Math 120 Practice exam 1 solutions

#1. (a) Using the formula: $\cos\theta = \dfrac{v \cdot w}{||v||||w||}$, one can compute

$$\cos\theta = \frac{-3}{\sqrt{6}\sqrt{6}} = \frac{-1}{2}.$$

Therefore $\theta = \boxed{\dfrac{2\pi}{3}}$.

(b) The vector projection of w onto v is

$$\text{proj}_v w = \frac{v \cdot w}{||v||^2}\, v = \frac{-3}{6} < 1, 2, 1 >= \boxed{\frac{-1}{2} < 1, 2, 1 >}.$$

#2. *Note that the plane is to contain the given line, not be perpendicular to it (this was a common error).*

One way to solve this is to find three points on the plane: one is given, and we can find (any) two on the line. Let $P = (1, -2, 5)$. Pick any two points on the given line: let

$$t = \frac{x-2}{2} = \frac{y+5}{-5} = \frac{z-4}{-1},$$

then $Q = (2, -5, 4)$ is a point on the line corresponding to $t = 0$, and $R = (0, 0, 5)$ is another point on the line corresponding to $t = -1$. (*Note that one can find different points by choosing different values of* t.) The plane which contains three points P, Q, and R has a normal vector:

$$\overrightarrow{RQ} \times \overrightarrow{RP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -5 & -1 \\ 1 & -2 & 0 \end{vmatrix} =< -2, -1, 1 >$$

An equation for the plane is

$$\boxed{-2(x-1) - (y+2) + (z-5) = 0} \quad \text{or} \quad \boxed{-2x - y + z = 5}.$$

(*There are other ways to solve this problem, for example noticing that any normal vector to the desired plane is perpendicular to the direction of the line and the direction vector through* $(2, -5, 4)$ *and* $(1, -2, 5)$. *Hence one can also find a normal vector by computing the cross product of* $< 2, -5, -1 >$ *and* $< -1, 3, 1 >$.)

 # 3. (a) No. If they intersect, there must exist a solution to the following system of equations:

$$\begin{cases} e^t = s..........(1) \\ e^{-t} = s........(2) \\ \sqrt{2}t = e^s......(3) \end{cases}$$

From equation (1) and (2), we know that $(s,t) = (1,0)$ is the solution. But it does not satisfy equation (3). Hence there is no solution to the system of equations, i.e. they do not intersect.

(b) At $t = 2$, the position vector is $< e^2, e^{-2}, 2\sqrt{2} >$ and the direction of the tangent line is parallel to

$$\frac{d}{dt}\gamma_A\Big|_{t=2} =< e^t, -e^{-t}, \sqrt{2} > \Big|_{t=2} =< e^2, -e^{-2}, \sqrt{2} > .$$

Parametric equations for the tangent line:

$$\boxed{\begin{cases} x = e^2 + e^2\ \alpha \\ y = e^{-2} - e^{-2}\alpha\ , \qquad \alpha \in \mathbb{R} \\ z = 2\sqrt{2} + \sqrt{2}\ \alpha \end{cases}}$$

(c) The point $(1,1,0)$ corresponds to $t = 0$ and the point $(e, \frac{1}{e}, \sqrt{2})$ corresponds to $t = 1$. The total distance traveled by particle A is the length of the space curve given by γ_A from $t = 0$ to $t = 1$. Hence

$$\begin{aligned} \int_0^1 ||\frac{d}{dt}\gamma_A(t)||\ dt &= \int_0^1 \sqrt{e^{2t} + e^{-2t} + 2}\ dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\ dt \\ &= \int_0^1 (e^t + e^{-t})\ dt = e^t - e^{-t}\Big|_0^1 = \boxed{e - e^{-1}} \end{aligned}$$

#4. <u>I: e. II: a. III: d.</u>

Explanations were not required, but here is a description of the curves, for you to see: Let k be any constant in the range of the given function. The level curves for (a) - (e):
(a) $k = (x+y)^2 \Rightarrow \sqrt{k} = x + y \Rightarrow$ Lines with slope -1
(b) $k = \cos(x - y^2) \Rightarrow \cos^{-1}(k) = x - y^2 \Rightarrow$ Parabolas opening to the right
(c) $k = x - y \Rightarrow k = x - y \Rightarrow$ Lines with slope 1
(d) $k = x + y^2 \Rightarrow$ Parabolas opening to the left
(e) $k = \ln(x^2 + 2y^2) \Rightarrow e^k = x^2 + 2y^2 \Rightarrow$ Ellipses

#5. (a) Because $|\sin(x)| \le 1$ and $0 \le \dfrac{y^2}{2x^2+y^2} \le \dfrac{y^2}{y^2} = 1$ for all (x,y) in the domain, we have

$$\left|\frac{y^3\sin(x)}{2x^2+y^2}\right| \le |y|.$$

This is equivalent to

$$-|y| \le \frac{y^3\sin(x)}{2x^2+y^2} \le |y|.$$

Moreover, $\lim\limits_{(x,y)\to(0,0)} |y| = |\lim\limits_{(x,y)\to(0,0)} y| = 0$. By Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{y^3\sin(x)}{2x^2+y^2} = \boxed{0}.$$

(b) Let $x = r\cos\theta$, $y = r\sin\theta$ (polar coordinates). Then

$$\lim_{(x,y)\to(0,0)} \frac{1-e^{x^2+y^2}}{x^2+y^2} = \lim_{r\to0^+} \frac{1-e^{r^2}}{r^2}.$$

By L'Hospital's Rule, $\lim\limits_{r\to0^+} \dfrac{1-e^{r^2}}{r^2} = \lim\limits_{r\to0^+} \dfrac{-2re^{r^2}}{2r} = \boxed{-1}$.

(c) $\boxed{\text{The limit does not exist}}$. First approach $(0,0)$ along the path $y = 0$. Then $h(x,0) = \dfrac{x^2}{x} = x$ for $x \ne 0$, so $h(x,y) \to 0$ as $(x,y) \to (0,0)$ along the line $y = 0$. Next approach $(0,0)$ along the path $x = y^2$. Then $h(y^2,y) = \dfrac{y^4+y^2}{y^2} = y^2+1$ for $y \ne 0$, so $h(x,y) \to 1$ as $(x,y) \to (0,0)$ along the path $x = y^2$. Since h has two different limits $(0 \ne 1)$ along two different paths, the limit does not exist.

(*Note 1: $h(x,y)$ is not defined at $x = 0$, so one cannot study the function $h(x,y)$ along the line $x = 0$.*

Note 2: When using polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x} = \lim_{r\to0} \frac{r}{\cos\theta}.$$

One cannot say that the limit exists and is equal to zero. This is because the function still has a denominator, and $\frac{0}{0}$ can happen if one approaches the origin along a path where $\theta \to \pi/2$, such as the parabola $x = y^2$ above.)

6. Let $F(x,y,z) = z - \ln(y\cos(x)) + e^{3x}$. Then $z = \ln(y\cos(x)) - e^{3x}$ is a level surface $F(x,y,z) = 0$.

$$\nabla F = < F_x, F_y, F_z > = < \frac{y\sin(x)}{y\cos x} + 3e^{3x}, -\frac{\cos(x)}{y\cos(x)}, 1 >$$

$\nabla F(0,1,-1) = < 3, -1, 1 >$. An equation for the tangent plane is $3x - (y - 1) + (z + 1) = 0$, or

$$\boxed{3x - y + z = -2}.$$