 1 | \varphi_2 \rangle = \langle \varphi_1 | \varphi_2 \rangle$

- The ***norm*** of $\varphi_i(x)$ with a *weight function* $s(x)$ is defined as:

$$\boxed{\|\varphi_i\| = \sqrt{\langle \varphi_i | s | \varphi_i \rangle} = \left[\int_a^b \varphi_i(x) \cdot s(x) \cdot \varphi_i(x) dx\right]^{1/2}}$$

Orthogonal functions

- Consider now a ***set*** of real-value functions:

$$\varphi_1(x)\ ,\varphi_2(x),\dots,\varphi_N(x)$$

- The functions in the set are said to be ***orthogonal*** *if and only if* :

$$n,m = 1,2,3,\dots,N$$

$$\langle\varphi_n|s|\varphi_m\rangle\ = 0 \qquad (n \neq m)$$

- The functions in the set are ***normalzied*** if:

$$\|\varphi_n\|^2 = \langle\varphi_n|s|\varphi_n\rangle = 1$$

- The functions in the set are said to be ***orthonormal*** if :

$$\delta_{nm} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

$$\langle\varphi_n|s|\varphi_m\rangle = \delta_{nm}$$

Example - Trigonometric functions

$$\varphi_n(x) = \sin(nx) \quad , \quad \varphi_m(x) = \sin(mx) \qquad \boxed{n, m = 1,2,3, \ldots}$$

- These functions are ***orthogonal*** with $s(x) = 1$ within $-\pi \leq x \leq \pi$:

$$\langle \varphi_n | 1 | \varphi_m \rangle = \langle \varphi_n | \varphi_m \rangle \quad = \int_{-\pi}^{\pi} \sin(nx) \cdot \sin(mx)\, dx$$

$$= \int_{-\pi}^{\pi} \frac{1}{2} \cdot \left[\cos\big((n-m)\cdot x\big) - \cos\big((n+m)\cdot x\big)\right] dx$$

$$= \frac{1}{2} \cdot \cancel{\int_{-\pi}^{\pi} \cos\big((n-m)\cdot x\big)\, dx} \quad - \frac{1}{2} \cdot \cancel{\int_{-\pi}^{\pi} \cos\big((n+m)\cdot x\big)\, dx} \quad = 0$$

$$\boxed{n \neq m}$$

- These functions, however, are ***not*** ***normalized***:

$$\|\varphi_n\|^2 = \langle \varphi_n | \varphi_n \rangle \quad = \frac{1}{2} \cdot \int_{-\pi}^{\pi} 1\, dx \quad = \pi \neq 1$$

- To make these functions ***orthonormal*** we must divide them by the norm!

$$\varphi_n(x) = \frac{1}{\sqrt{\pi}} \cdot \sin(nx) \quad , \quad \varphi_m(x) = \frac{1}{\sqrt{\pi}} \cdot \sin(mx)$$

$$\|\varphi_n\|^2 = \langle \varphi_n | \varphi_n \rangle = 1$$

Orthonormal trigonometric functions

$$\varphi_n(x) = \sqrt{\frac{1}{L}} \sin\left(\frac{\pi n x}{L}\right) \quad , \quad \psi_n(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi n x}{L}\right)$$

- These "famous" functions are ***orthonormal*** with $s(x) = 1(x)$ within $-L \leq x \leq L$

$$\langle \varphi_n | \varphi_m \rangle = \delta_{nm} \qquad \langle \psi_n | \psi_m \rangle = \delta_{nm} \qquad \langle \varphi_n | \psi_m \rangle = 0$$

$\boxed{\text{See H.W.}}$

Orthogonality of S-L eigenfunctions

Theorem

$$\frac{d}{dx}\left(p(x)\cdot\frac{dy}{dx}\right)+\left(q(x)+\lambda\cdot s(x)\right)\cdot y=0$$

Let the coefficients $p(x)$, $q(x)$ *and* $s(x)$ *of S-L system be continues in* $a \le x \le b$.

Let the eigenfunctions $\varphi_n(x)$ *and* $\varphi_m(x)$, *corresponding to* ***distinct*** *eigenvalues* λ_n *and* λ_m.

Then, $\varphi_n(x)$ *and* $\varphi_m(x)$ *are* ***orthogonal*** *with respect to a weight function* $s(x)$ *in* $a \le x \le b$.

$$\downarrow$$

$$\boxed{\langle\varphi_n|s|\varphi_m\rangle=\delta_{nm}}$$

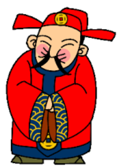
Proof

- The *eigenfunctions* $\varphi_n(x)$ *and* $\varphi_m(x)$, *and the corresponding to eigenvalues* λ_n *and* λ_m satisfy:

$(-)$

$$\frac{d}{dx}\left(p(x)\cdot\frac{d\varphi_n}{dx}\right)+\big(q(x)+\lambda_n\cdot s(x)\big)\cdot\varphi_n=0 \qquad /\cdot\ \varphi_m(x)$$

$$\frac{d}{dx}\left(p(x)\cdot\frac{d\varphi_m}{dx}\right)+\big(q(x)+\lambda_m\cdot s(x)\big)\cdot\varphi_m=0 \qquad /\cdot\ \varphi_n(x)$$

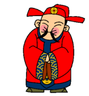


- *Rearranging:*

$$\underbrace{\frac{d}{dx}\left(p(x)\cdot\frac{d\varphi_n}{dx}\right)\cdot\varphi_m-\frac{d}{dx}\left(p(x)\cdot\frac{d\varphi_m}{dx}\right)\cdot\varphi_n}_{\frac{d}{dx}\left[p(x)\cdot\left(\frac{d\varphi_n}{dx}\cdot\varphi_m-\frac{d\varphi_m}{dx}\cdot\varphi_n\right)\right]}+q(x)\cdot(\varphi_n\cdot\varphi_m-\varphi_m\cdot\varphi_n)$$

$$+(\lambda_n-\lambda_m)\cdot(\varphi_n\cdot s(x)\cdot\varphi_m)=0$$

$\boxed{\textit{Verify !!!}}$



$$(\lambda_n - \lambda_m) \cdot (\varphi_n \cdot s(x) \cdot \varphi_m) = \frac{d}{dx}\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right] \qquad / \int_a^b [**]dx$$

$$\downarrow$$

$$(\lambda_n - \lambda_m) \cdot \int_a^b (\varphi_n \cdot s(x) \cdot \varphi_m)dx = \left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=b}$$

$$\underbrace{-\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=a}}_{\text{The B.C . terms of the S-L problem}} = 0$$

We will see soon the B.C. terms **truly** vanish for **all** S-L systems...

$$\downarrow$$

$$\boxed{(\lambda_n - \lambda_m) \cdot \int_a^b (\varphi_n \cdot s(x) \cdot \varphi_m)dx = 0}$$

$$(\lambda_n - \lambda_m) \cdot \int_a^b (\varphi_n \cdot s(x) \cdot \varphi_m) dx = 0$$

- *Recall that the eigenvalues are **distinctive** !*

$$\lambda_n \neq \lambda_m$$

$$\downarrow$$

$$\boxed{\langle \varphi_n | s | \varphi_m \rangle = \int_a^b (\varphi_n \cdot s(x) \cdot \varphi_m) dx = 0}$$

$\checkmark$

The eigenfunctions of S-L systems are indeed orthogonal !

The B.C. terms

- It is left to show (***as promised***) that the B.C. terms ***vanish*** !

$$\left[p(x)\cdot\left(\frac{d\varphi_m}{dx}\cdot\varphi_n-\frac{d\varphi_n}{dx}\cdot\varphi_m\right)\right]_{x=b}-\left[p(x)\cdot\left(\frac{d\varphi_m}{dx}\cdot\varphi_n-\frac{d\varphi_n}{dx}\cdot\varphi_m\right)\right]_{x=a}=0$$

- *We will show this for each of the following cases:*

Regular S-L
$$\begin{cases} a_1\cdot y(a)+a_2\cdot y'(a)=0 \\ b_1\cdot y(b)+b_2\cdot y'(b)=0 \end{cases}$$

Periodic S-L
$$\begin{cases} y(a)=y(b) \quad ; \quad y'(a)=y'(b) \\ p(x=a)=p(x=b) \end{cases}$$

Singular S-L
$$\begin{cases} \lim_{x\to a} y(x)<\infty \;\; ; \;\; p(x=a)=0 \\ b_1\cdot y(b)+b_2\cdot y'(b)=0 \end{cases}$$

(I) regular S-L system

Regular S-L

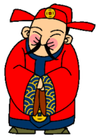
$$\begin{cases} a_1 \cdot y(a) + a_2 \cdot y'(a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

- *Focusing first on the B.C. on* $\boldsymbol{x = b}$, *which are satisfied by* ***both*** φ_n *and* φ_m.

$$b_1 \cdot \varphi_n(b) + b_2 \cdot \varphi'_n(b) = 0 \qquad /\cdot\ \varphi_m(b)$$

$(-)$

$$b_1 \cdot \varphi_m(b) + b_2 \cdot \varphi'_m(b) = 0 \qquad /\cdot\ \varphi_n(b)$$



$$\downarrow$$

$$b_2 \cdot \underbrace{\left[\varphi'_n(b) \cdot \varphi_m(b) - \varphi'_m(b) \cdot \varphi_n(b)\right]}_{=0} = 0$$

If $b_2 = 0$, *we can do the same procedure but with multiplying by* $\varphi'_n(b)$ *and* $\varphi'_m(b)$

$$\boxed{\varphi'_n(b) \cdot \varphi_m(b) - \varphi'_m(b) \cdot \varphi_n(b) = 0}$$

$$\boxed{\varphi'_n(b) \cdot \varphi_m(b) - \varphi'_m(b) \cdot \varphi_n(b) = 0}$$

- *Repeating the **same** procedure on* $x = a$ *yields:*

$$\boxed{\varphi'_n(a) \cdot \varphi_m(a) - \varphi'_m(a) \cdot \varphi_n(a) = 0}$$

- *Thus, these B.C. terms vanish !*

$$\cancel{\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=b}} - \cancel{\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=a}} = 0$$

$$\downarrow$$

Regular S-L ✓

(II) periodic S-L system

$\boxed{\text{Periodic S-L}}$

$$\begin{cases} y(a) = y(b) \;\;;\;\; y'(a) = y'(b) \\ p(x=a) = p(x=b) \end{cases}$$

- The B.C. are satisfied by **both** φ_n and φ_m.

$$\varphi_n(a) = \varphi_n(b) \qquad \varphi'_n(a) = \varphi'_n(b)$$

$$\varphi_m(a) = \varphi_m(b) \qquad \varphi'_m(a) = \varphi'_m(b)$$

- The B.C. term:

$$\left[p(x)\cdot\left(\frac{d\varphi_m}{dx}\cdot\varphi_n - \frac{d\varphi_n}{dx}\cdot\varphi_m\right)\right]_{x=b} - \left[p(x)\cdot\left(\frac{d\varphi_m}{dx}\cdot\varphi_n - \frac{d\varphi_n}{dx}\cdot\varphi_m\right)\right]_{x=a}$$

The same

- *Thus, the B.C. term produces:*

$$[p(b) - p(a)] \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)_{x=a \text{ or } x=b}$$

- *Recall that* $p(x)$ *is also periodic:*

$$\boxed{p(x=a) = p(x=b)}$$

- *Thus, this B.C. terms vanishes !*

$$\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=b} - \left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=a} = 0$$

$$\downarrow$$

$$\boxed{\textit{Periodic S-L}}$$

√

(III) singular S-L system

Singular S-L

$$\begin{cases} \lim_{x \to a} y(x) < \infty \ ; \ p(x = a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

- *The B.C. term:*

$$\left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=b} - \left[p(x) \cdot \left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)\right]_{x=a}$$

- *The "regular" B.C. on $x = b$ were shown before to produce:*

$$\boxed{\varphi'_n(b) \cdot \varphi_m(b) - \varphi'_m(b) \cdot \varphi_n(b) = 0}$$

- *Thus, the remaining B.C. term is:*

$$-\left[\cancel{p(x)} \cdot \underbrace{\left(\frac{d\varphi_m}{dx} \cdot \varphi_n - \frac{d\varphi_n}{dx} \cdot \varphi_m\right)}_{<\infty}\right]_{x=a} \qquad = 0$$

- *Using the second B.C. :*

$$\lim_{x \to a} y(x) < \infty$$

$$p(x = a) = 0$$

- *Thus, these B.C. terms also vanish !*

$$\downarrow$$

$\boxed{\textit{Singular S-L}}$ ✓

A comment on Orthonormal relations: discrete vs. continues function sets

- For a ***discrete*** ***set*** of ***orthonormal*** eigenfunctions:

$$\varphi_n(x) \qquad n = 1,2 \ldots .$$

$$\boxed{\langle \varphi_n | s | \varphi_m \rangle = \int_a^b \varphi_n(x) \cdot s(x) \cdot \varphi_m(x) dx = \delta_{nm}}$$

Discrete set

Generalization

- For a ***continues set*** of ***orthonormal*** eigenfunctions:

$$\varphi_k(x) = \varphi(x,k) \qquad k \in \mathcal{R}$$

$$\boxed{\langle \varphi_k | s | \varphi_p \rangle = \int_a^b \varphi(x,k) \cdot s(x) \cdot \varphi(x,p) dx = \delta(k-p)}$$

Continues set