

Yeast Signaling

July 30, 2024

We begin with the equation:

$$\frac{\partial C(x, z)}{\partial t} = D\nabla^2 C(x, z) + r\Theta(-x)\delta(z) - \Omega\delta(z)C(x, z) \quad (1)$$

Look for steady state solutions:

$$D\nabla^2 C(x, z) + r\Theta(-x)\delta(z) - \Omega\delta(z)C(x, z) = 0 \quad (2)$$

Define new coefficients:

$$\partial_{xx}C(x, z) + \partial_{zz}C(x, z) + \tilde{r}\Theta(-x)\delta(z) - \tilde{\Omega}\delta(z)C(x, z) = 0 \quad (3)$$

And drop the tildes from here forward.

What happens if we integrate over an ϵ region around $z = 0$:

$$\partial_z C(x, z)|_{z=+\epsilon} - \partial_z C(x, z)|_{z=-\epsilon} = -r\Theta(-x) + \Omega C(x, z=0) \quad (4)$$

This is the virtual flux of α factors through the z -plane, and we can turn it into a Robin BC:

$$\partial_z C(x, z)|_{z=0} = -r\Theta(-x) + \Omega C(x, z=0)$$

And just solve the Poisson equation on the positive half volume!

Assume structure of $C(x, z)$:

$$C(x, z) = X(x)Z(z) + \frac{r}{2\Omega} \quad (5)$$

We assume this structure due to the fact that one can deduce from the equation that:

$$C(-x, z) + C(x, z) = \frac{r}{\Omega} \quad (6)$$

$$\Rightarrow C(0, z) = \frac{r}{2\Omega} \quad (7)$$

And also:

$$C(x, z) = X(x)Z(z) \Rightarrow C(0, z) = X(0)Z(z) \quad (8)$$

Where $X(0)$ is a constant, and there is no way to satisfy the previous condition for $C(0, z)$ besides constant $Z(z)$, which cannot be a solution.

So we have BC:

$$\partial_z C(x, z)|_{z=L_z} = 0 \quad (9)$$

$$\partial_z C(x, z)|_{z=0} = -r\Theta(-x) + \Omega C(x, z=0) \quad (10)$$

$$C(x, z)|_{x=0} = \frac{r}{2\Omega} \quad (11)$$

$$\partial_x C(x, z)|_{x=-L_x} = 0 \quad (12)$$

And we solve for the $x < 0$ region, which dictates the $x > 0$ region.

This is the simplest structure that has the potential to adhere to the constraints, where $C(x, z)$ is still seperable in the Poisson equation.

Before the analytical solution, we perform a coarse-grained simulation of the above to see how results should look like (with finite L_x, L_z) in 2D (1), and plot $C(x, z=0)$ for different values of Ω (fig. 2).

Now solve:

$$\nabla^2 C(x, z) = 0 \quad (13)$$

$$\frac{1}{X(x)} \partial_{xx} X(x) = -k^2 = -\frac{1}{Z} \partial_{zz} Z(z) \quad (14)$$

$$X_n(x) = A_n e^{+ik_n x} + B_n e^{-ik_n x} \quad (15)$$

$$Z_n(z) = C_n e^{+k_n z} + D_n e^{-k_n z} \quad (16)$$

BC 3:

$$C(x, z)|_{x=0} \propto A_n + B_n + \frac{r}{2\Omega} = \frac{r}{2\Omega} \quad (17)$$

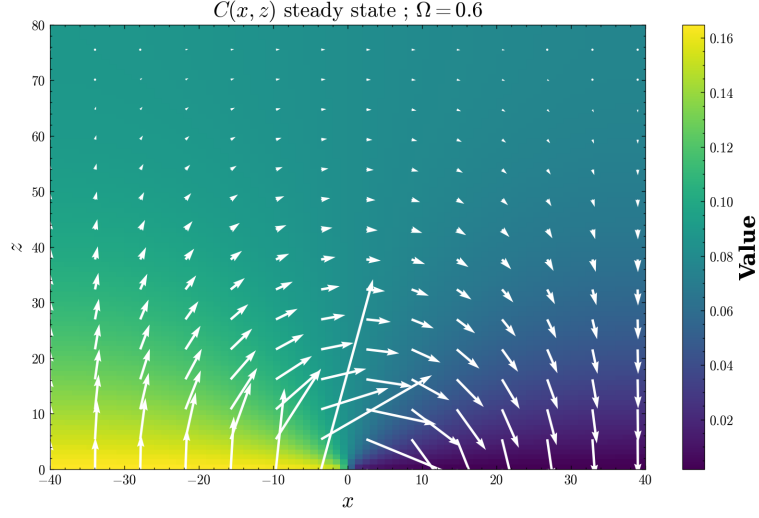


Figure 1: Heatmap of the $C(x, z)$ steady state (simulation taken to 10^6 steps); Arrows represent the flow field at every point; Parameter values are $r_0 = 0.1$, $L_x = 20$, $L_z = 30$.

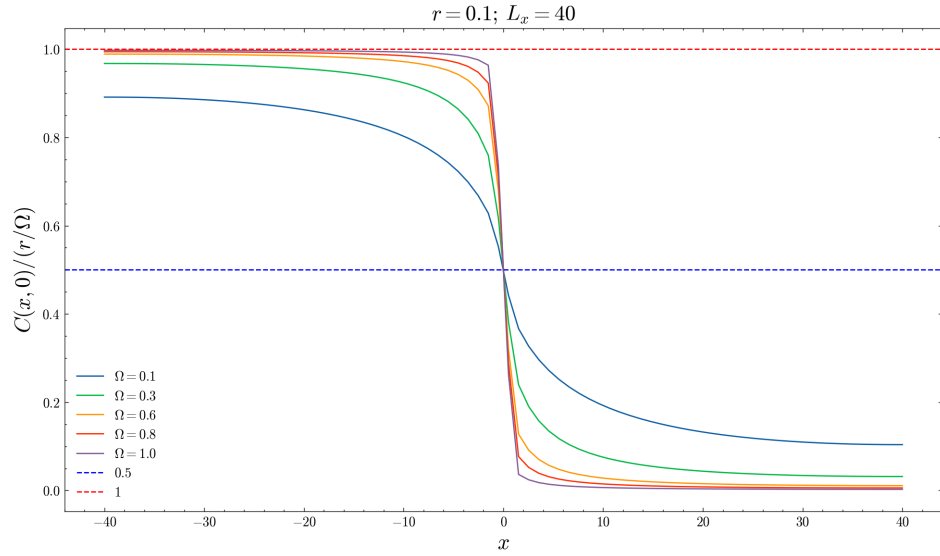


Figure 2: Simulation results for $C(x, z = 0)$ for different Ω values

$$\Rightarrow B_n = -A_n \quad (18)$$

$$\Rightarrow X_n(x) = A_n \sin(k_n x) \quad (19)$$

BC 4:

$$\partial_x C(x, z)|_{x=-L_x} \propto \cos(k_n L_x) = 0 \quad (20)$$

$$k_n = \frac{\pi}{2L_x}(2n+1) \Rightarrow k_{n_{odd}} = \frac{\pi}{2L_x}n_{odd} \quad (21)$$

Where we have only odd wavenumbers.

BC 1:

$$\partial_z C(x, z)|_{z=L_z} \propto C_n e^{+k_n L_z} - D_n e^{-k_n L_z} = 0 \quad (22)$$

$$D_n = C_n e^{+2k_n L_x} \quad (23)$$

$$\Rightarrow Z_n(z) \propto (e^{+k_n z} + e^{+2k_n L_x} e^{-k_n z}) \propto \cosh(k_n(z - L_z)) \quad (24)$$

So we now have for $x < 0$:

$$C(x, z) = \sum_{n_{odd}} A_n \sin(k_n x) \cosh(k_n(z - L_z))$$

And our final BC is (on $x < 0$):

$$\partial_z C(x, z)|_{z=0} = -r + \Omega C(x, z=0) \quad (25)$$

$$\sum -k_n A_n \cos(k_n x) \sinh(k_n L_z) = +\tilde{\Omega} \sum A_n \sin(k_n x) \cosh(k_n L_z) - \frac{r}{2} \quad (26)$$

$$\sum_{n_{odd}} A_n \sin(k_n x) [\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z)] = \frac{r}{2} \quad (27)$$

$$A'_n := A_n \frac{\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z)}{r/2} \quad (28)$$

$$\sum_{n_{odd}} A'_n \sin(k_n x) = 1 \quad (29)$$

Multiply by $\sin(k_m x)$ and integrate on $x < 0$:

$$A'_m \frac{L_x}{2} = \int_{-L_x}^0 \sin(k_m x) dx = -\frac{1}{k_m} \quad (30)$$

Where we get that for $m_{odd} \neq n_{odd}$ these sines are orthogonal on $[-L_x, 0]$.

$$-\frac{2}{k_n L_x} = A'_n = A_n \frac{\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z)}{r/2} \quad (31)$$

$$\Rightarrow A_n = -\frac{r}{L_x} \frac{1}{k_n (\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z))} \quad (32)$$

$$C_{<}(x, z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n (\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z))} \sin(k_n x) \cosh(k_n (z - L_z)) \quad (33)$$

And this actually agrees with the symmetry we found, so this is the solution on all x :

$$C(x, z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n (\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z))} \sin(k_n x) \cosh(k_n (z - L_z)) \quad (34)$$

Using some hyperbolic identities:

$$C(x, z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n (\Omega + k_n \tanh(k_n L_z))} \sin(k_n x) [\cosh(k_n z) - \sinh(k_n z) \tanh(k_n L_z)] \quad (35)$$

Graphic representation of this analytical result can be found in (3).

Now, we are most interested at what happens on $z = 0$ plane, so:

$$C(x, 0) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n (\Omega \cosh(k_n L_z) + k_n \sinh(k_n L_z))} \sin(k_n x) \cosh(k_n L_z) \quad (36)$$

$$= \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n (\Omega + k_n \tanh(k_n L_z))} \sin(k_n x) \quad (37)$$

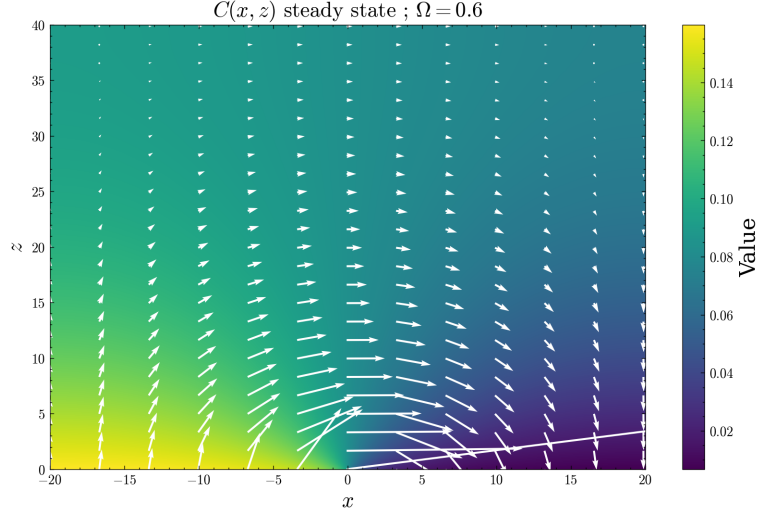


Figure 3: Plotted is the analytical result derived for $C(x, z)$ for the case of finite L_x, L_z ; Arrows represent the flow field at every point; Parameter values are $L_x = 20, L_z = 40, \Omega = 0.6, r = 0.1, N_{max} = 100$ where the last term represents the cutoff term for the series solution.

And at $L_z \gg 1$:

$$\approx \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n)} \sin(k_n x) \quad (38)$$

Which is exactly the same solution as with absorbing BC at $L_z = \infty$.

We can further simplify this solution:

$$\frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n)} \sin(k_n x) = \quad (39)$$

$$= \frac{r}{2\Omega} - \frac{r}{\Omega L_x} \sum_{n_{odd}} \left(\frac{1}{k_n} - \frac{1}{\Omega + k_n} \right) \sin(k_n x) \quad (40)$$

$$= \frac{r}{2\Omega} - \frac{r}{2\Omega} \sum_{n_{odd}} 4 \frac{\sin(k_n x)}{\pi n} + \frac{r}{\Omega L_x} \sum_{n_{odd}} \frac{\sin(k_n x)}{k_n + \Omega} \quad (41)$$

We know $\sum_{n_{odd}} 4 \frac{\sin(k_n x)}{\pi n}$ is the fourier transform of a sin square wave, so:

$$= \frac{r}{\Omega} (\Theta(-x) + \frac{1}{L_x} \sum_{n_{odd}} \frac{\sin(k_n x)}{k_n + \Omega}) \quad (42)$$

We plot this solution in (4)

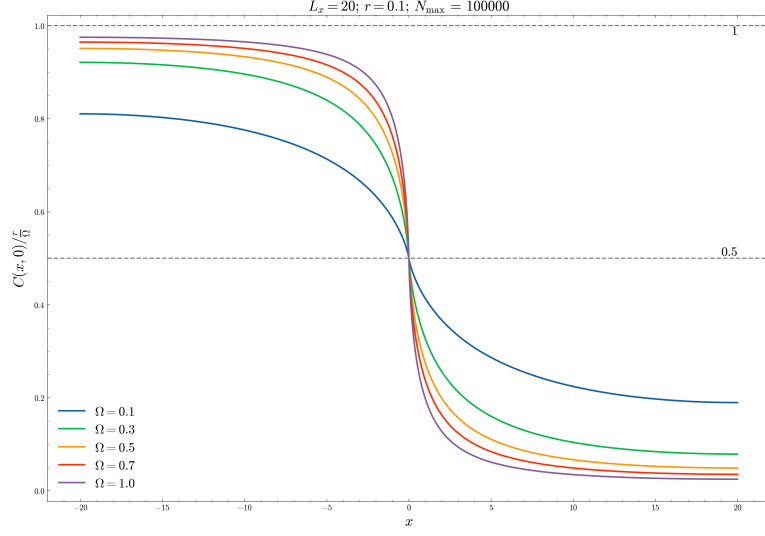


Figure 4: Analytical results for $C(x, 0)$ for varying values of Ω ; Parameter values are $L_x = 20, r = 0.1, N_{max} = 10^5$

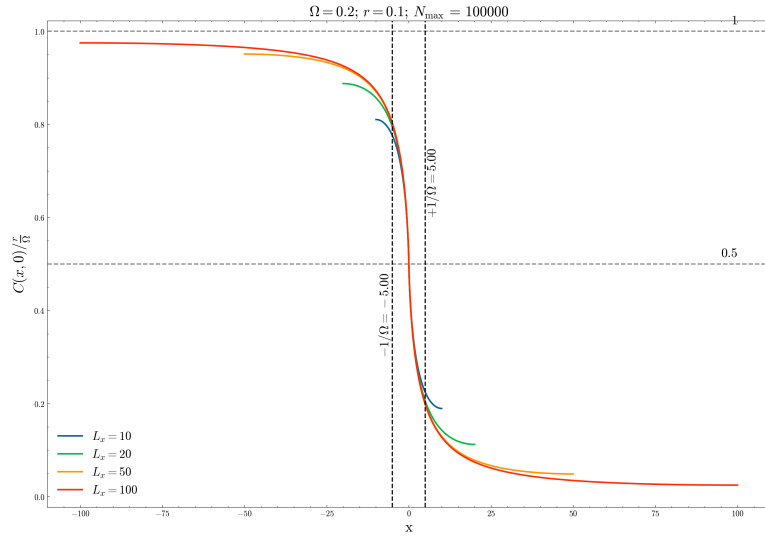


Figure 5: Analytical results for $C(x, 0)$ for varying values of L_x ; Parameter values are $\Omega = 0.2, r = 0.1, N_{max} = 10^5$; Can be seen length scale $1/\Omega$ controls decay scale where all results are identical

Now we can return to original notation of r, Ω :

$$= \frac{r}{\Omega} (\Theta(-x) + \frac{1}{L_x} \sum_{n_{odd}} \frac{\sin(k_n x)}{k_n + \frac{\Omega}{D}}) \quad (43)$$

Take to the continuum limit:

$$y = k_n = \frac{\pi n}{2L_x}; dy = \frac{\pi}{2L_x} dn; dn = 2 \quad (44)$$

$$\begin{aligned} \frac{1}{L_x} \sum_{n_{odd}} \frac{\sin(k_n x)}{k_n + \frac{\Omega}{D}} &= \frac{1}{\pi} \int_0^\infty \frac{\sin(yx)}{y + \frac{\Omega}{D}} dy = \frac{1}{\pi} \int_{\frac{\Omega}{D}}^\infty \frac{\sin(yx - \frac{\Omega}{D}x)}{y} dy \\ &= \frac{1}{\pi} \int_{\frac{\Omega}{D}}^\infty \frac{\sin(yx) \cos(\frac{\Omega}{D}x) - \cos(yx) \sin(\frac{\Omega}{D}x)}{y} dy = \frac{\cos(\frac{\Omega}{D}x)}{\pi} \int_{\frac{\Omega}{D}}^\infty \frac{\sin(yx)}{y} dy - \frac{\sin(\frac{\Omega}{D}x)}{\pi} \int_{\frac{\Omega}{D}}^\infty \frac{\cos(yx)}{y} dy \\ &= \frac{\cos(\frac{\Omega}{D}x)}{\pi} \text{sign}(x) [\frac{\pi}{2} - Si(\frac{\Omega}{D}x)] + \text{sign}(x) \frac{\sin(\frac{\Omega}{D}x)}{\pi} Ci(x \frac{\Omega}{D}) \end{aligned} \quad (45)$$

With $Ci(\omega) = -\int_\omega^\infty \frac{\cos(y)}{y} dy$ and $Si(\omega) = \int_0^\omega \frac{\sin(y)}{y} dy$ being the cosine integral and sine integral, respectively.

Therefore we have:

$$C(x, z=0) = \frac{r}{\Omega} \times \left(\Theta(-x) + \frac{\cos(\frac{\Omega}{D}x)}{\pi} \text{sign}(x) [\frac{\pi}{2} - Si(\frac{\Omega}{D}x)] + \text{sign}(x) \frac{\sin(\frac{\Omega}{D}x)}{\pi} Ci(x \frac{\Omega}{D}) \right) \quad (46)$$

Observe we have only 1 length scale of $D/\Omega := L_D$; Let us explore the 2 extreme regimes of x/L_D .

For $x \gg L_D$ we have the asymptotic expansions:

$$Si(\frac{x}{L_D}) \approx \frac{\pi}{2} - \frac{\cos(\frac{x}{L_D})}{\frac{x}{L_D}}; Ci(x) \approx \frac{\sin(\frac{x}{L_D})}{\frac{x}{L_D}} \quad (47)$$

Plugging this in to get the behaviour far from the origin (assume $x > 0$ for simplicity):

$$C(x, z=0) \approx \frac{r}{\Omega} \times \left(\frac{L_D}{\pi x} \right)$$

And for $x < 0$:

$$C(x, z=0) \approx \frac{r}{\Omega} \times \left(1 + \frac{L_D}{\pi x} \right)$$

And this is the approximate behaviour for $x \gg L_D$, which is different then the exponential asymptotic behaviour of the 1D version.

We plot this asymptotic behaviour in 6.

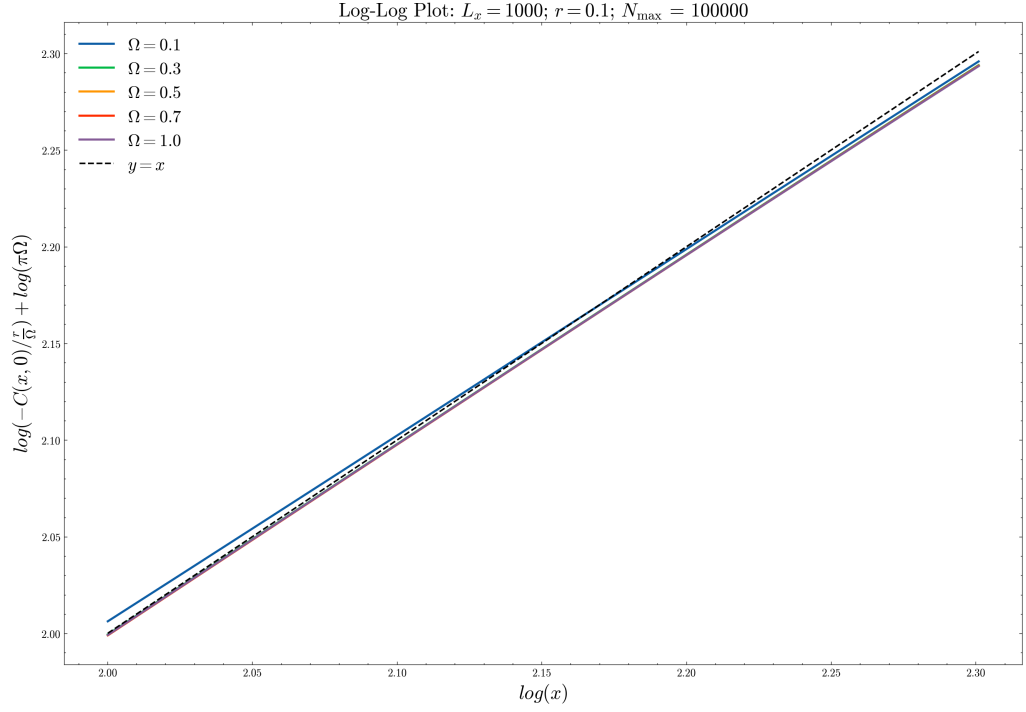


Figure 6: vertical axis is $\log(-C(x, z = 0)/\frac{r}{\Omega}) + \log(\pi\Omega)$. If we plot this in a region of $x \gg L_D$ we expect $\log(-C(x, z = 0)/\frac{r}{\Omega}) + \log(\frac{\pi}{L_D}) \approx \log(\frac{\pi}{L_D}) - \log(\frac{L_D}{\pi x}) = \log(x)$. Thus, plotting $\log(-C(x, z = 0)/\frac{r}{\Omega}) + \log(\pi\Omega)$ for different values of Ω vs $\log(x)$ for $x \gg L_D$ we expect all plots to fall on a 45 degree line, as can be seen.