# Candidate Statement

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**Research Plan.** The intricate relationship between mathematics and physics has been a driving force in scientific progress for centuries. In the late 1980s, a profound connection was uncovered between the principles of quantum physics and the topological properties of three-dimensional shapes. This led to the emergence of topological quantum field theories (TQFT), a mathematical framework that describes the behavior of quantum systems in a purely mathematical context. While TQFTs offered valuable insights, their broken connection between four and three dimensions presented a challenge in understanding our four-dimensional universe.

In response to this challenge, L. Crane and I. Frenkel introduced the innovative approach of "categorification" in the mid 1990s. Categorification involves elevating mathematical structures from one level of abstraction to a higher one. By applying this technique to TQFTs, Crane and Frenkel aimed to lift them from the three-dimensional realm into a four-dimensional one. This ambitious endeavor could provide a more direct and accurate representation of our universe, potentially leading to breakthroughs in our understanding of fundamental physics. It would also provide a computationally accessible approach to understand and reconstruct the celebrated works of the Fields Medalists Jones, Donaldson and Witten.

In the late 1990, an astonishing discovery of Khovanov served as one of the earliest evidence that the Crane-Frenkel proposal could actually be realized. In particular, Khovanov gave a categorification of the celebrated Jones polynomial into a higher structure, now known as "Khovanov homology". He further envisioned a means to categorify 3d TQFTs, which are parametrized by certain algebraic numbers called "roots of unity", via Hopfological algebra. Khovanov's vision has been gradually realized, starting with my PhD thesis [1] at Columbia University under his guidance. In my doctoral thesis, I delved into developing the foundations of hopfological algebra, a mathematical framework essential for categorifying key components of 3d Witten-Reshetikhin-Turaev (WRT) TQFTs. Since then, significant progress has been made on quantum groups at roots of unity ([2], [3], [5],[6], [9]), their representations ([7]), and their higher-level tensor product structures ([22], [23]). Through rigorous investigation, I successfully developed a link homology theory at certain roots of unity with my collaborator J. Suaan ([17], [18]). This has been a significant advancement in the field of categorifying topological quantum field theories.

Quantum groups, the central protagonist in the above development, were independently defined by Drinfeld and Jimbo. They capture the symmetries of quantum integrable systems and 3-dimensional TQFTs. While much progress has been made in understanding quantum groups at generic parameters, the center of small quantum groups defined at roots of unity remains a subject of active research and deep mystery. In joint works ([8], [10], [14]) with A. Lachowska, we have made surprising discoveries connecting the (derived) center of small quantum groups to double affine Hecke algebras, a key subject in modern representation theory. This discovery has generated significant interest among experts in geometric representation theory, opening up new avenues for exploration and potentially leading to further breakthroughs in the field.

The surprising symmetries enjoyed by the derived center of small quantum groups will shed new light on 3d TQFTs, which is being gradually revealed in my ongoing projects with Lachowska and Sussan. At the same time, the categorification of such small quantum groups will lift 3d theories to 4d TQFTs, eventually fulfilling Crane and Frenkel's dream. I plan to investigate quantum groups at prime roots of unity and their associated Hecke algebras via categorical and geometric methods. This plan fits partially into the my longer-term research program to categorify the WRT 3-dimensional TQFTs, producing 4-dimensional theories. A key ingredient in this program has been to find a hopfological enhancement of certain colored link homology theories, generalizing my previous work with J. Sussan. The main technical goal is to build a categorical homology theory of knotted links in our 3-dimensional space that remains unchanged under certain combinatorial moves classified by Kirby at prime roots of unity. A few first steps have been made in my work with L.-H. Robert, J. Sussan and E. Wagner in [24] and more recent preprint [2].

Below I will list the next milestones I plan to achieve.

- ▷ Firstly, with several of my collaborators, we propose to study the structural behavior of quantum groups at roots of unity, including Lusztig's "big" and "small" quantum group, and investigate their applications towards 3-dimensional TQFTs. In particular, I plan to further study the structure and symmetries on the (derived) center of these objects initiated in the joints works with Lachowska.
- ▷ The technical framework for categorification at a root of unity belongs to the subject of "hopfological algebra," as initiated by Khovanov and myself, as a generalization of the usual homological algebra that is an indispensable tool in modern mathematics and physics. My collaborators and I have been and will further utilize hopfological algebra towards studying categorical representation theory of quantum groups. In particular, we plan to give a categorification of the key objects of "fusion rings" that play a pivitol role in 3d WRT TQFTs.
- ▷ Finally, some hopfological colored link homology theories arising from categorical quantum groups and Hecke algebras will be constructed and studied. They will be further utilized to study knots and links in the 3-sphere, and their functorial behavior with respect to link cobordisms will also be investigated.

**Teaching.** My teaching philosophy is grounded in the belief that mathematics is not merely a collection of formulas and theorems but a dynamic and evolving field that offers profound insights into the nature of reality. Through my experiences at previous institutions at Columbia, Berkeley, Yale, Caltech, and my current home of University of Virginia, I have developed a pedagogical approach that emphasizes critical thinking, problem-solving, and the cultivation of a deep understanding of mathematical concepts. I have participated in efforts in redeveloping our current algebra curriculum, and modernizing our graduate algebra qualifying exams.

I have had the distinct privilege of mentoring students at various levels, from undergraduate to graduate, guiding them through challenging topics such as topological quantum field theories, representation theory, and quantum topology. My goal is to inspire students to appreciate the beauty and elegance of mathematics, and to equip them with the intellectual tools necessary to contribute to the advancement of the field. I have mentored several undergraduate students who went on to purse doctoral degrees at Standford, MIT and Georgia Tech, including my Master degree student David Winters at UVA.

Besides normal teaching duties, I have led the UVA Directed Reading Program for the in the past and current academic years (2023–2025). I have actively participated in working with students from many different backgrounds. In the summer graduate workshop "Soergel Bimodules" at the former Mathematical Science Research Institute at Berkeley (now SLMath) in 2017, I gave lectures to graduate students entering the field, and hosted problem sessions for the workshop. The lecture notes and exercise material were featured as a chapter in the introductory textbook *Introduction to Soergel Bimodules* published by Springer. In the summer of 2020 during the height of the pandemic, I gave a minicourse series on root of unity categorification to graduate students at the virtual QUACKS conference and wrote up exercises for problem sessions. Most of the attendees for these events found the lectures and problem sessions quite useful in bridging the gap between textbook knowledge and current research in the field. As a junior faculty member at Virginia, I am co-mentoring Kang Lu, a postdoctoral fellow at UVa together with Weiqiang Wang.

When the subject in the research plan is more properly developed in the next few years, it will be possible to extract workable examples for undergraduate students as mini research topics. In particular, I have taught REU research courses at Virginia in the summers of 2020–2023, and will continue to do so if our REU funding is approved. I hope we will expand the current Virginia REU with representation theory and quantum topology as a staple topic.

One of my key strategies is to create a supportive and intellectually stimulating learning environment where students feel comfortable asking questions, sharing their ideas, and engaging in rigorous mathematical discourse. I believe that collaborative learning fosters a deeper understanding of mathematical concepts and helps students develop essential communication skills. Additionally, when teaching and advising, I incorporate real-world examples and applications to illustrate the relevance and practicality of mathematics in various fields, from physics and engineering to computer science and finance.

Beyond the classroom, I have been actively engaged in research and outreach activities. I have coorganized two international conferences sponsored by the National Science Foundation at UVA, as well as co-orgnizing the Virginia Mathematical Lectures featuring the distinguished speaker Mikhial Khovanov. These events greatly benefited our local mathematical community from students to established researchers alike. I am particularly interested in exploring the connections between mathematics and other disciplines, such as physics and computer science. Additionally, I believe in the importance of giving back to the community and building up a UVA platform for learning and idea-exchanging.

In conclusion, my passion for mathematics and my commitment to effective teaching have shaped me into a dedicated educator. I am excited to continue my teaching career and to contribute to the intellectual growth and development of future generations of mathematicians and scientists