Chapter 21
Connections to Link Invariants

This chapter is based on expanded notes of a lecture given by

You Qi

and taken by

Johannes Flake

Abstract We relate the quantum group of $\mathfrak{gl}_n$ and the Hecke algebra of the symmetric group S_m through a quantum version of Schur–Weyl duality and we show how both can be used to systematically generate link invariants. In the case of the quantum group, braided tensor category concepts lead to the construction of the Reshetikhin–Turaev quantum invariants, which include the Jones polynomial as a special case. Hecke algebras are employed together with Markov traces to obtain the celebrated HOMFLYPT polynomial in two variables, which can be specialized to give the quantum invariants as special cases. Finally, we describe how the HOMFLYPT polynomial is categorified using Rouquier complexes and Khovanov's triply-graded link homology.

21.1 Temperley–Lieb Algebra

We explore once more the Temperley–Lieb algebra. Previously we described the Temperley–Lieb category by generators and relations in Sect. 7.4, and generalized this with a formal variable δ in Sect. 9.2. The endomorphism ring of the object m is called the Temperley–Lieb algebra $\mathrm{TL}_{m,\delta}$, and we discussed its Jones–Wenzl projectors in Sect. 9.3, and its trace map in Exercise 9.28. We now give another exposition which does not rely on this previous material, describing $\mathrm{TL}_{m,\delta}$ itself by generators and relations.

Definition 21.1 (Temperley–Lieb Algebra) For each $m \in \mathbb{Z}_{\geq 1}$, the *Temperley–Lieb algebra* $\mathrm{TL}_{m,\delta}$ is the $\mathbb{Z}[\delta]$-algebra generated by $u_1, \ldots, u_{m-1}$ modulo the

Y. Qi
Division of Physics, Mathematics and Astronomy, Caltech, Pasadena, CA, USA

J. Flake
Department of Mathematics, Rutgers University, Piscataway, NJ, USA

B. Elias et al., *Introduction to Soergel Bimodules*, RSME Springer Series 5,
https://doi.org/10.1007/978-3-030-48826-0_21

relations

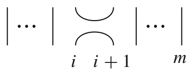
$$u_i^2 = \delta u_i, \tag{21.1}$$

$$u_i u_j u_i = u_i \quad \text{if} \quad |i-j| = 1, \tag{21.2}$$

$$u_i u_j = u_j u_i \quad \text{if} \quad |i-j| \geq 2, \tag{21.3}$$

for all $1 \leq i, j \leq m-1$.

Using graphical calculus, we can represent u_i by the diagram[1]



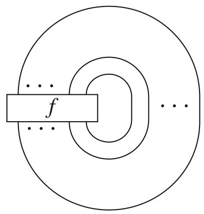
(21.4)

and $1 \in \mathrm{TL}_{m,\delta}$ by the diagram . Multiplication of diagrams is accomplished by stacking vertically and evaluating closed loops in the resulting diagrams to δ. The diagrams which can be obtained in this way are "crossingless matchings," and a general element in the Temperley–Lieb algebra is a $\mathbb{Z}[\delta]$-linear combination of crossingless matchings. The relation (21.1) evaluates particular closed loops to δ, while the other two relations imply that isotopic diagrams yield the same element of $\mathrm{TL}_{m,\delta}$.

The Temperley–Lieb algebra has a natural $\mathbb{Z}[\delta]$-linear *trace map*

$$\mathrm{Tr}\colon \mathrm{TL}_{m,\delta} \to \mathbb{Z}[\delta]. \tag{21.5}$$

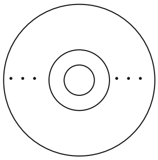
For a single diagram, the trace is obtained by "closing up" the diagram and by evaluating each closed loop in the resulting closed diagram to δ. Graphically, for $f \in \mathrm{TL}_{m,\delta}$, this can be expressed as

$\mathrm{Tr}(f) =$  .

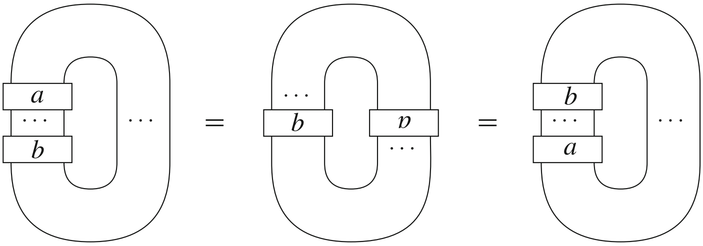
(21.6)

[1]This element u_i is not to be confused with the diagram u from Sect. 7.4. We used the letter u previously because it looks like a cup. Here, we name the generators u_i to match the conventions in the literature.

Notice that, in particular,

$$\mathrm{Tr}(1) =$$

$$= \delta^m. \tag{21.7}$$

A crucial feature of the trace map is that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for any two elements $a, b \in \mathrm{TL}_{m,\delta}$, just as for the ordinary trace map on matrices.



(21.8)

Example 21.2 Consider the standard $\mathfrak{gl}_2$-module $V = \mathbb{C}^2$ with the standard basis e_1, e_2. Then on $V^{\otimes 2}$, switching the two tensor factors naturally intertwines the $\mathfrak{gl}_2$-action on the tensor product space (recall that $x \in \mathfrak{gl}_2$ naturally acts as $\Delta(x) = x \otimes 1 + 1 \otimes x$ on any tensor product of $\mathfrak{gl}_2$-modules). On the other hand, we can assign $\mathbb{C}$-linear maps to the "cup" and "cap" diagrams in the Temperley–Lieb category for $\delta = -2$:

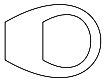
$$\begin{array}{llllll} & V \otimes V & e_1 \otimes e_2 - e_2 \otimes e_1 & & & \\ \bigcup : & \uparrow & \mapsto & & & \\ & \mathbb{C} & 1 & & & \\ & \mathbb{C} & -1 & 1 & 0 & 0 \\ \bigcap : & \uparrow & \mapsto & \mapsto & \mapsto & \mapsto \\ & V \otimes V & e_1 \otimes e_2 & e_2 \otimes e_1 & e_1 \otimes e_1 & e_2 \otimes e_2 \end{array} \tag{21.9}$$

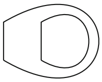
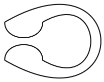
It can be checked that these linear maps are, in fact, $\mathfrak{gl}_2$-module maps, where $\mathbb{C}$ is a $\mathfrak{gl}_2$-module via the trace of 2×2-matrices, and that the endomorphism of $V \otimes V$ sending $v \otimes w \mapsto w \otimes v$ can be realized as

(21.10)

Consider the specialization $\mathrm{TL}_{2,-2} := \mathbb{C} \otimes_{\mathbb{Z}[\delta]} \mathrm{TL}_{2,\delta}$, where $\mathbb{Z}[\delta] \to \mathbb{C}$ sends $\delta \mapsto -2$. Then the assignment above yields an isomorphism of the group algebra $\mathbb{C}S_2$ with $\mathrm{TL}_{2,-2}$ that is compatible with the action of both algebras on $V \otimes V$.

We thus obtain a graphical way to compute the trace of any element from $\mathbb{C}S_2$ acting on $V^{\otimes 2}$, namely by closing up the corresponding Temperley–Lieb diagram and evaluating each closed loop to $\delta = -2$:

$$\mathrm{Tr}(\mid \quad \mid) = 4 = (-2)^2 =$$  ,

$$\mathrm{Tr}($$  $$) = 2 = (-2)^2 - 2 =$$  $+$  .

The previous example can be generalized to arbitrary $m \in \mathbb{Z}_{\geq 1}$ by observing that

- S_m, and thus the group algebra $\mathbb{C}S_m$, acts on $V^{\otimes m}$ by permutation of tensor factors;
- $\mathrm{TL}_{m,-2}$ acts on $V^{\otimes m}$, where we now use $m - 1$ different cup and cap maps;
- there is a surjective algebra map $\mathbb{C}S_m \to \mathrm{TL}_{m,-2}$ sending the transposition $(i, i+1)$ to $1 + u_i$ for all $1 \leq i \leq m - 1$;
- the action of $\mathbb{C}S_m$ on $V^{\otimes m}$ factors through the action of $\mathrm{TL}_{m,-2}$, that is, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}S_m \arrow[r, dotted, no head, "\curvearrowright"] \arrow[d, two heads] & V^{\otimes m} \\
\mathrm{TL}_{m,-2} \arrow[ur, dotted, no head, "\curvearrowright"] &
\end{tikzcd} \tag{21.11}$$

This generalization still applies to the case $\dim V = 2$ only. Finally, we can observe that the action of $\mathbb{C}S_m$ (or $\mathrm{TL}_{m,-2}$) on $V^{\otimes m}$ commutes with the action of $\mathfrak{gl}(V)$ on the same space.

Note however that the graphical trace Tr of a Temperley–Lieb diagram, viewed as an endomorphism of $V^{\otimes m}$, does not match the actual trace of the endomorphism, being off by a sign $(-1)^m$. After all, $\mathrm{Tr}(1) = (-2)^m$, while the dimension of $V^{\otimes m}$ is 2^m. This sign is discussed in the following remark.

Remark 21.3 The Temperley–Lieb algebras $\mathrm{TL}_{m,\delta}$ and $\mathrm{TL}_{m,-\delta}$ are isomorphic via the map sending $u_i \mapsto -u_i$, so it seems odd that we should choose to specialize $\delta = -2$ rather than $\delta = 2$. However, the Temperley–Lieb categories for δ and $-\delta$ are not isomorphic! For instance, scaling the cap by -1 and the cup by 1 would violate the isotopy relations. For the Temperley–Lieb category, it is only $\delta = -2$ (and not $\delta = 2$) which admits a functor to the category of $\mathfrak{gl}_2$-modules. People who study monoidal categories with duals are content to think that V actually has dimension -2, not dimension 2.

Exercise 21.4 Compute the trace of $s_1 s_2 \cdots s_{m-1}$ acting on $V^{\otimes m}$, where $V = \mathbb{C}^2$, in two ways: directly, and using the Temperley–Lieb algebra.

21.2 Schur–Weyl Duality

When $V = \mathbb{C}^n$ has dimension greater than two, we still want to study the action of S_m on $V^{\otimes m}$, together with the action of $\mathfrak{gl}(V)$. We will see that the actions are related by a double centralizer property duality, and in particular, the image of $\mathbb{C}S_m$ in $\mathrm{End}(V^{\otimes m})$ equals exactly the space of $\mathfrak{gl}(V)$-intertwiners.

We denote the universal enveloping algebra of $\mathfrak{gl}(V)$ by $U(\mathfrak{gl}(V))$. As an algebra, it is generated by the elements of $\mathfrak{gl}(V)$. The action of $\mathfrak{gl}(V)$ on tensor products is determined by the coproduct map $\Delta \colon U(\mathfrak{gl}(V)) \to U(\mathfrak{gl}(V)) \otimes U(\mathfrak{gl}(V))$, an algebra map which is defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{gl}(V)$, see Sect. 21.4.

Theorem 21.5 (Schur–Weyl Duality) *Let V be an n-dimensional vector space over $\mathbb{C}$, and let $m \in \mathbb{Z}_{\geq 1}$. Then the images of $U(\mathfrak{gl}(V))$ and $\mathbb{C}S_m$ in $\mathrm{End}(V^{\otimes m})$ are centralizers of each other. Furthermore, we have a decomposition*

$$V^{\otimes m} \simeq \bigoplus_{\lambda} V_\lambda \otimes L_\lambda \tag{21.12}$$

as a $U(\mathfrak{gl}(V))$-module and $\mathbb{C}S_m$-module, where V_λ (resp. L_λ) are mutually non-isomorphic irreducible modules of $U(\mathfrak{gl}(V))$ (resp. $\mathbb{C}S_m$), which can be indexed by the Young diagrams λ with m boxes and at most $n = \dim V$ rows.

Example 21.6 For $n \geq 2$ and $m = 2$, there are two Young diagrams $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ which correspond to the trivial and the sign representation of S_2, respectively.

In fact, $V^{\otimes m} = V \otimes V$ decomposes into symmetric and anti-symmetric tensors on which the non-trivial element of S_2 acts (by swapping the tensor factors) as 1 or -1, respectively. The corresponding submodules are spanned by elements of the form $v \otimes w \pm w \otimes v$ for $v, w \in V$. Both submodules are irreducible under the action of $\mathfrak{gl}(V)$; they are isomorphic to $\mathrm{Sym}^2 V$ and $\wedge^2 V$, respectively.

Remark 21.7 (Sketch of Proof of Theorem 21.5) Consider $x \in \mathfrak{gl}(V)$, then x acts as

$$\Delta^{m-1}(x) = x \otimes \mathrm{id}^{\otimes(m-1)} + \mathrm{id} \otimes x \otimes \mathrm{id}^{\otimes(m-2)} + \cdots + \mathrm{id}^{\otimes(m-1)} \otimes x$$

in $\mathrm{End}(V^{\otimes m}) \simeq (\mathrm{End}(V))^{\otimes m}$ on $V^{\otimes m}$. This tensor product is invariant under any permutation of tensor factors, so it commutes with the action of any $\sigma \in S_m$ on $V^{\otimes m}$. Thus, the actions of $U(\mathfrak{gl}(V))$ and $\mathbb{C}S_m$ commute.

In particular, the image of $U(\mathfrak{gl}(V))$ in $\mathrm{End}(V^{\otimes m})$ is in the centralizer of the image of $\mathbb{C}S_m$. We even claim that it is the full centralizer, that is, the space of invariants

$$(\mathrm{End}(V^{\otimes m}))^{S_m} \simeq ((\mathrm{End}(V))^{\otimes m})^{S_m} \simeq \mathrm{Sym}^m(\mathrm{End}(V)) \,.$$

It can be shown that the latter space is spanned by elements of the form $x^{\otimes m} = x \otimes x \otimes \cdots \otimes x$ for $x \in \mathrm{End}(V) = \mathfrak{gl}(V)$, and that every such $x^{\otimes m}$ is in the

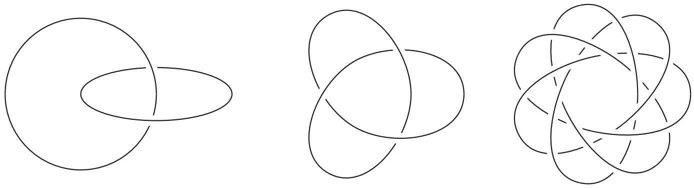


Fig. 21.1 Link diagrams of the Hopf link, the trefoil and the (4,7)-torus knot

subalgebra generated[2] by $\Delta^{m-1}(x), \Delta^{m-1}(x^2), \ldots, \Delta^{m-1}(x^m)$, in particular in the image of $U(\mathfrak{gl}(V))$. Hence, the centralizer of the image of $\mathbb{C}S_m$ is the image of $U(\mathfrak{gl}(V))$.

Finally, by Maschke's theorem, $\mathbb{C}S_m$ is semisimple and $V^{\otimes m}$ is a semisimple $\mathbb{C}S_m$-module. Now by the double commutant theorem, the image of $\mathbb{C}S_m$ is its own double commutant, i.e. it is the centralizer of the image of $U(\mathfrak{gl}(V))$.

The decomposition statement is a corollary.

To reiterate, the images of $\mathbb{C}S_m$ and $U(\mathfrak{gl}(V))$ inside $\mathrm{End}(V^{\otimes m})$ are commutants of each other, but neither the map $U(\mathfrak{gl}(V)) \to \mathrm{End}(V^{\otimes m})$ nor the map $\mathbb{C}S_m \to \mathrm{End}(V^{\otimes m})$ need be injective. When $n = 2$, the image of $\mathbb{C}S_m$ inside $\mathrm{End}(V^{\otimes m})$ is precisely the Temperley–Lieb algebra, via the map $\mathbb{C}S_m \to \mathrm{TL}_{m,-2}$ discussed above.

Exercise 21.8

1. Why does $(1 - s_i - s_{i+1} + s_i s_{i+1} + s_{i+1} s_i - s_i s_{i+1} s_i)$ in $\mathbb{C}S_n$ act as 0 on $V^{\otimes n}$ for $n \geq 3$ and $V = \mathbb{C}^2$?
2. What is the kernel of the action of $\mathbb{C}S_4$ on $V^{\otimes 4}$, when $V = \mathbb{C}^3$?

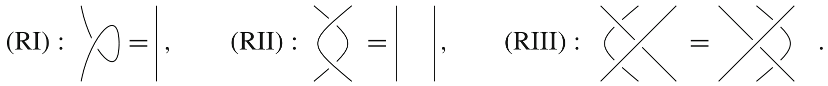
21.3 Trace and Link Invariants

We will discuss how the Temperley–Lieb algebra and its trace yield link invariants, including the celebrated Jones polynomial.

Definition 21.9 A *link* in $\mathbb{R}^3$ (or S^3) is a finite collection of smoothly embedded S^1 (circles). Links can be represented using *link diagrams*, that is, their projections onto suitable planes, where we allow only a finite number of crossing points, each of which corresponding to exactly two points of the link (Fig. 21.1). We will also discuss *oriented* links, where each embedded S^1 is given an orientation, indicated by an arrow in the link diagram.

[2] This follows from the statement that the complete homogeneous symmetric polynomials of degree 1 to m in m variables generate the algebra of symmetric polynomials in m variables.

It is a well-known fact that two link diagrams describe isotopic links if and only if they are related by a finite sequence of *Reidemeister moves*:



(21.13)

Note that these and similar pictures always refer to a small region of a possibly larger diagram, and we want to identify two diagrams if they are identical outside the small region, and related inside the small region as described.

Links can be obtained by taking closures of (type A) braid group elements. We recall this braid group from Chap. 19.

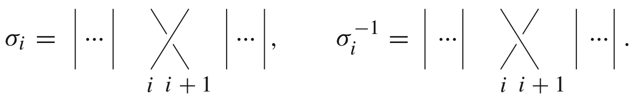
Definition 21.10 (Braid Group) The (type A) *braid group* on m strands is the group Br_m generated by $\sigma_1, \ldots, \sigma_{m-1}$ modulo the *braid relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if} \quad |i-j| \geq 2, \tag{21.14}$$

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \quad \text{if} \quad |i-j| = 1, \tag{21.15}$$

for all $1 \leq i, j \leq m-1$.

We represent σ_i and σ_i^{-1} diagrammatically by the following positive and negative simple crossing on m strands:



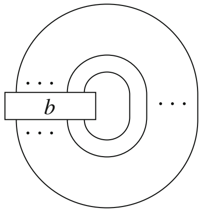
(21.16)

(The choice of upper strand at the crossing point is a mere convention.) Then an arbitrary element of Br_m is represented as an m-strand braid, in a way analogous to the strand diagram for the type A_{m-1} Coxeter group S_m (see Sect. 1.1.2). In fact, the assignment $\sigma_i \mapsto s_i = (i, i+1)$ defines a surjective group homomorphism

$$\mathrm{Br}_m \twoheadrightarrow S_m. \tag{21.17}$$

In this diagrammatics, the defining relations of Br_m become isotopy relations for braids. For example, (RII) says that $\sigma_i\sigma_i^{-1} = \mathrm{id}$. The relation (21.14) says that far-away crossings commute, while (21.15) becomes a version of (RIII) involving only positive crossings. Since two braids are isotopic if and only if they are related by a finite sequence of (RII) and (RIII) moves, it follows that we may view elements of Br_m as m-strand braids up to isotopy. The natural quotient map $\mathrm{Br}_m \twoheadrightarrow S_m$ sends a braid to the permutation induced on the strands.

The *closure* of a braid $b \in \mathrm{Br}_m$ is the link



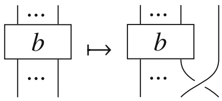
(21.18)

The following two results relate braids with links:

Theorem 21.11 (Alexander's Theorem) *Every (oriented) link is isotopic to the closure of an (oriented) braid.*

Theorem 21.12 (Markov) *The braid closures of $b' \in \mathrm{Br}_m$ and $b'' \in \mathrm{Br}_k$ are isotopic links if and only if b', b'' are related by a finite sequence of*

1. *conjugations in a fixed braid group: $b \mapsto aba^{-1}$;*
2. Markov moves*:*



(21.19)

Hence, the problem of finding an invariant for (oriented) links is equivalent to finding an invariant for (oriented) braids up to conjugation and Markov moves. We note that conjugation invariance is the reason why we should be looking for trace maps on the braid group, i.e. maps Tr for which $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$.

Lemma 21.13 *The assignment*

$$\sigma_i \mapsto vu_i + v^{-1}, \qquad 1 \leq i \leq m-1, \tag{21.20}$$

defines a group homomorphism

$$\mathrm{Br}_m \to \mathrm{TL}^{\times}_{m,-v^2-v^{-2}} \tag{21.21}$$

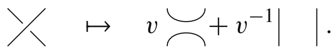
from Br_m *to the group of invertible elements in the Temperley–Lieb algebra at parameter* $\delta = -v^2 - v^{-2}$.

Proof One easily checks using (21.1) that $vu_i + v^{-1}$ is invertible (with inverse $v^{-1}u_i + v$). We need to check that the elements $vu_i + v^{-1}$ satisfy the braid relations. The far-away braid relation (21.14) follows from the far-away commutation relation (21.3) in the Temperley–Lieb algebra. For the nearby braid relation (21.15), compute

$$\begin{aligned}
&(vu_i + v^{-1})(vu_{i+1} + v^{-1})(vu_i + v^{-1}) \\
&\quad = v^3 u_i u_{i+1} u_i + v(u_i u_{i+1} + u_{i+1} u_i + u_i^2) + v^{-1}(2u_i + u_{i+1}) + v^{-3} \\
&\qquad = v(u_i u_{i+1} + u_{i+1} u_i) + v^{-1}(u_i + u_{i+1}) + v^{-3} + \underbrace{(v^3 + v\delta + v^{-1})u_i}_{=0},
\end{aligned}$$

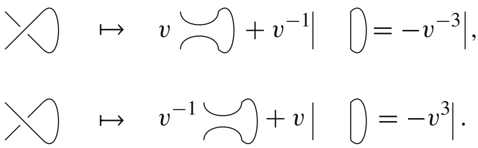
and note that the last expression is symmetric in u_i and u_{i+1}. Here, the last equality uses (21.1) and (21.2). □

Exercise 21.14 The assignment of Lemma 21.13 may be drawn diagrammatically as

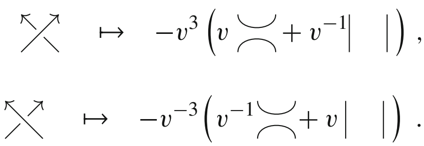


Verify the nearby braid relation for these images, now using diagrammatics. It may be helpful to use the computation of the inverse:

Thus to any braid we have assigned an element of some Temperley–Lieb algebra. Composing with the trace map, we obtain a braid invariant that is moreover invariant under conjugation. However, the assignment is not invariant under the Markov move:



To remedy this, we can pass to oriented braids and define:



(21.22)

(21.23)

Now both  and  are mapped to the diagram .

The constructed mapping assigns a diagram in $\mathrm{TL}_{m,-v^2-v^{-2}}$ to every oriented braid. Applying the trace map we obtain a polynomial in δ, or a Laurent polynomial in v, which we denote by J.

By construction, J is invariant under conjugation, Markov moves and braid isotopy, thus, it is invariant under link isotopy of the braid closure. Furthermore, it follows from (21.22) and (21.23) that J satisfies the *skein relation*

$$q^2 J(\nwarrow\!\!\!\!\nearrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) J(\uparrow\ \uparrow) \tag{21.24}$$

with $q = -v^{-2}$.

The skein relation allows us to identify J as the celebrated *Jones polynomial*, which is the link invariant determined by this specific skein relation up to a normalization. The normalization of J as just constructed is given by

$$J(\bigcirc) = \delta = -v^2 - v^{-2} = q + q^{-1}\,. \tag{21.25}$$

There are variations P_n (for all $n \in \mathbb{Z}$) of the Jones polynomial with the more general skein relation

$$q^n P_n(\nwarrow\!\!\!\!\nearrow) - q^{-n} P_n(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) P_n(\uparrow\ \uparrow) \tag{21.26}$$

and the normalization

$$P_n(\bigcirc) = \frac{q^n - q^{-n}}{q - q^{-1}}\,. \tag{21.27}$$

In this family of link invariants, P_0 corresponds to the *Alexander polynomial*, $P_1 \equiv 1$, and P_2 corresponds to the Jones polynomial. An even more general (two-parameter) version of this skein relation will be used to define the *HOMFLYPT polynomial* in Sect. 21.5.

References on link invariants and the Jones polynomial are Jones [93] and Kauffman [104].

21.4 Quantum Groups and Link Invariants

We recall that the module category of any Lie algebra $\mathfrak{g}$ is isomorphic to the module category of its universal enveloping algebra $U(\mathfrak{g})$. Since $U(\mathfrak{g})$ has the structure of a *cocommutative Hopf algebra*, the category of modules has the structure of a *symmetric tensor category*; in particular, the tensor product of two modules is naturally a module, where an element $x \in \mathfrak{g}$ acts as its image under the coproduct map

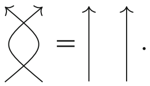
$$\Delta(x) = x \otimes 1 + 1 \otimes x \quad \in U(\mathfrak{g}) \otimes U(\mathfrak{g}) \ , \tag{21.28}$$

and for any pair of modules (V, W), the *flip map*

$$\tau \colon V \otimes W \to W \otimes V \ , \quad v \otimes w \mapsto w \otimes v \ , \tag{21.29}$$

is a module map satisfying $\tau^2 = \mathrm{id}$.

Given a symmetric tensor category, we can assign a morphism in the category to any oriented braid if we label the strands of the braid with an object from the category, thus obtaining a tensor product object in the category, and translate every crossing to a morphism of the tensor product using the flip map for the corresponding pair of tensor factors. This does not distinguish crossing points with different upper strands, and as $\tau^2 = \mathrm{id}$, we have



(21.30)

In other words, if we pick an object V as the label for all strands, then for any $m \in \mathbb{Z}_{\geq 1}$, the symmetric group S_m will act on $V^{\otimes m}$, where the action of a transposition is given by τ applied to the corresponding pair of tensor factors, and the morphism associated to a braid is actually the morphism which corresponds to the image of the braid group element under the quotient morphism to the symmetric group.

To make things more interesting, we can consider the quantum group $U_q(\mathfrak{g})$, which is a one-parameter deformation of $U(\mathfrak{g})$ as a Hopf algebra.

Example 21.15 ($U_q(\mathfrak{sl}_2)$) For $\mathfrak{g} = \mathfrak{sl}_2$, the quantum group $U = U_q(\mathfrak{sl}_2)$ is the (unital) $\mathbb{Z}[q^{\pm}]$-algebra generated by E, F, K, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K \ , \quad KEK^{-1} = q^2 E \ , \quad KFK^{-1} = q^{-2}E \ , \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}} \ ,$$

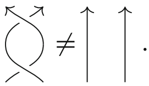
together with a counit $\varepsilon \colon U \to \mathbb{Z}[q^{\pm}]$, a coproduct $\Delta \colon U \to U \otimes U$ and an antipode $S \colon U \to U$ defined as algebra maps by

$$\begin{aligned}
\varepsilon(E) &= 0 \ , & \Delta(E) &= K \otimes E + E \otimes 1 \ , & S(E) &= -K^{-1}E \ , \\
\varepsilon(F) &= 0 \ , & \Delta(F) &= 1 \otimes F + F \otimes K^{-1} \ , & S(E) &= -FK \ , \\
\varepsilon(K) &= 1 \ , & \Delta(K) &= K \otimes K \ , & S(K) &= K^{-1} \ .
\end{aligned}$$

It can be checked that this defines a Hopf algebra structure.

The coproduct of the Hopf algebra $U_q(\mathfrak{g})$ is not cocommutative (invariant under permutations of the tensor factors) any longer, and the flip τ will not be a module map anymore. However, $U_q(\mathfrak{g})$ is still a *quasitriangular Hopf algebra* making its module category a *braided tensor category*, that is, we have a *braiding map* $c\colon V \otimes W \to W \otimes V$ for any pair of modules (V, W), but $c \neq \tau$ and $c^2 \neq \mathrm{id}$, in general.

As before, we can assign a morphism to oriented braids, but now (generally)



(21.31)

Since $c^2 \neq \mathrm{id}$, the symmetric group S_m will not act on $V^{\otimes m}$ for a module V; only the braid group Br_m, which lacks the quadratic relations, will act, where the braiding of adjacent tensor factors is given by c.

To generate actual link invariants, we also have to take care of the "cups" and "caps" in their diagrams, or equivalently, we have to find trace functions mapping the endomorphisms associated to braids to, say, polynomials in a way which is invariant under link isotopy. This is dealt with using *enhanced R-matrices* or *Ribbon categories*, and the link invariants obtained with this machinery (or its generalization to Lie superalgebras and their enveloping algebras) are sometimes called *Reshetikhin–Turaev* or *quantum invariants*.

Theorem 21.16 (Reshetikhin–Turaev, Kauffman–Saleur) *The invariants P_n from the end of Sect. 21.3 (including the Jones polynomial) are quantum invariants. For $n \geq 1$, P_n can be obtained as a quantum invariant from $U_q(\mathfrak{sl}_n)$.*

References for quantum groups and link invariants are Reshetikhin–Turaev [148], Turaev [170] and Kassel [102].

21.5 Ocneanu Trace and HOMFLYPT Polynomial

We fix $n, m \in \mathbb{Z}_{\geq 1}$. Then Schur–Weyl duality implies the existence of subalgebras $S(n,m)$, $Q(n,m)$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes m})$ which are centralizers of each other such that the following diagram commutes:

$$
\begin{tikzcd}
U(\mathfrak{gl}_n) \arrow[r, dotted, "\curvearrowright"] \arrow[d, two heads] \arrow[dr] & (\mathbb{C}^n)^{\otimes m} \arrow[d, dotted, "\circlearrowleft"] & \mathbb{C}S_m \arrow[l, dotted, "\curvearrowleft"'] \arrow[d, two heads] \arrow[dl] \\
S(n,m) \arrow[r, hook] & \mathrm{End}((\mathbb{C}^n)^{\otimes m}) & Q(n,m) \arrow[l, hook']
\end{tikzcd}
\tag{21.32}
$$

The algebra $S(n,m)$ is called the *Schur algebra*, the algebra $Q(n,m)$ is called the *n-row quotient* of $\mathbb{C}S_m$ for $n < m$. (Note that for $n \geq m$, the action map $\mathbb{C}S_m \to \mathrm{End}((\mathbb{C}^n)^{\otimes m})$ is injective.) As subalgebras of a finite-dimensional endomorphism algebra, both algebras are finite-dimensional. Finally, note that, as we have seen above, $Q(2, m) \simeq \mathrm{TL}_{m,-2}$, and that $\mathbb{C}S_m$ is the specialization of the Hecke algebra $\mathrm{H}(S_m)$ (see Chap. 3) at $q = 1$.

Passing to the quantum case, $U(\mathfrak{gl}_n)$ is replaced by the quantum group $U_q(\mathfrak{gl}_n)$. We can consider $\mathbb{Q}(q)^n$ as a $U_q(\mathfrak{gl}_n)$-module. Then the braid group Br_m acts on $(\mathbb{Q}(q)^n)^{\otimes m}$, and the action factors through the Hecke algebra $\mathrm{H}(S_m)$. Quantum Schur–Weyl duality now implies the existence of finite-dimensional subalgebras $S_q(n, m)$, $Q_q(n, m)$ in $\mathrm{End}((\mathbb{Q}(q)^n)^{\otimes m})$ which are centralizers of each other such that the following diagram commutes:

$$
\begin{tikzcd}
 & & \mathbb{Q}(q)\mathrm{Br}_m \arrow[d, two heads] \arrow[dl, dotted, no head, "\circlearrowright"'] \\
U_q(\mathfrak{gl}_n) \arrow[r, dotted, no head, "\circlearrowleft"] \arrow[d, two heads] \arrow[dr] & (\mathbb{Q}(q)^n)^{\otimes m} \arrow[r, dotted, no head, "\circlearrowright"] \arrow[d, dotted, no head, "\circlearrowleft"] & \mathrm{H}(S_m) \arrow[d, two heads] \arrow[dl] \arrow[ddl, dashed, bend left] \\
S_q(n, m) \arrow[r, hook] \arrow[dr] & \mathrm{End}((\mathbb{Q}(q)^n)^{\otimes m}) \arrow[d, "\mathrm{Tr}"] & Q_q(n, m) \arrow[l, hook'] \arrow[dl] \\
 & \mathbb{Q}(q) &
\end{tikzcd}
\tag{21.33}
$$

Here we have inserted the trace map on $\mathrm{End}((\mathbb{Q}(q)^n)^{\otimes m})$, its restrictions to $S_q(n, m)$ and $Q_q(n, m)$, and an additional dashed arrow for later reference. The algebra $S_q(n, m)$ is sometimes called the *quantum Schur algebra*, the algebra $Q_q(n, m)$ is a Hecke algebra (q-)version of the n-row quotient (it will be discussed further in Chap. 22) and in particular, $Q_q(2, m) \simeq \mathrm{TL}_{m,-[2]}$.

Remark 21.17 Technically, the algebras in the diagram should all be over $\mathbb{Q}(q)$, so they should be obtained by base change from the $\mathbb{Z}[q, q^{-1}]$ algebras $\mathrm{H}(S_m)$ and $U_q(\mathfrak{gl}_n)$. However, every algebra in the diagram has an integral form, and the maps in the diagram descend to maps between these integral forms. It is the diagram with integral forms which is more amenable to categorification.

Both sides of the duality can be exploited to generate link invariants. As discussed in Sect. 21.4, we can use the braided tensor category of $U_q(\mathfrak{gl}_n)$-modules to construct the quantum link invariants. Alternatively we can use the Hecke algebra, generalizing the approach outlined in Sect. 21.3, where a link was first mapped to an element in the Temperley–Lieb algebra, whose trace then turned out to be a link invariant.

To generalize this idea, we fix a system of embeddings $S_1 \subset S_2 \subset \cdots$, and thus a system of embeddings $\mathrm{H}(S_1) \subset \mathrm{H}(S_2) \subset \cdots$, so we can make the following definition.

Definition 21.18 A *Markov trace* on $\bigcup_{m\geq 1} \mathrm{H}(S_m)$ with parameter z is a linear map

$$\mathrm{Tr}\colon \bigcup_{m\geq 1} \mathrm{H}(S_m) \to \mathbb{Z}[q^{\pm 1}, z] \tag{21.34}$$

such that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ and $\mathrm{Tr}(M(b)) = z\,\mathrm{Tr}(b)$ for all $a, b \in \bigcup_{m\geq 1} \mathrm{H}(S_m)$, where $M(b)$ is the modification of b according to the Markov move (21.19).[3]

Theorem 21.19 (Ocneanu) *There is a unique Markov trace on* $\bigcup_{m\geq 1} \mathrm{H}(S_m)$ *with normalization* $\mathrm{Tr}(1) = 1$.

Of course, this unique Markov trace, when specialized at $z = 0$, agrees with the standard trace we introduced in Chap. 3, see Definition 3.12.

Remark 21.20 For a suitable specialization of q, z, this trace reproduces the trace on the n-row quotient algebra $Q_q(n, m)$ which comes from the endomorphism algebra $\mathrm{End}((\mathbb{Q}(q)^n)^{\otimes m})$. This is indicated by the dashed arrow in the commutative diagram (21.33).

Theorem 21.21 (HOMFLYPT[4]) *There is a unique* $\mathbb{C}[t^{\pm 1}, x^{\pm 1}]$*-valued invariant* I *of oriented links with the skein relation*

$$t I(\nwarrow\!\!\!\!\!\nearrow) - t^{-1} I(\nwarrow\!\!\!\!\!\nearrow) = x I(\uparrow\ \ \uparrow) \tag{21.35}$$

and normalization $I(\bigcirc) = 1$.

Definition 21.22 This invariant is called the *HOMFLYPT polynomial*.

Comparing the skein relations, we note that the HOMFLYPT polynomial generalizes the quantum invariants P_n which we have seen in Sect. 21.3 and in Theorem 21.16. In this sense, it is the "one polynomial to rule them all."

It turns out that the HOMFLYPT polynomial can, indeed, be constructed using Hecke algebras and a generalization of the ideas from Sect. 21.3.

Theorem 21.23 (Jones) *Every Markov trace yields an invariant for oriented links. The Ocneanu trace yields the HOMFLYPT polynomial (up to a change of variables).*

Remark 21.24 The proof follows the key ideas of Sect. 21.3: For any oriented link, we pick an oriented braid whose closure is the given link. Next, we assign an element in the Hecke algebra $\mathrm{H}(S_m)$ to the braid. Finally, we apply a modification of the Ocneanu trace map, which is still invariant under conjugation, and also invariant

[3] Alternatively, we can think of this as a family of trace maps which is compatible with the embeddings $\mathrm{H}(S_m) \subset \mathrm{H}(S_{m+1})$.

[4] HOMFLYPT is an acronym combining the initials of two independent groups of authors, Hoste–Ocneanu–Millett–Freyd–Lickorish–Yetter and Przytycki–Traczyk.

under the Markov move. This will yield a link invariant, and it remains to verify the skein relation.

It should be mentioned that the HOMFLYPT polynomial is not a complete invariant:

Theorem 21.25 (Kanenobu [97]) *There are infinitely many distinct links with the same HOMFLYPT polynomial.*

References for the HOMFLYPT polynomial and the Ocneanu trace are the original papers by HOMFLY [69] and Jones [92].

Exercise 21.26 Compute the Jones polynomial and the HOMFLYPT polynomial of the trefoil. The trefoil is the closure of σ_1^3 in the braid group on two strands.

21.6 Categorification of Braids and of the HOMFLYPT Invariant

21.6.1 Rouquier Complexes

We want to categorify link invariants and the HOMFLYPT polynomial. To this end, we start with the categorification of braid groups using Rouquier complexes.

Let us recall some material from Chap. 19. We fix $m \geq 1$ and consider the type A_{m-1} Coxeter system $W = S_m$ with generators $S = \{s_1, \ldots, s_{m-1}\}$. Let V be the geometric representation of W over $\Bbbk$. As usual, let $R := \mathrm{Sym}(V)$ be the symmetric algebra of V, and let $B_s := R \otimes_{R^s} R(1)$ for every $s \in S$. We consider the complexes of Soergel bimodules

$$
\begin{aligned}
F_s &:= \left(\cdots \longrightarrow 0 \longrightarrow \underline{B_s} \longrightarrow R(1) \longrightarrow 0 \longrightarrow \cdots \right) \\
F_s^{-1} &:= \left(\cdots \longrightarrow 0 \longrightarrow R(-1) \longrightarrow \underline{B_s} \longrightarrow 0 \longrightarrow \cdots \right)
\end{aligned}
\tag{21.36}
$$

for every $s \in S$, where the underline indicates that B_s is placed in cohomological degree 0 in both complexes.

As before, we denote the bounded homotopy category of Soergel bimodules by $K^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$, which is a monoidal category under tensor product of complexes. As discussed in Chap. 19, F_s and F_s^{-1} are inverse to each other in $K^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$. We have already seen the following theorem (see Theorem 19.36) in Chap. 19:

Theorem 21.27 (Rouquier) *The mapping* $F\colon \mathrm{Br}_m \to K^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$ *sending*

$$
\sigma_{j_1}^{\epsilon_1} \cdots \sigma_{j_t}^{\epsilon_t} \mapsto F_{s_{j_1}}^{\epsilon_1} \otimes_R \cdots \otimes_R F_{s_{j_t}}^{\epsilon_t} \tag{21.37}
$$

for all $t \geq 1$ *and* $j_i \in \{1, \ldots, m-1\}$, $\epsilon_i \in \{\pm 1\}$ *for all* $1 \leq i \leq t$ *is well-defined, i.e. it respects the braid relations, and it induces a group homomorphism from* Br_m *to the group of isomorphism classes of invertible objects in* $K^{\mathrm{b}}\mathbb{S}\mathrm{Bim}$.

Hence, we can view Rouquier complexes, which are defined precisely to be the image of the above mapping, as a categorification of braid groups.

21.6.2 Hochschild Homology

In order to obtain link invariants as in Sect. 21.3, we need to find an analogue of the trace map for Rouquier complexes. We will see that the role of the trace map will be played by a combination of the Hochschild homology functor, the homology functor of chain complexes and the Euler characteristic.

For any $\Bbbk$-algebra A, we define $A^e := A \otimes_{\Bbbk} A^{\mathrm{opp}}$, the *enveloping algebra* of A. The category of A^e-modules is equivalent to the category of A-bimodules, but rephrasing bimodules in terms of A^e-modules allows one to use familiar ideas from homological algebra. For instance, it may not be obvious what a "projective resolution" of A-bimodules means, but it is obvious what a projective resolution of A^e-modules is. "Free A-bimodules" are A^e-modules that are isomorphic to a direct sum of copies of A^e itself.

Meanwhile, the regular A-bimodule, namely A itself, is not free as an A^e-module. Instead, it is the "universal" bimodule on which the left and right actions of A agree. That is, for any A-bimodule M, $\mathrm{Hom}_{A^e}(A, M)$ is isomorphic to the largest submodule on which the left and right actions agree, while $A \otimes_{A^e} M$ is the largest quotient on which the left and right actions agree, and agrees with the quotient of M by the space of all vectors $am - ma$ for $a \in A$ and $m \in M$. An A-bimodule on which the left and right actions agree is perhaps best thought of just as an A-module.

Definition 21.28 (Hochschild Homology) The functor $M \mapsto A \otimes_{A^e} M$, from A^e-modules to A-modules, is called the *A-coinvariants functor*, and is right exact. Its i-th left derived functor $M \mapsto \mathrm{Tor}_i^{A^e}(A, M)$ is called the *i-th Hochschild homology* of M and is denoted by $\mathrm{HH}_i(A, M)$. The *Hochschild homology* of M is defined as

$$\mathrm{HH}_*(A, M) := \bigoplus_{i \geq 0} \mathrm{HH}_i(A, M)\ . \tag{21.38}$$

Hochschild homology is a contravariant functor from the category of A-bimodules to the category of $\mathbb{Z}_{\geq 0}$-graded A-modules. Similarly, the *Hochschild cohomology* of M is

$$\mathrm{HH}^*(A, M) := \bigoplus_{i \geq 0} \mathrm{HH}^i(A, M) \tag{21.39}$$

where $\mathrm{HH}^i(A, M)$ is the i-th right derived functor of the left exact *A-invariants functor* $M \mapsto \mathrm{Hom}_{A^e}(A, M)$.

Let us note that these definitions also make sense for graded modules over graded rings.

If $(\cdots \to P_2 \to P_1 \to P_0 \to A \to 0)$ is a projective resolution of A as an A-bimodule, then the Hochschild homology of any A-bimodule M can be computed as the homology of the complex

$$\cdots \to P_2 \otimes_{A^e} M \to P_1 \otimes_{A^e} M \to P_0 \otimes_{A^e} M \to 0\,. \tag{21.40}$$

Example 21.29 (Koszul Resolution for Polynomial Rings, One Variable) Assume $A = \Bbbk[x]$ is a polynomial ring in one variable. Then $A^{\mathrm{opp}} = A$, $A^e = A \otimes A$ (both is true for any commutative algebra A), and A has the projective (Koszul) resolution

$$0 \longrightarrow A \otimes A \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} A \otimes A \xrightarrow{\mu} A \longrightarrow 0\,, \tag{21.41}$$

where μ is the multiplication map.

Exercise 21.30 Suppose that M is a graded bimodule over $A = \Bbbk[x]$, where $\deg x = 2$. Deduce that $\mathrm{HH}_0(A, M)$ is the cokernel of the map $(x^l - x^r)\colon M(-2) \to M$, where x^l is the left action of x, and x^r is the right action of x. Similarly, $\mathrm{HH}_1(A, M)$ is the kernel of this map.

Example 21.31 (Koszul Resolution for Polynomial Rings, Many Variables) More generally, let A be the ring $R = \mathrm{Sym}(V)$ defined above; choosing a basis for V, we may identify R with a polynomial ring in $r = \dim V < \infty$ variables. Then the Koszul resolution of R as an R^e-module can be written in a basis-free way as

$$\Lambda^r V \otimes R^e \xrightarrow{d_r} \cdots \xrightarrow{d_3} \Lambda^2 V \otimes R^e \xrightarrow{d_2} V \otimes R^e \xrightarrow{d_1} R^e \xrightarrow{\mu} R \longrightarrow 0, \tag{21.42}$$

where for each $1 \le k \le r$, $\Lambda^k V$ denotes the k-th exterior power of V, and the differential $d_k : \Lambda^k V \otimes R^e \to \Lambda^{k-1} V \otimes R^e$ is determined by

$$d_k\left((r_1 \wedge \cdots \wedge r_k) \otimes (1 \otimes 1)\right) = \sum_{i=1}^{k} (-1)^{i+1} r_1 \wedge \cdots \widehat{r_i} \cdots \wedge r_k \otimes (r_i \otimes 1 - 1 \otimes r_i). \tag{21.43}$$

Exercise 21.32 As a warm-up, for $r \in \{1, 2, 3\}$, use a basis of V to write down (21.42) explicitly. Then show that (21.42) is a resolution for any $r \ge 1$.

Exercise 21.33 Let M be a graded bimodule over $R = \Bbbk[x_1, x_2]$, where $\deg x_i = 2$. Consider the complex

$$0 \longrightarrow M(-4) \xrightarrow{\begin{pmatrix} x_1^l - x_1^r \\ x_2^r - x_2^l \end{pmatrix}} \begin{matrix} M(-2) \\ \oplus \\ M(-2) \end{matrix} \xrightarrow{\begin{pmatrix} x_2^l - x_2^r & x_1^l - x_1^r \end{pmatrix}} \underline{M} \longrightarrow 0\,, \tag{21.44}$$

where the final M is in homological degree zero. Prove that $\mathrm{HH}_i(R, M)$ is the degree $-i$ cohomology of this complex.

Exercise 21.34 Let $R = \mathbb{R}[x_1, x_2]$, acted on by S_2, and define the Soergel bimodules R and B_s as usual. Apply the technique of the previous exercise to compute $\mathrm{HH}_*(R, R)$ and $\mathrm{HH}_*(R, B_s)$. For a future exercise, you should keep careful track of the grading shifts involved. (*Hint:* It is much easier to compute $\mathrm{HH}_*(R, B_s)$ if you think of R as $\mathbb{R}[x_1 + x_2, x_1 - x_2]$.)

For polynomial rings, there is a "Poincaré duality" relating Hochschild homology and cohomology. More precisely, if $A = R$ is a polynomial ring in d variables, then

$$\mathrm{HH}^i(R, M) \simeq \mathrm{HH}_{d-i}(R, M) \tag{21.45}$$

for any R-bimodule M. Note that this isomorphism exchanges the gradings in the same way as Poincaré duality for a closed oriented manifold of dimension d.

Exercise 21.35 Let $A = R$ be a polynomial ring in d variables, and let M be any R-bimodule. Show that the complexes obtained by applying the functors $- \otimes_{R^e} M$ and $\mathrm{Hom}_{R^e}(-, M)$ to the Koszul resolution (21.42) can be identified up to a cohomological shift by d. Deduce the isomorphism (21.45).

(*Hint:* Fix an isomorphism $\Lambda^d V \xrightarrow{\sim} \Bbbk$. For any $0 \leq i \leq d$, multiplication defines a perfect pairing

$$(-) \wedge (-) : \Lambda^i V \times \Lambda^{d-i} V \to \Lambda^d V \xrightarrow{\sim} \Bbbk,$$

inducing isomorphisms $(\Lambda^i V)^* \simeq \Lambda^{d-i} V$. Use these isomorphisms to obtain the desired identification.)

Remark 21.36 This exercise shows that the isomorphism (21.45) arises from a "self-duality" in the Koszul resolution (21.42) of the polynomial ring itself. For a more general class of rings, there is a version of the isomorphism (21.45) due to van den Bergh [171, 172] that involves twisting the bimodule structure on one side by an automorphism.

Exercise 21.37 Let M be a graded bimodule over $R = \Bbbk[x_1, x_2]$, as in Exercise 21.33. Applying the appropriate grading shift and cohomological shift to the complex (21.44), one can obtain a complex whose cohomology in degree i matches

$\mathrm{HH}^i(R, M)$. Work this out explicitly, finding the correct shifts involved. Use this to compute $\mathrm{HH}^*(R, R)$ and $\mathrm{HH}^*(R, B_s)$ as in Exercise 21.34.

21.6.3 Categorifying the Standard Trace

We wish to argue that taking the Hochschild cohomology of a Soergel bimodule is the categorification of the standard trace map ϵ in the Hecke algebra. Recall that the standard trace could be expressed as $\epsilon(b) = (1, b)$, where $(-, -)$ is the standard form on the Hecke algebra (see Sect. 3.2.1). By the Soergel Hom formula (Theorem 5.27), $(-, -)$ categorifies to $\mathrm{Hom}(-, -)$. Thus the standard trace categorifies to $\mathrm{Hom}_{R^e}(R, -)$, whose derived functor is Hochschild cohomology.

The crucial feature of trace maps which we used in this chapter is that $\epsilon(ab) = \epsilon(ba)$. Since multiplication categorifies to $\otimes_A$, we want a relationship between $\mathrm{HH}^*(A, M \otimes_A N)$ and $\mathrm{HH}^*(A, N \otimes_A M)$ for any A-bimodules M and N. Let us instead consider Hochschild homology. At the start we have four A-actions: the left/right action on M/N, which we can think of as four A-actions on $M \otimes_{\Bbbk} N$. Taking the quotient $M \otimes_A N$ will "identify" the right action on M with the left action on N. Then applying the functor $\mathrm{HH}_0(A, -) = A \otimes_{A^e} (-)$ (or any derived functor HH_i) will identify the left action on M with the right action on N. Now two different A-actions remain. Meanwhile, computing $\mathrm{HH}_0(A, N \otimes_A M)$ will also identify the same actions, but in a different order. In other words, we can think of both $\mathrm{HH}_0(A, M \otimes_A N)$ and $\mathrm{HH}_0(A, N \otimes_A M)$ as the same quotient of $M \otimes_{\Bbbk} N$, where the inner actions are identified and the outer actions are identified, and these two spaces are isomorphic as $\Bbbk$-vector spaces! However, $\mathrm{HH}_0(A, M \otimes_A N)$ and $\mathrm{HH}_0(A, N \otimes_A M)$ are not isomorphic as A-modules: the A-module structure on $\mathrm{HH}_0(A, M \otimes_A N)$ comes from the outer action on $M \otimes_{\Bbbk} N$, while the A-module structure on $\mathrm{HH}_0(A, N \otimes_A M)$ comes from the inner action on $M \otimes_{\Bbbk} N$. A similar argument applies to Hochschild cohomology.

To reiterate, there is an isomorphism

$$\mathrm{HH}^*(A, M \otimes_A N) \simeq \mathrm{HH}^*(A, N \otimes_A M), \tag{21.46}$$

but only when viewing HH^* as a functor to graded $\Bbbk$-vector spaces, not as a functor to graded A-modules. This motivates viewing HH^* as a functor to vector spaces, rather than A-modules.

Exercise 21.38 Observe that for any Soergel bimodule M, the invariant subring R^W acts the same on the left and right of M. Deduce that for any Soergel bimodules M and N the isomorphism $\mathrm{HH}^*(R, M \otimes_R N) \simeq \mathrm{HH}^*(R, N \otimes_R M)$, which is not an isomorphism of R-modules, is at least an isomorphism of R^W-modules. Despite this, we continue to view $\mathrm{HH}^*(R, -)$ as a functor to vector spaces, to match the literature.

Now let us specialize $\Bbbk$ to $\mathbb{Q}$ and A to R. For any $\sigma \in \mathrm{Br}_m$, we have a (bounded) complex of Soergel bimodules

$$F(\sigma) = \left(\cdots \to F^j(\sigma) \to F^{j+1}(\sigma) \to \cdots\right) .$$

After applying Hochschild cohomology, we obtain a complex of bigraded $\mathbb{Q}$-vector spaces

$$\mathrm{HH}^*(R, F(\sigma)) = \left(\cdots \to \mathrm{HH}^*(R, F^j(\sigma)) \to \mathrm{HH}^*(R, F^{j+1}(\sigma)) \to \cdots\right) . \tag{21.47}$$

Note that there are three gradings: the *internal grading* of each graded R-bimodule, the *homological grading* in the complex, and the *Hochschild grading* which states which summand HH^i of HH^* is being considered. Taking homology of this complex yields a triply graded $\mathbb{Q}$-vector space which we call $\mathrm{HHH}(\sigma)$, the *triply graded link homology*. It categorifies the HOMFLYPT polynomial in the following sense.

Theorem 21.39 (Khovanov [112]) *Up to an overall grading shift,* $\mathrm{HHH}(\sigma)$ *is an invariant of oriented links, that is, it depends only on the closure of σ as an oriented link up to isotopy. Taking the Euler characteristic of* $\mathrm{HHH}(\sigma)$ *yields the HOMFLYPT polynomial, after some renormalization.*

The Euler characteristic turns the homological grading into a sign. Relating the two remaining gradings to the parameters t, x from (21.35) is actually quite difficult, as it passes through both a renormalization and a change of variable, but the precise relationship can be found in [52, Appendix].

Exercise 21.40 This exercise is surprisingly hard, which is why it is difficult to compute triply graded link homology by elementary techniques.

1. Consider the unknot, which is the closure of the identity braid on one strand. One should think of its triply graded link homology as the total Hochschild cohomology of $R_1 = \mathbb{Q}[x_1]$. Compute this triply graded vector space. (*Hint:* The graded dimension is $\frac{1+AQ^{-2}}{1-Q^2}$, where Q denotes the internal grading and A denotes the Hochschild grading.)
2. The unknot is also the closure of σ_1 in the braid group on two strands. Compute its triply graded link homology. Your base ring should be $R_2 = \mathbb{Q}[x_1, x_2]$, acted on by S_2 in the usual way. Your answer should agree with the previous one up to a shift.