On the axioms of module algebras over Hopf algebras

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March 18, 2022

Abstract

The axiom of an H-module algebras can be simplified into a single one.

Let $\Bbbk$ be a commutative ground ring with unity, and let H be a Hopf algebra over $\Bbbk$, whose comultiplication, counit and antipode will be denoted Δ, ϵ and S respectively. We will adopt Sweedler's notation that, for any $h \in H$, $\Delta(h) = \sum_h h_1 \otimes h_2$, $(\Delta \otimes \mathrm{Id})(\Delta(h)) = (\mathrm{Id} \otimes \Delta)(\Delta(h)) = \sum_h h_1 \otimes h_2 \otimes h_3$ and so on. The Hopf algebra axioms include the following compatibility condition among multiplication, comultiplication, antipode and counit: $\sum_h h_1 S(h_2) = \epsilon(h) = \sum_h S(h_1) h_2$.

The notion of an *H-module algebra* is classical, and can be found, for instance, in [Mon93, Definition 4.1.1]. Traditionally, it is required to be a $\Bbbk$-algebra A equipped with an H-module structure

$$\cdot : H \times A \longrightarrow A, \quad (h, a) \mapsto h \cdot a, \tag{1}$$

such that the following axioms are satisfied:

$$h \cdot (ab) = \sum_h (h_1 \cdot a)(h_2 \cdot b). \tag{2}$$

for any two elements $a, b \in A$; and on the unit element 1_A of A,

$$h \cdot (1_A) = \epsilon(h) 1_A. \tag{3}$$

Lemma 1. Axiom (3) follows from Axiom (2).

Proof. We compute

$$\begin{array}{rcl} h \cdot 1_A & = & (h \cdot 1_A) 1_A \\ & = & \sum_h (h_1 \cdot 1_A)(h_2 S(h_3) \cdot 1_A) \\ & = & \sum_h h_1 \cdot (1_A (S(h_2) \cdot 1_A)) \\ & = & \sum_h h_1 \cdot (S(h_2) \cdot 1_A) \\ & = & \sum_h (h_1 S(h_2)) \cdot 1_A \\ & = & \epsilon(h) 1_A. \end{array}$$

The result follows. $\square$

There are similar reductions of the axioms for an *H-comodoule algebra* (see, for instance, [Mon93, Definition 4.1.2]) into a single one via the equivalence of H-comodules and rational H^*-modules.

References

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

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