Homework 3

The work handed in should be entirely your own. You can consult any abstract algebra textbook (e.g. Dummit and Foote, Artin), the course textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Mar. 2.

1. Let A and B be matrices that are orthogonally conjugate to each other, i.e., there is a $g \in O_n(\mathbb{R})$ such that $A = gBg^{-1}$.

   (a) Show that $\mathrm{Tr}(A^t A) = \mathrm{Tr}(B^t B)$.

   (b) Show that $\sum_{i,j=1}^{n} A_{ij}^2 = \sum_{i,j=1}^{n} B_{ij}^2$.

2. Let $S = \mathrm{M}_{m\times n}(\mathbb{R})$ be the set of $m \times n$-matrices with coefficients in $\mathbb{R}$, and consider the group
$$G = GL_m(\mathbb{R}) \times GL_n(\mathbb{R}) = \{(g,h) | g \in GL_n(\mathbb{R}), h \in GL_m(\mathbb{R})\}$$
the cross product of the groups. Consider
$$* : G \times S \longrightarrow S \quad , (g,h) * s := gsh^{-1}.$$

   (a) Show that $*$ defines a group action.

   (b) Describe the decomposition of S into G orbits.

   (c) Suppose $m \leq n$. What is the stablizer group of the matrix $[I_m|0]$?

3. Let G be an abelian group. In class we defined the left, right and adjoint action of G on itself. How many distinct orbits are there for each action?

4. The group of orthogonal transformations $O_n(\mathbb{R})$ acts on the set of lines in $\mathbb{R}^n$.

   (a) Write this action in terms of the formal definition of group actions on a set we defined in class.

   (b) Let $l = \{(a, 0, \ldots, 0) | a \in \mathbb{R}\}$ be the line of the first coordinate axis. Find the stablizer group of l.

5. Show that H_2^n has C_2^n as a normal subgroup inside it.

6. (a) Prove that, given any element of a group $g \in G$, there is a unique group homomorphism ϕ_g determined by g:
$$\phi_g : \mathbb{Z} \longrightarrow G, \quad n \mapsto g^n.$$

   (b) Consider $G = SO(2, \mathbb{R})$, and pick
$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
   Here θ is a fixed angle. Write down explicitly the group homomorphism determined by this element. When is this homomorphism injective?

7. In this exercise, you will show one important ingredient in the course of our classification of finite subgroups of $SO_3(\mathbb{R})$. Recall that a polyhedron in $\mathbb{R}^3$ is called *regular* if for the same number of edges emanate from each vertex, and each facet of the polyhedron has the same number of edges.

   Use Euler's formula
   $$V - E + F = 2,$$
   where V is the number of vertices of (any) polyhedron, E is the number of edges and F is the number of facets, to show that there are only five regular polyhedra in $\mathbb{R}^3$.