Exercises for Week 4

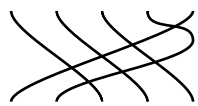
The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 2.

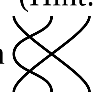
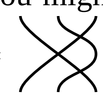
Reading. With Dummit and Foote, please read Section 1.6, 3.1, 3.2. Alternatively, read Artin 2.8, 2.5.

1. If G is a finite group, use Lagrange's theorem $|G| = [G : K]|K|$ to show the more general version: if $K \subset H \subset G$ are inclusions of subgroups, then $[G : K] = [G : H][H : K]$.

2. Prove that if H, K are finite subgroups of a group G whose orders are coprime, then $K \cap H = \{1_G\}$.

3. Find the minimal coset representative in S_5/S_4 of the element

    .

   Here by "minimal" we mean it should have the minimal number of crossings in its left coset, as we did in class. (Hint: You might find it useful that, locally, you can wiggle pictures using the relation  $=$ )

4. Recall we have shown in class that $\mathbb{Z}/(p)^* := \mathbb{Z}/(p)\backslash\{0\}$ is an abelian group of order $p-1$ under multiplication of modular numbers. Use this fact to show Fermat's little theorem: for any integer $a \in \mathbb{Z}$, $a^p \equiv a \pmod{p}$. (Hint: check that if $(a, p) = 1$, $a^{p-1} \equiv 1 \pmod{p}$. To do this remember what we have said about orders of elements in a finite group).

5. In class we discussed that picking any element $x \in G$ uniquely determines a group homomorphism $\mathbb{Z} \longrightarrow G$. Consider in $SO(2, \mathbb{R})$, and pick

   $$x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   Here θ is a fixed angle. Write down explicitly the group homomorphism determined by this element. When is this homomorphism injective?

6. Let K be a subgroup of a given group G.
   (a) Show that for a fixed $g \in G$, $gKg^{-1} := \{gkg^{-1} | k \in K\}$ is a subgroup of G.
   (b) Show that $|K| = |gKg^{-1}|$.
   (c) Show that if G has only one subgroup K of order n, then K is normal.

7. Determine under what condition the inverse map $(-)^{-1} : G \longrightarrow G, g \mapsto g^{-1}$ is a group homomorphism.