Research Statement of You Qi

1 Introduction

My current primary research goal is to study quantum groups at roots of unity and their representation theory via categorical methods, and, at the same time, to investigate their applications in constructing quantum topological invariants and beyond.

Quantum groups originated as symmetries of quantum integrable systems in statistical mechanics. Formalized by Drinfeld [9] and Jimbo [20], a quantum group is a Hopf algebra, depending on a formal parameter q, whose representation theory enjoys a braided monoidal structure. Quantum groups and their modules are ubiquitous in quantum physics, representation theory and low-dimensional topology. For instance, quantum groups give rise to sophisticated solutions to the highly non-linear Yang-Baxter equations [9, 20]; have surprising connections to representation theory of affine Lie algebras when q is specialized at roots of unity [21]; lead to (categorical) quantum topological invariants, such as the Witten-Reshetikhin-Turaev (WRT), the Turaev-Viro and the Henning invariants [50, 53, 18]; they give rise to examples of topological phases of matter in condensed matter physics [36].

When the parameter q is formal, a significant part of the structure and representation theory of quantum groups resembles the classical complex semisimple Lie algebras. However, when q is a root of unity, many fundamental questions about quantum groups remain unanswered, and their applications in low-dimensional topology remain largely unexplored. This statement will outline three closely interrelated projects that I plan to pursue in the next few years. The projects naturally stem out of my attempt to further understand root-of-unity quantum groups, their representations and categorification, as well as their associated (categorical) Hecke algebras.

- Quantum groups at roots of unity, including their centers, and connections to the WRT topological quantum field theories (TQFT) and their non-semisimple relatives.
- Categorification of fusion products including categorical braiding of categorified anyons.
- A categorification of the WRT 3-manifold invariant into a 4-dimensional TQFT.

2 Project I: Derived center of quantum groups

Given a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ with Chevalley generators

$$\{E_i, F_i, H_i | i = 1, \ldots, r = \mathrm{rank}(\mathfrak{g})\},$$

the *quantum group* $\mathrm{U}_v(\mathfrak{g})$ over the field of fractions $\mathbb{Q}(v)$ is generated, as an associative algebra, by $\{E_i, F_i, K_i | i = 1, \ldots, r\}$, subject to the relations that deform the classical relations for $\mathrm{U}(\mathfrak{g})$, the usual universal enveloping algebra of $\mathfrak{g}$. The explicit relations can be found, for instance, in [38]. Furthermore, a $\mathbb{Z}[v^{\pm}]$ lattice $\mathrm{U}_{\mathbb{Z}[v^{\pm}]}(\mathfrak{g}) \subset \mathrm{U}_v(\mathfrak{g})$ was introduced by Lusztig, using quantum analogues of usual divided powers of the Chevalley generators. This integral lattice serves as a quantum analogue of the Kostant integral form of $\mathrm{U}(\mathfrak{g})$ over $\mathbb{Z}$.

Fix $q \in \mathbb{C}^*$ a primitive odd lth root of unity that is greater than the Coxeter number of $\mathfrak{g}$. Also choose l to be coprime to the determinant of the Cartan matrix of $\mathfrak{g}$. For such a q, Lusztig introduced the *big quantum group* as the $\mathbb{C}$-algebra

$$\mathsf{U}_q(\mathfrak{g}) := \mathrm{U}_{\mathbb{Z}[v^{\pm}]}(\mathfrak{g}) \otimes_{\mathbb{Z}[v^{\pm}]} \mathbb{C}, \quad v \mapsto q. \tag{1}$$

Both $\mathrm{U}_v(\mathfrak{g})$ and $\mathsf{U}_q(\mathfrak{g})$ have Hopf algebra structures over $\mathbb{Q}(v)$ and $\mathbb{C}$ respectively, making their representations over these fields braided monoidal categories.

The quantum group $\mathsf{U}_q(\mathfrak{g})$ is infinite-dimensional over $\mathbb{C}$. However, much of its "quantum nature" is captured by a finite dimensional Hopf subalgebra sitting inside it. The *small quantum group* $\mathsf{u}_q(\mathfrak{g})$ is the Hopf subalgebra in $\mathrm{U}_q(\mathfrak{g})$ generated by $\{E_i, F_i, K_i\}$, and they satisfy the relations $E_i^l = 0$, $F_i^l = 0$ and $K_i^l = 1$. The Hopf subalgebra $\mathsf{u}_q(\mathfrak{g})$ is the quantum analogue of the classical restricted enveloping algebra for an algebraic group over a field of prime characteristic. It captures the quantum nature of $\mathsf{U}_q(\mathfrak{g})$ because of the following quantum analogue of the Frobenius homomorphism for algebraic groups in finite characteristic, known as the *quantum Frobenius sequence*, that is introduced by Lusztig [38]:

$$0 \longrightarrow \mathsf{u}_q(\mathfrak{g}) \longrightarrow \mathsf{U}_q(\mathfrak{g}) \xrightarrow{\phi} \widehat{\mathrm{U}}(\mathfrak{g}). \tag{2}$$

Here the hat on $\mathrm{U}(\mathfrak{g})$ indicates a certain completion of the universal enveloping algebra by power series in $\mathfrak{h}$. The finite-dimensional representation theory of $\widehat{\mathrm{U}}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g})$ are equivalent, and allows one to abuse notation and just write $\mathrm{U}(\mathfrak{g})$ in what follows.

The small quantum group $\mathsf{u}_q(\mathfrak{g})$ plays an important role in 3-dimensional topological quantum field theories (TQFTs), as shown in [50] and [18]. However, despite more than 30 years since its introduction, not much is known about the structure of small quantum groups. For instance, the commutative algebra structure or even the dimension of the center of $\mathsf{u}_q(\mathfrak{g})$ remain conjectural for almost all cases. Because of these limitations, topological applications have been restricted mostly to the case of $\mathfrak{g} = \mathfrak{sl}_2$.

Decompose the small quantum group into a direct product of indecomposable ideals, also known as *blocks*:

$$\mathsf{u}_q(\mathfrak{g}) = \prod_{\lambda} \mathsf{u}_{\lambda}. \tag{3}$$

Here λ ranges over the coset space of the extended affine Weyl group $P \rtimes W$ action on the weight lattice P of $\mathfrak{g}$. The special case of when λ equals the zero weight corresponds to the *principal block* of $\mathsf{u}_q(\mathfrak{g})$, and is usually denoted u_0.

The following problem, stated as conjectures in [33, 34] and established for small rank Lie algebras, has been a focal point in understanding the center of small quantum groups in recent years, and has attracted ample attention among experts in the representation theory community (see, for instance, [2, 5]). Before our previous work, however, even some top representation theorists were doubtful that such a beautiful connection between different branches of algebra could exist.

Conjecture 2.1. *The center $\mathsf{z}(\mathsf{u}_0)$ of the principal block u_0 of the small quantum group $\mathsf{u}_q(\mathfrak{sl}_n)$ affords an action by the Weyl group $\mathfrak{S}_n$ of $\mathfrak{sl}_n$. As a representation, there is an isomorphism of the block center*

$$\mathsf{z}(\mathsf{u}_0) \cong \frac{\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]}{\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_+^{\mathfrak{S}_n}},$$

with the diagonal coinvariant algebra of $\mathfrak{S}_n$ as defined by Haiman [17].

We now propose a further generatlization of this problem with connections towards more recent quantum topology developments in dimension 3. Recall that, given an algebra A, the center of A can be identified canonically with the ring $\mathrm{End}_{A\otimes A^{\mathrm{op}}}(A)$, where A^{op} denotes the opposite algebra of A. Extending this definition to the derived category, one has the notion of the *Hochschild cohomology* or *derived center* of A, which is the (differential graded) commutative algebra

$$\mathrm{HH}^\bullet(A) := \mathrm{Ext}^\bullet_{A\otimes A^{\mathrm{op}}}(A). \tag{4}$$

The quantum group $\mathsf{U}_q(\mathfrak{g})$ acts on $\mathsf{u}_q(\mathfrak{g})$ by the adjoint representation. This action preserves the small quantum group and descends to an action of $\mathrm{U}(\mathfrak{g})$ on the derived center $\mathrm{HH}^\bullet(\mathsf{u}_q(\mathfrak{g}))$ via the quantum Frobenius map (2).

In [35], we have initiated the study of the next two natural problems with A. Lachowska. When $\mathfrak{g} = \mathfrak{sl}_2$, the answer is explicitly presented in the work, motivating the following questions as working problems.

Problem 2.2. *(i) Determine the* $\mathrm{U}(\mathfrak{g})$*-module structure of* $\mathrm{HH}^\bullet(\mathsf{u}_q(\mathfrak{g}))$.

(ii) When $\mathfrak{g} = \mathfrak{sl}_n$*, show that the* $\mathrm{U}(\mathfrak{sl}_n)$*-action on* $\mathrm{HH}^0(\mathsf{u}_q(\mathfrak{sl}_n)) = \mathsf{z}(\mathfrak{sl}_n)$ *is trivial.*

In particular, the second part of the problem implies that central elements of $\mathsf{u}_q(\mathfrak{sl}_n)$ arise from restricting central elements in $\mathsf{U}_q(\mathfrak{sl}_n)$ to the small quantum group. Previous results from [34] reveal that this action of $\mathsf{U}(\mathfrak{g})$ on $\mathsf{z}(\mathfrak{g})$ is nontrivial in non-simply laced types, starting in type B_2.

Problem 2.3. *Show that the* $\mathrm{U}(\mathfrak{g})$ *action preserves the block decomposition of* $\mathsf{u}_q(\mathfrak{g})$*. Determine the derived center, as a module over* $\mathrm{U}(\mathfrak{g})$*, according to the block decomposition*

$$\mathrm{HH}^\bullet(\mathsf{u}_q(\mathfrak{g})) = \prod_\lambda \mathrm{HH}^\bullet(\mathsf{u}_\lambda).$$

The right hand side of the above equation can be explicitly computed via the algorithm of [33] and [34] by machine computation. Explicitly, one identifies $\mathrm{HH}^\bullet(\mathsf{u}_\lambda)$ with the geometric $\mathbb{C}^*$-equivariant Hochschild cohomology

$$\mathrm{HH}^\bullet(\mathsf{u}_\lambda) \cong \mathrm{HH}_{\mathbb{C}^*}(T^*(G/P_\lambda)), \tag{5}$$

similar to the main results of [3] for the principal block. Here the P_λ is a parabolic group of G whose Weyl group of its Levi subgroup stablizes λ. The algorithm then reduces the problem to a classical Lie algebra cohomology computation.

Via equation (5), the derived centers of small quantum groups carry interesting Gerstenhaber algebra structures arising from the geometry of the symplectic/Poisson varieties $T^*(G/P_\lambda)$'s.

Problem 2.4. *Determine the Gerstenhaber bracket on* $\mathrm{HH}^\bullet(\mathsf{u}_q(\mathfrak{g}))$.

As with the usual center, we expect determining the derived center of the small quantum group will have quantum topological applications. They should, optimistically, parametrize certain non-semisimple 3d TQFTs, just as the usual central elements do in semisimplified Witten-Reshetikhin-Turaev (WRT) 3d TQFTs (see [39]). A lot of attention has been directed over the last few years to the physical and practical importance of these 3d WRT TQFTs, their variants and non-semisimple generatlizations [15, 16]. Physically, the Turaev-Viro invariant has been shown to be connected to the Levin-Wen model in condensed matter physics and topological phases of matter [28, 31]. In another direction, 3d TQFTs from non-semisimple categories of representations associated to semi-restricted and unrolled quantum groups have been constructed [4, 14]. Such non-semisimple TQFTs use special central elements called the *integrals* of small quantum groups.

Along with Lachowska, we studied in [35] a new explicit family of central elements which constitute an ideal known as the *Higman ideal* in the derived center of a small quantum group. The ideal consists of central elements that are contained in projective summands inside the Hopf adjoint representation of the small quantum group $\mathsf{u}_q(\mathfrak{g})$. In the decomposition (3), the ideal intersects each block in a one-dimensional subspace. Lachowska has previously shown in [32] that there is an analogue of the celebrated *Verlinde algebra* structure on projective characters of $\mathsf{u}_q(\mathfrak{g})$. The following problem, therefore, connects the study of small quantum group derived centers with that of WRT TQFTs and their recent non-semisimple generalizations, and should play a fundamental role in the representation theoretic understanding of such field theories.

Problem 2.5. *(i) Exhibit a Verlinde type product on the Higman ideal of* $\mathsf{u}_q(\mathfrak{g})$*.*

(ii) Utilize the derived center and the Higman ideal of small quantum groups to provide a represenation theoretical explanation of generalized WRT TQFTs.

This problem amounts to finding a Higman ideal analogue of the Kirby color utilized in the original WRT construction, and proving its invariance under two Kirby moves. A categorical analogue of this question will also be addressed in Project III below.

3 Project II: Categorifying fusion products

As the original motivation for categorification and arguably one of the most important problems in quantum topology, Crane and Frenkel [8] proposed to construct computationally accessible 4d TQFTs by lifting combinatorially defined WRT 3d TQFTs. A key ingredient of WRT 3d TQFTs is the input of a *fusion product*, which, for quantum groups at roots of unity, arise from tensor products of tilting modules modulo negligible morphisms. I will outline in the next two projects some approaches to attack the Crane-Frenkel conjecture by categorifying the 3d TQFTs in Project I.

Denote by $V_r^{\mathbb{Z}[v^\pm]}$ the $(r+1)$-dimensional irreducible representation of $\mathrm{U}_v(\mathfrak{sl}_2)$ over $\mathbb{Q}(v)$. The representation, just like the quantum group, has a $\mathbb{Z}[v^\pm]$-lattice, allowing one to base change from the Laurent polynomial ring into $\mathbb{C}$:

$$V_r := V_r^{\mathbb{Z}[v^\pm]} \otimes_{\mathbb{Z}[v^\pm]} \mathbb{C}, \quad v \mapsto q. \tag{6}$$

These modules are called *Weyl modules* over $\mathsf{U}_q(\mathfrak{sl}_2)$. In particular, V_1 is the deformed vector space representation of $\mathfrak{sl}_2$. A *tilting module* for $\mathsf{U}_q(\mathfrak{sl}_2)$ is a direct summand of $V_1^{\otimes r}$ for some $r \in \mathbb{N}$. In particular, when $0 \leq r \leq l-1$, all modules V_r appears as summands of $V_1^{\otimes r}$, and thus are tilting modules. The rest of tilting modules not contained in $\{V_0, \ldots, V_{l-1}\}$ are called *negligible tilting modules.*

The *fusion ring* of $\mathsf{U}_q(\mathfrak{sl}_2)$ is the additive monoidal category generated by taking direct sums, tensor products and direct summands of tilting modules, modulo the ideal generated by negligible tilting modules. It is equipped with the quotient tensor product called the *fusion product.*

An important feature of this ring is that it is semisimple, and carries a projective $\mathrm{SL}_2(\mathbb{Z})$-action, generated by the S and T matrices:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The fusion ring has been uniformly defined by Andersen [1], in a similar manner, for any big quantum group $\mathsf{U}_q(\mathfrak{g})$ when q is a primitive root of unity as in the previous subsection.

In order to construct a categorical analogue of the fusion ring, a first step is to categorify $V_r \otimes V_s$ and $V_1^{\otimes r}$. In [25, 46], categorifications of such tensor product representations over $\mathsf{U}_q(\mathfrak{sl}_2)$ were constructed, which will be briefly recalled here.

Let $\Bbbk$ be a field of finite characteristic p. A *p-differential graded (p-DG) algebra* A is a graded $\Bbbk$-algebra $A = \oplus_{k\in\mathbb{Z}} A^k$, equipped with a degree-one endomorphism $\partial : A^k \longrightarrow A^{k+1}$, such that $\partial(ab) = \partial(a)b + a\partial(b)$, $\partial^p(a) = 0$ holds for any $a, b \in A$.

The study of the seemingly exotic homological object was introduced by Khovanov [23], who coined the terminology "*hopfological algebra*", in relation to categorification at prime roots of unity. It was further extended by the my collaborators and I in [40, 41, 42]. Many familiar homological properties of usual differential graded algebras can be incorporated and generalized in the framework of hopfological algebra. In particular, to a p-DG algebra A, there is the notion of p-DG modules over A. The collection of all p-DG modules over A constitutes an abelian category, denoted (A, ∂)-mod. One can then pass to the homotopy category $\mathcal{C}(A, \partial)$ and the derived category $\mathcal{D}(A, \partial)$, which are triangulated, just as for usual DG algebras. The fact that the hopfological machinery plays an important role in categorification at roots of unity is due to the following foundation result ([23, 40]).

Theorem 3.1. *Given a p-DG algebra A, the Grothendieck groups $K_0(\mathcal{C}(A, \partial))$ and $K_0(\mathcal{D}(A, \partial))$ of the triangulated categories $\mathcal{C}(A, \partial)$ and $\mathcal{D}(A, \partial)$ are modules over the cyclotomic ring of integers*

$$\mathcal{O}_p = \mathbb{Z}[q]/(1 + q + \cdots + q^{p-1}).$$

In the joint works [11, 12, 24], the quantum groups $\mathsf{u}_q(\mathfrak{sl}_2)$ and $\mathsf{U}_q(\mathfrak{sl}_2)$ were categorified by enhancing Lauda's generic categorification $\mathcal{U}$ of $\mathrm{U}_v(\mathfrak{sl}_2)$ into a p-DG category $(\mathcal{U}, \partial)$. Here the multiplication by $q \in \mathcal{O}_p$ is obtained as decategorification of the grading shift functor on the p-DG categories. This resolves parts of the conjectures raised in [8] on categorifying quantum $\mathfrak{sl}_2$ at roots of unity. Furthermore, in [25, 46], the following categorical tensor product modules are realized, building on the generic categorification work of Hu and Mathas via *quiver Schur algebras* [19] and Webster [54].

Theorem 3.2. *(i) There are p-DG algebras $(S_{r,s}, \partial)$ and (S_r, ∂), known as (truncated) p-DG quiver Schur algebras, whose p-DG Grothendieck groups are isomorphic to the $\mathcal{O}_p$-modules*

$$K_0(\mathcal{D}(S_{r,s}, \partial)) \cong V_r \otimes V_s \qquad K_0(\mathcal{D}(S_r, \partial)) \cong V_1^{\otimes r}.$$

*(ii) There is a natural categorical action by $(\mathcal{U}, \partial)$-*mod *on $(S_{r,s}, \partial)$-*mod *and (S_r, ∂)-*mod*, which, after passing to the Grothendieck group, exhibit the action of $\mathsf{U}_q(\mathfrak{sl}_2)$ on $V_r \otimes V_s$ and $V_1^{\otimes r}$.*

The following problem constitutes the categorical core of Crane-Frenkel's conjecture.

Problem 3.3. *(i) Exhibit a categorical braid group action on $\mathcal{D}(S_r, \partial)$.*

(ii) Construct categorical automorphism functors $\mathcal{S}$ and $\mathcal{T}$ on $\oplus_{r,s}\mathcal{D}(S_{r,s}, \partial)$ that induce the S and T matrix action on $\oplus_r V_r$.

Here the categorical analogue $\mathcal{S}$ generator of the modular group should come from computing the matrix of Hopf links "colored" by $\mathcal{D}(S_{r,s}, \partial)$ categorfying $V_r \otimes V_s$. The $\mathcal{T}$ generator arises from categorical framing corrections of partially closing braid functors.

The derived categories $\mathcal{D}(S_r, \partial)$ decompose into a direct sum $\mathcal{D}(S_r, \partial) = \oplus_{i=0}^{n}\mathcal{D}(S_r, \partial)_i$. The simplest nontrivial case, $\mathcal{D}(S_r, \partial)_1$ has been studied in great detail [44, 45]. There, the second highest weight space of $V_1^{\otimes r}$ has been realized as a special case of quiver Schur algebras known as the *dual zig-zag algebra* $A_r^!$. On the derived category level, the braid group action was constructed by directly resolving, in the hopfological algebra sense, the simple p-DG bimodules over $A_r^! \otimes (A_r^!)^{\mathrm{op}}$, which are utilized as Fourier-Mukai kernels for convolution. This method, however, does not generalize readily from this relatively simple weight space to other lower weight spaces, as the complexity of simple modules for quiver Schur algebras grows quite drastically when the weight decreases.

One possible approach, as shown in [46], is that quiver Schur algebras enjoy some desirable algebraic properties, known as being *cellular quasi-hereditary algebras*. Furthermore, this structure on quiver Schur algebras is compatible with the p-differential, giving rise to examples of *p-DG cellular quasi-hereditary algebras*. In particular, there is a class of p-DG modules over S_r called *cell or standard* p-DG modules which interpolates between cofibrant (analogues of projective in the usual homological algebra) and simple p-DG modules.

Problem 3.4. *Construct reflection functors on module categories over the p-DG algebra (S_r, ∂), via convolution with p-DG standard bimodules. Show that the reflection functors give rise to a categorical braid group Br_r action on $\mathcal{D}(S_r, \partial)$.*

Another natural direction generalizing Theorem 3.2 is the following two-part problem ([25, Conjecture 10.14]).

Problem 3.5. *(i) Construct a categorification of $V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_k}$, where $r_1, \ldots, r_k \in \mathbb{N}$, and show that the braid group Br_k acts naturally on this tensor product categorification.*

(ii) Construct functors realizing the natural embedding and projection of categories, which correspond to, on the Grothendieck group level, the projections and embeddings of $\mathsf{U}_q(\mathfrak{sl}_2)$-modules, when $r_1 + \cdots + r_k = r$,

$$V_1^{\otimes r} \twoheadrightarrow V_{r_1} \otimes V_{r_2} \otimes \cdots \otimes V_{r_k} \hookrightarrow V_1^{\otimes r}. \tag{7}$$

The braiding described above is important in topological quantum computation and is a mathematical counterpart of braiding of anyons in condensed matter physics. A candidate p-DG algebra is proposed in [25]. However, in general, it is no longer a p-DG cellular quasi-hereditary unless $r_i = 1$ for all i (when all $r_i = 1$, the candidate algebra is just S_r). To deal with the general case, a plausible approach is to find a hopfological analogue of *(affine) properly stratified algebras* in the sense of Kleshchev [29]. In addition to cofibrant, simple and standard modules, this machinery will need the input of yet another class of p-DG modules, namely a p-DG analogue of *proper standard modules*. The interplay between these four classes of p-DG modules enables one, for instance, to compute the size of p-DG Grothendieck groups.

Curiously enough, in contrast to the case when the differential is absent, the Koszul dual of the quiver Schur algebra S_r, over a field of characteristic 2, carries an A_∞-algebra structure that has infinitely many nonzero higher differentials which arise from the Koszul dual of the 2-differential. The algebra is not formal as an A_∞-algebra, as can already be seen from the particular case of the zig-zag algebra $A_r^!$. This naturally leads to the following question.

Problem 3.6. *(i) Develop the p-DG analogue of A_∞-algebras and their homotopy cateogories.*

(ii) Show that the Koszul dual $S_r^!$ of S_r carries this infinity algebra structure coming from taking the Koszul dual of the p-differential.

I expect that many interesting objects such as $S_r^!$, with the induced infinity algebra structure, should provide analogues of certain (localized) Fukaya categories attached to unit disks with r punctures at prime roots of unity. Such infinity algebras can be regarded as p-DG analogues of the Lipshitz-Ozsváth-Thurston DG algebras in bordered Heegaard-Floer theory [37]. This is already evident, as computed by the me and collaborators [7] in the special case of the dual zig-zag algebra $A_r^!$ discussed earlier and its Koszul dual.

Another important duality phenomenon occurs already on the decategorified level of quantum group modules. As representations, the left action of $\mathsf{U}_q(\mathfrak{sl}_2)$ on $V_1^{\otimes r}$ naturally commutes with the right action by the Iwarohi-Hecke algebra for the symmetric group on r letters $\mathfrak{S}_r$. Observe that the compositions of the projections and embeddings in (7) naturally sit inside this algebra.

The Iwahori-Hecke algebra admits a combinatorial categorification by Soergel bimodules [52]. A diagrammatic description of Soergel bimodules by generators and relations for $\mathfrak{S}_r$ is given by Elias-Khovanov [10]. Denote the diagrammatic algebra by $\mathcal{SB}_r$.

Problem 3.7. *Over a field of positive characteristic p, show that a p-DG enhanced Soergel category for $\mathfrak{S}_r$, denoted $(\mathcal{SB}_r, \partial)$, categorifies the Iwahori-Hecke algebra of $\mathfrak{S}_r$ at a prime root of unity.*

Some partial progress towards this problem was given in [13], where an essentially unique p-differential is exhibited on $\mathcal{SB}_r$, giving rise to categorical Hecke algebra relations at a pth root of unity. To fully address this question, though, a p-DG analogue of (affine) cellular algebras [30] is needed, further generalizing the study of p-DG cellular algebras in [25, 46].

From the above discussion, one sees that Problem 3.7 is intimately related to Problem 3.5. The solution of this problem is a work in progress by the myself and B. Elias, where partial results are given in [13]. More generally, we expect, similar to Problem 3.5, that the development of p-DG properly stratified algebras will be utilized in establishing the more general cases.

4 Project III: Categorical WRT invariants

A family of homological invariants of links were introduced by M. Khovanov and L. Rozansky in the seminal works [22, 26, 27], now bearing the name of *(Khovanov-Rozansky)* $\mathfrak{sl}_n$ and *HOMFLYPT homology*. The $\mathfrak{sl}_n$-homology may be regarded as arising from categorical representation theory of quantum $\mathfrak{sl}_n$ or the corresponding Soergel bimodules, while the HOMFLYPT homology is then a limit theory as n tends to infinity. The $\mathfrak{sl}_n$-homology theories enjoy functoriality properties with respect to link cobordisms. However, functoriality does not hold for the HOMFLYPT homology.

A simple construction of HOMFLYPT homology was given by Khovanov using Soergel bimodules in [22] whose construction is now briefly recalled. Let $R = \Bbbk[x_1, \ldots, x_m]$ be the polynomial ring in m variables over a ground field $\Bbbk$, and R^i be the subring consisting of polynomials invariant under permuting the ith and $(i+1)$st variables. Set the degrees of each x_i to be two.

Suppose a link $L = \widehat{\beta}$ is the braid closure of a braid $\beta \in \mathrm{Br}_m$ with m strands. Then β can be written as a product of the Coxeter generators σ_i, $i = 1, \ldots, m-1$, which satisfy the *braid relations* $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$. For each i, define the complex of Soergel bimodules, known as the *Rouquier complex*

$$B_i := \left(0 \longrightarrow R \otimes_{R^i} R \xrightarrow{M} R \longrightarrow 0\right), \tag{8}$$

where M indicates the multiplication map of R. Then one can show that B_i are invertible complexes in the homotopy category of (R,R)-bimodules, and they satisfy the above braid relations:

$$B_i \otimes_R B_{i+1} \otimes_R B_i \cong B_{i+1} \otimes_R B_i \otimes_R B_{i+1}, \qquad B_i \otimes_R B_j \cong B_j \otimes_R B_i \text{ if } |i-j| > 1. \tag{9}$$

For the link $L = \widehat{\beta}$, one takes the tensor product of $B_i^{\pm 1}$ according to the presentation of β as a word in the generators $\sigma_i^{\pm 1}$, and in this way produces a chain complex of graded (R,R)-bimodules. Taking Hochschild homology of every term sitting in each homological degree, individually, as an (R,R)-bimodule, one obtains a chain complex of bigraded homology groups (Hochschild grading and the natural grading on R). The homology of this complex, denoted $\mathrm{HHH}(L)$ (taking $\mathrm{H}_\bullet$ of $\mathrm{HH}_\bullet$), can be shown to be independent of the way L is written as a braid closure $\widehat{\beta}$. This is the HOMFLYPT homology of L. The bigraded Euler characteristic of HHH recovers the HOMFLYPT polynomial of L.

In finite characteristic p, there is a p-differential graded algebra structure on R over a finite field $\Bbbk$ that induces a p-DG chain complex structure on $\mathrm{HHH}(L)$. More specifically, define a p-DG algebra structure on R by $\partial(x_i) := x_i^2$, $i = 1, \ldots, m$ and extended to R via the Leibniz rule. Then the multiplication map in (8) is a map of p-DG modules. Taking the cone of M in the hopfological algebra sense:

$$B_i' := \left(0 \longrightarrow R \otimes_{R^i} R[1] \xrightarrow{M} R \longrightarrow 0\right) \tag{10}$$

one obtains a braiding p-complex in the homotopy category of p-DG bimodules over R. Similar as above, one can define the hopfological version of Hochschild homology of (R,R)-bimodules, and the p-DG structure on a bimodule descends to a p-differential on its *hopfological Hochschild homology*.

By developing the above hopfological algebra of p-Hochschild homology Sussan and I gave a construction of a p-complex analogue of the triply graded homology theory in [48], which will be denoted as pHHH. Furthermore, the new homology theory pHHH categorifies the HOMFLYPT polynomial of links specialized at a $2p$th root of unity. Curiously, in this construction, the triply-graded structure on HHH is forced to collapse onto a bigrading on pHHH.

Problem 4.1. *Does there exist a triply-graded link invariant of* pHHH *in the homotopy category of p-complexes?*

Work in progress by myself and collaborators indicates that the answer is no to this problem. This is quite an intriguing feature of the pHHH theory compared with the original work of Khovanov-Rozansky.

Up until this point, the construction seems quite formal and parallel to Khovanov's construction, but a crucial difference is that $p\mathrm{HHH}(L)$ is quasi-isomorphic to a finite-dimensional p-complex.

Problem 4.2. *Determine the functoriality of* $p\mathrm{HHH}(L)$*.*

Finite-dimensionality of the homology theory is crucial here, as it is a basic requirement for a TQFT to be functorial. This is a major obstruction for functoriality for the usual HOMFLYPT homology.

Next, I and my collaborators utilized an observation by Cautis in [6], modified in the p-DG context. The usual Hochschild homology of R is isomorphic to the space of differential forms on $\mathrm{Spec}(R) \cong \mathbb{A}^m$. On the Hochschild homology groups, there are contraction maps with vector fields

$$x_1^n \frac{\partial}{\partial x_1} + x_2^n \frac{\partial}{\partial x_2} + \cdots + x_m^n \frac{\partial}{\partial x_m} \tag{11}$$

on $\mathbb{A}^m$ that serve as (super) differentials. It is readily seen that contractions with certain vector fields commute with the p-differential. Therefore, the sum of the p-differential and contraction operation for $n = 2$ in (11) defines a new p-DG structure on the bigraded $p\mathrm{HHH}(L)$, at the cost of further collapsing the p-differential and Hochschild homological gradings. One thus obtains a singly graded p-DG object $p\mathrm{H}(\beta)$ for a braid β, as done in [48]. The new invariant, computed for $(2, n)$ torus links, appears to be more refined than their Khovanov homologies.

Theorem 4.3. *Let L be a link presented as the closure of a braid β and p be an odd prime. The object $p\mathrm{H}(\beta)$ is independent of the choice of β giving rise to a finite-dimensional link invariant $p\mathrm{H}(L)$ whose Euler characteristic is the Jones polynomial evaluated at a $2p$th root of unity.*

Problem 4.4. *(i) Under some restrictions on n in* (11)*, show that the singly graded p-DG link homology groups are quasi-isomorphic to finite-dimensional p-complexes, and they categorify quantum $\mathfrak{sl}_n$-polynomials at prime roots of unity.*

(ii) As for the triply graded case, determine the functoriality of the singly graded theory.

For technical reasons, I have only been able to resolve the first problem above when $n = kp + 2$. The construction was given in [47]. It was shown there that it indeed categorifies the $\mathfrak{sl}_n$ polynomial at a $2p$th root of unity. When k is even, this $\mathfrak{sl}_n$-polynomial at a $2p$th root of unity is actually just the Jones polynomial at this specialization of the quantum parameter. Thus [47] provides infinitely many distinct categorifications of the Jones polynomial evaluated at a $2p$th root of unity.

Along with Sussan, Robert and Wagner, I have extended this categorification of the Jones polynomial to the colored case [43]. The procedure follows the sketch given above. The main technical difference is that Soergel bimodules are replaced by singular Soergel bimodules. Let L be a link colored by irreducible representations of the small quantum group presented as the closure of some colored braid β. Then in [43], it is shown that there is a p-DG object $p\mathrm{H}^{col}(\beta)$ giving rise to a link invariant $p\mathrm{H}^{col}(L)$.

Theorem 4.5. *There is finite-dimensional framed link invariant $p\mathrm{H}^{col}(L)$ whose Euler characteristic is the colored Jones polynomial evaluated at a $2p$th root of unity.*

Problem 4.6. *Generalize this colored construction to higher rank $\mathfrak{sl}_n$ homologies at roots of unity.*

To summarize, the previous projects and discussion in this proposal is to extend the constructions of Theorem 4.3 and 4.5 to a homological 3-manifold invariant, which would then give a constructive solution to Crane and Frenkel's conjecture.

Problem 4.7. *Utilize the colored link invariant from Theorem 4.5 to construct a categorification of the WRT* 3*-manifold invariant when the quantum parameter is a prime root of unity.*

Naively, one would take a direct sum of the homological link invariants above. A more sophisticated assembly construction, building upon the hopfological analogue of homotopy colimits, of the colored Jones homologies is needed. A categorification of the so-called *Kirby color*, that is, a certain quantum $\mathfrak{sl}_2$ representations at a root of unity, where each non-negligible tilting module appears with multiplicity equal to its quantum dimension, would be constructed as a finite homotopy colimit of p-DG colored homology groups in resolving the above problem.

In a parallel direction, the works of Robert-Wagner [51] and Queffelec-Rose-Satori [49] gave a more topological description of the $\mathfrak{sl}_n$ and HOMFLYPT homology theories. There, the maps of Soergel bimodules are replaced by topological "foams", which are interpretated as cobordisms between singular link diagrams. The above p-DG structure on Soergel bimodules naturally induces a p-DG structure on the foam category, and exhibits another parallel way to construct p-DG link homologies. This is closely connected to studying p-DG Soergel calculus of $\mathfrak{S}_m$ proposed in the previous part.

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Teaching Statement of You Qi

Throughout the more than twenty years of education I received, and personally being taught by great masters in mathematics, a question constantly comes to my mind as *why and how do we teach students mathematics*? Walking on the Grounds of University of Virginia and seeing students rushing by from class to class makes me wonder why students and their families spend so much fortune and effort on learning some classical subjects like calculus that are more than three hundred years old? Why not just learn them online from videos of lectures in such modern times?

Let me put these questions aside first and describe some of my teaching experiences, which have led to some thoughts of my own on these issues.

My first experience as an entire course instructor was at Columbia when I was a graduate student. The very first course I taught was an entry level course *College Algebra and Analytic Geometry* in the Spring of 2012. The students' backgrounds were very diverse and interesting. Some were post-back students for medical schools, some came back to school after working for a few years, while some were veterans who had just returned from battlefields! One thing in common, as the students told me, though, was that their memory of high school math, which they had left behind for several years, was just boring and abstract tricks in problem solving. With these students, I definitely needed the more hands-on way of teaching, which was the "backwards" way compared to the textbook's Theorem-Example presentation. Typically, a class of two hours consisted of me working on several examples with them, discussing with them what properties we did use in tackling the problem, keeping on until finally the some student could shout out the gist of the reasoning of the main result of the class. Then together we laughed at the old saying of Johann Wolfgang von Gothe:

> "Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different."

After graduation, I have taught many different courses at UC Berkeley, Yale, Caltech and Virginia. The courses range from basic calculus, linear algebra, abstract algebra and complex analysis, to more advanced ones like a graduate level representation theory, algebraic topology, as well as topic courses on categorification and Khovanov homology. Teaching graduate courses is usually quite enjoyable since students are motivated enough and there are many communications between me and the students, both during class and afterwards. When instructing undergraduate courses, however, I have found that many students tend to memorize techniques and formulas that help them resolving specific types of problems designed for exams, rather than trying to understand the basic principles of mathematics that underlie the techniques. Such principles are usually much simpler and more intuitive to understand, and can be applied, with a little ingenuity, to many other situations not dealt with in textbooks. When teaching these students, I usually first make sure they understand basic concepts well by throwing in simple minded quizzes during classes. Although the quiz problems are usually quite basic, they would help me telling students what the most fundamental concepts are there that they need to understand. Once they are past this step, which is like overcoming a language barrier, the students are more comfortable thinking on their own during classes, which I tend to make slightly different from textbooks. I usually like to present the mathematical reasoning while adding some mathematical history and anecdotes to make students understand it is in fact a lively subject. After lectures, when contrasted with the textbook, students then realize that they can really understand the textbook rather than memorizing the rules, and then start to appreciate the simplicity and beauty of the subjects.

While teaching at Yale, I also had the pleasure of working with many truly talented students and communicating with them. One of them was Alexandros Musatov, who was an excellent student at Yale interested in mathematics and physics. In his last year at Yale, Alex wrote his senior thesis on Witten-Reshetikhin-Turaev (WRT) topological field theories under my guidance. We had weekly meetings for an hour over the academic year, and I guided him through reading textbooks on Hopf algebras, quantum groups and how to construct three-manifold invariants out of quantum groups. After he had understood the basics, we moved on to understanding more specific Kuperberg and Hennings' three-manifold invariants and compared them with those arising from WRT. Alex wrote a clear and concise summary of his readings as a thesis, and his work has already helped other students that I had taught approaching the subject more easily. After graduation with distinction from Yale, Alex went on to study solid state physics at Stanford. He recently told me that, to his delight, his study with me on topological field theories had already helped him in theoretical aspects of his study at Stanford. I am also very delighted to learn that I have contributed to this talented student's passion for exploring theoretical physics and mathematics. Similarly, I had also directed Andrew Salmon on studying representation theory at Yale, who later started his PhD program at MIT. At University of Virginia, I mentored a master degree student David Winters, who was a talented student interested in mathematics and theoretical physics, on topological quantum field theories in dimensions 3 and 4. David went on to continue his PhD studies at Georgia Tech in mathematics.

After these experiences and connections with such a diverse body of students, I am starting to appreciate the importance of college education as an apprenticeship rather than mechanical information passing-on. Mathematics, usually only a small part in most students' curriculum, is not only a useful subject of knowledge, but also a logical and artistic way of thinking. The knowledge may fade away as time goes by, but the capability of facing problems without fear and resolving them rationally and coherently will definitely benefit students in more than one way in their lives later on. This latter virtue, which is probably far more important, is not to be found in textbooks. It can only be imprinted on one from communicating with professors and watching them practicing the cycle of learning a problem, getting puzzled about it, thinking deeper and tackling the problem, and of course exercising this cycle again and again by the student him/herself. I believe my experience at Columbia, Berkeley, Yale, Caltech and University of Virginia has deeply reshaped my understanding about teaching, and will keep helping me become a better and better teacher in the years to come, and my passion in mathematics will continue to make me a better teacher in the future.