Math 120, Practice final exam solutions

1. The normal vector for the plane $x+2y-z=4$ is $\langle 1,2,-1\rangle$, and the one for the plane $x-y+z=1$ is $\langle 1,-1,1\rangle$. Hence the direction vector for the common line is

$$\langle 1,2,-1\rangle \times \langle 1,-1,1\rangle = \langle 1-2,-3\rangle.$$

Then one should find a common point, one that solves both plane equations. The line of intersection is not horizontal (we can see that from the direction vector above), so it intersects the xy-plane at some point with $z=0$. Plugging that into the equations, we get $x+2y=4$ and $x-y=1$. Subtracting the two equations gives $3y=3$ or $y=1$, and then $x=2$. Our solution is $(2,1,0)$.

The vector equation is given by

$$r(t)=\langle 2,1,0\rangle + t\langle 1,-2,-3\rangle.$$

2. When $(x,y)=(1,0)$, we have $u=1$ and $v=0$.

By the chain rule,

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=\frac{1}{u-v}\cos(y)+\frac{-1}{u-v}y^2\cos(xy^2)=1$$

and

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{u-v}(-x\sin(y))+\frac{-1}{u-v}2xy\cos(xy^2)=0$$

3. Let $g(x,y,z)=x^2+y^2+z^2$. To find the absolute maximum and minimum of the function $f=x-y^2+z$ which is subject to the constraint $g(x,y,z)=1$, one should first try to find the local max/min of f by Lagrange multiplier method, i.e.

$$\nabla f=\lambda\nabla g,$$

$$g(x,y,z)=1.$$

The gradients are $\nabla f=\langle 1,-2y,1\rangle$ and $\nabla g=\langle 2x,2y,2z\rangle$. We get the following equations:

$$\begin{aligned} 1 &= 2\lambda x && (1)\\ -2y &= 2\lambda y && (2)\\ 1 &= 2\lambda z && (3)\\ x^2+y^2+z^2 &= 1 && (4) \end{aligned}$$

We can see from equation (1) that $\lambda \neq 0$, and then (1) and (3) give $x = \frac{1}{2\lambda} = z$.

Equation (2) reads $2y(\lambda + 1) = 0$, which has two solutions: $y = 0$ or $\lambda = -1$.

The first option is $y = 0$. Equation (4) becomes $x^2 + 0^2 + x^2 = 1$ so $x = z = \pm 1/\sqrt{2}$.

The second option is $\lambda = -1$. Then (1) and (3) say $x = z = -1/2$. Equation (4) says $1/4 + y^2 + 1/4 = 1$ so $y^2 = 1/2$ and $y = \pm 1/\sqrt{2}$.

Now we evaluate the function at our candidate points, and compare values:

$$\begin{aligned}
f(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}) &= \sqrt{2} \\
f(-\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}) &= -\sqrt{2} \\
f(-\frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2}) &= -\frac{3}{2} \\
f(-\frac{1}{2}, -\frac{\sqrt{2}}{2}, -\frac{1}{2}) &= -\frac{3}{2}
\end{aligned}$$

We conclude that f achieves absolute maximum $\sqrt{2}$ at $(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$ and absolute minimum $-\frac{3}{2}$ at the points $(-\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, -\frac{1}{2})$.

4. (a) By Fubini theorem,

$$\iint_R f'(x)g'(y)\mathrm{d}A = \int_a^b \int_c^d f'(x)g'(y)dydx = \int_a^b f'(x)(g(d) - g(x)) = (f(b) - f(a))(g(d) - g(c)).$$

   (b) The inequality holds because D_1 contains D_2, and the function $f(x, y) = e^{x^2} + e^{y^2}$ is positive everywhere.

5. (a) When $\theta = 0$, $r = \sin(3\theta) = 0$. Hence the graph on the right-hand side is the right one.

   (b) We can choose the region D enclosed by one leaf to be

$$0 \leq \theta \leq \pi/3,\ 0 \leq r \leq \sin(3\theta)$$

   in terms of polar coordinates. Hence the area of this region is

$$\iint_D dA = \int_0^{\pi/3} \int_0^{\sin(3\theta)} rdrd\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1 - \cos(6\theta)}{2}d\theta = \frac{\pi}{12}$$

6. We should use Stokes theorem for this problem. First we parametrize the plane which consists of points $P = (0, -1, 0), Q = (-1, 0, 2)$ and $R = (0, 1, 0)$. We have vectors $\vec{PQ} = \langle -1, 1, 2 \rangle$ and $\vec{RP} = \langle 0, -2, 0 \rangle$. Then the normal vector of this plane can be given by $\vec{PQ} \times \vec{RP} = \langle 4, 0, 2 \rangle$. So the the linear equation for the plane is $4x + 2z = 0$. Hence we can parametrize our plane by

$$\boldsymbol{r}(x, y) = (x, y, -2x),$$

where (x, y) is in the region D given by

$$D = \{(x, y)| -1 \leq x \leq 0, -x - 1 \leq y \leq x + 1\}.$$

Hence $\boldsymbol{r}_x \times \boldsymbol{r}_y = \langle 2, 0, 1 \rangle$. Note that the orientation of the plane given by this parametrization is upward. It induces counter clock-wise orientation on the curve C consisting of the three segments, which is opposite to the given one. Then by Stokes theorem,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D \mathrm{curl}(\boldsymbol{F}) \cdot \langle 2, 0, 1 \rangle dxdy,$$

where D is the region given above, and $\mathrm{curl}\boldsymbol{F} = \langle 0, -2x + cos(e^z) \cdot e^z, -2 \rangle$. Then $\mathrm{curl}(\boldsymbol{F}) \cdot (\boldsymbol{r}_x \times \boldsymbol{r}_y) = 2$. Hence

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D 2dxdy = 2A$$

where A is the area of the triangle underneath the plane, with vertices $(0, -1)$, $(0, 1)$, $(-1, 0)$. The area is $A = 1$, so $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 2$.

7. There are two solutions to this problem.

   It is quite easy to see that the vector field $\boldsymbol{F}(x, y, z)$ is conservative, since it is defined everywhere, and $Q_x = P_y$. The initial point of the given curve is $\boldsymbol{r}(0) = (0, 0)$ and the terminal point is $\boldsymbol{r}(1) = (0, -1)$.

   Solution 1. Find the potential function f, i.e. $\nabla f = \boldsymbol{F}$. One can take $f = (y + 1)x^2 + y^3(x + 1)$. Then by fundamental theorem of line integral, $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f(0, -1) - f(0, 0) = -1$.

   Solution 2. Since $\boldsymbol{F}$ is conservative, the line integral does not dependent on the choice of path. So one can replace the complicated the curve by some simpler curve C', for example the straight line from $(0, 0)$ to $(0, -1)$. The equation is given by $\boldsymbol{r}(t) = (1 - t)\langle 0, 0 \rangle + t\langle 0, -1 \rangle = \langle 0, -t \rangle$, where $0 \leq t \leq 1$. So

$$\int_C \boldsymbol{F} \cdot \boldsymbol{r} = \int_{C'} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 \left(\boldsymbol{F} \cdot \boldsymbol{r}'\right) dt = \int_0^1 \left(\langle -t^3, 3t^2 \rangle \cdot \langle 0, -1 \rangle\right) dt = \int_0^1 (-3t^2) dt = -1.$$

8. (a) i. positive. We have $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, and $\boldsymbol{F} \cdot \boldsymbol{n} = \langle 0, 0, z^5 \rangle \cdot \langle x, y, z \rangle = z^6$ is positive on the sphere.

   (b) i. 4. By Green theorem and taking into account of orientation, $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = -\iint_D (-2) dA = 2 \text{ area } (D) = 4$.

   (c) iii. zero. This is because $(\mathbf{v} \times \mathbf{w})$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

   (d) iii. lines. Setting $f(x, y) = c$ gives $y = cx - cy$ or $(1 + c)y = cx$.

   (e) i. positive. You can see that the magnitude of the field increases in the direction of the positive y axis, so there is more field coming out to the top than coming in from the bottom.

9. (a) The parametric equation for the surface D is given by

$$\boldsymbol{r}(x, \theta) = \langle x, (4 - x^2) \cos\theta, (4 - x^2) \sin\theta \rangle,$$

   where $-2 \le x \le 2$, $0 \le \theta \le 2\pi$.

   (b) Since the vector field $\boldsymbol{F} = \left\langle -\frac{z}{x^2 + y^2 + z^2}, 0, \frac{x}{x^2 + y^2 + z^2} \right\rangle$ has singularity at origin, we cannot apply divergence theorem to the solid which is enclosed by the surface.

   We need to cut out the origin. We choose unit sphere S', which is contained in S, and consider the solid E between S and S', which does not contain the origin. We choose the outward orientation on S'. Then by a general version of divergence theorem, we have

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} - \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \operatorname{div}(\boldsymbol{F}) dV.$$

   Easy to compute that $div(\boldsymbol{F}) = 0$. Hence $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S}$.

   For unit sphere, we have standard spherical coordinate,

$$\boldsymbol{r}(\phi, \theta) = \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle,$$

   where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Moreover, by computation $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \sin\phi\, \boldsymbol{r}(\phi, \theta)$ gives the outward orientation.

   Now we compute $\boldsymbol{F} \cdot (r_\phi \times \boldsymbol{r}_\theta) = -\cos\phi \sin^2\phi \cos\theta + \sin\phi \cos\theta \sin\phi \cos\phi = 0$.

   Hence

$$\iint_{S'} \boldsymbol{F} \cdot d\boldsymbol{S} = \int_0^{2\pi} \int_0^{\pi} \boldsymbol{F} \cdot (r_\phi \times \boldsymbol{r}_\theta)\, d\phi\, d\theta = 0.$$

   Note that the dot product $\langle -\cos\phi, 0, \sin\phi \cos\theta \rangle \cdot \sin\phi \boldsymbol{r}(\phi, \theta)$ is zero.

10. (a) We first compute the normal vector of the surface given by the parametrization,

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \langle \cos v, \sin v, 2\rangle \times \langle -u \sin v, u\cos v, 0\rangle = \langle -2u\cos v, -2u\sin v, u\rangle.$$

Hence $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{5}u$. Then

$$\iint_S x^2 dS = \int_0^2 \int_0^{2\pi} u^2 \cos^2 v |\boldsymbol{r}_u \times \boldsymbol{r}_v| dv du = \sqrt{5}\int_0^2 \int_0^{2\pi} u^3 (\cos v)^2 dv du = 4\sqrt{5}\pi.$$

(b) We can compute $\operatorname{curl}(\boldsymbol{F}) = \langle 0, 0, -1\rangle$. Now we can use the work done in part (a), where we have $\boldsymbol{r}_u \times \mathbf{r}_v = \langle -2u\cos v, -2u\sin v, u\rangle$. However this orientation is upward (the given orientation is downward), one should take $-\boldsymbol{r}_u \times \boldsymbol{r}_v = \langle 2u\cos v, 2u\sin v, -u\rangle$. Then

$$\iint_S \operatorname{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_0^2 \int_0^{2\pi} \langle 0,0,-1\rangle \times \langle 2u\cos v, 2u\sin v, -u\rangle\, dv du = \int_0^2 \int_0^{2\pi} u\, dv du = 4\pi.$$

11. The region E in terms of spherical coordinate is given by

$$\{(\rho, \phi, \theta) | 0 \le \rho \le 1, \pi/2 \le \phi \le \pi, 0 \le \theta \le \pi\}.$$

We convert the spherical coordinate into rectangular coordinate,

$$x = \rho \sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi.$$

The field is defined eerywhere, the surface is closed and oriented outward, so we can use divergence theorem. The divergence of F is $z+1$, which gives

$$\begin{aligned}
\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} &= \iiint_E \operatorname{div}(\boldsymbol{F}) dV \\
&= \int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^1 (\rho\cos\phi + 1)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{\pi} \int_{\pi/2}^{\pi} \left(\frac{1}{4}\cos\phi\sin\phi + \frac{1}{3}\sin\phi\right) d\phi\, d\theta \\
&= \int_0^{\pi} \left(-\frac{1}{8} + \frac{1}{3}\right) d\theta \\
&= \frac{5}{24}\pi.
\end{aligned}$$