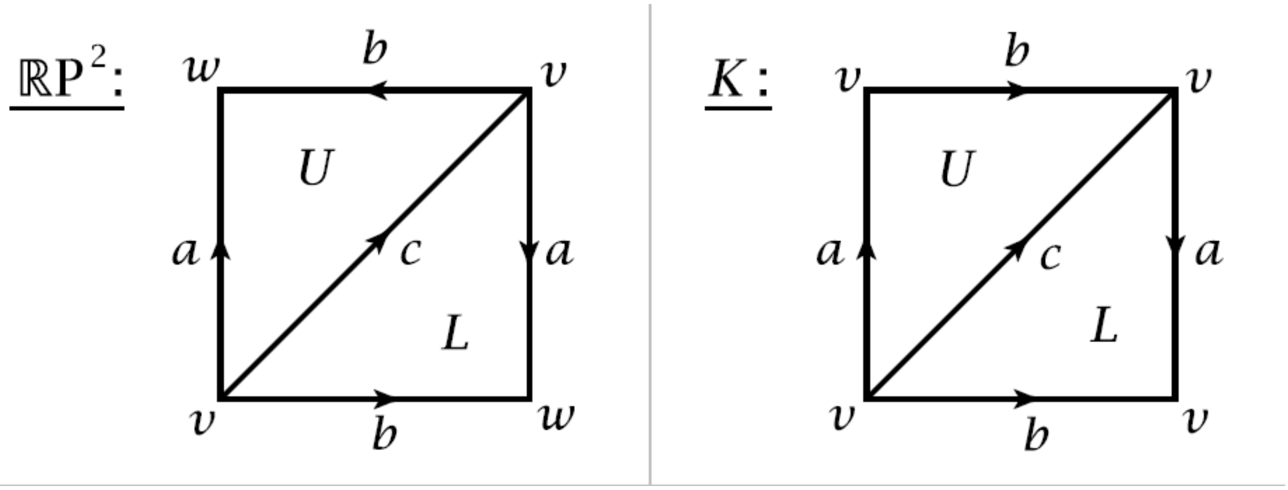
Solution to Homework 2

1 Cohomology ring of $\mathbb{R}P^2$ and the Klein bottle

(a) A triangulation of $\mathbb{R}P^2$ and the Klein bottle is given in Hatcher, p.102. Or any other valid triangulations would work.



(b) We have a triangulation from part (a). And the cohomology groups as abelian groups were computed in the previous homework, so I will just use the computation there. Note that in both cases the top cohomology occurs in dimension 2, so the only nontrivial cup products can happen when you multiply two generators in dimension 1.

- For $\mathbb{R}P^2$, $H^1(\mathbb{R}P^2; \mathbb{Z}/2)$ is generated by one element, which is dual to c. Denote a representative for it as ϕ. Now, from the definition of cup products on the level of cochains, we have that

$$(\phi \cup \phi)(U) = \phi(c)\phi(b), (\phi \cup \phi)(L) = \phi(c)\phi(a).$$

Moreover, as ϕ is a cocycle, we have $\phi(c) + \phi(b) = \phi(a)$. And our choice of ϕ guarantees that $\phi(c) = 1$. Thus,

$$(\phi \cup \phi)(U) = \phi(c)\phi(b) = \phi(b),$$
$$(\phi \cup \phi)(L) = \phi(c)\phi(a) = \phi(a)$$

hold, meaning $(\phi \cup \phi)(U + L) = \phi(a) + \phi(b) = \phi(c) = 1$. Note that $H^2(\mathbb{R}P^2; \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}/2}(H_2(\mathbb{R}P^2; \mathbb{Z}/2), \mathbb{Z}/2) \cong H_2(\mathbb{R}P^2; \mathbb{Z}/2)$ by the universal coefficient theorem, we have that a cohomology element is uniquely determined from its evaluation on homology classes. (In fact, this argument applies to any field coefficients.) As $U + L$ generates $H_2(\mathbb{R}P^2; \mathbb{Z}/2)$, $\phi \cup \phi$ represents the generator of $H^2(\mathbb{R}P^2; \mathbb{Z}/2)$($\because$ their evaluation on $[U + L]$ is the same). Hence, $[\phi] \cup [\phi] = [\phi]^2$ generates the second cohomology group, giving a presentation of the cohomology ring as $H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^3)$.

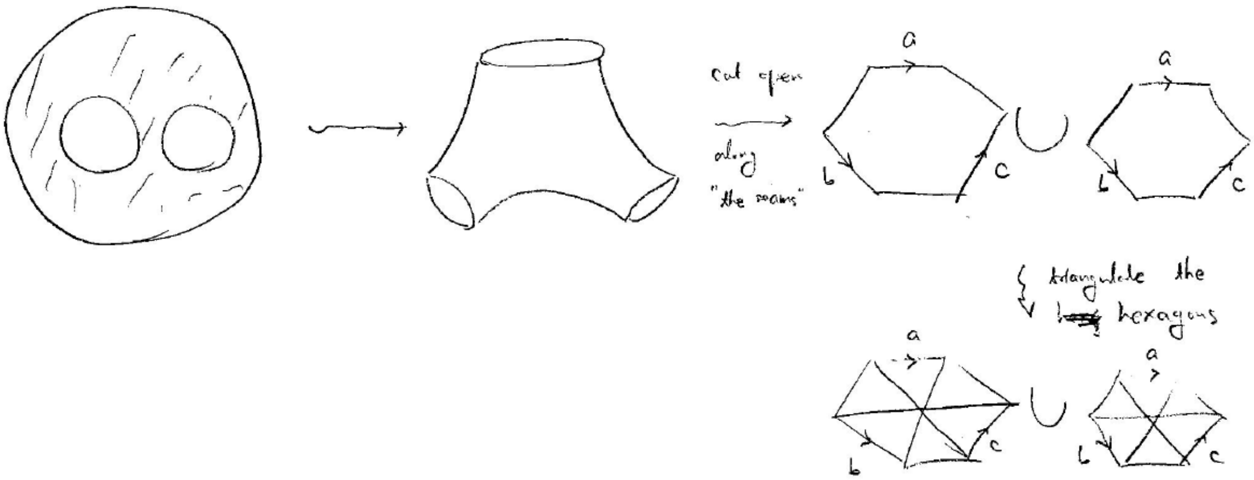
– For the Klein bottle K, the first cohomology group is freely generated by two elements which are dual to a and b, resp. Let a representative for them be ϕ and ψ, resp. Especially we can choose so that $\phi(a) = 1, \phi(b) = 0, \psi(a) = 0, \psi(1) = 1$ hold. We can then determine the cocycles ϕ and ψ from the cocycle condition: $\phi(c) = \phi(b) - \phi(a) = 1, \psi(c) = \psi(b) - \psi(a) = 1$. Then

$$(\phi \cup \phi)(U) = \phi(a)\phi(b) = 0, (\phi \cup \phi)(L) = \phi(c)\phi(a) = 1,$$
$$(\phi \cup \psi)(U) = \phi(a)\psi(b) = 1, (\phi \cup \psi)(L) = \phi(c)\psi(a) = 0,$$
$$(\psi \cup \psi)(U) = \psi(a)\psi(b) = 0, (\psi \cup \psi)(L) = \psi(c)\psi(a) = 0.$$

As before, from the universal coefficient theorem, a cohomology element is uniquely determined by its evaluation on homologies, and $U + L$ generates the second homology group. Hence we obtain that $[\phi]^2 = [\phi] \cup [\phi] = [\phi] \cup [\psi]$ is the generator of $H^2(K; \mathbb{Z}/2)$ and $[\psi]^2 = [\psi] \cup [\psi] = 0$, giving a presentation for the cohomology ring as $H^*(K; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x, y]/(x^3, y^2, x^2 - xy)$.

2 Cohomology ring of the unit disk minus 2 small disks in it

Denote the space by X. First, let me give my favorite triangulation. Note that the unit disk minus 2 small disks in it is homeomorphic to a pair of pants, see the figure below. (Imagine that you pull the two holes out of the screen to make them two ankles and the boundary of the big disk the waist.)



Then you cut open along "the seams of the pair of pants" to obtain two hexagons which are glued to give a pair of pants. Now it is easy to triangulate the hexagons with 6 triangles, giving a triangulation of X with 12 triangles. (Of course, any other valid triangulation is equally good.)

With a triangulation, we can compute its cohomology groups using the simplicial complex. The result of this tedious computation is: $H^0(X) \cong R, H^1(X) \cong R \oplus R$, and all higher cohomology groups vanishes. (Remark. As professor Qi did not make the coefficient ring clear, I suppose that the coefficient ring is arbitrary, but a use of a particular ring in your solution would not redue a point.) (Another remark. One easier way to see this is by deformation retracting X to a wedge of two circles.) As all the higher cohomologies vanish, the cup product is trivial, giving a presentation of the cohomology ring of X by $H^*(X) \cong R[x,y]/(x^2, xy, y^2)$.

3 Exercise 3.2.2, Hatcher

As A and B are contractible, $H^k(X, A; R) \cong H^k(X, pt; R) \cong \tilde{H}^k(X; R)$ and $H^l(X, B; R) \cong H^k(X, pt; R) \cong \tilde{H}^l(X; R)$. Thus, for any pair of positive-dimensional cohomology elements α and β, we can identify them with an element in $H^k(X, A; R)$ and $H^l(X, B; R)$, resp. Now consider the maps $(X, \varnothing) \to (X, A)$, $(X, \varnothing) \to (X, B)$, and $(X, \varnothing) \to (X, A \cup B)$, which induce the following commutative diagram by the naturality of cup products:

$$\begin{array}{ccc}
H^k(X, A; R) \times H^l(X, B; R) & \xrightarrow{\cup} & H^{k+l}(X, A \cup B; R) \\
\cong\downarrow \quad \times \quad \cong\downarrow & & \downarrow \\
H^k(X; R) \ \times \ H^l(X; R) & \xrightarrow{\cup} & H^{k+l}(X; R)
\end{array}$$

. Here, the first two vertical arrows are the identification that we mentioned. This diagram explains that the cup product of α and β factors through $H^{k+l}(X, A \cup B; R)$. But as $X = A \cup B$ by the hypothesis,

$H^{k+l}(X, A \cup B; R) \cong 0$. Hence this factorization of cup products implies that $\alpha \cup \beta = 0$ as desired.
As a suspension is a union of two cones, each of which are contractible, we can apply this to any suspensions. Thus the cup product in a suspsension is always trivial.
Now it is obvious to apply the argument above to n-fold cup product map:

$$\begin{array}{ccc} \prod_{i=1}^{n} H^{k_i}(X, A_i; R) & \xrightarrow{\cup} & H^{\sum_{i=1}^{n} k_i}(X, \cup_{i=1}^{n} A_i; R) \\ \cong \downarrow & & \downarrow \\ \prod_{i=1}^{n} H^{k_i}(X; R) & \xrightarrow{\cup} & H^{\sum_{i=1}^{n} k_i}(X; R) \end{array}$$

to obtain the desired generalization of the previous proposition.

4 Excersie 3.2.3, Hatcher

(a) Recall that $H^*(\mathbb{R}P^m; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^{m+1})$ where $\deg x = 1$. Suppose that $f : \mathbb{R}P^m \to \mathbb{R}P^n$ induces a nontrivial map on the first cohomology groups. This implies that $f^*(x) = y$ where y is the generator of the cohomology ring of $H^*(\mathbb{R}P^n; \mathbb{Z}/2)$, i.e., $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[y]/(y^{n+1})$. But then we obtain the identity $0 = f^*(0) = f^*(x^{m+1}) = y^{m+1}$, forcing that $m + 1 \geq n + 1$, or equivalently $n \leq m$. Thus there is no map $f : \mathbb{R}P^m \to \mathbb{R}P^n$, $n > m$, inducing a nontrivial map on the first cohomology groups with coefficients $\mathbb{Z}/2$.
The very same argument with the isomorphism $H^*(\mathbb{C}P^m; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{m+1})$ where $\deg x = 2$ applies so that (now we can only say that $f^*(x) = ky$ for a nonzero integer k) $0 = f^*(0) = f^*(x^{m+1}) = k^{m+1}y^{m+1}$, implying that $n \leq m$ as before. Thus there is no map $f : \mathbb{C}P^m \to \mathbb{C}P^n$ inducing a nontrivial map on the second cohomology groups with integer coefficients.

(b) Suppose on the contrary that $f : S^n \to \mathbb{R}^n$ such that $f(x) = -f(-x)$ for all x exists. Let h be as in the textbook. From part (a), we deduce that $h^* : H^1(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ is trivial. But $\mathbb{Z}/2$ is a field, so if a map h induces a trivial map on cohomologies, then it must also induce a trivial map on homologies. But $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ is already abelian, so h induces a trivial map on the fundamental groups. On the other hand, the generator of $\pi_1(\mathbb{R}P^n)$ lifts to a non-closed path in S^n. But g is antipodal, so the lifted path in S^n is mapped to a non-closed path in S^{n-1}, which projects down to the generator of $\mathbb{R}P^{n-1}$. This argument shows that h must send the generator of $\pi_1(\mathbb{R}P^n)$ to the generator of $\pi_1(\mathbb{R}P^{n-1})$, which is a contradiction.

5 Excersice 3.2.7, Hatcher

Note that $\widetilde{H}^*(X \vee Y) \cong \widetilde{H}^*(X) \times \widetilde{H}^*(Y)$. And we know that $\widetilde{H}^*(\mathbb{R}P^3; \mathbb{Z}/2) \cong x(\mathbb{Z}/2)[x]/(x^4)$, $\widetilde{H}^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong x(\mathbb{Z}/2)[x]/(x^3)$, and $\widetilde{H}^*(S^3; \mathbb{Z}/2) \cong y(\mathbb{Z}/2)[y]/(y^2)$ where $\deg x = 1$ and $\deg y = 3$. Hence, there is no nilpotent element of index 4 in $H^*(\mathbb{R}P^2 \vee S^3; \mathbb{Z}/2)$, but x is such an element in $H^*(\mathbb{R}P^3; \mathbb{Z}/2)$. As the cohomology ring is a homotopy invariant, we conclude that $\mathbb{R}P^3 \not\simeq \mathbb{R}P^2 \vee S^3$.

6 Excersice 3.2.11, Hatcher

As $f : S^{k+l} \to S^k \times S^l$ induces multiplication by d map on top homology and top cohomology where d is the degree of f, f induces a trivial map on top cohomology if and only if f induces a trivial map on top homology. Thus let me show that $f^* : H^{k+l}(S^k \times S^l) \to H^{k+l}(S^{k+l})$ is trivial. Now, from the Künneth formula, $H^*(S^k \times S^l) \cong H^*(S^k) \otimes H^*(S^l)$ holds. (We can apply the Künneth formula because the nontrivial cohomology groups of spheres are all $\mathbb{Z}$, which is clearly finitely generated.) Hence, the generator $[S^k \times S^l]$ of $H^{k+l}(S^k \times S^l)$ is the cup product of the generators α and β of $H^k(S^k)$ and $H^l(S^l)$, resp. But the cohomology groups $H^k(S^{k+l})$ and $H^l(S^{k+l})$ vanish, therefore $f^*(\alpha)$ and $f^*(\beta)$ are both 0. Thus $f^*([S^k \times S^l]) = f^*(\alpha) \cup f^*(\beta) = 0$, which is the desired result.