Exercises for Week 1 and 2

The work handed in should be entirely your own. You can consult any abstract algebra textbook (e.g. Dummit and Foote, Artin) and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Feb. 2.

1. Let $G := \{a + b\sqrt{2} | (a, b) \in \mathbb{Q}^2 \backslash (0, 0)\}$. Show that G is a group under multiplication.
2. Prove that if all elements of a group G satisfy $x^2 = 1$, then G is abelian.
3. List all subgroups of S_3.
4. Let G be a group and g_0 be a fixed element of G. Show that the set $Z_G(g_0) := \{g \in G | g g_0 g^{-1} = g_0\}$ is a subgroup of G. This is called the *centralizer* of g_0 in G.
5. If (a), (b) are two given subgroups of $\mathbb{Z}$, we will define
$$(a) + (b) := \{n \in \mathbb{Z} | n = x + y,\ x \in (a),\ y \in (b)\}.$$
Show that
    - (i) $(a) + (b)$ is a subgroup of $\mathbb{Z}$.
    - (ii) Explicitly list the first 10 positive elements of the group $(4) + (6)$.
    - (iii) By a Theorem we proved in class, $(a) + (b) = (d)$ for some non-negative integer d. Show that $d = gcd(a, b)$. (Hint: Use the fact that, if $d = gcd(a, b)$, then, by the Euclidean algorithm, one can find integers $r, s \in \mathbb{Z}$ such that $d = ra + sb$.)
6. In class, we have define the group $\mathbb{Z}/(n)$ with the group addition given by
$$\overline{a} + \overline{b} := \overline{a + b}.$$
Show that this addition is well-defined (i.e. it is independent of choices representative for the cosets). How many elements are there in the group? Justify your answer.
7. Let $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ be the map $f(a) = e^{2\pi i a}$. Prove that it's a group homomorphism and determine its kernel and image.
8. Consider the group homomorphism $\det_n : GL_n(\mathbb{R}) \longrightarrow (\mathbb{R} \backslash \{0\}, \times)$. Find its kernel and image.