Exercises for Week 8

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 30.

Reading.

- With Dummit and Foote, please read Section 6.3. Alternatively, read Artin Sections 7.9, 7.10.
- **Important**! Make sure you are comfortable with linear algebra. Read through Artin Sections 4.1-4.4, 5.1 and try some exercises from these sections. Alternatively, revisit your old linear algebra textbook on the corresponding material.

1. Let G and H be two groups. Show that if G can be generated by n elements, H can be generated by m elements, then $G \times H$ can be generated by $n+m$ elements.

2. In class, we have shown that picking two elements in a group G is equivalent to defining a homomorphism ϕ from the free group on two letters $F\{x, y\}$ to G. Now let $G = U(1) := \{e^{i\theta} | \theta \in [0, 2\pi)\}$, and pick the elements $x = e^{2\pi i/3}$ and $y = e^{2\pi i/4}$. Determine what a general element $x^{a_1}y^{b_1}x^{a_2}y^{b_2}\dots x^{a_r}y^{b_r}$ ($a_i, b_i \in \mathbb{Z}\backslash\{0\}$, b_r may be zero) is mapped to under ϕ. Determine the size of the image group $\mathrm{Im}(\phi)$.

3. Show that the (lattice) Heisenberg group
$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}$$
has a group presentation as $\langle p, q, z | pz = zp, qz = zq, pqp^{-1}q^{-1} = z\rangle$. (Hint: Find explicitly what matrices $p,\ q,\ z$ should be.)

4. Present the groups
(a) $\mathbb{Z} \times \mathbb{Z}$,
(b) $\mathbb{Z}/(5) \times \mathbb{Z}/(7)$
by two generators and relations.

5. In class we introduced the braid group on n strands. Let $n = 3$. Construct five elements in terms of the braid generators σ_1 and σ_2 that are in the kernel of the homomorphism
$$Br_3 \mapsto S_3,\ \sigma_i \mapsto (i, i+1), \quad (i = 1, 2).$$
Try to give your answer in terms of braid pictures. For more introductory information on the braid group, watch the nice Youtube video:
https://www.youtube.com/watch?v=u3Gt578803I