Exercises for Week 10

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Nov. 13.

Reading. Review Artin Sections 4.1-4.4 as we go along. Alternatively, consult any linear algebra book for the corresponding material. Read Artin Section 6.3-6.4.

You may have wondered what are all possible finite subgroups of $\mathrm{Iso}(\mathbb{R}^2)$ rather than just $O(2,\mathbb{R})$. We'll show through the following exercises that any finite subgroup $G \subset \mathrm{Iso}(\mathbb{R}^2)$ is also isomorphic to either the cyclic group or the dihedral group.

In what follows, U stands for a Euclidean vector space with the standard metric.

1. In class we have shown that the subgroup of $\mathrm{Iso}(U)$ consisting of isometries fixing the origin of U is the orthogonal group $O(U)$. Prove that if $\mathbf{a} \in U$ is an arbitrary vector in U, the subgroup of isometries fixing $\mathbf{a}$ is also isomorphic to $O(U)$ via conjugation by a translation.

2. Let G be a finite subgroup of $\mathrm{Iso}(U)$, and $\mathbf{a}$ be an arbitrary point in U. Recall that we say a point $\mathbf{b} \in U$ is in the *orbit* of $\mathbf{a}$ under the action of G, if there is an isometry $\phi \in G$ such that $\phi(\mathbf{a}) = \mathbf{b}$. The collection of all point in U that is obtainable via applying isometries in G to $\mathbf{a}$ is called the *orbit* of $\mathbf{a}$ under the G-action, which we will denote by $O_{\mathbf{a}}$:
$$O_{\mathbf{a}} := \{\mathbf{x} \in U | \exists \phi \in G,\ \phi(\mathbf{a}) = \mathbf{x}\}.$$
Also recall that the center of mass of $O_{\mathbf{a}}$, by definition, is located at (imagine each point in the orbit carries unit mass)
$$\mathbf{c} = \frac{\sum_{\mathbf{x}\in O_{\mathbf{a}}} \mathbf{x}}{|O_{\mathbf{a}}|},$$
where the sum is under the usual addition of vectors. Prove that G fixes the center of mass $\mathbf{c}$ of the orbit $O_{\mathbf{a}}$.

3. Combining the previous two exercises, show that G is isomorphic to a subgroup of $O(U)$. In particular, if $U \cong \mathbb{R}^2$, then either $G \cong C_n$ or $G \cong D_n$ by the classification theorem we proved in class.

   The next few exercises are independent of the previous ones.

4. Prove that under addition, $\mathbb{R}^n$ does not contain any non-trivial finite subgroup. (Hint: Use induction on n and the isomorphism $\mathbb{R}^n/\mathbb{R}^{n-1} \cong \mathbb{R}$).

5. Simplify the expression $\rho^2 r \rho^{-1} r^{-1} \rho^3 r^3$ in the dihedral group
$$D_n = \langle \rho, r | \rho^n,\ r^2,\ \rho r \rho r\rangle$$

6. Let D_n be the dihedral group of the symmetries of a regular n-gon. What's the stablizer of a vertex? of an edge?

7. Let $GL_n(\mathbb{R})$ act on $\mathbb{R}^n$ by left multiplication. How many orbits are there in $\mathbb{R}^n$ under the group action? What are the stablizers of e_1 and 0?