Part B – Expending Green's function method

(non-homogeneous B.C.)

- So far we ***focused*** on boundary-value problems with ***homogeneous*** boundary conditions:

$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

$$U_a[y] = 0 \qquad\qquad U_b[y] = 0$$

- In real life, however, many problems involve with ***inhomogeneous*** boundary conditions.

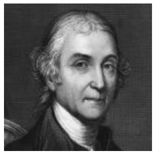
$$L[y] = y'' + p(x) \cdot y' + q(x) \cdot y = f(x)$$

$$U_a[y] = \alpha \qquad\qquad U_b[y] = \beta$$

We will now employ ***different derivation*** for the Green's function method - to handle ***any general*** B.C. state.

Reminder: Lagrange and Green's identities

➢ For ***any*** linear differential operator (L) - ***exists*** an *adjoint operator* ($\tilde{L}$), such that:



$$\boxed{z \cdot L[y] - y \cdot \tilde{L}[z] = \frac{d}{dx} F(x, y, y', z, z')}$$

Lagrange identity

$$\boxed{\int_a^b \{z \cdot L[y] - y \cdot \tilde{L}[z]\} dx = [F(x, y, y', z, z')]_{x=a}^{b}}$$

Green's identity

Second-order differential equations:

$$L[y] = a_0(x) \cdot \frac{d^2y}{dx^2} + a_1(x) \cdot \frac{dy}{dx} + a_2(x)$$

$$\tilde{L}[z] = \frac{d^2}{dx^2}[a_0(x) \cdot z] - \frac{d}{dx}[a_1(x) \cdot z] + a_2(x) \cdot z$$

$$\downarrow$$

$$\boxed{F = a_0 \cdot (z \cdot y' - z' \cdot y) + (a_1 - a_0') \cdot z \cdot y}$$

(I) Modifying Green's function formulation

- We are looking for a ***solution*** $\varphi(x)$ that satisfies:

$$U_a[\varphi] = \alpha \qquad U_b[\varphi] = \beta$$

$$\boxed{L[\varphi] = f(x)}$$

$$/\cdot \tilde{G}(x, x_1) \qquad / \int_a^b [**] dx$$

- Introducing ***Green's function*** of the *adjoint problem* - defined via:

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0$$

$$\boxed{\tilde{L}[\tilde{G}(x, x_1)] = \delta(x - x_1)}$$

$$/\cdot \varphi(x) \qquad / \int_a^b [**] dx$$

- ***Note that:*** The B.C. for $\tilde{G}(x, x_0)$ are ***not the same*** as for $\varphi(x)$

Manipulating these equations via:

- *Multiplying* the above by $\tilde{G}(x, x_1)$ and $\varphi(x)$ respectively.
- *Integrating* both equations along $a \leq x \leq b$ (not along x_1 - as previously done).
- *Subtracting* the equations.

- The resulting expression (***after some algebra…***) is:

$$\varphi(x_1) = \int_a^b \tilde{G}(x,x_1)\cdot f(x)dx + \int_a^b \underbrace{\left\{\varphi(x)\cdot\tilde{L}\left[\tilde{G}(x,x_1)\right] - \tilde{G}(x,x_1)\cdot L[\varphi(x)]\right\}}_{=\frac{d}{dx}F\left(x,\varphi,\frac{d\varphi}{dx},\tilde{G},\frac{d\tilde{G}}{dx}\right)}dx$$

- Noting that - the terms in the right brackets fulfil - *Lagrange identity*.

- These brackets can thus be expressed as a ***full differential***.

- After integration it is obtained that:

$$\boxed{\varphi(x_1) = \int_a^b \tilde{G}(x,x_1)\cdot f(x)dx + \left[F\left(x,\varphi,\frac{d\varphi}{dx},\tilde{G},\frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}}$$

Similar *to the homogeneous case*

(but - integration over x)

Boundary-condition term

The boundary conditions term

$$\left[F\left(x,\varphi,\frac{d\varphi}{dx},\tilde{G},\frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

Boundary-condition term

- Explicitly, for a second-order differential operator:

$$[F]_{x=a}^{b}=\left[a_0\cdot\left(\tilde{G}(x,x_1)\cdot\frac{d\varphi(x)}{dx}-\frac{d\tilde{G}(x,x_1)}{dx}\cdot\varphi(x)\right)+(a_1-a_0')\cdot\tilde{G}(x,x_1)\cdot\varphi(x)\right]_{x=a}^{b}$$

- We must know ***both*** $\varphi(x)$ and $\frac{d\varphi(x)}{dx}$ - in ***both*** $x=a$ and $x=b$.

__Four__ linear independent B.C. are __needed__ !

- Unfortunately, we have ***only*** ***two*** ***available***:

$$U_a[\varphi]=a_1\cdot\varphi(a)+a_2\cdot\varphi'(a)=\alpha \qquad U_b[\varphi]=b_1\cdot\varphi(b)+b_2\cdot\varphi'(b)=\beta$$

From where can we fill the __missing__ __two__ ?????????

- ***Luckily***, we have in hand two more conditions ***available*** on $\tilde{G}(x, x_1)$!

$$\tilde{U}_a[\tilde{G}] = \left[a_1 \cdot \tilde{G}(x, x_1) + a_2 \cdot \frac{d\tilde{G}(x, x_1)}{dx}\right]_{x=a} = 0$$

$$\tilde{U}_b[\tilde{G}] = \left[b_1 \cdot \tilde{G}(x, x_1) + b_2 \cdot \frac{d\tilde{G}(x, x_1)}{dx}\right]_{x=b} = 0$$

- These two conditions – are ***yet to be determined*** so far – and can be ***freely selected***…

The B.C on $\tilde{U}_a[\tilde{G}]$ and $\tilde{U}_b[\tilde{G}]$ will be chosen to eliminate the unknown terms in $[F]_{x=a}^{b}$

- With these B.C. we can ***now construct*** the $\tilde{G}(x, x_1)$ and completely define the solution:

$$\varphi(x_1) = \int_a^b \tilde{G}(x, x_1) \cdot f(x)dx + \left[F\left(x, \varphi, \frac{d\varphi}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

An example:

$$L[y] = \frac{d^2y}{dx^2} + \frac{dy}{dx} = f(x)$$

$$y(0) = 1 \qquad\qquad y'(1) = 2$$

- For this equation $a_0 = 1$, $a_1 = 1$ and $a_2 = 0$, and the operator is

$$L = \frac{d^2}{dx^2} + \frac{d}{dx}$$

- The ***adjoint operator*** is:

$$\tilde{L} = \frac{d^2}{dx^2} - \frac{d}{dx}$$

- And the ***B.C. term*** is:

$$[F]_{x=0}^{1} = \left[\tilde{G}(x,x_1)\cdot\frac{d\varphi(x)}{dx} - \frac{d\tilde{G}(x,x_1)}{dx}\cdot\varphi(x) + \tilde{G}(x,x_1)\cdot\varphi(x)\right]_{x=0}^{1}$$

- Substituting the B.C. for $\varphi(x)$: $\boxed{y(0) = 1}$ $\boxed{y'(1) = 2}$

$$[F]_{x=0}^{1} = \left(\tilde{G}(x,x_1)\cdot 2\right)\Big|_{x=1} - \cancel{\frac{d\tilde{G}(x,x_1)}{dx}\Big|_{x=1}}\cdot\varphi(x=1) + \cancel{\tilde{G}(x,x_1)\Big|_{x=1}}\cdot\varphi(x=1)$$

Unknown B.C. on $\varphi(x)$

$$-\left(\cancel{\tilde{G}(x,x_1)}\cdot\frac{d\varphi(x)}{dx}\right)\Big|_{x=0} + \frac{d\tilde{G}(x,x_1)}{dx}\Big|_{x=0}\cdot 1 - \tilde{G}(x,x_1)\Big|_{x=0}\cdot 1$$

- To ***eliminate*** the ***unknowns*** - we set the following B.C. on $\tilde{G}(x,x_1)$

$$\boxed{\tilde{U}_a[\tilde{G}] = \tilde{G}(x,x_1)\Big|_{x=0} = 0}$$

$$\boxed{\tilde{U}_b[\tilde{G}] = \tilde{G}(x,x_1)\Big|_{x=1} - \frac{d\tilde{G}(x,x_1)}{dx}\Big|_{x=1} = 0}$$

- Now - the ***B.C. term*** is ***fully defined***:

$$[F]_{x=0}^{1} = 2\cdot\tilde{G}(x,x_1)\Big|_{x=1} + \frac{d\tilde{G}(x,x_1)}{dx}\Big|_{x=0} - \tilde{G}(x,x_1)\Big|_{x=0}$$

- To finalize - we calculate Green's function $\tilde{G}(x,x_1)$, that satisfies:

The same way as we did for the homogeneous B.C.

$$\tilde{L}\left[\tilde{G}(x,x_1)\right] = \frac{d^2\tilde{G}(x,x_1)}{dx^2} - \frac{d\tilde{G}(x,x_1)}{dx} = \delta(x-x_1)$$

Try at home…

$$\tilde{G}(x,x_1)\Big|_{x=0} = 0 \qquad \tilde{G}(x,x_1)\Big|_{x=1} - \frac{d\tilde{G}(x,x_1)}{dx}\Big|_{x=1} = 0$$

- And the solution is given by:

$$\varphi(x_1) = \int_a^b \tilde{G}(x,x_1)\cdot f(x)dx + \left[F\left(x,\varphi,\frac{d\varphi}{dx},\tilde{G},\frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

(II) Formulation check - degeneration into the homogeneous B.C. solution

- A ***particular case*** of the inhomogeneous boundary-value problem is the ***homogeneous case***:

$$U_a[\varphi] = \alpha = 0$$
$$U_b[\varphi] = \beta = 0$$

$$L[\varphi] = f(x)$$

- In the former Green's function formulation (*homogeneous B.C.*), the solution was obtained by:

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0) dx_0$$

- While in the modified formulation (*generalized B.C.*), the solution is given by:

$$\varphi(x_1) = \int_a^b \tilde{G}(x, x_1) \cdot f(x) dx + \left[F\left(x, \varphi, \frac{d\varphi}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx} \right) \right]_{x=a}^{b}$$

- ***Clearly***, both expressions must ***coincide*** for the homogenous B.C. state above !

The boundary conditions term – should vanish…

- Two homogeneous B.C. are given for $\varphi\,(x)$:

$$U_a[\varphi] = 0 \qquad\qquad U_b[\varphi] = 0$$

- Two ***additional*** homogeneous B.C. are chosen for $\tilde{G}(x, x_1)$ – to ***eliminate*** the unknowns…

$$\tilde{U}_a[\tilde{G}] = 0 \qquad\qquad \tilde{U}_b[\tilde{G}] = 0$$

- These ***four*** ***homogeneous*** ***linear-independent*** relations make the B.C. term ***vanish!***

$$\left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b} =$$

$$= \left[a_0 \cdot \left(\tilde{G}(x, x_1) \cdot \frac{d\varphi(x)}{dx} - \frac{d\tilde{G}(x, x_1)}{dx} \cdot \varphi(x)\right) + (a_1 - a_0') \cdot \tilde{G}(x, x_1) \cdot \varphi(x)\right]_{x=a}^{b} = 0$$

The integral relation – should be the same…

- After eliminating the B. C. terms – it is left to show that the following are equivalent:

$$\varphi(x_1) = \int_a^b \tilde{G}(x, x_1) \cdot f(x)dx + 0$$

Modified formulation

$$\varphi(x) = \int_a^b G(x, x_0) \cdot f(x_0)dx_0$$

Original formulation

- Note the ***integration parameter*** is ***different*** between the formulation.

- The two formulations are thus ***equivalent*** if the following condition is fulfilled:

$$\tilde{G}(x_0, x_1) = G(x_1, x_0)$$

?

- This is shown in the following…

Relations between $G(x, x_0)$ and $\tilde{G}(x, x_1)$

- ***Green's function*** of the original formulation (*homogeneous B.C.*) satisfies:

$$U_a[G] = 0 \qquad U_b[G] = 0$$

$$L[G(x, x_0)] = \delta(x - x_0) \qquad /\cdot \tilde{G}(x, x_1) \qquad / \int_a^b [**]dx$$

- The *adjoint* ***Green's function*** of the modified formulation satisfies:

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0$$

$$\tilde{L}[\tilde{G}(x, x_1)] = \delta(x - x_1) \qquad /\cdot G(x, x_0) \qquad / \int_a^b [**]dx$$

- Repeating the same manipulation as before…

$$\tilde{G}(x_0, x_1) - G(x_1, x_0) = \int_a^b \{\tilde{G} \cdot L[G] - G \cdot \tilde{L}[\tilde{G}]\}dx \quad = \left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

- We have in hand ***four*** ***homogeneous*** ***linear-independent*** B.C. relations:

$$U_a[G] = 0 \qquad U_b[G] = 0$$

Same as for $\varphi(x)$

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0$$

Eliminate the remaining terms in $[F]_a^b$

- The B.C. term is thus eliminated.

$$\tilde{G}(x_0,x_1) - G(x_1,x_0) = \underbrace{\left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}}_{=0}$$

$$\downarrow$$

Equivalence validation – completed

$$\boxed{\tilde{G}(x_0,x_1) = G(x_1,x_0)}$$

✓

Relations between $G(x, x_0)$ and $\tilde{G}(x, x_1)$

- ***Green's function*** of the original formulation (*homogeneous B.C.*) satisfies:

$$U_a[G] = 0 \qquad U_b[G] = 0 \qquad L[G(x, x_0)] = \delta(x - x_0) \qquad / \cdot \tilde{G}(x, x_1) \qquad / \int_a^b [**] dx$$

- The *adjoint* ***Green's function*** of the modified formulation satisfies:

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0 \qquad \tilde{L}[\tilde{G}(x, x_1)] = \delta(x - x_1) \qquad / \cdot G(x, x_0) \qquad / \int_a^b [**] dx$$

- Repeating the same manipulation as before…

$$\tilde{G}(x_0, x_1) - G(x_1, x_0) = \int_a^b \{\tilde{G} \cdot L[G] - G \cdot \tilde{L}[\tilde{G}]\} dx \quad = \left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

- We have in hand ***four*** ***homogeneous*** ***linear-independent*** B.C. relations:

$$U_a[G] = 0 \qquad U_b[G] = 0$$

Same as for $\varphi(x)$

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0$$

Eliminate the remaining terms in $[F]_a^b$

- The B.C. term is thus eliminated.

$$\tilde{G}(x_0, x_1) - G(x_1, x_0) = \cancelto{0}{\left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}}$$

$$\downarrow$$

Equivalence validation – completed

$$\boxed{\tilde{G}(x_0, x_1) = G(x_1, x_0)}$$

This is an extraordinary outcome (!)

$$\boxed{\tilde{G}(x_0, x_1) = G(x_1, x_0)}$$

- The ***adjoint*** Green's function is thus the ***same*** (!!) as the ***original*** – with variable change (!!)

$$G(x, x_0) = \begin{cases} G_I(x, x_0) & (a \le x < x_0) \\ G_{II}(x, x_0) & (x_0 < x \le b) \end{cases}$$

$$(x \to x_1)$$
$$(x_0 \to x)$$

$$\tilde{G}(x, x_1) = \begin{cases} G_I(x_1, x) & (a \le x_1 < x) \\ G_{II}(x_1, x) & (x < x_1 \le b) \end{cases}$$

- We thus ***don't need*** to trouble finding $\tilde{G}(x, x_1)$ at al….

- We can simply find $G(x, x_0)$ and ***invert*** its *variables*…

(III) Methodology: solution of inhomogeneous B.C. problems via Green's function

$$U_a[y] = \alpha$$
$$U_b[y] = \beta$$

$$L[y] = f(x)$$

Step I: Find Green's function for the original case of *homogeneous B.C.*:

$$U_a[G] = 0$$
$$U_b[G] = 0$$

$$L[G(x, x_0)] = \delta(x - x_0)$$

Step II: Find the *adjoint* Green's function via:

$$\tilde{G}(x_0, x_1) = G(x_1, x_0)$$

Step III: Find the solution via:

$$\varphi(x_1) = \int_a^b \tilde{G}(x, x_1) \cdot f(x) dx + \left[F\left(x, \varphi, \frac{d\varphi}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx} \right) \right]_{x=a}^{b}$$

Example 1 - revised – Inhomogeneous B.C.

$$L[y] = y'' = e^x$$

$$y(0) = 1 \qquad\qquad y(L) = 3$$

- For this equation $a_0 = 1$ and $a_1 = a_2 = 0$
- We already found ***before*** that for the ***homogeneous*** B.C. case:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = \left(\dfrac{x_0}{L} - 1\right) \cdot x & (0 \le x < x_0) \\ G_{II}(x, x_0) = \left(\dfrac{x}{L} - 1\right) \cdot x_0 & (x_0 < x \le L) \end{cases}$$

$$(x \to x_1)$$
$$(x_0 \to x)$$

- Thus, the ***adjoint*** Green's function is:

$$\tilde{G}(x, x_1) = \begin{cases} G_I(x_1, x) = \left(\dfrac{x}{L} - 1\right) \cdot x_1 & (0 \le x_1 < x) \\ G_{II}(x_1, x) = \left(\dfrac{x_1}{L} - 1\right) \cdot x & (x < x_1 \le L) \end{cases}$$

- The solution is given by:

$$\varphi(x_1) = \int_0^L \tilde{G}(x, x_1) \cdot f(x)dx + \left[F\left(x, \varphi, \frac{d\varphi}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=0}^{L}$$

- We need to ***integrate*** and ***differentiate*** $\tilde{G}(x, x_1)$ *over* x.

- It is desired to change the "*running variable*" in $\tilde{G}(x, x_1)$ from $x_1 \to x$:

$$\tilde{G}(x, x_1) = \begin{cases} \left(\frac{x_1}{L} - 1\right) \cdot x & (0 \le x < x_1) \\ \left(\frac{x}{L} - 1\right) \cdot x_1 & (x_1 < x \le L) \end{cases}$$

- And the derivative of $\tilde{G}(x, x_1)$ is thus:

$$\frac{d\tilde{G}(x, x_1)}{dx} = \begin{cases} \frac{x_1}{L} - 1 & (0 \le x < x_1) \\ \frac{x_1}{L} & (x_1 < x \le L) \end{cases}$$

The boundary conditions term

- Substituting $a_0 = 1$ and $a_1 = a_2 = 0$ into the B.C. term yields:

$$\left[F\left(x, \tilde{G}, \frac{d\tilde{G}}{dx}, \varphi(x), \frac{d\varphi(x)}{dx}\right)\right]_{x=0}^{L} = \left[\tilde{G}(x, x_1) \cdot \frac{d\varphi(x)}{dx} - \frac{d\tilde{G}(x, x_1)}{dx} \cdot \varphi(x)\right]_{x=0}^{L}$$

- Substituting the ***given*** *inhomogeneous B.C.* for the solution $\varphi(x)$ yields:

$$[F]_{x=0}^{L} = \left[\tilde{G}(x, x_1)\Big|_{x=L} \cdot \underbrace{\frac{d\varphi(x)}{dx}\Big|_{x=L}}_{\boxed{?}} - \frac{d\tilde{G}(x, x_1)}{dx}\Big|_{x=L} \cdot 3\right] \qquad \boxed{\varphi(L) = 3}$$

$$- \left[\tilde{G}(x, x_1)\Big|_{x=0} \cdot \underbrace{\frac{d\varphi(x)}{dx}\Big|_{x=0}}_{\boxed{?}} - \frac{d\tilde{G}(x, x_1)}{dx}\Big|_{x=0} \cdot 1\right] \qquad \boxed{\varphi(0) = 1}$$

- Note that $\tilde{G}(x, x_1)|_{x=0} = 0$ ***and*** $\tilde{G}(x, x_1)|_{x=L} = 0$ - see previous slide…

- Thus, the ***unknowns*** in the B.C. term are ***eliminated*** !

$$[F]_{x=0}^{L} = -\left.\frac{d\tilde{G}(x,x_1)}{dx}\right|_{x=L} \cdot 3 + \left.\frac{d\tilde{G}(x,x_1)}{dx}\right|_{x=0} \cdot 1$$

- By substituting the derivatives of $\tilde{G}(x,x_1)$ - see previous slide… - the B.C. term yields:

$$\boxed{[F]_{x=0}^{L} = -1 - 2 \cdot \frac{x_1}{L}}$$

- The solution is thus given by:

$$\varphi(x_1) = \int_0^L \tilde{G}(x,x_1) \cdot f(x)dx + [F]_{x=0}^{L}$$

$[F]_{x=0}^{L} = -1 - 2 \cdot \frac{x_1}{L}$

$f(x) = e^x$

$$= \int_0^{x_1} \left(\frac{x_1}{L} - 1\right) \cdot x \cdot e^x dx + \int_{x_1}^{L} \left(\frac{x}{L} - 1\right) \cdot x_1 \cdot e^x dx + 1 - 2 \cdot \frac{x_1}{L}$$

$\int x \cdot e^x dx = e^x \cdot (x - 1)$

$$\longrightarrow \boxed{\varphi(x_1) = e^{x_1} - 2 - \frac{x_1}{L} \cdot (1 + e^L)}$$

Part C – Symmetry of Green's function

(Self-adjoint problems)

Symmetry of Green's function ?

Example 1

$$L[y] = y'' = f(x)$$

$$y(0) = y(1) = 0$$

$$G(x, x_0) = \begin{cases} (x_0 - 1) \cdot x & (0 \leq x < x_0) \\ (x - 1) \cdot x_0 & (x_0 < x \leq 1) \end{cases}$$

$$\boxed{G(x, x_0) = G(x_0, x)}$$

Symmetric !

Example 2

$$L[y] = y'' - y' = f(x)$$

$$y(0) = y'(1) = 0$$

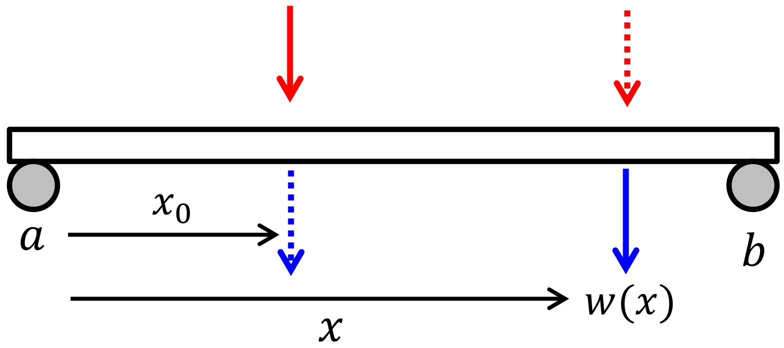
$$G(x, x_0) = \begin{cases} \mathrm{e}^{-x_0} \cdot (1 - \mathrm{e}^{x}) & (0 \leq x < x_0) \\ \mathrm{e}^{-x_0} \cdot (1 - \mathrm{e}^{x_0}) & (x_0 < x \leq 1) \end{cases}$$

$$\boxed{G(x, x_0) \neq G(x_0, x)}$$

Not symmetric !

Meaning of Green's function symmetry ?

An example: beam deflections



- $G(x, x_0)$ represents the deflection at a point x due to a ***singular*** *unit-force* – applied at x_0

$$w(x) = G(x, x_0)$$

- $G(x_0, x)$ represents the deflection at a point x_0 due to a ***singular*** *unit-force* – applied at x

$$w(x_0) = G(x_0, x)$$

- (Maxwell's) reciprocal principle - strength of materials:

Symmetric !

$$\boxed{w(x) = w(x_0)} \longrightarrow \boxed{G(x, x_0) = G(x_0, x)}$$

General view - Green's function symmetry

$$\boxed{G(x, x_0) = G(x_0, x\ )}$$

- Green's function - the ***fundamental*** solution of a ***physical problem*** - is ***invariant*** under the exchange:

x_0

("Source/load" point)

$\longleftrightarrow$

x

("Observation" point)

- Such symmetry appears in ***many*** physical systems – e.g. mechanical, heat transfer, electrostatics..

Maxwell's reciprocity: *The response of a field at an observation point (x) due to a source point (at x_0) is the same as the response at x_0 due to a source at x*

What are the conditions for a symmetric Green's function?

Reminder: Self-adjoint operator

- A general second-order differential operator (L) is:

$$L = a_0(x) \cdot \frac{d^2}{dx^2} + a_1(x) \cdot \frac{d}{dx} + a_2(x)$$

- The adjoint operator ($\tilde{L}$) is:

$$\tilde{L} = a_0 \cdot \frac{d^2}{dx^2} + (2a_0' - a_1) \cdot \frac{d}{dx} + (a_0'' - a_1' + a_2)$$

- The operator L is said to be ***self-adjoint*** if:

$$\boxed{L = \tilde{L}}$$

- A self-adjoint second-order differential operator is at the form of:

Self-adjoint differential operator (second-order)

$$\boxed{L = \tilde{L} = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)}$$

Strum-Liouville operator

$$\boxed{L = \tilde{L} = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)}$$

Lagrange and Green identities (self-adjoint operator):

$$z \cdot L[y] - y \cdot L[z] = \frac{d}{dx} F(x, y, z, y', z')$$

Lagrange identity

$$\int_a^b \{z \cdot L[y] - y \cdot L[z]\} dx = [F(x, y, z, y', z')]_a^b$$

Green's identity

- Where the B.C. term of a *self-adjoint* ***second-order*** operator is:

$$\boxed{F(x, y, z, y', z') = p(x) \cdot (z \cdot y' - z' \cdot y)}$$

Green's function – a self-adjoint operator

$\boxed{L = \tilde{L}}$

- We are now looking for a ***solution*** – for the problem with a ***self-adjoint operator***:

$$U_a[y] = \alpha \qquad U_b[y] = \beta$$

$$L[y] = f(y)$$

- ***Green's function*** for the *homogeneous B.C.* case satisfies:

$$U_a[G] = 0 \qquad U_b[G] = 0$$

$$L[G(x,x_0)] = \delta(x - x_0) \qquad /\cdot \tilde{G}(x,x_1) \qquad / \int_a^b [**]dx$$

- The *adjoint* ***Green's function*** for the *non-homogeneous B.C* case satisfies:

$$\tilde{U}_a[\tilde{G}] = 0 \qquad \tilde{U}_b[\tilde{G}] = 0$$

$$\tilde{L}\left[\tilde{G}(x,x_1)\right] = \delta(x - x_1)$$

$$\downarrow$$

$$L\left[\tilde{G}(x,x_1)\right] = \delta(x - x_1) \qquad /\cdot G(x,x_0) \qquad / \int_a^b [**]dx$$

- Using the *red* fellow…

$$\tilde{G}(x_0,x_1) - G(x_1,x_0) = \int_a^b \{\tilde{G}\cdot L[G] - G\cdot L[\tilde{G}]\}dx$$

Green's identity

$$= \left[F\left(x, G, \frac{dG}{dx}, \tilde{G}, \frac{d\tilde{G}}{dx}\right)\right]_{x=a}^{b}$$

$$L = \frac{d}{dx}\left(p(x)\cdot\frac{d}{dx}\right) + q(x)$$

$$= \left[p(x)\cdot\left(\tilde{G}\cdot\frac{dG}{dx} - \frac{d\tilde{G}}{dx}\cdot G\right)\right]_{x=a}^{b}$$

Using B.C. on G and $\tilde{G}$

$$= \cancel{\left[p(x)\cdot\left(\tilde{G}\cdot\frac{dG}{dx} - \frac{d\tilde{G}}{dx}\cdot G\right)\right]_{x=b}} - \cancel{\left[p(x)\cdot\left(\tilde{G}\cdot\frac{dG}{dx} - \frac{d\tilde{G}}{dx}\cdot G\right)\right]_{x=a}} = 0$$

$U_a[G] = 0$ $\quad$ $\tilde{U}_a[\tilde{G}] = 0$

$U_b[G] = 0$ $\quad$ $\tilde{U}_b[\tilde{G}] = 0$

$$\downarrow$$

$$\boxed{\tilde{G}(x_0,x_1) = G(x_1,x_0)}$$

Same as before..

Nothing new yet…

The B.C. term – a closer look:

$$\left[p(x)\cdot\left(\tilde{G}\cdot\frac{dG}{dx}-\frac{d\tilde{G}}{dx}\cdot G\right)\right]_{x=b}-\left[p(x)\cdot\left(\tilde{G}\cdot\frac{dG}{dx}-\frac{d\tilde{G}}{dx}\cdot G\right)\right]_{x=a}=0$$

- Lets see which B.C. are required to make it vanish…

$$U_a[G]=0 \qquad \tilde{U}_a[\tilde{G}]=0$$

$$U_b[G]=0 \qquad \tilde{U}_b[\tilde{G}]=0$$

?????

- An example:

$$G(x=a)=\tilde{G}(x=a)=0$$

$$G(x=b)=\tilde{G}(x=b)=0$$

The same B.C. for G and $\tilde{G}$!!!

Is it a luck or a rule ?

- In fact - the B.C. term vanishes when ***both*** G and $\tilde{G}$ satisfy the ***following*** B.C.:

$$\boxed{\begin{aligned} U_a[G] &= \widetilde{U}_a[\tilde{G}] \\ U_b[G] &= \widetilde{U}_b[\tilde{G}] \end{aligned}}$$

We will see this in details on the next lecture

"Regular" B.C.

$$\begin{cases} a_1 \cdot y(a) + a_2 \cdot y'(a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

Any general a_i and b_i values (including 0 and ∞)

Periodic B.C.

$$\begin{cases} y(a) = y(b) \;\; ; \;\; y'(a) = y'(b) \\ p(x=a) = p(x=b) \end{cases}$$

"Singular" B.C.

$$\begin{cases} \lim_{x \to a} y(x) < \infty \;\; ; \;\; p(x=a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

Bounded in $x = a$ and/or in $x = b$

The bounded B.C may be at $x \to \pm\infty$

Self-adjoint boundary-value problem

$$\begin{aligned} U_a[y] &= \alpha \\ U_b[y] &= \beta \end{aligned}$$

$$L[y] = f(x)$$

- A ***self-adjoint*** boundary-value *problem* includes:

  - A *self-adjoint* operator:

$$L = \tilde{L} = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)$$

  - The boundary conditions: either (I) regular, (II) periodic or (III) singular

Green's functions of S-A boundary-value problems are symmetric (!)

Green's function

The adjoint Green's function

$$L[G(x,x_0)] = \delta(x-x_0) \longleftrightarrow L[\tilde{G}(x,x_0)] = \delta(x-x_0)$$

The same !!

$$U_a[G]=0 \qquad U_b[G]=0 \longleftrightarrow U_a[\tilde{G}]=0 \qquad U_b[\tilde{G}]=0$$

$$\downarrow$$

$$\boxed{\tilde{G}(x,x_0) = G(x,x_0)}$$

Symmetric !

$$\boxed{\tilde{G}(x_0,x) = G(x,x_0)}$$

Recall that: we already found (from *Green's identity*) that:

$$\boxed{\tilde{G}(x,x_0) = G(x_0,x)}$$

Example 1

$$\boxed{\begin{gathered} L[y] = y'' - y' = f(x) \\ y(0) = y'(L) = 0 \end{gathered}}$$

$\boxed{L \neq \tilde{L}}$

$$L = \frac{d^2}{dx^2} - \frac{d}{dx} \longrightarrow \tilde{L} = \frac{d^2}{dx^2} + \frac{d}{dx}$$

$$G(x, x_0) = \begin{cases} e^{-x_0} \cdot (1 - e^{x}) & (0 \leq x < x_0) \\ e^{-x_0} \cdot (1 - e^{x_0}) & (x_0 < x \leq 1) \end{cases}$$

$$\boxed{G(x, x_0) \neq G(x_0, x)}$$

Not symmetric !

Example 2

$$\boxed{\begin{aligned} L[y] &= y'' = f(x) \\ y(0) &= y'(0) = 0 \end{aligned}}$$

✓ $\boxed{L = \tilde{L}}$

✗

Not really a boundary-value problem...

$$G(x, x_0) = \begin{cases} 0 & (0 \le x < x_0) \\ x - x_0 & (x_0 < x \le 1) \end{cases}$$

$$\boxed{G(x, x_0) \neq G(x_0, x\,)}$$

Not symmetric !

Example 3

Self-adjoint boundary-value problem

$$\boxed{\begin{array}{c} L[y] = y'' = f(x) \\ y(0) = y\,(1) = 0 \end{array}}$$

$\checkmark$

$\checkmark$

$\boxed{L = \tilde{L}}$

$$G(x, x_0) = \begin{cases} (x_0 - 1) \cdot x & (0 \le x < x_0) \\ (x - 1) \cdot x_0 & (x_0 < x \le 1) \end{cases}$$

$$\boxed{G(x, x_0) = G(x_0, x\,)}$$

Symmetric !