Yeast Signaling

August 28, 2024

We begin with the equation:

$$\frac{\partial C(x,z)}{\partial t} = D\nabla^2 C(x,z) + r\Theta(-x)\delta(z) - \Omega\delta(z)C(x,z) \tag{1}$$

Look for steady state solutions:

$$D\nabla^2 C(x,z) + r\Theta(-x)\delta(z) - \Omega\delta(z)C(x,z) = 0 \tag{2}$$

Define new coefficients:

$$\partial_{xx} C(x,z) + \partial_{zz} C(x,z) + \tilde{r}\Theta(-x)\delta(z) - \tilde{\Omega}\delta(z)C(x,z) = 0 \tag{3}$$

And drop the tildes from here forward.
What happens if we integrate over an ϵ region around $z = 0$:

$$\partial_z C(x,z)|_{z=+\epsilon} - \partial_z C(x,z)|_{z=-\epsilon} = -r\Theta(-x) + \Omega C(x, z=0) \tag{4}$$

This is the virtual flux of α factors through the z-plane, and we can turn it into a Robin BC:

$$\partial_z C(x,z)|_{z=0} = -r\Theta(-x) + \Omega C(x, z=0)$$

And just solve the Laplace equation on the positive half volume.
Assume structure of $C(x,z)$:

$$C(x,z) = X(x)Z(z) + \frac{r}{2\Omega} \tag{5}$$

We assume this structure due to the fact that one can deduce the following symmetry from the equation / BC:

$$C(-x,z) + C(x,z) = \frac{r}{\Omega} \tag{6}$$

$$\Rightarrow C(0,z) = \frac{r}{2\Omega} \tag{7}$$

And also:

$$C(x,z) = X(x)Z(z) \Rightarrow C(0,z) = X(0)Z(z) \tag{8}$$

Where $X(0)$ is a constant, and there is no way to satisfy the previous condition for $C(0,z)$ besides constant $Z(z)$, which cannot be a solution.

The symmetry can be seen as follows - if $C(x,z)$ is a solution, for $x>0$ we have:

$$\partial_z C(x,z)|_{z=0} = \Omega C(x, z=0)$$

And:

$$\partial_z C(-x,z)|_{z=0} = -r + \Omega C(-x, z=0)$$

Now take the function $\frac{r}{\Omega} - C(x,z)$, it satisfies the Robin BC for $x<0$ as well:

$$-\partial_z C(x,z)|_{z=0} = \Omega\left(\frac{r}{\Omega} - C(x,z)\right) = r - \Omega C(x, z=0)$$

It also satisfies the rest of the reflecting BC and is a solution of the Laplace equation as well, thus $\frac{r}{\Omega} - C(x,z)$ is also a solution on $x<0$.

From uniqueness we can deduce $\frac{r}{\Omega} - C(x,z) = C(-x,z)$ and we get our symmetry.

So, we have BC:

$$\partial_z C(x,z)|_{z=L_z} = 0 \tag{9}$$

$$\partial_z C(x,z)|_{z=0} = -r\Theta(-x) + \Omega C(x, z=0) \tag{10}$$

$$C(x,z)|_{x=0} = \frac{r}{2\Omega} \tag{11}$$

$$\partial_x C(x,z)|_{x=-L_x} = 0 \tag{12}$$

And we solve for the $x<0$ region, which dictates the $x>0$ region.

This is the simplest structure that has the potential to adhere to the constraints, where $C(x,z)$ is still seperable in the Poisson equation.

Before the analytical solution, we perform a coarse-grained simulation of the above to see how results should look like (with finite L_x, L_z) in 2D (1), and plot $C(x, z=0)$ for different values of Ω (fig. 2).

Now solve:

$$\nabla^2 C(x,z) = 0 \tag{13}$$

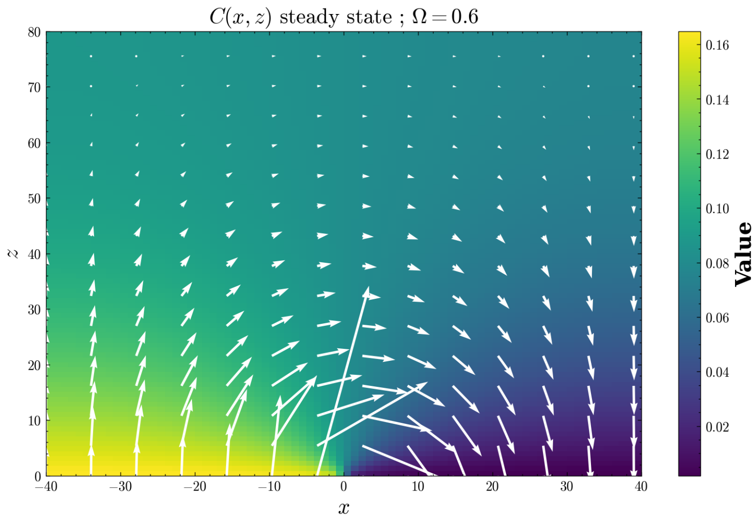

Figure 1: Heatmap of the $C(x,z)$ steady state (simulation taken to 10^6 steps); Arrows represent the flow field at every point; Parameter values are $r_0 = 0.1$, $L_x = 20$, $L_z = 30$.

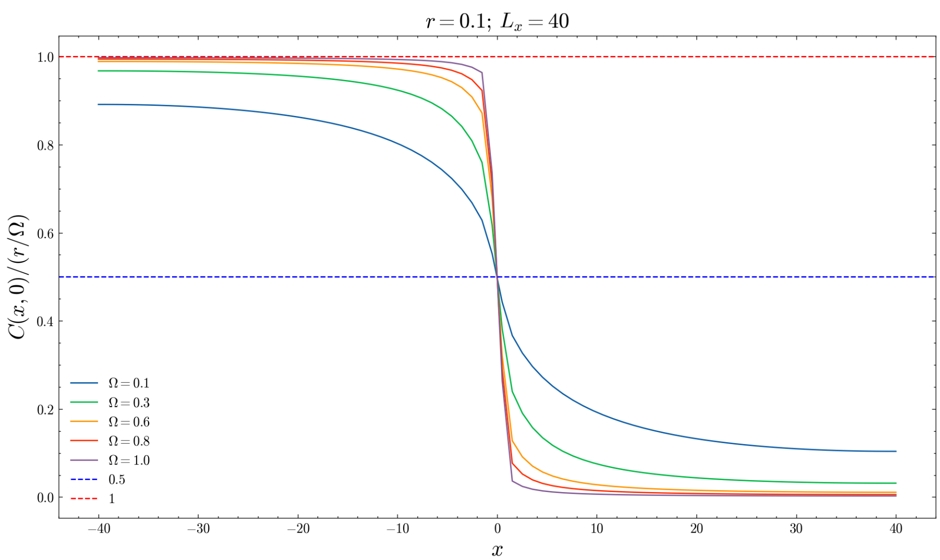

Figure 2: Simulation results for $C(x, z = 0)$ for different Ω values

$$\frac{1}{X(x)}\partial_{xx}X(x) = -k^2 = -\frac{1}{Z}\partial_{zz}Z(z) \tag{14}$$

$$X_n(x) = A_n e^{+ik_n x} + B_n e^{-ik_n x} \tag{15}$$

$$Z_n(z) = C_n e^{+k_n z} + D_n e^{-k_n z} \tag{16}$$

BC 3:

$$C(x,z)|_{x=0} \propto A_n + B_n + \frac{r}{2\Omega} = \frac{r}{2\Omega} \tag{17}$$

$$\Rightarrow B_n = -A_n \tag{18}$$

$$\Rightarrow X_n(x) = A_n sin(k_n x) \tag{19}$$

BC 4:

$$\partial_x C(x,z)|_{x=-L_x} \propto cos(k_n L_x) = 0 \tag{20}$$

$$k_n = \frac{\pi}{2L_x}(2n+1) \Rightarrow k_{n_{odd}} = \frac{\pi}{2L_x}n_{odd} \tag{21}$$

Where we have only odd wavenumbers.
BC 1:

$$\partial_z C(x,z)|_{z=L_z} \propto C_n e^{+k_n L_z} - D_n e^{-k_n L_z} = 0 \tag{22}$$

$$D_n = C_n e^{+2k_n L_x} \tag{23}$$

$$\Rightarrow Z_n(z) \propto (e^{+k_n z} + e^{+2k_n L_x} e^{-k_n z}) \propto cosh(k_n(z - L_z)) \tag{24}$$

So we now have for $x < 0$:

$$C(x,z) = \sum_{n_{odd}} A_n sin(k_n x) cosh(k_n(z - L_z))$$

And our final BC is (on $x < 0$):

$$\partial_z C(x,z)|_{z=0} = -r + \Omega C(x, z=0) \tag{25}$$

$$\sum -k_n A_n cos(k_n x) sinh(k_n L_z) = +\tilde{\Omega} \sum A_n sin(k_n x) cosh(k_n L_z) - \frac{r}{2} \tag{26}$$

$$\sum_{n_{odd}} A_n sin(k_n x)[\Omega cosh(k_n L_z) + k_n sinh(k_n L_z)] = \frac{r}{2} \tag{27}$$

$$A_n^{'} := A_n \frac{\Omega cosh(k_n L_z) + k_n sinh(k_n L_z)}{r/2} \tag{28}$$

$$\sum_{n_{odd}} A_n^{'} sin(k_n x) = 1 \tag{29}$$

Multiply by $sin(k_m x)$ and integrate on $x < 0$:

$$A_m^{'} \frac{L_x}{2} = \int_{-L_x}^{0} sin(k_m x) dx = -\frac{1}{k_m} \tag{30}$$

Where we get that for $m_{odd} \neq n_{odd}$ these sines are orthogonal on $[-L_x, 0]$.

$$-\frac{2}{k_n L_x} = A_n^{'} = A_n \frac{\Omega cosh(k_n L_z) + k_n sinh(k_n L_z)}{r/2} \tag{31}$$

$$\Rightarrow A_n = -\frac{r}{L_x} \frac{1}{k_n(\Omega cosh(k_n L_z) + k_n sinh(k_n L_z))} \tag{32}$$

$$C_<(x,z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega cosh(k_n L_z) + k_n sinh(k_n L_z))} sin(k_n x) cosh(k_n(z - L_z)) \tag{33}$$

And this actually agrees with the symmetry we found, so this is the solution on all x:

$$C(x,z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega cosh(k_n L_z) + k_n sinh(k_n L_z))} sin(k_n x) cosh(k_n(z - L_z)) \tag{34}$$

Using some hyperbolic identities:

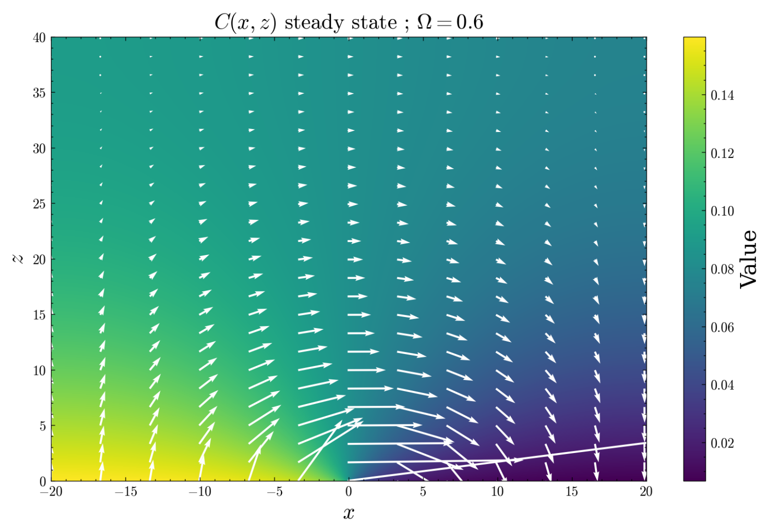

Figure 3: Plotted is the analytical result derived for $C(x,z)$ for the case of finite L_x, L_z ; Arrows represent the flow field at every point; Parameter values are $L_x = 20, L_z = 40, \Omega = 0.6, r = 0.1, N_{max} = 100$ where the last term represents the cutoff term for the series solution.

$$C(x,z) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n tanh(k_n L_z))} sin(k_n x)[cosh(k_n z) - sinh(k_n z) tanh(k_n L_z)] \tag{35}$$

Grapic representation of this analytical result can be found in (3).

Now, we are most interested at what happens on $z = 0$ plane, so:

$$C(x,0) = \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega cosh(k_n L_z) + k_n sinh(k_n L_z))} sin(k_n x) cosh(k_n L_z) \tag{36}$$

$$= \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n tanh(k_n L_z))} sin(k_n x) \tag{37}$$

And at $L_z \gg 1$:

$$\approx \frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n)} sin(k_n x) \tag{38}$$

Which is exactly the same solution as with absorbing BC at $L_z = \infty$.

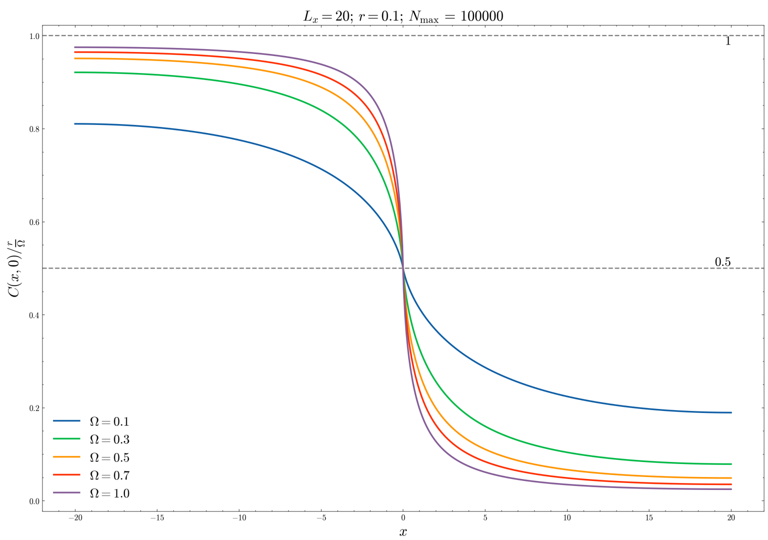

Figure 4: Analytical results for $C(x, 0)$ for varying values of Ω; Parameter values are $L_x = 20, r = 0.1, N_{max} = 10^5$

We can further simplify this solution:

$$\frac{r}{2\Omega} - \frac{r}{L_x} \sum_{n_{odd}} \frac{1}{k_n(\Omega + k_n)} sin(k_n x) = \tag{39}$$

$$= \frac{r}{2\Omega} - \frac{r}{\Omega L_x} \sum_{n_{odd}} (\frac{1}{k_n} - \frac{1}{\Omega + k_n}) sin(k_n x) \tag{40}$$

$$= \frac{r}{2\Omega} - \frac{r}{2\Omega} \sum_{n_{odd}} 4\frac{sin(k_n x)}{\pi n} + \frac{r}{\Omega L_x} \sum_{n_{odd}} \frac{sin(k_n x)}{k_n + \Omega} \tag{41}$$

We know $\sum_{n_{odd}} 4\frac{sin(k_n x)}{\pi n}$ is the fourier transform of a sin square wave, so:

$$= \frac{r}{\Omega}(\Theta(-x) + \frac{1}{L_x} \sum_{n_{odd}} \frac{sin(k_n x)}{k_n + \Omega}) \tag{42}$$

We plot this solution in (4)
Now we can return to original notation of r, Ω :

$$= \frac{r}{\Omega}(\Theta(-x) + \frac{1}{L_x} \sum_{n_{odd}} \frac{sin(k_n x)}{k_n + \frac{\Omega}{D}}) \tag{43}$$

Take to the continuum limit:

$$y = k_n = \frac{\pi n}{2L_x}; dy = \frac{\pi}{2L_x} dn; dn = 2 \tag{44}$$

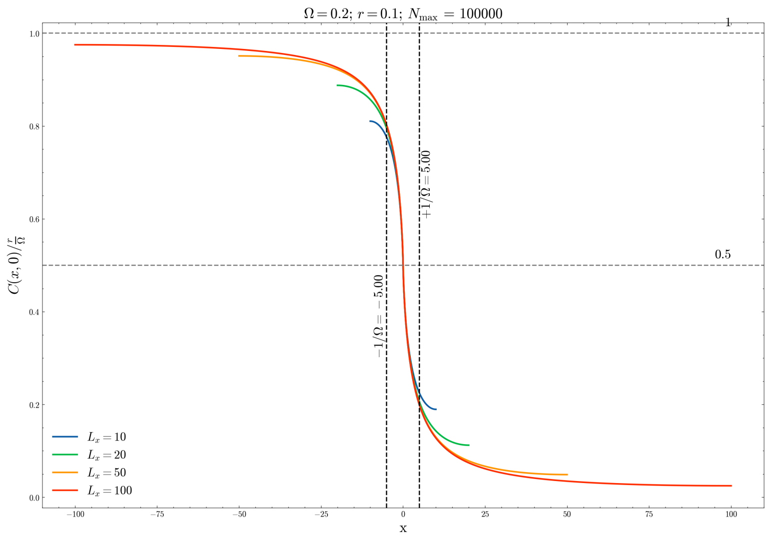

Figure 5: Analytical results for $C(x, 0)$ for varying values of L_x; Parameter values are $\Omega = 0.2, r = 0.1, N_{max} = 10^5$; Can be seen length scale $1/\Omega$ controls decay scale where all results are identicle

$$\frac{1}{L_x}\sum_{n_{odd}}\frac{sin(k_n x)}{k_n + \frac{\Omega}{D}} = \frac{1}{\pi}\int_0^\infty \frac{sin(yx)}{y + \frac{\Omega}{D}}dy = \frac{1}{\pi}\int_{\frac{\Omega}{D}}^\infty \frac{sin(yx - \frac{\Omega}{D}x)}{y}dy$$

$$= \frac{1}{\pi}\int_{\frac{\Omega}{D}}^\infty \frac{sin(yx)cos(\frac{\Omega}{D}x) - cos(yx)sin(\frac{\Omega}{D}x)}{y}dy = \frac{cos(\frac{\Omega}{D}x)}{\pi}\int_{\frac{\Omega}{D}}^\infty \frac{sin(yx)}{y}dy - \frac{sin(\frac{\Omega}{D}x)}{\pi}\int_{\frac{\Omega}{D}}^\infty \frac{cos(yx)}{y}dy$$

$$= \frac{cos(\frac{\Omega}{D}x)}{\pi}sign(x)[\frac{\pi}{2} - Si(\frac{\Omega}{D}|x|)] + \frac{sin(\frac{\Omega}{D}x)}{\pi}Ci(x\frac{\Omega}{D}) \tag{45}$$

With $Ci(\omega) = -\int_\omega^\infty \frac{cos(y)}{y}dy$ and $Si(\omega) = \int_0^\omega \frac{sin(y)}{y}dy$ being the cosine integral and sine integral, respectively.

Therefore we have:

$$C(x, z = 0) = \frac{r}{\Omega} \times \left(\Theta(-x) + \frac{cos(\frac{\Omega}{D}x)}{\pi}sign(x)[\frac{\pi}{2} - Si(\frac{\Omega}{D}|x|)] + \frac{sin(\frac{\Omega}{D}x)}{\pi}Ci(x\frac{\Omega}{D})\right) \tag{46}$$

Observe we have only 1 length scale of $D/\Omega := L_D$; Let us explore the 2 extreme regimes of x/L_D.

For $x \gg L_D$ we have the asymptotic expansions:

$$Si(\frac{x}{L_D}) \approx \frac{\pi}{2} - \frac{cos(\frac{x}{L_D})}{\frac{x}{L_D}}; Ci(x) \approx \frac{sin(\frac{x}{L_D})}{\frac{x}{L_D}} \tag{47}$$

Plugging this in to get the behaviour far from the origin (assume $x > 0$ for simplicity):

$$C(x, z = 0) \approx \frac{r}{\Omega} \times \left(\frac{L_D}{\pi x}\right) \tag{48}$$

And for $x < 0$:

$$C(x, z = 0) \approx \frac{r}{\Omega} \times \left(1 + \frac{L_D}{\pi x}\right) \tag{49}$$

And this is the approximate behaviour for $x \gg L_D$, which is different then the exponential asymptotic behaviour of the $1D$ version.

We plot this asymptotic behaviour in 6.

For $x \ll L_D$we have the expansions to 1st order:

$$Si(\frac{x}{L_D}) \approx \frac{x}{L_D}; Ci(x) \approx ln\left(\frac{|x|}{L_D}\right) + \gamma \tag{50}$$

Where $\gamma \approx 0.577$ is the the Euler-Mascheroni constant. Plugging this in and assuming $x > 0$ for brevity, we get:

$$C(x, 0) \approx \frac{r}{\Omega\pi} \times \left(sign(x)[\frac{\pi}{2} - \frac{|x|}{L_D}] + \frac{x}{L_D}\left(ln\left(\frac{|x|}{L_D}\right) + \gamma\right)\right) \tag{51}$$

Therefor, for $0 < x \ll L_D$:

$$C(x, 0) \approx \frac{r}{\Omega\pi} \times \left(\frac{\pi}{2} + \frac{x}{L_D}\left(ln\left(\frac{x}{L_D}\right) + \gamma - 1\right)\right) \tag{52}$$

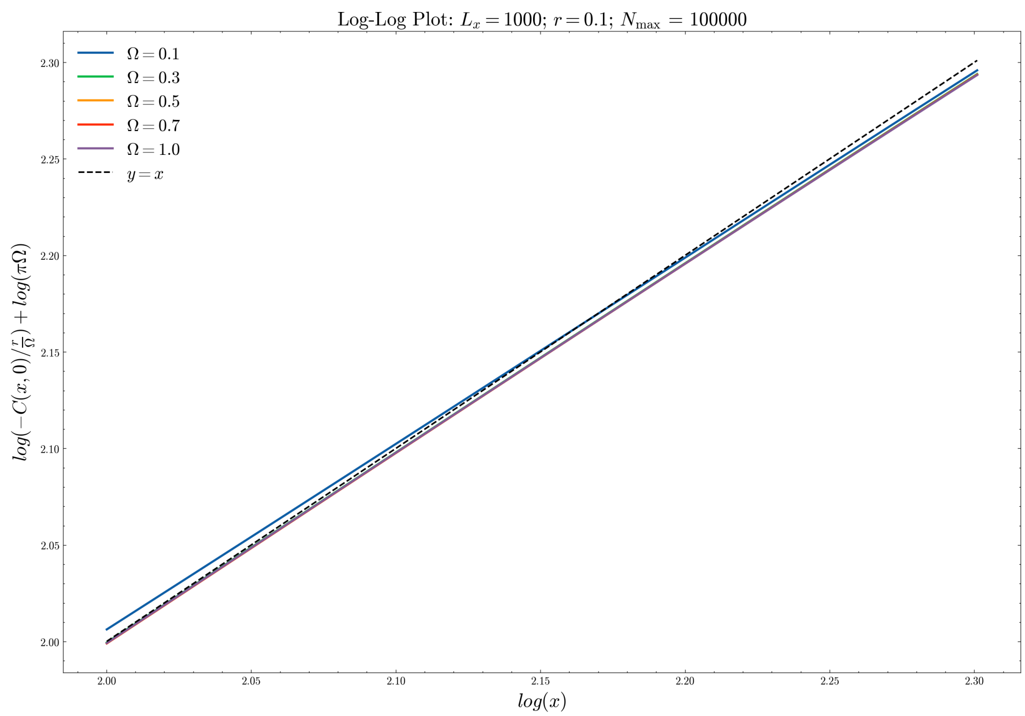

Figure 6: Vertical axis is $log(-C(x, z = 0)/\frac{r}{\Omega}) + log(\pi\Omega)$. If we plot this in a region of $x \gg L_D$ we expect $log(-C(x, z = 0)/\frac{r}{\Omega}) + log(\frac{\pi}{L_D}) \approx log(\frac{\pi}{L_D}) - log(\frac{L_D}{\pi x}) = log(x)$. Thus, plotting $log(-C(x, z = 0)/\frac{r}{\Omega}) + log(\pi\Omega)$ for different values of Ω vs $log(x)$ for $x \gg L_D$ we expect all plots to fall on a 45 degree line, as can be seen.