Solution to homework 4

1 Exercise 4.2.2, Hatcher

Note that the statement that the behavior of $\pi_1(\mathbb{R}P^n)$-action changes according to the parity of n hints a close relation with the behavior of antipodal maps.

Lemma 1.1 (Exercise 4.1.4, Hatcher)**.** Let $p : \tilde{X} \to X$ be the universal cover of a path-connected space X. Under the isomorphism $\pi_n(X) \cong \pi_n(\tilde{X})$, whjich holds for $n \geq 2$, the action of $\pi_1(X)$ on $\tilde{X}$ induced by the action of $\pi_1(X)$ on $\tilde{X}$ as deck transformations. More precisely, a formula like $\gamma{\cdot}p_*(\alpha) = p_*(\beta_{\tilde{\gamma}}(\gamma_*(\alpha)))$ holds where $\gamma \in \pi_1(X, x_0), \alpha \in \pi_n(\tilde{X}, \tilde{x}_0)$, and γ_* denotes the homomorphism induced by the action of γ on $\tilde{X}$.

Proof. For the sake of simplicity, we abuse notations and write γ for either the homotopy class in $\pi_1(X)$ or the deck transformation it induces on $\tilde{X}$. Then it is clear that $p_*(\beta_{\tilde{\gamma}}\gamma_*(\alpha)) = \beta_\gamma(p_* \circ \gamma_*(\alpha)) =: \beta_\gamma p_*(\alpha) = \gamma \cdot p_*(\alpha)$, where the first identity is simply saying that the homomorphism induced from a continuous map is π_1-equivariant and the second identity reflects the fact that γ is a deck transformation so that $p \circ \gamma = p$. $\square$

Assume for a moment that $n \geq 2$. And recall that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ is generated by a loop which induces the antipodal map on the universal cover(=S^n). Thus applying the lemma in the special case that $p : \tilde{X} \to X$ is $\pi : S^n \to \mathbb{R}P^n$, we know that the action of the (unique) generator γ on $\pi_n(S^n)$, which can be identified with that on $\pi_n(\mathbb{R}P^n)$ via the projection map, is homotopic to what is induced from the antipodal map. And we know that the antipodal map is homotopic to the identity if n is odd and to a reflection if n is even. Thus we obtain the desired result for $n \geq 2$. For $n = 1$, the action of $\pi_1(\mathbb{R}P^1)$ on $\pi_1(\mathbb{R}P^1)$ is simply the conjugation action, which is trivial as $\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}$ is abelian. Hence we get the same conclusion for $n = 1$.

2 Exercise 4.2.12, Hatcher

By Whitehead's theorem, it suffices to show that such an f is a weak homotopy equivalence, that is, $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \geq 1$. When $n = 1$, it follows from the assumption. Since a covering map $\tilde{X} \to X$ induces an isomorphism on the level of homotopy groups for $n \geq 2$, f induces an isomorphism on π_n if an only if its lift $\tilde{f} : \pi_n(\tilde{X}) \to \pi_n(\tilde{Y})$ induces an isomorphism on π_n, for $n \geq 2$. Now it follows from Corollary 4.33 in Hatcher.

3 Exercise 4.2.17, Hatcher

Notice that we can assume Y to be a CW complex, as from CW approximation theorem there exists a bijection

$$\langle X, \Gamma Y\rangle \to \langle X, Y\rangle$$

induced from the weak homotopy equivalence $\Gamma Y \to Y$, where ΓY is a CW approximation of Y. This allows us to assume without loss of generality that the elements $f \in \langle X, Y\rangle$ are cellular by cellular approximation. Furthermore, we can assume that X consists of cells with dimension greater than or equal to n except for the basepoint as the unique 0-cell.

- (injectivity) Suppose $f, g \in \langle X, Y\rangle$ such that $f_* = g_*$ on π_n. Note that $X^{(n)}$ is simply a wedge sum of n-spheres. Hence the condition that $f_* = g_*$ on π_n implies that f and g are homotopic restricted to $X^{(n)}$. Now an obstruction theoretic argument applies. That is, for an $(n+1)$-th cell e_α, we have a homotopy between $f|_{\partial e_\alpha}$ and $g|_{\partial e_\alpha}$. Think of this homotopy as a map from $\partial e_\alpha \times [0,1]$ to Y. As f and g are defined globally over X, we have an extension of this map defined on $\partial e_\alpha \times [0,1] \cup e_\alpha \times \{0,1\}$. But $\partial e_\alpha \times [0,1] \cup e_\alpha \times \{0,1\}$ is (homeomorphic to) an $(n+1)$-sphere, thus $\pi_{n+1}(Y) = 0$ means we can extend the map to a map defined on $e_\alpha \times [0,1]$, i.e., we have extended the homotopy on $X^{(n)}$ to $X^{(n)} \cup e_\alpha$. We can repeat this process to extend the homotopy on $X^{(n)}$ to a homotopy on X. Thus $f = g \in \langle X, Y\rangle$ as desired.
- (surjectivity) This is essentially done in Lemma 4.31 in Hatcher. For a homomorphism $\phi : \pi_n(X) \to \pi_n(Y)$, we can find a map $f : X^{(n+1)} \to Y$ inducing ϕ by Lemma 4.31. And the same obstruction theoretic argument as above allows us to extend the map $f : X^{(n+1)} \to Y$ to $f : X \to Y$. And as the inclusion $X^{(n+1)} \to X$ induces an isomorphism $\pi_n(X^{(n+1)} \cong \pi_n(X)$, we conclude that $f : X \to Y$ induces ϕ on π_n as desired.

Now, consider two $K(G, n)$ spaces X and Y. The above shows that $\langle X, Y\rangle \cong \mathrm{Hom}(\pi_n(X), \pi_n(Y))$ where $\pi_n(X) \cong \pi_n(Y) \cong G$. Thus, the homotopy class of a map f that corresponds to an isomorphism in $\mathrm{Hom}(\pi_n(X), \pi_n(Y))$ is a homotopy equivalence by the Whitehead's theorem.

4 Exercise 4.2.26, Hatcher

As the homotopy groups preserves products, X and $\tilde{X} \times K(\pi_1(X), 1)$ have isomorphic homotopy groups as desired. To show that the two spaces are not homotopy equivalent, we apply the Künneth formula to $\tilde{X} \times K(\pi_1(X), 1)$ so that we have an injective map

$$\bigoplus H_i(\tilde{X}) \times H_j(K(\pi_1(X), 1)) \to H_{i+j}(\tilde{X} \times K(\pi_1(X), 1)).$$

If $\pi_1(X)$ has a torsion element, then $K(\pi_1(X), 1)$ has infinitely many dimensions in which the homology group is nontrivial. Hence we know that $\tilde{X} \times K(\pi_1(X), 1)$ has the same property as well. But X is finite dimensional, hence there are only finitely many dimensions in which the homology group is nontrivial. Hence the two spaces are not homotopy equivalent.

5 Exercise 4.3.1, Hatcher

Note that the homology groups of $\mathbb{R}P^\infty$ are all torsion, but those for $\mathbb{C}P^\infty$ are torsion-freee. Thus, every map from $\mathbb{R}P^\infty$ to $\mathbb{C}P^\infty$ induces a trivial homomorphism on homology. On the other hand, as $\mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$, we have the fundamental class $\alpha \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$. As the Brown repreentability theorem gives us the isomorphism $\langle \mathbb{R}P^\infty, \mathbb{C}P^\infty \rangle \cong H^2(\mathbb{R}P^\infty; \mathbb{Z})$ by pulling α back by $f \in \langle \mathbb{R}P^\infty, \mathbb{C}P^\infty \rangle$, we know that there exists a map $f \in \langle \mathbb{R}P^\infty, \mathbb{C}P^\infty \rangle$ such that $f^*(\alpha)$ is nontrivial in $H^2(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}/2$. This is one example that the problem asks us to find.
Note that the universal coefficient theorem for cohomology gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_1(\mathbb{C}P^\infty;\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d] & H^2(\mathbb{C}P^\infty;\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{Hom}(H_2(\mathbb{C}P^\infty;\mathbb{Z}),\mathbb{Z}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_1(\mathbb{R}P^\infty;\mathbb{Z}),\mathbb{Z}) \arrow[r] & H^2(\mathbb{R}P^\infty;\mathbb{Z}) \arrow[r] & \mathrm{Hom}(H_2(\mathbb{R}P^\infty;\mathbb{Z}),\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

which becomes

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

after pluging the computation of the homology and cohomology groups into the diagram. Thus the map f induces a zero map on the first and third vertical arrows and nontrivial map on the second vertical arrow. This do not give any contradiction as the two squares commutes nontheless the middle arrow is trivial or not.

6 Exercise 4.3.5, Hatcher

For the construction of a $K(\mathbb{Z}, n)$, one can start with S^n (with its minimal CW structure) and then add $(n+2)$-cells to kill π_{n+1}, $(n+3)$-cells to kill π_{n+2}, and so on, to kill off all higher homotopy groups. Thus we have

$$H^n(X; \mathbb{Z}) \cong [X, K(\mathbb{Z}, n)] \cong [X, S^n]$$

where the second isomorphism comes from cellular approximation.

7 Exercise 4.3.7, Hatcher

Note that $\mu(f,g) = \mu \circ (f \times g) \circ \Delta$ where $\Delta : X \to X \times X$ is the diagonal map $x \mapsto (x,x)$.

Lemma 7.1. For the fundamental class $\alpha \in H^n(K(G,n);G)$, we have

$$\mu^*(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha,$$

where $\alpha \otimes 1 + 1 \otimes \alpha$ is thought of as an element in $\bigoplus_i H^i(K(G,n);G) \otimes H^{n-i}(K(G,n);G) \subseteq H^n(K(G,n) \times K(G,n);G)$.

Proof. First recall that a $K(G,n)$ is $(n-1)$-connected. Thus the direct sum of tensor products as above consists only of two summands, $H^n(K(G,n);G) \otimes H^0(K(G,n);G)$ and $H^0(K(G,n);G) \otimes H^n(K(G,n);G)$. And it also let us conclude that $K(G,n) \times K(G,n)$ is $(n-1)$-connected as well. Thus there is no non-trivial Tor part in the Künneth formula, hence the inclusion as above is in fact an isomorphism. This almost proves the claim, but let me be more precise.
From the definition of an H-space, the inclusion $\iota_1 : K(G,n) \to K(G,n) \times K(G,n), x \mapsto (x,e)$ satisfies $\mu \circ \iota_1 = id$. And it is clear that the induced map

$$\iota_1^* : H^n(K(G,n) \times K(G,n);G) \cong H^n(K(G,n);G) \otimes H^0(K(G,n);G) \oplus H^0(K(G,n);G) \otimes H^n(K(G,n);G) \to H^n(K(G,n);G)$$

is the projection onto the first factor (and forgeting about the second tensorand). Thus a cohomology class in $H^n(K(G,n) \times K(G,n);G)$, which can be represented by a form of $x \otimes 1 + 1 \otimes y$ from the discussion above, is mapped under ι_1^* to x. But $\iota_1^* \circ \mu^* = id$, so we know that for $\mu^*(\alpha) = x \otimes 1 + 1 \otimes y$, $x = \iota_1^*(\mu^*(\alpha)) = \alpha$. Similarly $y = \alpha$ holds. $\square$

From the lemma, we conclude that

$$\begin{aligned}
\mu(f,g)^*(\alpha) &= \Delta^* \circ (f \times g)^* \circ \mu^*(\alpha) \\
&= \Delta^* \circ (f \times g)^*(\alpha \otimes 1 + 1 \otimes \alpha) \\
&= \Delta^*(f^*(\alpha) \otimes 1 + 1 \otimes g^*(\alpha)) \\
&= f^*(\alpha) + g^*(\alpha),
\end{aligned}$$

which is exactly what we desire to show.