Exercises for Week 2

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Sept. 18.

Reading. With Dummit and Foote, please read Sections 0.2, 0.3, 2.1, 2.3. Alternatively read Artin Sections 2.3, 2.4, 2.7.

1. Show by example for subgroups in S_3 that, if H and K are subgroups, then $H \cup K$ is not necessarily a subgroup.

2. Show that if $(A, +)$ is an abelian group and H, K are subgroups, then

$$H + K := \{x \in A | x = h + k \text{ for some } h \in H, k \in K\}$$

   is a subgroup, and it is the *smallest* subgroup that contains both H and K.

3. (a) Prove that, if $(G, \star_G)$, $(H, \star_H)$ are given two groups, then $G \times H$ with the product structure $\star$ defined by

$$(g_1, h_1) \star (g_2, h_2) := (g_1 \star_G g_2, h_1 \star_H h_2)$$

   for any $(g_1, h_1), (g_2, h_2) \in G \times H$ is a group. This is called the *product group* of G and H.

   (b) Show that $G \times \{e_H\}$ is a subgroup of $G \times H$.

   (c) Consider $G = H = (\mathbb{Z}, +)$. Construct five different subgroups of $\mathbb{Z} \times \mathbb{Z}$.

4. Perform the Euclidean algorithm to find the $\gcd$ of

   (1). 90 and 300.

   (2). 60 and 17.

5. Consider the set H of matrices

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} | a \in \mathbb{R} \right\} \subset M_2(\mathbb{R})$$

   under the usual matrix multiplication.

   (a) Prove that it is a subgroup of $GL(2, \mathbb{R})$. Is it abelian? Can you identify it with a more familiar group?

   (b) Find a countably infinite subgroup of H.

6. Consider the maps
   (1) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}, (x, y) \mapsto x^2 + y^2$.
   (2) $f : \mathbb{R} \longrightarrow \mathbb{R}, x \mapsto x^3$
   (3) $f : \mathbb{C} \longrightarrow \mathbb{C}, z \mapsto z^3$
   Determine the equivalence relation on the respective domains determined by f. Namely, explicitly describe equivalence classes coming from the map f.

7. Show that if $a = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$, where each $a_i \in \{0, 1, 2, \ldots, 9\}$. Then
$$a \equiv a_n + a_{n-1} + \cdots + a_1 + a_0 \pmod{9}.$$

8. Use the theorem on classification of subgroups of $\mathbb{Z}$ to prove that, if $a_1, a_2, \ldots, a_n \in \mathbb{Z}$, then
$$\gcd(a_1, a_2, \ldots, a_n) = \gcd(\gcd(a_1, \ldots, a_k), \gcd(a_{k+1}, \ldots, a_n))$$
for any $1 \leq k \leq n$.