Homework 1

January 28, 2016

Exercise 1. Show that, if U and V are simple representations of G and H respectively, then $U \otimes V$ is a simple representation of $G \times H$.

Exercise 2. Use character theory to prove the following result.

Proposition. Let U and V be finite-dimensional representations of a finite group G. Let

$$V \cong \oplus_{i=1}^n V_i^{r_i}, \quad U \cong \oplus_{i=1}^n V_i^{s_i}$$

be a decomposition into simple factors.

(i) For any $i, j \in \{1, \ldots, n\}$, $\langle \chi_i | \chi_j \rangle = \delta_{ij}$.

(ii) $\langle \chi_U | \chi_V \rangle = \sum_{i=1}^n r_i s_i$.

(iii) V is irreducible if and only if $\langle \chi_V | \chi_V \rangle = 1$.

Exercise 3. Let G be a finite group and $\{L_1, \ldots, L_n\}$ be its full list of pairwise non-isomorphic irreducible representations. For the characters $\chi_i := \chi_{L_i}$, ($i = 1, \ldots, n$), define the elements in the group algebra

$$e_i := \frac{\dim(V_i)}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g \in \mathbb{C}G.$$

Show that $\{e_i | i = 1, \ldots, n\}$ is a maximal set of central orthogonal idempotents in the group algebra.

Exercise 4. Consider the symmetric group S_n acting on the n-dimensional vector space $\mathbb{C}^n \cong \oplus_{i=1}^n \mathbb{C}e_i$ by permuting the indices of $\{e_i\}$. Evidently S_n preserves the subspace of vectors

$$V := \{(a_1, \ldots, a_n) | \sum_{i=1}^n a_i = 0\}.$$

Show that V is an irreducible representation.

Exercise 5. A representation V of a group G is called *self-dual* if $V \cong V^*$ as G-representations, where V^* is equipped with the group action determined as follows. Given any $g \in G$ and $f \in V^*$,

$$(g \cdot f)(v) := f(g^{-1}v).$$

(1) Show that V is self-dual if and only if $V \otimes V$ contains the trivial representation $\mathbb{C}$ as a direct summand.

(2) Show that any irreducible representation of S_n over the complex number are self-dual.

Exercise 6. Work out explicitly all the irreducible representations of D_{2n} when n is odd and compute the character table.