Linear Algebra and Matrix Theory Midterm Exam

October 11, 2016

Don't forget to write down clearly your

Name: ____________________ **Net ID:** ____________________

Instructions.

- The exam book contains 6 basic problems, worth 120 points. Out of the last five problems, please choose **four** to answer and receive a grading. **Please circle the problems you choose to be graded in the front page marking table below**.
- The total time for the exam is 75 minutes.
- No books, notes or calculators are allowed.
- Read the following story before opening the exam book!

  A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train.

  "Aha," says the engineer, "I see that Scottish sheep are black."

  "Hmm," says the physicist, "You mean that some Scottish sheep are black."

  "No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

  So, please be precise with your answers just as the mathematician in the story!
- Good luck with the exam!

CIRCLE THE PROBLEMS TO BE GRADED BELOW

Problem Number	Points
1	
2	
3	
4	
5	
6	
Total Points	

1. True or False (20 points). Mark "T" (True) in front of a correct statement and "F" (False) in front of a wrong one. No justification needed. U, V stand for vector spaces over some field $\mathbb{F}$.

(1) ____ Every vector space contains a unique zero vector.

(2) ____ The union of two subspaces is another subspace.

(3) ____ If U is a subspace of V, then any linearly independent subset of U is also linearly independent in V.

(4) ____ $\dim M_{3\times 3}(\mathbb{R}) = 6$.

(5) ____ If $R, S : U \longrightarrow V$ are linear maps that agree on a basis of U, then $R = S$.

(6) ____ If $\dim(U) = \dim(V)$, then any linear map $T : U \longrightarrow V$ is an isomorphism.

(7) ____ The linear map $\int : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R}),\ f \mapsto \int_0^x f(t)dt$, is one-to-one.

(8) ____ Any linear map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ has a one-dimensional kernel.

(9) ____ If $T : V \longrightarrow V$ is an isomorphism and β is an ordered basis for V, then we have the equality of matrices $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

(10) ____ $L_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$ is an isomorphism if and only if A is an $n \times n$ invertible matrix.

2. Basis (20 points) Consider the following collection of vectors for $P_2(\mathbb{C})$:

$$G = \{f_1(x) = x + 1, f_2(x) = x - 1, f_3(x) = 3x + 1, f_4(x) = 2, f_5(x) = x^2 + x + 1\}.$$

Find a maximal linearly independent subset of G, and determine if it is a basis for $P_2(\mathbb{C})$ or not.

3. Ordered basis and matrix (20 points) Let $\beta = \{e_1 = (1, 0), e_2 = (0, 1)\}$ be the standard basis of the plane $\mathbb{R}^2$.

(a) Let R be the counterclockwise rotation of vectors in the plane by 60 degrees, and S be the reflection of vectors about the y-axis (e_2-axis). Find the matrices for R and S with respect to the standard basis β.

(b) Compute the matrices for the composition operations $R \circ S$ and $S \circ R$ with respect to the standard basis β.

4. Linear Maps and Subspaces (20 points). Consider the space of 2×2 matrices over $\mathbb{F}$

$$M_2(\mathbb{F}) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{F} \right\}.$$

(a) Prove that, the following operation on matrices,

$$T : M_2(\mathbb{F}) \longrightarrow M_2(\mathbb{F}), \qquad A \mapsto T(A) := \frac{1}{2}(A + A^t),$$

is a linear transformation from $M_2(\mathbb{F})$ to itself.

(b) Show that the collection of symmetric matrices

$$V_1 := \{A \in M_2(\mathbb{F}) | A = A^t\}$$

and anti-symmetic matrices

$$V_2 := \{A \in M_2(\mathbb{F}) | A = -A^t\}$$

are subspaces of $M_2(\mathbb{F})$.

5. Polynomials (20 points). Let P be the space of polynomials

$$P := P_3(\mathbb{R}) = \{f(x) | \deg(f) \leq 3\}$$

and consider the evaluation map

$$T : P \longrightarrow \mathbb{R}^4, \quad f(x) \mapsto T(f) := (f(0), f(2), f(4), f(6)).$$

(a) Compute the matrix $[T]_\beta^\gamma$ for T with respect to the standard bases for P and $\mathbb{R}^4$ respectively, i.e., $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{e_1, e_2, e_3, e_4\}$.

(b) Is the matrix $[T]_\beta^\gamma$ in part (a) an invertible matrix? Justify your answer.

6. Isomorphism and Dimension (20 points). Let $T : U \longrightarrow V$ be an isomorphism of vector spaces. Follow the steps below to show that $\dim(U) = \dim(V)$.

(a) Let $\beta = \{v_1, \ldots, v_n\}$ be any linearly independent subset of U. Show that, for any injective map $S : U \longrightarrow V$, the colloection of vectors

$$S(\beta) := \{S(v_1), \ldots, S(v_n)\}$$

is still linearly independent.

(b) Use part (a) to prove that, if $T : U \longrightarrow V$ is an isormorphism, then $\dim(U) = \dim(V)$.