Math H1b Midterm

Don't forget to write down clearly your **Name**: _____ and **ID number**: _____

1. Fill in the blanks (10 points). Answer the following questions. No explanation is needed.

(1) The integral $\int_0^5 \sqrt{25 - x^2} =$ _____ .

(2) When doing trigonometric substitution $x = \cos\theta$ for $\sqrt{1 - x^2}$, the range of θ can be taken to be _____ .

(3) The improper integral $\int_{-\infty}^{\infty} xe^{-x^2} =$ _____ .

(4) When performing long division of $P(x) = x^2 + 4x + 4$ by $Q(x) = x + 2$, the remainder is $R(x) =$ _____ .

(5) If a thin plate with uniform density is bounded by the curve $x^2 - 2x + 1 + y^2 = 100$, the coordinate of the center of mass should be $(x_0, y_0) =$ _____ .

2. Multiple choices (10 points). Mark the box in front of the correct answer.

(1) The integration substitution rule is a consequence of the Fundamental Theorem of Calculus together and which of the following differentiation rule that is the most relevant?

☐ $(f(g(x)))' = f'(g(x))g'(x)$ ☐ $(f(x) + g(x))' = f'(x) + g'(x)$

☐ $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$ ☐ $(af(x))' = af'(x)$ if $a \in \mathbb{R}$.

(2) The value of the improper integral $\int_{-1}^1 x^{-1}dx$ is

☐ 0   ☐ -1 ☐ 1   ☐ divergent

(3) What is the arclength integral for $f(x) = \sqrt{16 - x^2}$ from $x = 0$ to $x = 4$?

☐ 1   ☐ π ☐ 2   ☐ 2π

(4) Which of the following is **not** a separable differential equation?

☐ $y' = \dfrac{x^2}{y^2}$ ☐ $\dfrac{dy}{dx} = x^2 y^2$ ☐ $y' = x^2 + y^2$ ☐ $\dfrac{dy}{dx} = x^2$

(5) Which of the following predictions does the logistic population model

$$P'(t) = kP(1 - P/M)$$

make to us? Here $k, M > 0$ are some fixed constants.

☐ The population will decrease if the population exceeds M.

☐ The population will increase if the population exceeds M.

☐ The population will always grow exponentially.

☐ The population will always decrease exponentially.

3. Mathematical induction. (10 points) Prove, using mathematical induction, that the following integration formula holds.

$$\int_0^\infty x^n e^{-x} = n! \ .$$

Choose one of the next two problems to solve. (10 points)

4. Differential equations in chemistry.   In an elementary chemical reaction, single molecules of two reactants A and B form a molecule of the product C: $A + B \rightarrow C$. The law of mass action states that the rate of reaction is proportional to the product of the concentrations of A and B:

$$\frac{d[C]}{dt} = k[A][B].$$

Thus if the initial concentrations are $[A] = a$ mol/L and $[B] = b$ mol/L, and suppose $x = [C]$ mol/L, we then have

$$\frac{dx}{dt} = k(a - x)(b - x).$$

   (a) Assume the reaction starts at $t = 0$, and $a = 2$ mol/L while $b = 1$ mol/L. Find the concentration of C as a function of time.

   (b) Assume the reaction starts at $t = 0$, and $a = b = 1$ mol/L. Find the concentration of C as a function of time.

5. Probability and hydrogen atom. The hydrogen atom, at ground state, has a spherical symmetric probability distribution function in terms of the distance from the nucleus. At ground state, it is described by

$$p(r) = \frac{4}{a_0^3} r^2 e^{-2r/a_0} \quad (r \geq 0)$$

where $a_0 \approx 5.59 \times 10^{-11} m$ is the *Bohr radius*.

(a) Show that $p(r)$ is a probability density function. (Hint: You may use a statement you've been given or shown within the exam paper.)

(b) Use your favorite way to show that, the probability of finding the electron with in the distance R from the nucleus is

$$P = -\frac{1}{2}(2R/a_0)^2 e^{-2R/a_0} - (2R/a_0)e^{-2R/a_0} - e^{-2R/a_0} + 1$$