Instructions:

Submission can be done either in pairs or singles.

Write your id on the submission.

You should implement your code either in Matlab or numpy and submit it.

When you are asked to draw a function then do it using simulation (not by hand).

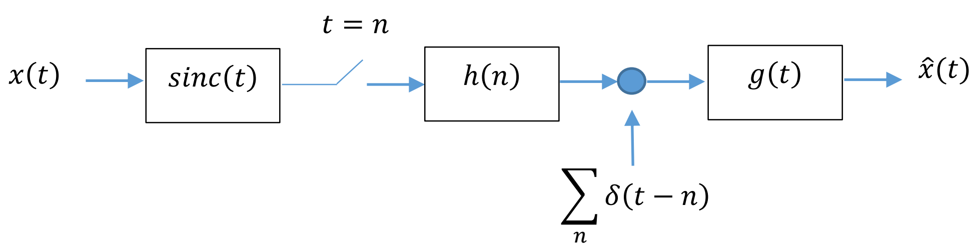
Question 1

Consider a signal $x(t)$ formed by a RC circuit with an impulse response $g(t) = \tau^{-1} \exp\left(-\frac{t}{\tau}\right) u(t)$, where $u(t)$ is the step function. Therefore,

$$x(t) = \tau \sum_{n=-\infty}^{\infty} a_n g(t-n)$$

where a_n are the coefficients of $x(t)$ in the Riesz basis $\{g(t-n)\}$.

Assume that $x(t)$ is point-wise sampled



1. Prove that $g(t-n)$ is a Riesz basis
2. Draw $x(t)$ both in time and frequency domain
3. Draw $\hat{x}(t)$ in the case of regular Nyquist reconstruction: $h(t) = \delta(t)$ and $sinc(t)$ is used instead of $g(t)$ as the interpolation filter.
4. What should be the digital correction filter $h(n)$ that guarantees perfect reconstruction?
5. Draw $\hat{x}(t)$ for the $h(n)$ calculated in 4.

Question 2 - Spline Interpolation

Splines are piecewise polynomials with pieces that are smoothly connected together. The joining points of the polynomials are called knots. For a spline of degree n, each segment is a polynomial of degree n. At the knots, the spline and its derivatives up to order (n - 1) are continuous. Here we only consider splines with uniform knots and unit spacing. These splines are uniquely characterized in terms of a B-spline expansion

$$x(t) = \sum_{k=-\infty}^{\infty} d[k]\beta^n(t-k)$$

where $\beta^n(t)$ is a B-spline of degree n, which is constructed from the (n + 1)-fold convolution of a rectangular pulse $\beta^0(t)$:

$$\beta^0(t) = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ \frac{1}{2}, & |t| = \frac{1}{2} \\ 0, & otherwise \end{cases}$$

$$\beta^n(t) = \underbrace{\beta^0(t) * \beta^0(t) * \ldots * \beta^0(t)}_{(n+1)\ times}$$

1. One appealing property of splines is that differential and integral operations on them can be carried out by simple manipulation of the expansion coefficients {d[n]}.

   (a) Prove that $\frac{d\beta^n(t)}{dt} = \beta^{n-1}\left(t+\frac{1}{2}\right) - \beta^{n-1}\left(t-\frac{1}{2}\right)$.

   (b) Prove that $\int_{-\infty}^{t} \beta^n(\tau)d\tau = \sum_{k=0}^{\infty} \beta^{n+1}\left(t-\frac{1}{2}-k\right)$.

2. Write a Matlab function *SplineExpansion(d,t,n)* that accepts a vector of expansion coefficients $\{d[k]\}_{k=1}^{N}$, a vector of time instances $\{t_l\}_{l=1}^{L}$, and a scalar $0 \leq n \leq 3$ indicating spline order. The function should return the values of a spline of order n at the location $\{t_l\}_{l=1}^{L}$, i.e. $\{x(t_l)\}_{l=1}^{L}$.

(a) Generate a random sequence $\{d[k]\}_{k=1}^{N}$, and evaluate the corresponding cubic spline (n = 3) on a dense grid $\left\{x\left(\frac{l}{M}\right)\right\}_{l=-M}^{(N+2)M}$, where M=20. Plot the result.

(b) Plot the derivative of the spline on the same grid, i.e. $\left\{x'\left(\frac{l}{M}\right)\right\}_{l=-M}^{(N+2)M}$. Use the relations in question 1.

Compare the result to numeric approximation of the derivative using differences.

3. Suppose that a spline x(t) of order n_1 is sampled at the integers after going through the filter s(t) =$\beta^{n_2}(t)$, (t), which is a B-spline of order n_2. Denote the sequence of samples by c[k].

(a) Write the frequency response of the digital correction filter h[k] that outputs the expansion coefficients (d[k]} when fed with the samples c[k]. Show that this filter is the same as long as $n_1 + n_2 = C$, where C is some constant.

(b) Show that the same correction filter h[k] is also optimal when there is no sampling filter and $n_1 = C + 1$.

(c) For the special case C = 2, express the Z-transform of h[k] as the product of a causal IIR filter $H_c(z)$ and a non-causal IIR filter $H_{nc}(z)$ (hint: examine the convolutional inverse of h[k]).

4. Write a Matlab function *interpCubic(c,t)* that accepts a vector $\{c[k]\}_{k=1}^{K}$ of pointwise samples of a cubic spline and a vector of time instances $\{t_l\}_{l=1}^{L}$. The samples are assumed to be taken at the integers, i.e. c[k] = x(k), and are assumed to be zero outside the range [1,K]. The function should return the values of x(t) at the locations $\{t_l\}_{l=1}^{L}$, i.e. $\{x(t_l)\}_{l=1}^{L}$. The correction filter h[k] should be implemented by first filtering the result with the recursive formula corresponding to $H_c(z)$ (running from left to right) and then filtering the result with the recursive formula corresponding to $H_{nc}(z)$ (running from right to left).

(a) Generate a cubic spline using the function *SplineExpansion* and plot it on a fine grid as in question 2. Evaluate the pointwise samples of the spline at the integers and user the function *interpCubic* to reconstruct it. Plot the reconstructed spline on the same fine grid.

5. It is sometimes helpful to modify the basis functions such that no digital processing is needed. In the case of splines, there functions are called cardinal splines.

(a) Express the cardinal spline of order n, $\eta^n(t)$ in terms of the correction filter h[k] and the B-spline $\beta^n(t)$.

(b) What sequence c[k] should be supplied to the function *SplineExpansion* in order for it to output $\eta^n(t)$? Use this method to plot $\eta^n(t)$ on a fine grid. Compare with the sinc kernel.

6. We want to up-sample an image with a spline of order 2 (which is applying the 1-D spline separably on each dimension).
    a. What should be the sampling filter?
    b. Please write a code that downsamples an image and then upscale it. Present the images.
7. Write a Matlab function that up-sample an image with a spline of order 3 and downsample it with a spline of order 1.
    a. What should be the correction filter?
    b. Please implement it and present the results (like in 6.b).