# Linear Algebra and Matrix Theory Final Exam

December 15, 2014

Don't forget to write down clearly your

**Name**: ______________________ and **ID number:** ______________________

**Instructions.**

- The exam book contains 7 basic problems, worth 100 points, and an extra-credit problem for 10 points.
- For problems 3-8, please show necessary reasoning and/or computation.
- The total time for the exam is 120 minutes.
- No books, notes or calculators are allowed.
- Good luck with the exam, and wish you all a happy holiday season!

**1. Multiple choices (10 points).** Choose the correct answer for each question.

1. Which of the following maps $f(x,y) : \mathbb{C}^2 \longrightarrow \mathbb{C}$ is linear? Answer: ____.
   A. $f(x,y) = 5x - y$, B. $f(x,y) = 5x - y + 1$,
   C. $f(x,y) = 5x - y - 1$, D. $f(x,y) = 5x - y + i$.

2. If $f(x) = x^2 + x + 1$ and $g(x) = x - 1$, which of the following vectors in $P_2(\mathbb{R})$ is linearly independent of $\{f(x), g(x)\}$? Answer:____.
   A. $x^2 + 2$, B. $x^2 + 2x + 2$, C. $x^2 + 2x$, D. $2x^2 + 2x + 2$.

3. Let $V, W$ be a finite dimensional vector spaces of equal dimension, and $T : V \longrightarrow W$ be a linear transformation. Which of the following statements is **not** equivalent to the rest? Answer:____.
   A. $T$ is one-to-one. B. The zero vector is in $\mathrm{Im}(T)$.
   C. $T$ is an isomorphism D. $T$ is surjective onto $W$.

4. Which of the following matrix is **not** an elementary matrix? Answer:____.
   A. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, C. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, D. $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

5. Which of the following matrix is **not** invertible? Answer:____.
   A. $(1)$ B. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, C. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, D. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

**2. True or False (20 points).** Mark "T" (True) in front of a correct statement and "F" (False) in front of a wrong one. No justification needed.

(1) ____ Any non-zero matrix has rank greater or equal to one.

(2) ____ If $V, W$ are two finite-dimensional vector spaces, the space of linear maps $\mathcal{L}(V, W)$ is also finite-dimensional.

(3) ____ Any spanning set of a finite-dimensional vector space must be finite.

(4) ____ The identity map $\mathrm{Id}_V$ has its matrix $[\mathrm{Id}_V]_\alpha^\beta = I_n$ for any choices of two bases $\alpha$ and $\beta$ for an $n$-dimensional vector space $V$.

(5) ____ There are linear maps from $\mathbb{R}^2$ to $\mathbb{R}^3$ that are invertible.

(6) ____ If $\mathrm{rank}(A) = k$, then the augmented matrix $\mathrm{rank}(A|b) = k+1$ where $b$ is a column vector of the same size as columns of $A$.

(7) ____ Any invertible matrix can be written as a finite product of elementary matrices.

(8) ____ If $A = 5B$, where $A$, $B \in M(2, \mathbb{R})$, then $\det(A) = 5\det B$.

(9) ____ The sum of two eignevectors of a linear operator may not be an eigenvector of the same operator.

(10) ____ A matrix $A \in M_n(\mathbb{F})$ is diagonlizable if and only if its characteristic polynomial splits over $\mathbb{F}$.

**3. Linear equations (10 points).** (a) Use your favorite way to find the inverse of the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(b) Solve for the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14, \\ x_1 + x_3 &= 4, \\ x_1 - x_2 + x_3 &= 2. \end{aligned}$$

**4. Trace (10 points).** Let $A \in M_{n\times m}(\mathbb{F})$ and $B \in M_{m\times n}(\mathbb{F})$ be matrices which are not necessarily square.

(a) Prove that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

(b) Now suppose $A, B \in M_n(\mathbb{F})$ are square matrices. Show that $\mathrm{Tr}(A) = \mathrm{Tr}(B)$ if $A$ and $B$ are similar.

**5. Determinant (15 points).** Prove the following statement.

If $M$ is the following $(n+m) \times (n+m)$-matrix written in the block form

$$M = \begin{pmatrix} A_{n\times n} & B_{n\times m} \\ 0_{m\times n} & C_{m\times m} \end{pmatrix},$$

then $\det M = \det A \cdot \det C$.

**6. Determinant and characteristic polynomial (15 points).** Let $A$ be the $4 \times 4$-matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{pmatrix}$$

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(a) Compute the characteristic polynomial of $A$.

(b) Let $g(t) = t^4 - 2t^3 + 4t^2 - 6t$, find a $4 \times 4$-matrix whose characteristic polynomial equals $g(t)$, and compute its determinant.

**7. Diagonalizing matrices (20 points).** Let $A$ be the $2 \times 2$-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d$ are real numbers.

(a) Prove that the characteristic polynomial of $A$ equals

$$f_A(t) = t^2 - \mathrm{Tr}(A)t + \det(A).$$

(b) Show that the characteristic polynomial $f_A(t)$ of $A$ splits over $\mathbb{R}$ if and only if

$$(a-d)^2 + 4bc \geq 0;$$

while it always splits over the complex numbers $\mathbb{C}$.

(c) Let $A$ be the rotational matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi/2$. Show that $A$ can not be diagonalized over real numbers.

(d) Prove that if $(a-d)^2 + 4bc > 0$, then $A$ can be diagonalized over real numbers.

**8. Extra Credit Problem (10 points).** Let $V$ be a real vector space and $T : V \longrightarrow V$ be a linear operator. Suppose we have three eigenvectors $v_1$, $v_2$, $v_3$ of $T$ in $V$ such that

$$T(v_1) = v_1, \quad T(v_2) = 2v_2, \quad T(v_3) = 3v_3.$$

Let $w = v_1 + v_2 + v_3$, and $W \subset V$ be a $T$-invariant subspace that contains $w$. Show that $W$ is at least three-dimensional.