Homework 5

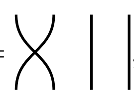
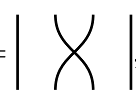
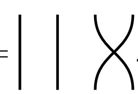
The work handed in should be entirely your own. You can consult any abstract algebra textbook (e.g. Dummit and Foote, Artin), the course textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due April. 6.

1. Read the textbook Chapter 4 on root systems.

2. Prove the statement we made in class: Let $P \subset V$ be a hyperplane orthogonal to the vector $\mathbf{r} \in V$:
$$P_{\mathbf{r}} = \{\mathbf{v} \in V | (\mathbf{v}, \mathbf{r}) = 0\}.$$
Let $g : V \longrightarrow V$ be an orthogonal transformation. Show that the hyperplane orthogonal to $g(\mathbf{r})$ coincides with
$$P_{g(\mathbf{r})} = g(P_{\mathbf{r}}) := \{g(\mathbf{v}) | \mathbf{v} \in P_{\mathbf{r}}\}.$$

3. Let V_1 and V_2 be two subspaces of a Euclidean vector space V. Prove that
   - $(V_1^{\perp})^{\perp} = V_1$,
   - $(V_1 + V_2)^{\perp} = V_1^{\perp} \cap V_2^{\perp}$,
   - $(V_1 \cap V_2)^{\perp} = V_1^{\perp} + V_2^{\perp}$.

4. In this exercise, you will be asked to check the reflection group structure of S_4, the symmetric group on four letters.
   (1) Recall that S_4 acts on the set $\{1, 2, 3, 4\}$ by permuting the letters. We "linearize" the action by constructing a (Euclidean) vector space $\mathbb{R}^4$ with the standard coordinate bases $\{e_1, e_2, e_3, e_4\}$, and let S_4 permute the indices of the basis vector. Show that in this way, S_4 is a subgroup of $O_4(\mathbb{R})$.
   (2) Using the diagrammatic notation we have developed in class, justify that any element of S_4 can be generated by the following three elements

$\sigma_1 :=$ , $\sigma_2 :=$ , $\sigma_3 :=$ .

   (3) How many reflection elements are there in S_4? Can you find an order-two element that is not a reflection?
   (4) Find roots for σ_i ($i = 1, 2, 3$).
   (5) Find all roots for S_4. (Hint: Use the group action on the roots you have found in part (4)).
   (6) Is this action of S_4 on $\mathbb{R}^4$ effective? What is the point-wise fixed space
$$V_0 = \{\mathbf{v} \in \mathbb{R}^4 | s(\mathbf{v}) = \mathbf{v},\ \forall s \in S_4\}?$$