Homework 1 Solution

1 projective resolutions

As you can find the prototype in the textbook, I will only point out the necessary modifications.
In constructing α_i, the argument using basis in Hatcher has no advantages in our situation as we should further consider the nontrivial relations on the generators which makes the argument lengthy. Rather, we can use the projectiveness of each modules to find α_i's:
Assume we already construct maps α_j for $j \leq i$. Obviously we want to lift the map $\alpha_i \circ f_{i+1}$ along f'_{i+1} to get a map α_{i+1}. The problem is that f'_{i+1} is not in general surjective. But we can still use the projectiveness if we can show that $\mathrm{im}(\alpha_i \circ f_{i+1}) \subseteq \mathrm{im}(f'_{i+1})$ by passing through $F'_{i+1} \overset{f'_{i+1}}{\to} \mathrm{im}(f'_{i+1})$. From the exactness at F'_i, we have the equality $\mathrm{im}(f'_{i+1}) = \ker(f'_i)$, so it is enough to check whether $f'_i \circ \alpha_i \circ f_{i+1} = 0$. But from the commutativity, we have $f'_i \circ \alpha_i \circ f_{i+1} = \alpha_{i-1} \circ f_i \circ f_{i+1} = \alpha_{i-1} \circ 0 = 0$.

$$\begin{tikzcd}
F_{i+1} \arrow[r, "f_{i+1}"] \arrow[d, dashed] \arrow[dr] & F_i \arrow[r, "f_i"] \arrow[d, "\alpha_i"] & F_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] & \cdots \\
F_{i+1} \arrow[r, "f'_{i+1}"] & F_i \arrow[r, "f'_i"] & F_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Thus we can lift the map $\alpha_i \circ f_{i+1}$ along f'_{i+1} to get a dashed arrow α_{i+1} which makes the left square in the above diagram commutes, finishing the induction step.
Caveat: The initial step $i = 0$ in the induction step needs a slight modification of notations, but still the same argument(even easier, as we have already assumed the surjectivity of f'_0) applies.
For building up the homotopy λ, the identity $f'_i(\beta_i - \lambda_{i-1} \circ f_i) = 0$ which comes from the commutative diagrams $f'_i \circ \beta_i = \beta_{i-1} \circ f_i$ and the induction hypothesis $\beta_{i-1} = f'_i \circ \lambda_{i-1} + \lambda_{i-2} \circ f_{i-1}$ is the criterion to lift $\beta_i - \lambda_{i-1} \circ f_i$ to λ_i by the projectiveness of F_i, so the part needs no altering except we cannot start from a basis element anymore.
In part (b), Hatcher didn't use the freeness in the proof, so we are safe.

2 continuity of Ext functor

Choose resolutions

$$\cdots \to P_{i,2} \to P_{i,1} \to P_{i,0} \to M_i \to 0$$

for each M_i. Then, by taking componentwise direct sum, we obtain a resolution

$$\cdots \to \oplus_i P_{i,2} \to \oplus_i P_{i,1} \to \oplus_i P_{i,0} \to \oplus_i M_i \to 0$$

This is indeed a projective resolution as each components are a direct sum of projective A-modules which are projective, and the exactness is guaranteed as a direct sum of exact sequences.
Hence, the Ext functors are the cohomology groups of the (truncated) dual of this resolution:

$$0 \to \prod_i Hom(P_{i,0}, N) \to \prod_i (P_{i,1}, N) \to \prod_i Hom(P_{i,2}, N) \to \cdots$$

, which is the product of the cohomology groups of each factor resolutions, i.e., the desired identity holds.

3 Excersie 3.1.3, Hatcher

A free resolution of $\mathbb{Z}/2$ is as follows:

$$\cdots \to \mathbb{Z}/4 \overset{\times 2}{\to} \mathbb{Z}/4 \overset{\times 2}{\to} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

, where the last nontrivial map is the natural mod 2 map.
Thus, the Ext groups $\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2)$ are the cohomology groups of

$$0 \to Hom_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2) \overset{\times 2}{\to} Hom_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2) \overset{\times 2}{\to} \cdots .$$

But $Hom_{\mathbb{Z}/4}(\mathbb{Z}/4, \mathbb{Z}/2) \cong \mathbb{Z}/2$, meaning all the maps in the above chain complex are trivial. Thus $\mathrm{Ext}^n_{\mathbb{Z}/4}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2 \neq 0$ for all $n \neq 0$.

4 Excersice 3.1.6, Hatcher

For part (a), start from the simplicial chain complex of $S^1 \times S^1$:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0.$$

Thus, for $A = \mathbb{Z}$ and $\mathbb{Z}/2$, by taking $Hom(-, A)$, we obtain

$$0 \to A \xrightarrow{0} A^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}} A^2 \to 0.$$

As usual, the normal form of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$ determines the cohomology groups, which is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence,

$$H^n(S^1 \times S^1; A) = \begin{cases} A & n = 0, 2 \\ A^2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly for (b), the simplicial chain complex of $\mathbb{R}P^2$ is:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}} \mathbb{Z}^2 \to 0.$$

By taking $Hom(-, A)$,

$$0 \to A^2 \xrightarrow{\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} A^2 \to 0.$$

The normal form of two matrices are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, thus the cohomology groups are

$$H^n(\mathbb{R}P^2; A) = \begin{cases} A & n = 0 \\ \mathbb{Z}/2 & n = 1 \text{ and } A = \mathbb{Z}/2 \\ A/2A & n = 2 \\ 0 & \text{otherwise} \end{cases}.$$

For Klein bottle, the simplicial chain complex is:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \to 0.$$

Thus the cochain complex is

$$0 \to A \xrightarrow{0} A^3 \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \mathbb{Z}^2 \to 0.$$

The normal form of the matrix $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$, thus the cohomology groups are

$$H^n(K;A) = \begin{cases} A & n=0 \\ (\mathbb{Z}/2)^2 & n=1 \text{ and } A=\mathbb{Z}/2 \\ \mathbb{Z} & n=1 \text{ and } A=\mathbb{Z} \\ A/2A & n=2 \\ 0 & \text{otherwise} \end{cases}.$$

5 Excersice 3.1.8, Hatcher

(a) For the initial case, we know that

$$\tilde{H}^i(S^0;G) = \begin{cases} G & i=0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus it only remains to show that $\tilde{H}^i(S^n) \cong \tilde{H}^{i+1}(S^{n+1})$. This can be shown in two different ways:
The first method uses the LES associated to the pair (D^{n+1}, S^n). We have

$$\cdots \to \tilde{H}^i(D^{n+1}) \to \tilde{H}^i(S^n) \to \tilde{H}^{i+1}(D^{n+1}, S^n) \to \tilde{H}^{i+1}(D^{n+1}) \to \cdots$$

Using the fact that D^{n+1} is contractible, we have the isomorphisms $\tilde{H}^i(S^n) \cong \tilde{H}^{i+1}(D^{n+1}, S^n) \cong \tilde{H}^{i+1}(S^{n+1})$ where the last isomorphism comes from the homeomorphism $D^{n+1}/S^n \cong S^{n+1}$.(This part requires part (b) as a prerequisite.)
The second uses the Mayer-Vietoris sequence. From the decomposition $S^{n+1} = D^{n+1} \cup_{S^n} D^{n+1}$, the Mayer-Vietoris sequence

$$\cdots \to \tilde{H}^i(D^{n+1}) \oplus \tilde{H}^i(D^{n+1}) \to \tilde{H}^i(S^n) \to \tilde{H}^{i+1}(S^{n+1}) \to \tilde{H}^{i+1}(D^{n+1}) \oplus \tilde{H}^{i+1}(D^{n+1}) \to \cdots$$

gives the isomorphism $\tilde{H}^i(S^n) \cong \tilde{H}^{i+1}(S^{n+1})$ as D^{n+1} is contractible.

(b) The reasoning is exactly the same as the corresponding result (proposition 3.22 in Hatcher) in homology, so I will just adopt the same notation as in there. The commutative diagram becomes

$$\begin{array}{ccccc} H^n(X,A) & \longleftarrow & H^n(X,V) & \longrightarrow & H^n(X-A,V-A) \\ \uparrow & & \uparrow & & \uparrow \\ H^n(X/A,A/A) & \longleftarrow & H^n(X/A,V/A) & \longrightarrow & H^n(X/A-A/A,V/A-A/A) \end{array}$$

All the maps are isomorphisms by the same reason as in homology case, except that all the arrows are reversed.

(c) This is the cohomology version of the remark that follows the splitting lemma in Hatcher. If $r : X \to A$ is a retraction, then the relation $\iota^* r^* = id_A$ implies ι^* is a retraction of r^*. In particular, as in the original case, ι^* is surjective, so we have the following SES from the LES associated to the pair (X, A):

$$0 \to H^n(X,A) \to H^n(X) \to H^n(A) \to 0$$

And this is split (from the relation $\iota^* r^* = id$), giving the decomposition $H^n(X) \cong H^n(X,A) \oplus H^n(A)$.

6 Excersice 3.1.11, Hatcher

(a) First of all, as $\tilde{H}^i(S^{n+1}) = \tilde{H}_i(S^{n+1}) = 0$ except for $i = n+1$, we only need to consider $i = n+1$. But the Moore space $M(\mathbb{Z}_m, n)$ has trivial homology group in dimension $n+1$, so the quotient map induced trivial maps on homology. But from the universal coefficient theorem for cohomology, we have $H^{n+1}(X) = Hom(H_{n+1}(X), \mathbb{Z}) \oplus \operatorname{Ext}(H_n(X), \mathbb{Z}) \cong \operatorname{Ext}(\mathbb{Z}/m, \mathbb{Z}) \cong \mathbb{Z}/m$, thus there is still a chance that the quotient map induces a nontrivial homomorphism. To investigate this map, consider the LES associated to the pair (X, S^n):

$$\cdots \to H^{n+1}(X, S^n) \to H^{n+1}(X) \to H^{n+1}(S^n) \to \cdots$$

The second map $H^{n+1}(X) \to H^{n+1}(S^n)$ is never injective since the target is trivial and the source is nontrivial from the previous computation. But this implies that the first map $H^{n+1}(X, S^n) \to H^{n+1}(X)$, which is innduced from the quotient map $X \to X/S^n$ after the canonical isomorphism $H^{n+1}(X, S^n) \cong \tilde{H}^{n+1}(X/S^n)$, is nontrivial (from the exactness of the sequence).
Now assume that the splitting $H^{n+1}(X) = \operatorname{Hom}(H_{n+1}(X), \mathbb{Z}) \oplus \operatorname{Ext}(H_n(X), \mathbb{Z})$ is natural. This means in particular that there exists a commutative diagram

$$\begin{array}{ccccc} H^{n+1}(X) & = & \operatorname{Hom}(H_{n+1}(X),\mathbb{Z}) & \oplus & \operatorname{Ext}(H_n(X),\mathbb{Z}) \\ \uparrow & & \uparrow & \oplus & \uparrow \\ H^{n+1}(S^{n+1}) & = & \operatorname{Hom}(H_{n+1}(S^{n+1}),\mathbb{Z}) & \oplus & \operatorname{Ext}(H_n(S^{n+1}),\mathbb{Z}) \end{array}$$

where the vertical arrows in the RHS is induced from the map on the homology induced by $X \to S^{n+1}$, which we know must be trivial. This is a contradiction, so the splitting in the universal coefficient theorem for cohomology is not natural.

(b) From the above computation, we know that X has nontrivial cohomology only in dimension $n+1$, but the cohomology of S^n is trivial in dimension $n+1$. Thus the inclusion $S^n \to X$ induces a trivial map on cohomology. On the other hand, in the following LES associated to the pair (X, S^n):

$$\cdots \to H_n(S^n) \to H_n(X) \to H_n(X, S^n) \to \cdots$$

The second map is not injective from the same reasoning as in part (a), hence the first map is nontrivial.