Homework 2

February 11, 2016

Exercise 1. Find the foundamental groups of the following Lie groups $O(n, \mathbb{R})$, $U(n)$, $SU(n)$ and $Sp(n)$. Here $Sp(n)$ is defined as the group that preserves the standard inner product on the n-dimensional quaternionic space $\mathbb{H}^n$:

$$Sp(n) := \{A \in \mathrm{M}(n, \mathbb{H}) | \langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{H}^n\}.$$

Exercise 2. Let A, B be any matrix in $\mathrm{M}(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Prove the following identities.

- $\exp(BAB^{-1}) = B\exp(A)B^{-1}$ if B is invertible.
- $\exp(A^*) = (\exp(A))^*$, where $*$ can either be the transpose, conjugation (on $\mathbb{C}$ and $\mathbb{H}$) or the composition of these two operations.
- $\exp : \mathrm{M}(n, \mathbb{F}) \longrightarrow \mathrm{M}(n, \mathbb{F})$ is real analytic, and the differential $d(\exp)|_0$ is nondegenerate at $T_0(\mathrm{M}(n, \mathbb{F})) \longrightarrow T_{\mathrm{Id}}(\mathrm{M}(n, \mathbb{F}))$.
- $\det(\exp(A)) = e^{\mathrm{tr}(A)}$.
- Use these properties to find the tangent space $T_{Id}G$ for the following matrix groups:

  $$G = GL(n, \mathbb{F}), \quad SO(n, \mathbb{R}), \quad U(n), \quad SU(n), \quad Sp(n),$$

  and compute their dimensions over $\mathbb{R}$.

Exercise 3. Let U be a charted open set of a manifold M, and let ξ η be two vector fields on M whose restriction on U are given by

$$\xi|_U = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}, \quad \eta|_U = \sum_{i=1}^{n} b_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i}.$$

Show that, if we define the commutator vector field $[\xi, \eta]$ locally by

$$[\xi, \eta]|_U := \sum_{i,j=1}^{n} (a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_j}) \frac{\partial}{\partial x_i},$$

then $[\xi, \eta]$ is a well-defined global vector field (i.e. it is independent of choices of the chart U).

Exercise 4. Let A be a finite-dimensional algebra over $\mathbb{R}$, and let D be a derivation on A. Then

$$\exp(D) : A \longrightarrow A, \quad a \mapsto \sum_{k=0}^{\infty} \frac{D^n(a)}{n!}$$

is an algebra automorphism of A.

Exercise 5. Humphrey's book. Page 5, Exercise 6, 9. Page 10, Exercise 4, 11.