Linear Algebra and Matrix Theory Midterm Exam

October 14, 2014

Don't forget to write down clearly your

Name: ____________________ and **ID number:** ____________________

Instructions.

- The exam book contains 5 basic problems, worth 100 points, and an extra-credit problem for 10 points.
- The total time for the exam is 75 minutes.
- No books, notes or calculators are allowed.
- Read the following story before opening the exam book!

  A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train.

  "Aha," says the engineer, "I see that Scottish sheep are black."

  "Hmm," says the physicist, "You mean that some Scottish sheep are black."

  "No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

  So, please be precise with your answers just as the mathematician in the story!

- Good luck with the exam!

1. True or False (20 points). Mark "T" (True) in front of a correct statement and "F" (False) in front of a wrong one. No justification needed.

(1) ____ The set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is a field.

(2) ____ If U_1, U_2 are two subspaces of a vector space V, then so is $U_1 \cap U_2$ a subspace of V.

(3) ____ Every generating set for a finite dimensional vector space is finite.

(4) ____ If v is a non-zero vector of a vector space, then the set $\{v\}$ is linearly independent.

(5) ____ The dimension of the space $P_5(\mathbb{F})$ of polynomials of degree less or equal to 5 is 5.

(6) ____ Every finite dimensional vector space has finitely many different bases.

(7) ____ The map $f(x, y, z) = x - y - 6z : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is linear.

(8) ____ A linear map between vector spaces is an isomorphism if and only if it is both one-to-one and onto.

(9) ____ If V and W are vector spaces of dimension 5 and 6 respectively, then any linear map $T : V \longrightarrow W$ is one-to-one.

(10) ____ The linear map $\mathrm{Tr} : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ has rank one.

2. Basis (20 points). Construct bases for the following vector spaces as required.

(a) Find two different bases of the space $P_2(\mathbb{R})$ of polynomials of degree less or equal to two (10 points).

(b) Find two different bases for the vector space of all 2×2 upper triangular matrices over a field (10 points):

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{F} \right\}.$$

3. Ordered basis and matrix (20 points) Consider the collection of 2×2 traceless matrices over a field $\mathbb{F}$

$$V := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \Big| a, b, c \in \mathbb{F} \right\}.$$

(a) Prove that

$$\beta := \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for V (5 points).

(b) In class, we have defined the transpose of a matrix of an $n \times n$ matrix. Show that if an $n \times n$ matrix A has $\mathrm{Tr}(A) = 0$, then $\mathrm{Tr}(A^t) = 0$ as well (5 points).

(c) By part (b), taking transpose defines a map on the set of 2×2 traceless matrices

$$T : V \longrightarrow V,\ A \mapsto A^t.$$

Compute the matrix of T with respect to the basis β of V given in (a).

4. Lagrangian interpolation (20 points). Consider four numbers $\{1, 2, 3, 4\} \subset \mathbb{N}$, and let $i, j \in \{1, 2, 3, 4\}$.

(a) Please find polynomials $e_i(x) \in P_3(x)$ ($i = 1, 2, 3, 4$) (i.e., they are of degree less than or equal to three), which satisfies

$$e_i(j) = \delta_{i,j},$$

where $i, j \in \{1, 2, 3, 4\}$ (10 points).

(b) Please find a degree less than or equal to three polynomial $f(x)$, which satisfy

$$f(1) = 5, \quad f(2) = 6, \quad f(3) = 7, \quad f(4) = 8.$$

You may express your answer for $f(x)$ in terms of the polynomials you have constructed in part (a) (10 points).

5. Composition of linear maps (20 points). Let $T : V \longrightarrow W$ and $U : W \longrightarrow Z$ be maps between vector spaces.

(a) Prove that, if $U \circ T$ is onto, then U is also onto (5 points).

(b) Give an example that T is **not** onto, yet $U \circ T$ still is onto (5 points).

(c) Show that if U and T are both isomorphisms, then $U \circ T$ is also an isomorphism (10 points).

6. Extra Credit Problem (10 points). Let $T_1, \ldots, T_n \in \mathcal{L}(V, W)$ be rank-one linear maps between vector spaces V and W. In class we have seen that $\mathcal{L}(V, W)$ is a vector space. Prove that, if there exist vectors $w_i \in \mathrm{Im}(T_i)$ for each $i = 1, \ldots, n$ such that $\{w_1, \ldots, w_n\}$ is a linearly independent subset of W, then $\{T_1, \ldots, T_n\} \subset \mathcal{L}(V, W)$ is linearly independent.