Complex Analysis Mid-term Exam

March 20, 2014

Don't forget to write down clearly your

Name: ____________________ and **ID number:** ____________________

Instructions.

- The exam book contains 5 basic problems, worth 100 points, and an extra-credit problem of 10 points.
- The total time for the exam is 1.5 hours.
- No books, notes or calculators are allowed.
- Read the following story before opening the exam book!

  A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train.

  "Aha," says the engineer, "I see that Scottish sheep are black."

  "Hmm," says the physicist, "You mean that some Scottish sheep are black."

  "No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

  So, please be precise with your answers just as the mathematician in the story!

- Good luck with the exam, and make sure you enjoy spring break afterwards!

1. True or False (20 points). Mark “T” (True) in front of a correct statement and “F” (False) in front of a wrong one. No justification needed.

(1) ____ The equation $x^3 + 1 = 0$ has three distinct solutions over $\mathbb{C}$.

(2) ____ Any complex-valued smooth function on $\mathbb{C}$ is complex analytic.

(3) ____ The function $f(z) = \sqrt[3]{z}$ is well-defined on $\mathbb{C}$.

(4) ____ The map $f(z) = \overline{z}$ is conformal from $\mathbb{C}$ to $\mathbb{C}$.

(5) ____ If $u(x, y)$ is smooth, then its value on the center of a disk equals its average on the boundary of the disk.

(6) ____ If $f(z)$ and $g(z)$ are both analytic on a domain D, then their difference $f(z) - g(z)$ is also analytic on the domain.

(7) ____ The function $u(x, y) = x^2 - y^2$ is harmonic on $\mathbb{C}$.

(8) ____ Any harmonic function defined on the unit disk $\{|z| < 1\}$ has a harmonic conjugate on the unit disk.

(9) ____ Every closed differential one-form on $\mathbb{C}\backslash\{0\}$ is exact.

(10) ____ There is a complex analytic function $F(z)$ on $\{z|0 < |z| < 1\}$ such that $F'(z) = 1/z$.

2. Analytic functions (20 points). Let $h(w)$ be a continuous function defined on the unit circle $\{|w| = 1\}$, and define

$$H(z) := \oint_{|w|=1} \frac{h(w)}{z - w} dw.$$

Show that $H(z)$ is an analytic function in the domain $D := \{z||z| > 1\}$, and find its complex derivative in terms of an integral involving $h(w)$.

3. Integral evaluations (20 points) Find the following integrals in your favorite way.

(a)

$$\oint_{\partial D} y dx,$$

where ∂D is the boundary of the half disk $D := \{z | |z| \leq 1, \ \mathrm{Im}(z) \geq 0\}$ oriented counter-clockwise.

(b)

$$\oint_{|z|=1} \frac{\cos z}{z^5} dz.$$

4. Harmonic Functions (20 points). Recall the differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(a) Prove that, for any smooth complex-valued function $f(z)$ on $\mathbb{C}$,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4\frac{\partial^2 f}{\partial z \partial \overline{z}}.$$

(b) Show that, if $f(z)$ is a complex-valued harmonic function on a domain D, and $zf(z)$ is also harmonic there, then $f(z)$ is analytic on D.

5. Cauchy's theorem (20 points). Consider a complex-valued function $f(z)$ defined on neighborhood containing a bounded domain D, whose boundary ∂D is piecewise smooth.

(a) Show that $f(z)$ is complex analytic on D if and only if the complex-valued differential one-form $f(z)dz$ is closed.

(b) Prove Cauchy's theorem

$$\oint_{\partial D} f(z)dz = 0.$$

6. Extra Credit Problem (10 points). Show that, if an analytic $f(z)$ on a domain D has its image inside either a circle or straight line of $\mathbb{C}$, then $f(z)$ is a constant-valued function.