Solution to the midterm

1 Induced map to suspension

It suffices to construct $h^*(X) \to H^{*+1}(\Sigma X)$ and show that this is an isomorphism. ($\because$ by iterating this isomorphism we can get the desired map.) We know that $\Sigma X \simeq SX = CX/X$, so let's apply the LES associated to the pair (CX, X) for h:

$$\cdots \to h^k(\Sigma X) \to h^k(CX) \to h^k(X) \xrightarrow{\delta} h^{k+1}(\Sigma X) \to \cdots$$

Using the fact that a cone of a space is contractible, we obtain that $h^k(CX) = 0, \forall k$. Thus by the LES above, we get the desired isomorphism $h^k(X) \to h^{k+1}(\Sigma X)$, which is the coboundary map δ for the pair (CX, X). Now it is clear from the naturality of the coboundary map (note that this is in the axioms for reduced cohomology theories) and the fact that the homotopy $\Sigma X \simeq CX/X$ is natural that this isomorphism enjoys the desired naturality property. More precisely, any map $f : X \to Y$ induces $Cf : CX \to CY$, which can be thought of as a map between pairs $Cf : (CX, X) \to (CY, Y)$. and by quotienting X and Y in CX and CY resp., it induces $Sf : SX \to SY$. By quotienting out the segment $\{pt\}\times I$, we obtain the map $\Sigma f : \Sigma X \to \Sigma Y$, and by the construction we have the commutative diagrams

$$\begin{tikzcd}
(CX, X) \arrow[r] \arrow[d, "f"'] & (SX, *) \arrow[r] \arrow[d, "f"'] & (\Sigma X, *) \arrow[d, "f"'] \\
(CY, Y) \arrow[r] & (SY, *) \arrow[r] & (\Sigma Y, *)
\end{tikzcd}$$

Thus the naturality of the coboundary map δ means

$$\begin{tikzcd}
h^*(X) \arrow[r, "\delta"] & h^{*+1}(\Sigma X) \\
h^*(Y) \arrow[r, "\delta"] \arrow[u, "f^*"] & h^{*+1}(\Sigma Y) \arrow[u, "(\Sigma f)^*"']
\end{tikzcd}$$

commutes, which is the desired naturality property.

2 Universal Coefficient Theorem over a field $\Bbbk$

(a) The key fact is that $\text{Ext}_{\Bbbk}(-,-) = 0$ for any field $\Bbbk$, hence in the proof of the UCT as in Hatcher, we obtain the exact sequence

$$0 \to 0 \to H^n(C) \to \text{Hom}(H_n(C), \Bbbk) \to 0.$$

If the homology groups (hence cohomology groups) consist of finite dimensional vector spaces in each dimensions, then the double dual coincides with the original vector space, allowing the identification $\text{Hom}(\text{Hom}(H_n(C), \Bbbk), \Bbbk) \cong H_n(C)$. Thus $H_n(C) \cong \text{Hom}(H^n(C), \Bbbk)$ in this case. (In general, the double dual of a vector space may not be the same as the vector space, so we cannot say that the homology groups are dual of cohomology groups.)

(b) Applying the proof of Theorem 2.44 in Hatcher to our situation, we can see that $\dim_{\Bbbk} C_n = \dim_{\Bbbk} Z_n + \dim_{\Bbbk} B_{n-1}$ and $\dim_{\Bbbk} Z_n = \dim_{\Bbbk} B_n + \dim_{\Bbbk} H_n$ hold. Thus, $\chi(X) = \sum_m (-1)^m \dim_{\Bbbk} H_m(X; \Bbbk) = \sum_m (-1)^m \dim_{\Bbbk} H^m(X; \Bbbk)$ where the second equality is obtained from part (a).

3 maps between real projective spaces

This essentially follows from Problem 4 of Homework 2. As $n > m$, they cannot induce a nontrivial map on $H^1(-; \mathbb{Z}/2)$. But the cohomology ring of real projective spaces are generated by a single element in the first cohomology group. Hence, letting $H^*(\mathbb{R}P^m; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x]/(x^{m+1})$, $f^* : H^k(\mathbb{R}P^m; \mathbb{Z}/2) \to H^k(\mathbb{R}P^n; \mathbb{Z}/2)$ maps 0 to 0 and x^k to $f^*(x^k) = f^*(x)^k = 0$, which means f^* is zero in every positive dimensions.

4 Excersie 3.3.6, Hatcher

(a) Note that $M_1 \# M_2$ is a union of two open subsets $U_1 \cup U_2$, where U_i is essentially M_i but removing even smaller open ball along the connected sum region. From the definition, $U_1 \cap U_2 \cong S^{n-1} \times \mathbb{R}$. Thus, consider the Mayer-Vietoris sequence associated to the decomposition $M_1 \# M_2 = U_1 \cup U_2$:

$$\cdots \to \tilde{H}_k(S^{n-1} \times \mathbb{R}) \to \tilde{H}_k(U_1) \oplus \tilde{H}_k(U_2) \to \tilde{H}_k(M_1 \# M_2) \to \tilde{H}_{k-1}(S^{n-1} \times \mathbb{R}) \to \cdots$$

Now, compare the homology groups of U_i to that of M_i. To be more precise, consider the Mayer-Vietoris sequence for $M = (M \setminus \overset{\circ}{B^n}) \cup B^n$:

$$\cdots \to H_k(S^{n-1}) \to H_k(M \setminus \overset{\circ}{B^n}) \oplus H_k(B^n) \to H_k(M) \to H_{k-1}(S^{n-1}) \to \cdots$$

As B^n is contractible, we can reduce the sequence above as

$$\cdots \to \tilde{H}_k(S^{n-1}) \to \tilde{H}_k(M \setminus \overset{\circ}{B^n}) \to \tilde{H}_k(M) \to \tilde{H}_{k-1}(S^{n-1}) \to \cdots$$

Now, $\tilde{H}_k(S^{n-1})$ is nonzero only when $k = n-1$. Hence we obtain the equality $\tilde{H}_k(M \setminus \overset{\circ}{B^n}) \cong \tilde{H}_k(M)$ for $k \neq n, n-1$. The nontrivial part of the above LES is

$$0 \to \tilde{H}_n(M \setminus \overset{\circ}{B^n}) \to \tilde{H}_n(M) \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(M \setminus \overset{\circ}{B^n}) \to \tilde{H}_{n-1}(M) \to 0.$$

If M were orientable, then the map $\tilde{H}_n(M) \to \tilde{H}_{n-1}(S^{n-1})$ is an isomorphism, as ti coincides with

$$\tilde{H}_n(M) \overset{\cong}{\to} H_n(M, M \setminus \{pt\}) \overset{\cong}{\to} H_n(M, M \setminus \overset{\circ}{B^n}) \overset{\cong}{\to} H_n(B^n, B^n \setminus \overset{\circ}{B^n}) \cong \tilde{H}_{n-1}(S^{n-1}).$$

Hence the last nontrivial map in the 5-term exact sequence above is an isomorphism, and it is clear that $\tilde{H}_n(M \setminus \overset{\circ}{B^n}) = 0$. On the other hand, if M were nonorientable, then $\tilde{H}_n(M) = 0$, which gives a SES $0 \to \mathbb{Z} \to \tilde{H}_{n-1}(M \setminus \overset{\circ}{B^n}) \to \tilde{H}_{n-1}(M) \to 0$. Therefore,

$$\tilde{H}_k(U_i) = \begin{cases} \tilde{H}_k(M_i) & \text{for } k \neq n, n-1 \\ 0 & \text{for } k = n \\ \tilde{H}_k(M_i) & \text{for } k = n-1, M \text{ is orientable} \\ \text{determined from } 0 \to \mathbb{Z} \to \tilde{H}_{n-1}(U_i) \to \tilde{H}_{n-1}(M_i) \to 0 & \text{for } k = n-1, M \text{ is nonorientable} \end{cases}$$

Returning to the Mayer-Vietoris sequence associated to $M_1 \# M_2 = U_1 \cup U_2$, we know from the deformation retract $S^{n-1} \times \mathbb{R} \simeq S^{n-1}$ that $\tilde{H}_k(S^{n-1} \times \mathbb{R}) \cong 0$ except for $k = n-1$. Hence, we obtain the equality $\tilde{H}_k(M_1 \# M_2) \cong \tilde{H}_k(M_1) \oplus \tilde{H}_k(M_2)$ for $k \neq n, n-1$. Now, the nontrivial part of the LES is as follows:

$$0 \to \tilde{H}_n(M_1 \# M_2) \to \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2) \to \tilde{H}_{n-1}(M_1 \# M_2) \to 0.$$

When both of M_1, M_2 are orientable, then from the previous argument regarding the homology of U_i we know that the inclusion induces a trivial map $\tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(U_i)$. ($\because$ $\tilde{H}_n(M) \to \tilde{H}_{n-1}(S^{n-1})$ is an isomorphism.) Thus, the midle map on the 4-term exact sequence above is zero, which gives the isomorphism for $k = n-1$.

When only one of M_1 or M_2 is orientable, let us assume without loss of generality that M_1 is nonorientable. Then the part $\tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2)$ is identified with the sum of two maps $\tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(U_1)$ and $0 \to \tilde{H}_{n-1}(U_2)$, which are a part of the SES $0 \to \mathbb{Z} \to \tilde{H}_{n-1}(U_1) \to \tilde{H}_{n-1}(M_1) \to 0$ and $0 \to 0 \to \tilde{H}_{n-1}(U_2) \to \tilde{H}_{n-1}(M_2) \to 0$, resp. Note that we have the map $\tilde{H}_{n-1}(M_1 \# M_2) \to \tilde{H}_{n-1}(M_1) \oplus \tilde{H}_{n-1}(M_2)$ which is obtained by collapsing M_2 part and

M_1 part, resp., which makes the diagram

$$\begin{tikzcd}
\tilde{H}_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\cong"'] & \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2) \arrow[r] \arrow[d, "\cong"'] & \tilde{H}_{n-1}(M_1 \# M_2) \arrow[r] \arrow[d] & 0 \\
\tilde{H}_{n-1}(S^{n-1}) \arrow[r] & \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2) \arrow[r] & \tilde{H}_{n-1}(M_1) \oplus \tilde{H}_{n-1}(M_2) \arrow[r] & 0
\end{tikzcd}$$

commutes. Thus, from the five lemma, we obtain the equality $\tilde{H}_{n-1}(M_1 \# M_2) \cong \tilde{H}_{n-1}(M_1) \oplus \tilde{H}_{n-1}(M_2)$.

Finally, if both of M_1 and M_2 are nonorientable, then by adding one more $\tilde{H}_{n-1}(S^{n-1})$ on the above commutative diagram, we obtain

$$\begin{tikzcd}
\tilde{H}_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "\text{diagonal}"'] & \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2) \arrow[r] \arrow[d, "\cong"'] & \tilde{H}_{n-1}(M_1 \# M_2) \arrow[r] \arrow[d] & 0 \\
\tilde{H}_{n-1}(S^{n-1}) \oplus \tilde{H}_{n-1}(S^{n-1}) \arrow[r] & \tilde{H}_{n-1}(U_1) \oplus \tilde{H}_{n-1}(U_2) \arrow[r] & \tilde{H}_{n-1}(M_1) \oplus \tilde{H}_{n-1}(M_2) \arrow[r] & 0
\end{tikzcd}$$

, and from this, we can easily obtain an exact sequence

$$0 \to \mathbb{Z} \to \tilde{H}_{n-1}(M_1 \# M_2) \to \tilde{H}_{n-1}(M_1) \oplus \tilde{H}_{n-1}(M_2) \to 0.$$

Now, from the assumption we know that the torsion subgroup of $\tilde{H}_{n-1}(M_1 \# M_2), \tilde{H}_{n-1}(M_1)$ and $\tilde{H}_{n-1}(M_2)$ are all isomorphic to $\mathbb{Z}/2$ (as they are nonorientable). Thus, writing each homology groups in terms of free-torsion decomposition, we obtain

$$0 \to \mathbb{Z} \to \mathbb{Z}/2 \oplus \mathbb{Z}^{\text{rank } H_{n-1}(M_1 \# M_2)} \to (\mathbb{Z}/2)^2 \oplus \mathbb{Z}^{\text{rank } H_{n-1}(M_1) + \text{rank } H_{n-1}(M_2)} \to 0.$$

Thus, from the rank identity $1 + \text{rank } H_{n-1}(M_1) + \text{rank } H_{n-1}(M_2) = \text{rank } H_{n-1}(M_1 \# M_2)$, we obtain the desired result.

(b) Recalling the definition of the Euler characteristic,

$$\begin{aligned}
\chi(M_1 \# M_2) &= \sum_{k=0}^{n} (-1)^k \text{rank } H_k(M_1 \# M_2) \\
&= 1 + (-1)^{n-1} \text{rank } H_{n-1}(M_1 \# M_2) + (-1)^n \text{rank } H_n(M_1 \# M_2) + \sum_{k=1}^{n-2} (-1)^k \text{rank } H_k(M_1 \# M_2)
\end{aligned}$$

holds. Now, from part (a), we know that

- If at least one of M_1 or M_2 is orientable, then $\text{rank } H_k(M_1 \# M_2) = \text{rank } H_k(M_1) + \text{rank } H_k(M_2)$ holds for $k = 1, \cdots, n-1$. Observe that if both are orientable, then the top homology groups for $M_1, M_2, M_1\#M_2$ are all isomorphic to $\mathbb{Z}$, and if only one is orientable, then only one of the top homology groups for $M_1, M_2, M_1\#M_2$ is nonzero and isomorphic to $\mathbb{Z}$. This gives the identity rank $H_n(M_1\#M_2)$ = rank $H_n(M_1)$ + rank $H_n(M_2) - 1$. Inserting this identities, we obtain

$$\begin{aligned}\chi(M_1\#M_2) &= 1 + (-1)^{n-1}\text{rank } H_{n-1}(M_1\#M_2) + (-1)^n\text{rank } H_n(M_1\#M_2) + \sum_{k=1}^{n-2}(-1)^k\text{rank } H_k(M_1\#M_2) \\ &= 1 + (-1)^n(\text{rank } H_n(M_1) + \text{rank } H_n(M_2) - 1) + \sum_{k=1}^{n-1}\text{rank } H_k(M_1) + \text{rank } H_k(M_2) \\ &= \chi(M_1) + \chi(M_2) - 1 - (-1)^n = \chi(M_1) + \chi(M_2) - \chi(S^n)\end{aligned}$$

as desired.

– If both of M_1 and M_2 are nonorientable, then all the top homology groups vanish, and $1 +$ rank $H_{n-1}(M_1)$ + rank $H_{n-1}(M_2)$ = rank $H_{n-1}(M_1\#M_2)$ holds from part (a). Hence,

$$\begin{aligned}\chi(M_1\#M_2) &= 1 + (-1)^{n-1}\text{rank } H_{n-1}(M_1\#M_2) + (-1)^n\text{rank } H_n(M_1\#M_2) + \sum_{k=1}^{n-2}(-1)^k\text{rank } H_k(M_1\#M_2) \\ &= 1 + (-1)^{n-1}(\text{rank } H_{n-1}(M_1) + \text{rank } H_{n-1}(M_2) + 1) + \sum_{k=1}^{n-1}\text{rank } H_k(M_1) + \text{rank } H_k(M_2) \\ &= \chi(M_1) + \chi(M_2) - 1 - (-1)^n = \chi(M_1) + \chi(M_2) - \chi(S^n)\end{aligned}$$

as desired.

5 Excersice 3.3.26, Hatcher

First, we compute the cohomology ring of factor spaces $S^2 \times S^8$ and $S^4 \times S^6$. This is easy; we can apply the Künneth formula to have

$$H^*(S^2 \times S^8) \cong H^*(S^2) \otimes H^*(S^8), H^*(S^4 \times S^6) \cong H^*(S^4) \otimes H^*(S^6).$$

Hence $H^*(S^2 \times S^8)$ has a generator α_2 of degree 2, which must squares to 0 as $H^4(S^2 \times S^8) \cong 0$, another generator α_8 of degree 8, and their cup product $\alpha_2 \cup \alpha_8$ generates $H^{10}(S^2 \times S^8)$. Similarly, $H^*(S^4 \times S^6)$ has two generators β_4 and β_6 which square to 0 and their cup product $\beta_4 \cup \beta_6$ generates $H^{10}(S^4 \times S^6)$. Now, from the previous problem, the cohomology groups of $S^2 \times S^8\#S^4 \times S^6$ consists of $\mathbb{Z}$ on even dimensions from 0 to 10, and all others are 0. Moreover, we can identify the generators in each dimension from 2 to 8 with $\alpha_2, \beta_4, \beta_6$, and α_8, resp.
Now, let us see what the Poincare duality tells us. The nontrivial cup products which result in top dimension are $\alpha_2 \cup \alpha_8$ and $\beta_4 \cup \beta_6$. The nonsingularity of Poincare duality tells us that $\alpha_2 \cup \alpha_8$ and $\beta_4 \cup \beta_6$ generates $H^{10}(S^2 \times S^8\#S^4 \times S^6)$. Consider other cup products.

Lemma 5.1. The cup product of cohomology classes that come from a single factor space is the image of the cup product of the cohomology classes in the factor space, i.e., for the quotient map $q : M_1 \# M_2 \to M_1$,

$$q^*(\alpha) \cup q^*(\beta) = q^*(\alpha \cup \beta).$$

Proof. This is simply the naturality of the cup product. □

By this lemma, we know that the cup products of two α's or two β's are zero. Now the only pairings that might be nontrivial are $\alpha_2 \cup \beta_4$ and $\alpha_2 \cup \beta_6$. Suppose that $\alpha_2 \cup \beta_4 = k\beta_6$. Then $k\beta_4 \cup \beta_6 = \beta_4 \cup (\alpha_2 \cup \beta_4) = \alpha_2 \cup (\beta_4 \cup \beta_4) = 0$, forcing k to be 0. Similarly, for the integer k for which $\alpha_2 \cup \beta_6 = k\alpha_8$, $k\alpha_2 \cup \alpha_8 = \alpha_2 \cup (\alpha_2 \cup \beta_6) = (\alpha_2 \cup \alpha_2) \cup \beta_6 = 0$ implies $k = 0$. Thus all other cup products which cannot be dictated from the Poincare duality are 0.