三、在 $P_3(x)$ 中，定义内积 $(f(x),g(x))=\int_{-1}^{1} f(x)g(x)\,dx \quad \forall f(x),g(x)\in P_3(x)$

求 $P_3(x)$ 中与多项式 1、x 都正交的单位向量

解：设 $p(x)=a_0+a_1x+a_2x^2$

$$\begin{cases}(1,p(x))=0\\(x,p(x))=0\\(p(x),p(x))=1\end{cases}\Rightarrow\begin{cases}\int_{-1}^{1}(a_0+a_1x+a_2x^2)dx=0\\\int_{-1}^{1}x(a_0+a_1x+a_2x^2)dx=0\\\int_{-1}^{1}(a_0+a_1x+a_2x^2)^2dx=0\end{cases}$$

$$\begin{cases}2a_0+\frac{2}{3}a_2=0\\\frac{2}{3}a_1=0\end{cases}\qquad\begin{cases}a_2=-3a_0\\a_1=0\end{cases}$$

$$\therefore \int_{-1}^{1}(a_0-3a_0x^2)^2dx=\int_{-1}^{1}(a_0^2-6a_0^2x^2+9a_0^2x^4)dx$$

$$=a_0^2\int_{-1}^{1}(1-6x^2+9x^4)dx=a_0^2\left(x-2x^3+\frac{9}{5}x^5\Big|_{-1}^{1}\right)$$

$$=a_0^2\left(2-4+\frac{18}{5}\right)=1 \qquad a_0^2=\frac{5}{8}\Rightarrow a_0=\pm\frac{\sqrt{10}}{4}$$

$\therefore a_0=\frac{\sqrt{10}}{4}\quad a_1=0\quad a_2=-\frac{3\sqrt{10}}{4}$ 或 $a_0=-\frac{\sqrt{10}}{4}$，$a_1=0\quad a_2=\frac{3\sqrt{10}}{4}$

$\therefore p(x)=\frac{\sqrt{10}}{4}-\frac{3\sqrt{10}}{4}x^2$ 或 $p(x)=-\left(\frac{\sqrt{10}}{4}-\frac{3\sqrt{10}}{4}x^2\right)$

四、已知 $A=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}$ 求 (1) A 的奇异值分解 (2) 求 $\|A\|_F$

解：(1) $A^HA=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}$

$|\lambda I-A^HA|=\begin{vmatrix}\lambda-2&-1\\-1&\lambda-2\end{vmatrix}=[(\lambda-2)^2-1]=(\lambda-3)(\lambda-1)=0$

$\therefore \lambda_1=3\quad\lambda_2=1$ 对应的特征向量为 $\alpha_1=(1,1)^T\quad\alpha_2=(1,-1)^T$ $\quad V=\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{pmatrix}$

$AA^H=\begin{pmatrix}1&0\\0&1\\1&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&2\end{pmatrix}$

$|\lambda I-AA^H|=\begin{vmatrix}\lambda-1&0&-1\\0&\lambda-1&-1\\-1&-1&\lambda-2\end{vmatrix}=\lambda(\lambda-1)(\lambda-3)=0\quad\lambda_1=3\quad\lambda_2=1\quad\lambda_3=0$

对应的特征向量为 $\alpha_3=(1,1,2)^T\quad\alpha_4=(1,-1,0)^T\quad\alpha_5=(-1,-1,1)^T$

$\therefore U=\begin{pmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&-\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&\frac{1}{\sqrt3}\end{pmatrix}\qquad\therefore A=\begin{pmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&-\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&\frac{1}{\sqrt3}\end{pmatrix}\begin{pmatrix}\sqrt3&0\\0&1\\0&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\end{pmatrix}$

(2) $\|A\|_F=\sqrt{tr(A^HA)}=\sqrt{2+2}=2$