Solution to homework 3

1 Topological groups are n-simple

To distinguish between the two binary operators, I will use $\cdot$ to denote the action of a homotopy class of pathes in G and $*$ be denote the multiplication in the group G. Choose a representative $f : (S^n, \mathrm{pt}) \to (G, e)$ of $[f] \in \pi_n(G, e)$ and $\gamma : [0, 1] \to G$ of $[\gamma] \in \pi_1(G, e)$. Then the map

$$t \mapsto \gamma|_{[0,t]} \cdot (\gamma(t) * f)$$

defines a homotopy from f to $\gamma \cdot f$. Note that the pointwise formula for the above map depends on the explicit formula for $\cdot$ which Hatcher do not provide, so I avoid writing it down here.

2 When A is contractible

Note that the fact that A is contractible to a point in X is equivalent to saying that the inclusion map $i : A \to X$ is nullhomotopic. Thus $i_* = 0$, and the LES for the triple (X, A, x_0),

$$\cdots \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \cdots,$$

implies that j_* is injective and ∂ is surjective as desired. Moreover, the SES

$$0 \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

splits by the existence of the retraction $r : X \to X$ contracting A to x_0, which is equivalent to the retraction $r : (X, A, x_0) \to (X, x_0, x_0)$ of j.

3 Exercise 4.1.3, Hatcher

Note first that the statement of the problem can be equivalently rephrased as showing that $[\mu(f, g)] = [f + g] \in \pi_n(X, x_0)$. And for the expression to make sense, we should set x_0 to be the (homotopy) identity element, otherwise $\mu(f, g)$ does not even belong to $\pi_n(X, x_0)$ in general. Hence, x_0 will be the identity element from now on.

First observe that $(f, g) \mapsto \mu(f, g)$ defines a well-defined binary operation on $\pi_n(X, x_0)$. (I do not claim that this is in fact a group multiplication for a moment.)

For $f, g : (S^n, s_0) \to (X, x_0)$ representing $[f], [g] \in \pi_n(X, x_0)$, $x \mapsto \mu(f(x), g(x))$ for $x \in S^n$ maps s_0 to $\mu(x_0, x_0) = x_0$. Thus we can take the homotopy class $[\mu(f, g)]$ in $\pi_n(X, x_0)$. Moreover, if we take two representatives f, f' of $[f]$, then it is clear that the homotopy H between f and f' extends to $\mu(H, g)$ to give a homotopy between $\mu(f, g)$ and $\mu(f', g)$. Samely we have the independence of the choice of representatives for $[g]$, thus the operation is well-defined.
Then we need to construct a homotopy between $\mu(f, g)$ and $f + g$. Let $f, g : (I^n, \partial I^n) \to (X, x_0)$ be two maps representing $[f]$ and $[g]$ in $\pi_n(X, x_0)$, resp. Recall from the definition of H-spaces that $f = \mu(f, x_0)$ and $g = \mu(x_0, g)$ hold. (Fancier words: H-spaces are magmas in the homotopy category of based topological spaces.) Thus, we try to build a homotopy between $\mu(f, g)$ and $\mu(f, x_0) + \mu(x_0, g)$. And the last map is the same as $\mu(f + x_0, x_0 + g)$:

Lemma 3.1. For any maps $a, b, c, d : (I^n, \partial I^n) \to (X, x_0)$, we have that

$$\mu(a + b, c + d) = \mu(a, c) + \mu(b, d)$$

holds.

Proof. Just write down the pointwise formula for these two maps and you can't fail. □

(Aside: If you know the fancy "Eckmann-Hilton argument", then you may be able to finish the proof just by saying that the desired result follows from the Eckmann-Hilton argument.)
Now, a homotopy that one can naturally come up with is for example constructed from

$$f_t(x_1, \cdots, x_n) = \begin{cases} f((2-t)x_1, x_2, \cdots, x_n) & \text{if } x_1 \in [0, \frac{1+t}{2}] \\ x_0 & \text{if } x_1 \in [\frac{1+t}{2}, 1] \end{cases}$$

$$g_t(x_1, \cdots, x_n) = \begin{cases} g((2-t)x_1, x_2, \cdots, x_n) & \text{if } x_1 \in [\frac{1-t}{2}, 1] \\ x_0 & \text{if } x_1 \in [0, \frac{1-t}{2}] \end{cases}$$

by $\mu(f_t, g_t)$.

4 Exercise 4.1.5, Hatcher

Each elements in $\pi_1(X, A, x_0)$ is represented by a path from x_0 ending in A. Thus, the action of $\pi_1(X, x_0)$ on $\pi_1(X, A, x_0)$ is transitive as we can just close the path representing an element in $\pi_1(X, A, x_0)$ up to a loop based at x_0 by attaching a path in A connecting the two endpoint of the path. Here the condition that A is path-connected is exploited. But the stabilizer of the constant path in $\pi_1(X, A, x_0)$ now becomes an isomorphic copy of $\pi_1(A, x_0)$ (obvious from the construction above). Thus $\pi_1(X, A, x_0)$ is isomorphic to the set of cosets of $\pi_1(A, x_0)$.

5 Exercise 4.1.15, Hatcher

I will use the same enumeration of the steps as in Hatcher.

(a) We can consider f as a map from $(I^n, \partial I^n)$ to (S^n, s_0). Applying Lemma 4.10, we know that there exists a polyhedron K on which f is PL and a nonempty open subset $U \subseteq S^n$ satisfying $f^{-1}(U) \subseteq K$. Now we need to find a point $q \in S^n$ where f is an invertible linear map near $f^{-1}(q)$. This follows from the following dimension argument: I will show that the "dimension" of the points in U at which f is not an invertible linear map is less than n. The failure of this property comes in two different ways. One possibility is when the point is not comming from the interior of the n-dimensional facet in K. (i.e., Failure of linearity) The points corresponding to this case is the image of the $(n-1)$th skeleton of K under f, which clearly has dimension less than n. The other possibility is when f is linear but failed to be a linear isomorphism. Of course, the points corresponding to this case is locally the image under a noninvertible linear map between two vector space of equal dimension, thus has dimension less than n. Thus there exists a point $q \in U$ such that f is a linear isomorphism near $f^{-1}(q)$. Of course, K is compact and $f^{-1}(U) \subseteq K$, so $f^{-1}(q)$ is finite.

(b) Note that g as in Hatcher is homotopic to the identity map $id : S^n \to S^n$. Thus f and gf are homotopic. And gf is a map where the regular neighborhood of each of the points p_i in $f^{-1}(q)$ is mapped homeomorphically onto the complement of the basepoint, and which maps the complement of these neighborhoods to the basepoint. Thus, we can factor gf by the following form: $S^n \to \vee_{i=1}^k S^n \to S^n$ where the first map collapses the complement of the regular neighborhood of $f^{-1}(q)$ to a point and the second map is the quotient of gf. Considering each factor sphere at a time, it suffices to show the claim when $k = 1$. Note furthermore that this argument also reduce the case to when the map is a linear isomorphism except for the basepoint.

(c) Let me summarize the reduction procedure above. We have a linear isomorphism, say $L : \mathbb{R}^n \to \mathbb{R}^n$, and the map f is (homotopic to) the extension of L to $S^n = \mathbb{R}^n \cup \{\infty\}$. Call this extension of L to be $\hat{L}$ (this is sometimes called a one-point compactification of L). Then it is obvious that a homotopy of linear maps induces a homotopy between their extensions: If L_t is a continuous family of linear isomorphisms, then $\hat{L_t}$ is a continuous family of maps between S^n. Thus the only task left is to show that any linear isomorphism is homotopic to the identity(which extends to the identity map on the sphere) or the reflection of the first coordinate(which extends to the reflection of the sphere, homotopic to the negative of the identity). That is,

Lemma 5.1. *$GL(n, \mathbb{R})$ has exactly two path components, each of which contains the identity and the reflection of the first coordinate, resp.*

By composing with the reflection, we can assume without loss of generality that $\det L > 0$. The usual Gram-Schmidt process can be promoted (by inserting a time parameter in front of each projection) to give a path from L to a special orthogonal matrix. And $SO(n)$ is path connected. (You can either explicitly construct the path between any two orthogonal matrices, or use the fact that $S^n \cong SO(n)/SO(n-1)$ and $SO(1) \cong \{\mathrm{pt}\}$ and induct.)

6 Exercise 4.1.17

From the CW approximation type argument(i.e., by Proposition 4.15 in Hatcher), X and Y allow a CW structure having only one 0-cell and missing the cells in all the intermediate dimensions from 1 to m (or n), resp. Recall that then $X \times Y$ has the product CW structure, whose cells are

- The 0-cell which is the product of 0-cells of X and Y
- The cells having dimension $> n$ which are the product of a cell in X with the 0-cell in Y
- The cells having dimension $> m$ which are the product of a cell in Y with the 0-cell in X
- The other cells which are products of cells in X and Y of positive dimensions.

The cells that corresponds to the last case have dimension at least $m+1+n+1 = m+n+2$. Similarly, the wedge sum $X \vee Y$ has the canonical CW structure, whose cells are

- The 0-cell which is the 0-cell of X, identified with the 0-cell of Y
- The cells having dimension $> n$ which are a cell in X
- The cells having dimension $> m$ which are a cell in Y.

Hence, in the pair $(X \times Y, X \vee Y)$, all the cells in dimension less than $m+n+2$ in $X \times Y$ lie on $X \vee Y$. Therefore, (by Proposition 4.12 in Hatcher) $(X \times Y, X \vee Y)$ is $(m+n+1)$-connected.
The smash product $X \wedge Y$ is simply $X \times Y / X \vee Y$, so the claim for the smash product follows from above.