1 Example 1 - Homogeneous Boundary Conditions

Given equation:

$$L[y] = y'' = f(x)$$
$$y(0) = y(L) = 0$$

Need to find Green function for the equation.

Solution:

Green function is defined by solution of:

$$L[G(x, x_0)] = \delta(x - x_0)$$

$$\frac{d^2}{dx^2} G(x, x_0) = \delta(x - x_0)$$

for $x \neq x_0$:

$$\frac{d^2}{dx^2} G(x, x_0) = 0$$

Basic solutions:

$$\varphi(x) = 1 \qquad \varphi(x) = x$$

Therefore:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = A_1 + A_2 \cdot x & \{0 \leq x \leq x_0\} \\ G_{II}(x, x_0) = B_1 + B_2 \cdot x & \{x_0 \leq x \leq L\} \end{cases}$$

Continuity at x_0:

$$G_I(x = x_0, x_0) = G_{II}(x = x_0, x_0)$$

$$\boldsymbol{A_1 + A_2 \cdot x_0 = B_1 + B_2 \cdot x_0}$$

Derivative jump at x_0:

$$\frac{d}{dx} G_{II}(x = x_0, x_0) = \frac{d}{dx} G_I(x = x_0, x_0) + 1$$

$$\boldsymbol{B = A_2 + 1}$$

Boundary conditions:

$$y(x = 0) = G_I(0, x_0) = 0 \qquad \Rightarrow \qquad \boldsymbol{A_1 = 0}$$

$$y(x = L) = G_{II}(L, x_0) = 0 \qquad \Rightarrow \qquad \boldsymbol{B_1 + B_2 \cdot L = 0}$$

Now we have 4 equations with for unknowns (shown in bold above). Solution yields:

$$A_1 = 0$$

$$A_2 = -1$$

$$B_1 = -x_0$$

$$B_2 = {}^{x_0}/_{L}$$

And the Green function is:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = \left(\frac{x_0}{L} - 1\right) \cdot x & \{0 \le x- \le x_0\} \\ G_{II}(x, x_0) = \left(\frac{x}{L} - 1\right) \cdot x_0 & \{x_0 \le x \le L\} \end{cases}$$

2 Example 2 - Periodic Boundary Conditions

Given equation:

$$L[y] = y'' - k^2 \cdot y = f(x)$$

$$y(0) = y(L)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \left.\frac{dy}{dx}\right|_{x=L}$$

Need to find Green function for the equation.

Solution:

Green function is defined by solution of:

$$L[G(x, x_0)] = \delta(x - x_0)$$

$$\frac{d^2}{dx^2} G(x, x_0) - k^2 \cdot G(x, x_0) = \delta(x - x_0)$$

Basic solutions:

$$\varphi(x) = e^{kx} \qquad \varphi(x) = e^{-kx}$$

Therefore:

$$G(x, x_0) = \begin{cases} G_I(x, x_0) = A_1 \cdot e^{kx} + A_2 \cdot e^{-kx} & \{0 \le x \le x_0\} \\ G_{II}(x, x_0) = B_1 \cdot e^{kx} + B_2 \cdot e^{-kx} & \{x_0 \le x \le L\} \end{cases}$$

Continuity at x_0:

$$G_I(x = x_0, x_0) = G_{II}(x = x_0, x_0)$$

$$\boldsymbol{A_1 \cdot e^{kx_0} + A_2 \cdot e^{-kx_0} = B_1 \cdot e^{kx_0} + B_2 \cdot e^{-kx_0}}$$

Derivative jump at x_0:

$$\frac{d}{dx} G_{II}(x = x_0, x_0) = \frac{d}{dx} G_I(x = x_0, x_0) + 1$$

$$B_1 \cdot k \cdot e^{kx_0} + B_2 \cdot k \cdot e^{-kx_0} = A_1 \cdot k \cdot e^{kx_0} + A_2 \cdot k \cdot e^{-kx_0} + 1$$

$$\boldsymbol{k\left(B_1 \cdot e^{kx_0} + B_2 \cdot e^{-kx_0} - A_1 \cdot e^{kx_0} + A_2 \cdot e^{-kx_0}\right) = 1}$$

Boundary conditions:

$$y\left(x=0\right)=y\left(x=L\right)$$
$$G_I\left(0,x_0\right)=G_{II}\left(L,x_0\right)$$

$$\boldsymbol{A_1+A_2=B_1\cdot e^{kL}+B_2\cdot e^{-kL}}$$

$$\left.\frac{dy}{dx}\right|_{x=0}=\left.\frac{dy}{dx}\right|_{x=L}$$
$$\left.\frac{d}{dx}G_I\left(0,x_0\right)\right|_{x=0}=\left.\frac{d}{dx}G_{II}\left(L,x_0\right)\right|_{x=L}$$
$$\boldsymbol{A_1-A_2=B_1\cdot e^{kL}-B_2\cdot e^{-kL}}$$

Now we have 4 equations with for unknowns (shown in bold above). Solution is shown in the presentation.