**The concentration distribution of -factors without**

According to the experimental setup, -factors are secreted at the plane for at a rate . They can diffuse into the positive half-plane and are taken up by cells at both and on the plane at a rate . Assuming translational symmetry along the -axis (parallel to the interface), we can write the concentration of the -factor in the plane as follows

where is the Heaviside step function and is the Dirac delta function. In the steady state, where , we have

It is simple to show that has the following symmetry

If we substitute into , assuming is a solution, we find is also a solution, and thus from uniqueness we can conclude . Specifically, we note that

and we also note

meaning that the function is odd with respect to . Now, to solve for , we have

and we use to make an informed ansatz on

We can now rewrite the production and degradation terms from as a boundary condition at

and use and reflecting boundary conditions on to determine , and , and finally obtain:

with

In figure , we show a heat-map of , and the corresponding flow field . Clearly, the uptake of the -factor is not only due to the flow along the plane; the space also plays an essential role, leading to a quantitatively different result than a model of diffusion along . The importance of the dimensionality on diffusion in such problems is systemically studied in [1], [2].

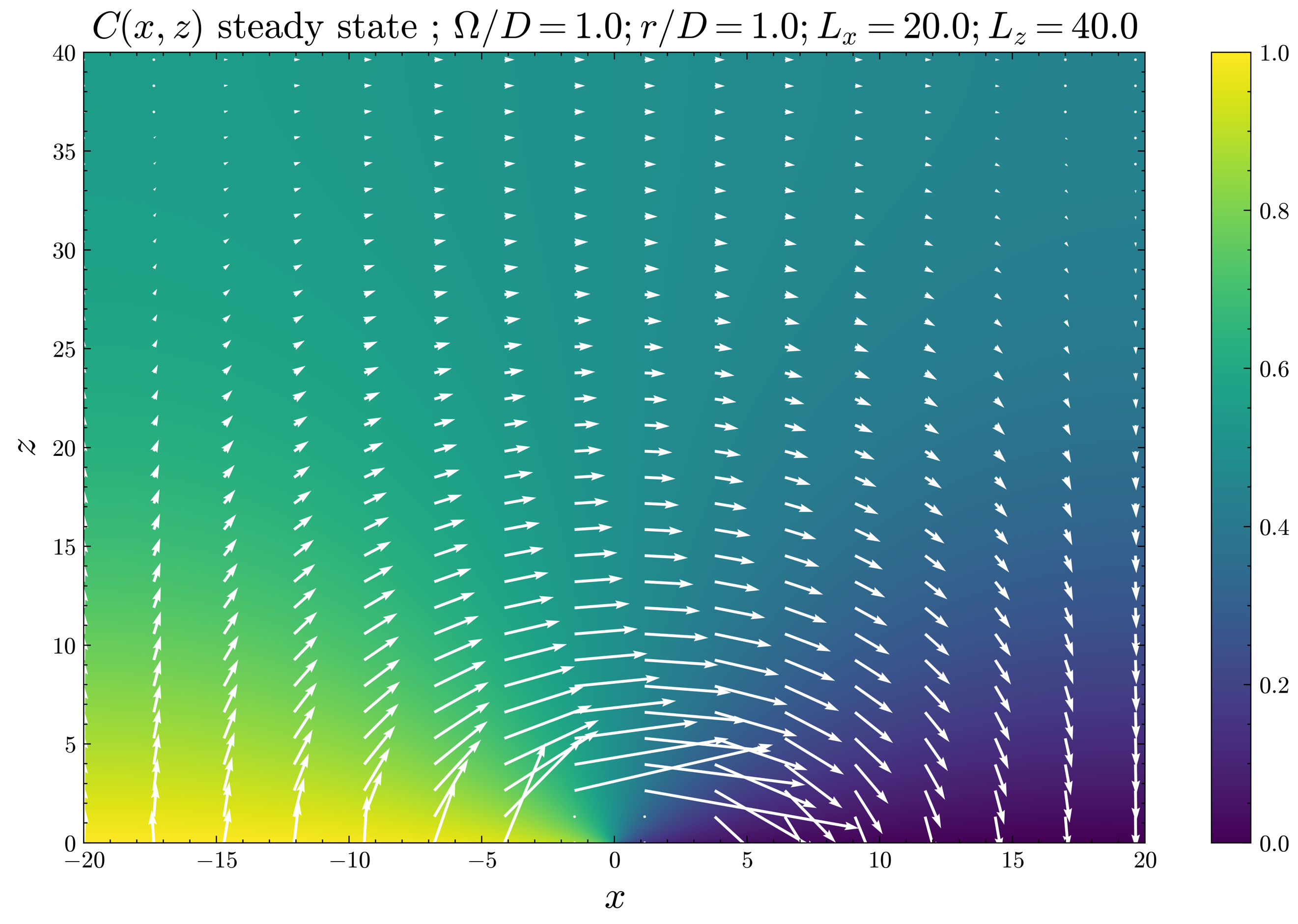


Figure S1: Heat-map of and the corresponding flow field. We omit a small neighborhood around as the flow becomes very large there. Colors represent the value of .

For , in the limit and using the known Fourier series expansion for square waves, we have

We see (figure ) that is accurate even for of the order of . We can then consider the continuous limit of as , with which we obtain

where and are the sine and cosine integrals, respectively. There is a length scale , which is different from the case of diffusion occurring along the -axis. Asymptotically expanding such that , we obtain

and we see decays as in the far-field, instead of exponential decay as in the diffusion case. In the experiment we find that , while , and it is evident (figure ) that this is a regime where is already quite accurate. When , we have to 1st order

where is the Euler-Mascheroni constant. This is, again, different than the exponential -dependence in the case.

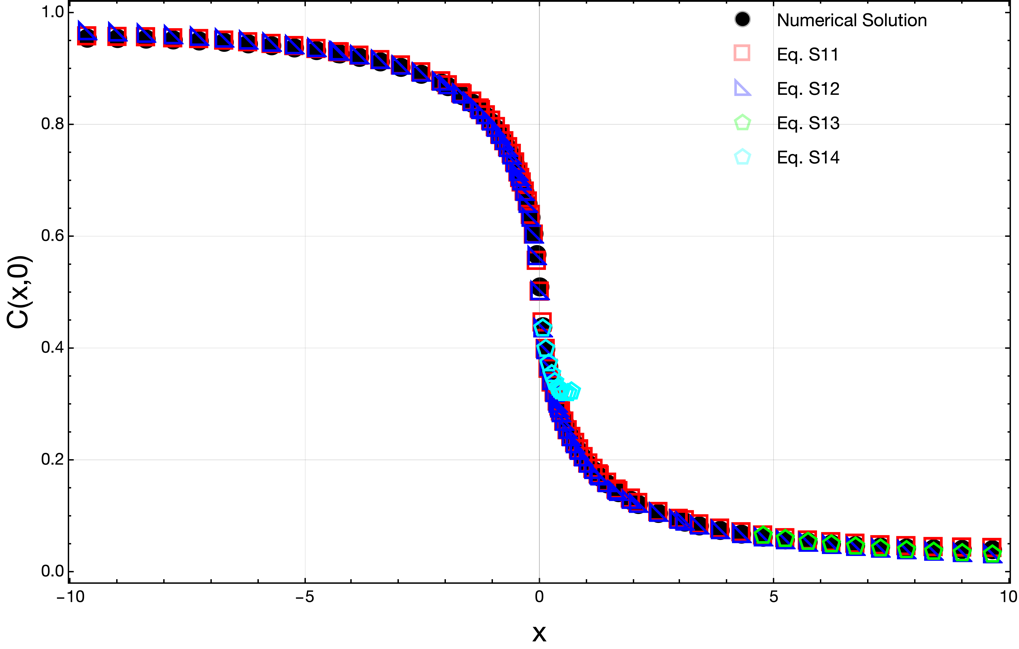


Figure S2: Analytical and numerical solutions , for parameter values , . Numerical solution is for , based on stated boundary conditions. Far-field and near-field approximations are plotted in appropriate regimes for the case.

# Bibliography

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| [1] | P. B. Dieterle and A. Amir, "Diffusive wave dynamics beyond the continuum limit," *Physical Review E,* 2021. |
| [2] | P. B. Dieterle, J. Min, D. Irimia and A. Amir, "Dynamics of diffusive cell signaling relays," *eLife,* 2020. |