Homework 4

The work handed in should be entirely your own. You can consult any abstract algebra textbook (e.g. Dummit and Foote, Artin), the course textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Mar. 30.

1. The most important exercise is that you should find some time during the break to review what we have learnt so far. Also, read Chapter 4 on fundamental regions.

2. Prove the following Proposition we stated in class.

   Proposition 0.1. *Let G be a group acting on a set S, and let $s, t \in S$ be two elements lying in the same orbit, i.e., there is a $g \in G$ such that $g * s = t$. Then there is an isomorphism of stablizer subgroups*

   $$\Phi_g : Z_G(s) \longrightarrow Z_G(t), \quad x \mapsto gxg^{-1}.$$

   In particular, if the stablizer groups are finite, then $|Z_G(s)| = |Z_G(t)|$.

3. Use the stablizer-orbit counting formula $|G| = |Z_G(s)||O_s|$ for a finite transitive group action to give an alternative count of the number of elements in the dihedral group D_2^n. (Hint: Consider the action of the dihedral group on the set of sides or vertices of a regular $n$gon and use the formula) .

4. Use the following inductive procedure to prove that $|S_n| = n!$.

   (1) Show that the group S_n acts transitively on the set $I_n = \{1, \ldots, n\}$, and determine the stablizer subgroup of the element n.

   (2) Use part (1) and induction (S_1 obviously has only one element) to prove the desired formula.

5. Show that the product of two reflections of $\mathbb{R}^2$ is a rotation through twice the angle between their reflecting lines. More precisely, say that S_i has reflecting line at angle θ_i with the positive x-axis, with $0 \leq \theta_i < \pi$, and say that $\theta_1 < \theta_2$, Then S_2S_1 is a counterclockwise rotation through angle $2(\theta_1 - \theta_2)$, and S_1S_2 is a clockwise rotation through angle $2(\theta_2 - \theta_1)$.