Homework 2

The work handed in should be entirely your own. You can consult any abstract algebra textbook (e.g. Dummit and Foote, Artin), the course textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Feb. 16.

1. Construct an injective group homomorphism from the cyclic group C_4 to the symmetric group S_4. Describe its image in S_4 in terms of the cycle notation. How many different injective homomorphisms from C_4 to S_4 can you define?

2. Let V be a Euclidean vector space, and let $\mathrm{Iso}(V) := \{\phi | \phi \text{ is an isometry of } V\}$.

   (a) Prove that $\mathrm{Iso}(V)$ forms a group under composition of maps.

   (b) Let $\mathrm{Tran}(V) \subset \mathrm{Iso}(V)$ be the subset of maps that are translations. Recall that a translation on V is a map $t_{\mathbf{v}_0} : V \longrightarrow V$, $\mathbf{v} \mapsto \mathbf{v} + \mathbf{v}_0$ for some fixed vector $\mathbf{v}_0$ determined by $t_{\mathbf{v}_0}$. Show that $\mathrm{Tran}(V)$ is a normal subgroup of $\mathrm{Iso}(V)$.

3. Prove the following claim we have made in class.

   Let V be a Euclidean vector space with an inner product $\cdot$ (to differentiate with the standard inner product on $\mathbb{R}^n$, let's use a different notation here):

$$\cdot : V \times V \longrightarrow \mathbb{R}, \quad \mathbf{u}, \mathbf{v} \mapsto \mathbf{u} \cdot \mathbf{v}.$$

   Suppose $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an orthonormal basis for V, i.e., it satisfies

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{i,j}.$$

   Prove that the parametrization isomorphism

$$\Psi_\beta : V \longrightarrow \mathbb{R}^n, \quad \mathbf{v} = \sum_{i=1}^{n} a_i \mathbf{v}_i \mapsto (a_1, \dots, a_n)^t$$

   is an *isometry* in the sense that, for any $\mathbf{u}, \mathbf{v} \in V$, we have

$$\mathbf{u} \cdot \mathbf{v} = (\Psi_\beta(\mathbf{u}), \Psi_\beta(\mathbf{v}))_{\mathbb{R}^n},$$

   where the right hand side stands for the standard inner product for vectors in $\mathbb{R}^n$.

4. Recall that to prove a statement that involves a natural number n, the method of induction can be used:

   (a) Show that the statement holds for $n = 1$.

   (b) Assuming that the statement holds for a given natural n, show that it also holds for $n + 1$.

From here it follows that the statement holds for all natural n.

Prove by induction that for all natural numbers n, and a given angle $0 \le \vartheta < 2\pi$, the following matrix equality holds:

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}^n = \begin{pmatrix} \cos n\vartheta & -\sin n\vartheta \\ \sin n\vartheta & \cos n\vartheta \end{pmatrix}.$$

Give a geometric interpretation of this identity.

5. Diagonalize the matrix

$$S_1 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

6. Consider the reflections in $\mathbb{R}^2$ given by the matrices

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_1 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Check that S_0 and S_1 are reflections and find the reflection axes. What group do they generate? Find all group elements and write down a multiplication table between them.

Hint: Since $S_0^2 = S_1^2 = 1$, the only nontrivial elements of the group generated by S_0 and S_1 are the products where S_0 and S_1 alternate. Find all distinct elements of this form and determine their products.

7. In this exercise, we will fill out the details of a Lemma we claimed in class.

Consider $\mathbb{R}^n$ with the standard Euclidean inner product. An operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called *normal* if $T^tT = TT^t$. For instance, if T is orthogonal, it is normal.

(i) If $\mathbf{u}$ is an eigenvector for T with eigenvalue $\lambda \in \mathbb{R}$, i.e.,

$$T(\mathbf{u}) = \lambda\mathbf{u}$$

show that $\mathbf{u}$ is also an eigenvector for T^t with the same eigenvalue. (Hint: Prove that $||T^t(\mathbf{u}) - \lambda\mathbf{u}||| = 0$.)

(ii) Show that if $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of T with distinct eigenvalues λ and μ respectively, then $\mathbf{u} \perp \mathbf{v}$.