Exercises for Week 11

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Nov. 20.

Reading. Read Artin Sections 6.8-6.9, 7.1.

1. Consider the $GL(n, \mathbb{R}) \times GL(m, \mathbb{R})$ action on $n \times m$ matrices $M_{n,m}$ given by

$$(GL(n) \times GL(m)) \times M_{n,m} \to M_{n,m}, \quad ((P, Q), M) \mapsto PMQ^{-1}.$$

   (a) Show that this is a group action.
   (b) Assume $n \leq m$. Describe all the orbits. (Hint: Review Gaussian elimination.)
   (c) Find the stablizer of the element $(I_n|0)_{n,m}$

2. Show that $D_3 \cong S_3$. (Hint: Consider the D_3 action on the three vertices of a equilateral triangle. Find out where to send the rotation and reflection generators of D_3).

3. Let G be a finite group and H a subgroup. Describe all the H-orbits, where H acts on G by left translation.

4. Let G be a group and H a subgroup. Consider the G action on G/H as we described in class. Find $\mathrm{Z}_G(gH)$.

5. In this exercise, we will give a geometric description of $SO(3)/SO(2)$, where we identify

$$SO(2) \cong \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3) \middle| \theta \in [0, 2\pi) \right\}.$$

   (a) Let $S^2 \subset \mathbb{R}^3$ be the set of all vectors of unit length (the unit sphere). Prove that $SO(3)$ acts transitively on S^2.
   (b) Find the stablizer of the vector $e_3 = (0, 0, 1)$.
   (c) Conclude that $SO(3)/SO(2) \cong S^2$.

6. This is a generalization of a problem we worked on before. Show that if G is a finite group, and H is a subgroup whose index $[G : H] = p$ is a smallest prime divisor of $|G|$, then H is normal in G. (Hint: Consider the G action on G/H. This gives rises to a group homomorphism $G \to \mathrm{Perm}_{G/H} \cong S_p$. Analyze the kernel of this homomorphism).