A braid group action on a p-DG homotopy category

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A R T I C L E   I N F O

A B S T R A C T

We construct a categorical braid group action on a homotopy category of p-DG modules of a deformed Webster algebra.

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Contents

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1. Introduction

An approach to categorification of quantum groups, their representations, and quantum invariants at a prime p root of unity was outlined in [5] and further developed in [13]. These works suggest that one should look for p-differentials on structures categorifying objects at generic values of the quantum parameter. There has been some progress in this program for quantum $\mathfrak{sl}_2$ [3,4,8,9,14–16].

In particular, a categorification of the braid group action on the Burau representation at a prime root of unity was constructed in [14]. The authors considered a p-DG structure on the algebra $A_n^!$ which describes a singular block of category $\mathcal{O}(\mathfrak{gl}_n)$ corresponding to the Young subgroup $S_1 \times S_{n-1}$. Ignoring the p-DG structure, this categorical action is a consequence of Koszul duality and Khovanov and Seidel's action of the braid group on the homotopy category of modules over the zigzag algebra A_n. Whereas the projective modules play the role of the Temperley-Lieb algebra in [11], the simple objects form an exceptional sequence of objects on the other side of Koszul duality. As a result, one must find projective resolutions of the simple objects of $A_n^!$ in order to construct the braid group action directly. In the context of the p-differential, one must find cofibrant replacements of the simple objects. This was the main technical step in [14].

Partially motivated by the construction in [14], a deformation $W = W(n, 1)$ of $A_n^!$ was considered in [12] and the authors showed that there is a categorical braid group action on the homotopy category of W-modules. This result was extended in [7] to a categorical braid group action on the homotopy category of a deformation of more general Webster algebras for $\mathfrak{sl}_2$, for which W is a special case (hence the notation).

In this note, we return to the simplified setting of W. There is a p-DG structure on this algebra (see [22]). We show that there are braiding complexes in a homotopy category of p-DG W-modules using key ideas from [10]. Using some results from p-DG theory we extend the main result of [12] to the p-DG setting.

Theorem. There is a categorical action of the braid group on n strands on the relative p-DG homotopy category of W.

Deforming the algebra $A_n^!$ allows us to replace the p-DG derived category in [14] with the relative p-DG homotopy category here. This theorem should be compared to [14, Theorem 5.14], albeit in the weaker context of the relative homotopy category. However, the authors believe that the results will become useful towards building a p-DG link homology theory, as proposed in [17]. These further questions will be addressed in subsequent works of the authors.

In [7] a braid group action on the homotopy category of deformed Webster algebras was constructed by exhibiting an action of the Khovanov-Lauda-Rouquier 2-category [6,18], and then using symmetric Howe duality. We expect the main result here holds on the level of generality of [7]. One would need to consider a p-DG version of the Khovanov-Lauda-Rouquier category and show the braiding of Rickard complexes [1] holds in the presence of the p-differential.

In recent work [20], the main result of [7] was proved by relating the deformed algebra to various objects in Lie theory and geometry. It would be interesting to import the p-DG structure to the categories of Gelfand-Tsetlin modules and perverse sheaves studied there.

1.1. Acknowledgments

2. Some exact functors on p-complexes

In this section, we gather some necessary background material for later use. Further details of this section can be found in [8, Section 2.1] and [17, Section 2.1].

2.1. Extension functors

Let $\Bbbk$ be a field of finite characteristic $p > 0$ and A be a $\Bbbk$-algebra. We think of A as a graded algebra by setting all elements of A to be of degree zero. A p-complex of A-modules $U = \oplus_{i \in \mathbb{Z}} U^i$ is then a graded A-module, equipped with a degree-two endomorphism ∂_U satisfying $\partial_U^p = 0$. A morphism of p-complexes $f : U \longrightarrow V$ consists of A-module maps $f_i : U^i \longrightarrow V^i$, $i \in \mathbb{Z}$, that commute with the p-differentials on U and V. A morphism $f : U \longrightarrow V$ of p-complexes is called *null-homotopic* if there is a collection of A-linear maps $h_i : U^i \longrightarrow V^{i-2p+2}$ such that, for any $i \in \mathbb{Z}$,

$$f_i = \sum_{k=0}^{p-1} \partial_U^{p-1-k} h_{i+2k} \partial_V^k. \tag{2.1}$$

When $p = 2$, these notions reduce to the usual notion of (co)chain complexes of A-modules over characteristic two, with the differential degree doubled. Furthermore, just as the usual homotopy category of chain complexes is triangulated, the *homotopy category of p-complexes*, obtained from the (abelian) category of p-complexes modulo the class of null-homotopic morphisms, is also triangulated. Let us denote this p-homotopy category by $\mathcal{C}(A, \partial)$.

In this subsection, we recall a functor relating the usual homotopy category $\mathcal{C}(A, d)$ of A with its p-homotopy category $\mathcal{C}(A, \partial)$.

To do this, recall that a usual chain complex of A-modules M consists of a collection of A-modules and homorphisms $d_i : M^i \longrightarrow M^{i+1}$ called coboundary maps

$$\cdots \xrightarrow{d_{i-2}} M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1} \xrightarrow{d_{i+1}} M^{i+2} \xrightarrow{d_{i+2}} \cdots ,$$

satisfying $d_i \circ d_{i-1} = 0$ for all $i \in \mathbb{Z}$. A morphism of chain complexes $f : M \longrightarrow N$ consists of a sequence $f_i : M^i \longrightarrow N^i$ that commute with the differentials. A map is null-homotopic if there is a sequence of A-module maps $f_i : M^i \longrightarrow N^i$, $i \in \mathbb{Z}$, of A-modules, as depicted in the diagram below,

which satisfy $f_i = d_{i+1} \circ h_i + h_{i+1} \circ d_i$ for all $i \in \mathbb{Z}$. The homotopy category $\mathcal{C}(A, d)$, by construction, is the quotient of chain complexes over A by the ideal of null-homotopic morphisms.

We define the *p-extension functor*

$$\mathcal{P} : \mathcal{C}(A, d) \longrightarrow \mathcal{C}(A, \partial) \tag{2.2}$$

as follows. Given a complex of A-modules, we repeat every term sitting in odd homological degrees $(p - 1)$ times while keeping terms in even homological degrees unaltered. More explicitly, for a given complex

$$\cdots \xrightarrow{d_{2k-2}} M^{2k-1} \xrightarrow{d_{2k-1}} M^{2k} \xrightarrow{d_{2k}} M^{2k+1} \xrightarrow{d_{2k+1}} M^{2k+2} \xrightarrow{d_{2k+2}} \cdots ,$$

the extended complex looks like

Likewise, for a chain-map

$$\cdots \xrightarrow{d_{2k-3}} M^{2k-2} \xrightarrow{d_{2k-2}} M^{2k-1} \xrightarrow{d_{2k-1}} M^{2k} \xrightarrow{d_{2k}} \cdots$$

with vertical maps f_{2k-2}, f_{2k-1}, f_{2k} to

$$\cdots \xrightarrow{d_{2k-3}} N^{2k-2} \longrightarrow N^{2k-1} \longrightarrow N^{2k} \longrightarrow \cdots$$

the obtained morphism of p-DG A-modules is given by

$$\cdots \xrightarrow{d_{2k-3}} M^{2k-2} \xrightarrow{d_{2k-2}} M^{2k-1} = \cdots = M^{2k-1} \xrightarrow{d_{2k-1}} M^{2k} \xrightarrow{d_{2k}} \cdots$$

with vertical maps f_{2k-2}, f_{2k-1}, f_{2k-1}, f_{2k} to

$$\cdots \xrightarrow{d_{2k-3}} N^{2k-2} \xrightarrow{d_{2k-2}} N^{2k-1} = \cdots = N^{2k-1} \xrightarrow{d_{2k-1}} N^{2k} \xrightarrow{d_{2k}} \cdots .$$

This is clearly a functor from the abelian category of cochain complexes over A into the category of p-complexes of A-modules, which we call $\widehat{\mathcal{P}}$.

Next, one may check that $\widehat{\mathcal{P}}$ preserves ideals of null-homotopic morphisms. Indeed, suppose $f = dh + hd$ is a null-homotopic morphism in $\mathcal{C}(A, d)$. We first extend, for any $i \in \mathbb{Z}$ and $h_i : M_i \longrightarrow N_{i-1}$, to a collection of maps

$$\hat{\mathcal{P}}(h)_i : \hat{\mathcal{P}}(M)_i \longrightarrow \hat{\mathcal{P}}(N)_{i-2p+2}$$

as follows. On unrepeated terms, $\hat{\mathcal{P}}(h)$ sends M^{2k} to the copy of N^{2k-1} sitting as the leftmost term in the repeated N^{2k-1}'s, while on the repeated terms, it only sends the rightmost M^{2k+1} to the unrepeated N^{2k} and acts by zero on the other repeated M^{2k+1}'s. Schematically, this has the effect:

$$\cdots = M^{2k-3} \xrightarrow{d_{2k-3}} M^{2k-2} \xrightarrow{d_{2k-2}} M^{2k-1} = \cdots = M^{2k-1} \xrightarrow{d_{2k-1}} M^{2k} \xrightarrow{d_{2k}} M^{2k+1} = \cdots .$$

with diagonal maps labeled 0, h_{2k-1}, h_{2k}, 0 to

$$\cdots = N^{2k-3} \xrightarrow{d_{2k-3}} N^{2k-2} \xrightarrow{d_{2k-2}} N^{2k-1} = \cdots = N^{2k-1} \xrightarrow{d_{2k-1}} N^{2k} \xrightarrow{d_{2k}} N^{2k+1} = \cdots$$

Lemma 2.1. The functor $\widehat{\mathcal{P}}$ sends null-homotopic morphisms in $\mathcal{C}(A, d)$ to null-homotopic morphisms in $\mathcal{C}(A, \partial)$.

Proof. The proof now is an easy exercise. See [17, Lemma 2.2] for more details. $\square$

This lemma implies that $\widehat{\mathcal{P}}$ descends to a functor

$$\mathcal{P} : \mathcal{C}(A, d) \longrightarrow \mathcal{C}(A, \partial), \tag{2.3}$$

which we call the *p-extension functor*. A key property of this functor that we will use is the following.

Proposition 2.2. [17, Proposition 2.3] The functor $\mathcal{P}$ is exact.

2.2. Relative homotopy categories

For any graded or ungraded algebra B over $\Bbbk$, denote by d_0 the zero super differential and by ∂_0 the zero p-differential on B, while letting B sit in homological degree zero. When B is graded, the homological grading is independent of the internal grading of B. For a graded module M over a graded algebra B, we let $M\{n\}$ denote the module M, where the internal grading has been shifted up by n.

Suppose (A, ∂_A) is a p-DG algebra, i.e., a graded algebra equipped with a differential ∂_A of degree two, satisfying

$$\partial_A^p(a) \equiv 0, \qquad \partial_A(ab) = \partial_A(a)b + a\partial_A(b), \tag{2.4}$$

for all $a, b \in A$. In other words, A is an algebra object in the module category of the graded Hopf algebra $H = \Bbbk[\partial]/(\partial^p)$, where the primitive degree-two generator $\partial \in H$ acts on A by the differential ∂_A.

Then, we may form the *smash product algebra* $A\#H$ in this case. As a $\Bbbk$-vector space, $A\#H$ is isomorphic to $A \otimes H$, subject to the multiplication rule determined by

$$(a \otimes \partial)(b \otimes \partial) = ab \otimes \partial^2 + a\partial_A(b) \otimes \partial. \tag{2.5}$$

Notice that, by construction, $A \otimes 1$ and $1 \otimes H$ sit in $A\#H$ as subalgebras.

A module over $A\#H$ is also called a *p-DG module*. We will consider the usual cochain complex or p-complexes of p-DG modules. In such a complex, the usual differential or the p-differential are required to respect the H-actions. Likewise, the null-homotopies in these cases intertwine $A\#H$-actions.

There is an exact forgetful functor between the usual homotopy categories of chain complexes of graded $A\#H$-modules

$$\mathcal{F}_d : \mathcal{C}(A\#H, d_0) \longrightarrow \mathcal{C}(A, d_0).$$

An object $K^\bullet$ in $\mathcal{C}(A\#H, d_0)$ lies inside the kernel of the functor if and only if, when forgetting the H-module structure on each term of $K^\bullet$, the complex of graded A-modules $\mathcal{F}_d(K_\bullet)$ is null-homotopic. The null-homotopy map on $\mathcal{F}_d(K^\bullet)$, though, is not required to intertwine H-actions.

Likewise, there is an exact forgetful functor

$$\mathcal{F}_\partial : \mathcal{C}(A\#H, \partial_0) \longrightarrow \mathcal{C}(A, \partial_0).$$

Similarly, an object $K^\bullet$ in $\mathcal{C}(A\#H, \partial_0)$ lies inside the kernel of the functor if and only if, when forgetting the H-module structure on each term of $K^\bullet$, the p-complex of A-modules $\mathcal{F}(K^\bullet)$ is null-homotopic. The null-homotopy map on $\mathcal{F}(K^\bullet)$, though, is not required to intertwine H-actions.

Definition 2.3. Given a p-DG algebra (A, ∂_A), the *relative homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial}(A, d_0) := \frac{\mathcal{C}(A\#H, d_0)}{\mathrm{Ker}(\mathcal{F}_d)}.$$

Likewise, the *relative p-homotopy category* is the Verdier quotient

$$\mathcal{C}^{\partial}(A, \partial_0) := \frac{\mathcal{C}(A\#H, \partial_0)}{\mathrm{Ker}(\mathcal{F}_\partial)}.$$

The superscripts in the definitions are to remind the reader of the H-module structures on the objects.

The categories $\mathcal{C}^{\partial}(A, d_0)$ and $\mathcal{C}^{\partial}(A, \partial_0)$ are triangulated with the triangulated structures inherited from those of $\mathcal{C}(A\#H, d_0)$ and $\mathcal{C}(A\#H, \partial_0)$. For instance, distinguished triangles in the quotient category are declared to be those that are isomorphic to distinguished triangles in $\mathcal{C}(A\#H, d_0)$ and $\mathcal{C}(A\#H, \partial_0)$ respectively. Recall that the latter distinguished triangles arise from short exact sequences of (p-)cochain complexes over $A\#H$ that are termwise split exact [13, Lemma 4.3].

By construction, there is a factorization of the forgetful functor

$$
\begin{array}{ccc}
\mathcal{C}(A\#H, d_0) \xrightarrow{\mathcal{F}_d} \mathcal{C}(A, d_0) & & \mathcal{C}(A\#H, \partial_0) \xrightarrow{\mathcal{F}_\partial} \mathcal{C}(A, \partial_0) \\
\searrow \quad \nearrow & , & \searrow \quad \nearrow \\
\mathcal{C}^{\partial}(A, d_0) & & \mathcal{C}^{\partial}(A, \partial_0)
\end{array} .
$$

Proposition 2.4. [17, Proposition 2.13] The p-extension functor $\mathcal{P} : \mathcal{C}(A\#H, d_0) \longrightarrow \mathcal{C}(A\#H, \partial_0)$ descends to an exact functor, still denoted $\mathcal{P}$, between the relative homotopy categories:

$$\mathcal{P} : \mathcal{C}^{\partial}(A, d_0) \longrightarrow \mathcal{C}^{\partial}(A, \partial_0) .$$

3. Deformed Webster algebras

3.1. The p-DG algebra

We begin by recalling the definition of a particular deformed Webster algebra $W(n, 1)$. More general versions of these algebras $W(\mathbf{s}, n)$ can be found in [12,7]. The p-DG structures on these algebras were introduced in [22]. The algebras are deformations of the algebras introduced by Webster in [21].

Definition 3.1. Let $n \geq 0$ be an integer. Let $\mathrm{Seq}(1^n, 1)$ be the set of all sequences $\mathbf{i} = (i_1, ..., i_{n+1})$ where n of the entries are 1 and the other entry is the symbol $\mathfrak{b}$. Therefore, we

have $|\mathrm{Seq}(1^n, 1)| = n + 1$. Denote by $\mathbf{i}_j$ the j-th entry of $\mathbf{i}$. Each transposition $\sigma_j \in S_{n+1}$ naturally acts on the set of sequences.

$W = W(n, 1)$ is the graded algebra over the ground field $\Bbbk$ generated by $e(\mathbf{i})$, where $\mathbf{i} \in \mathrm{Seq}(1^n, 1)$, x_j, where $1 \leq j \leq n$, y and ψ_j, where $1 \leq j \leq n$, satisfying the relations below.

$$\sum_{\mathbf{i} \in \mathrm{Seq}(1^n, 1)} e(\mathbf{i}) = 1 \tag{3.1}$$

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}) \tag{3.2}$$

$$\psi_j e(\mathbf{i}) = e(\sigma_j(\mathbf{i}))\psi_j \tag{3.3}$$

$$\psi_j e(\mathbf{i}) = 0 \quad \text{if } \mathbf{i}_j = \mathbf{i}_{j+1} = 1 \tag{3.4}$$

$$\psi_j \psi_\ell = \psi_\ell \psi_j \quad \text{if } |j - \ell| > 1 \tag{3.5}$$

$$x_j \text{ and } y \text{ are central} \tag{3.6}$$

$$\psi_j^2 e(\mathbf{i}) = (x_j - y)e(\mathbf{i}) \quad \text{if } (\mathbf{i}_j, \mathbf{i}_{j+1}) = (1, \flat), (\flat, 1) \tag{3.7}$$

The degrees of the generators are

$$\deg(e(\mathbf{i})) = 0, \quad \deg(x_j) = 2, \quad \deg(y) = 2, \quad \deg(\psi_j) = 1 .$$

In some contexts, it is natural to impose the so-called cyclotomic relation

$$e(\mathbf{i}) = 0 \quad \text{if} \quad \mathbf{i}_1 = \flat. \tag{3.8}$$

Quotienting W by the cyclotomic relation yields an algebra denoted by $\overline{W}$.

We now recall the diagrammatic description of the deformation of the Webster algebra $W = W(n, 1)$. Consider collections of smooth arcs in the plane connecting n red points and 1 black point on one horizontal line with n red points and 1 black point on another horizontal line. The red points correspond to the 1's in the sequence $\mathbf{i}$ of $e(\mathbf{i})$ and the black point corresponds to $\flat$ in the sequence $\mathbf{i}$. The n red points and 1 black point on the line appear in the order in which they appear in $\mathbf{i}$ of $e(\mathbf{i})$. The arcs are colored in a manner consistent with their boundary points. Arcs are assumed to have no critical points (in other words no cups or caps). Arcs are allowed to intersect (as long as they are both not solid red), but no triple intersections are allowed. Arcs can carry dots. Two diagrams that are related by an isotopy that does not change the combinatorial types of the diagrams or the relative position of crossings are taken to be equal. We give $W(n, 1)$ the structure of an algebra by concatenating diagrams vertically as long as the colors of the endpoints match. If they do not, the product of two diagrams is taken to be zero. For two diagrams D_1 and D_2, their product $D_1 D_2$ is realized by stacking D_1 on top of D_2.

The generator $e(\mathbf{i})$ is represented by vertical strands comprised of n red strands and one black strand. For instance, in the case $\mathbf{i} = (1, 1, \flat, 1)$, the generator $e(\mathbf{i})$ is represented by

The generator y is represented by a black dot and x_j is represented by a red dot on the jth red strand:

The generator $\psi_j e(\mathbf{i})$ is represented by a black-red crossing on the left of the diagram below if $\mathbf{i}_j = 1$ and $\mathbf{i}_{j+1} = \flat$ or a black-red crossing on the right if $\mathbf{i}_j = \flat$ and $\mathbf{i}_{j+1} = 1$:
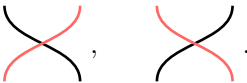
Note that a red-red crossing does not appear due to relation (3.4).

Far away generators commute. The relations (3.6) and (3.7) involving red-black strands are

 (3.9)

 (3.10)

 (3.11)

The cyclotomic relation (3.8) translates to: a black strand, appearing on the far left of any diagram, annihilates the entire picture:

 (3.12)

Note that setting red dots to be zero, we recover Webster's algebra, and so we may think of the polynomial algebra generated by red dots as a polynomial deformation space of the Webster algebra.

For $i = 0, \ldots, n$, there is a sequence $(1^i, \flat, 1^{n-i})$. Denote by e_i the idempotent $e(1^i, \flat, 1^{n-i})$;

$$e_i = e(1^i, \mathfrak{b}, 1^{n-i}) = \left| \underset{1}{} \cdots \underset{i}{} \underset{i+1}{} \cdots \underset{n}{} \right|. \tag{3.13}$$

Over a base field $\Bbbk$ of finite characteristic $p > 0$, a p-differential graded (p-DG) algebra structure was introduced on a generalization of W in [22].

Definition 3.2. The p-derivation $\partial : W \to W$ of degree 2 satisfying the Leibniz rule:

$$\partial(ab) = \partial(a)b + a\partial(b)$$

for any $a, b \in W$ is defined on the generators of the algebra W by

$$\partial(e_i) = 0, \quad \partial(x_i) = x_i^2, \quad \partial(y) = y^2, \quad \partial(\psi_j) = x_j\psi_j e_{j-1} + y\psi_j e_j$$

and extended by the Leibniz rule to the entire algebra.

An easy exercise shows that $\partial^p \equiv 0$.
In the diagrammatic description, we have

$$\partial \left(\begin{array}{c} \end{array} \right) = \begin{array}{c} \end{array}^2 \,, \qquad \partial \left(\begin{array}{c} \end{array} \right) = \begin{array}{c} \end{array}^2 \,, \tag{3.14a}$$

$$\partial \left(\begin{array}{c} \end{array} \right) = \begin{array}{c} \end{array} \,, \qquad \partial \left(\begin{array}{c} \end{array} \right) = \begin{array}{c} \end{array} \,. \tag{3.14b}$$

3.2. A basis

A basis for the cyclotomic deformed Webster $\overline{W}(n, 1)$ was given in [12] (see also [20] and [19]). We slightly modify this basis and a representation of the algebra for the case where the cyclotomic condition is omitted.

Proposition 3.3. Let

$$R_n = \Bbbk[x_1, \ldots, x_n], \qquad V_{n,i} = R_n[y_i], \qquad V_n = \bigoplus_{i=0}^{n} V_{n,i}.$$

There is an action of $W(n, 1)$ on V_n determined by

$$e_i : f(\mathbf{x}, y_j) \in V_{n,j} \mapsto \begin{cases} f(\mathbf{x}, y_i) \in V_{n,i} & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

$$x_k^{a_k} e_i : f(\mathbf{x}, y_i) \in V_{n,i} \mapsto x_k^{a_k} f(\mathbf{x}, y_i) \in V_{n,i},$$

$$y^{a_k} e_i : f(\mathbf{x}, y_i) \in V_{n,i} \mapsto y_i^{a_k} f(\mathbf{x}, y_i) \in V_{n,i},$$

$$\psi_i e_i : f(\mathbf{x}, y_i) \in V_{n,i} \mapsto f(\mathbf{x}, y_{i-1}) \in V_{n,i-1}$$

$$\psi_{i+1} e_i : f(\mathbf{x}, y_i) \in V_{n,i} \mapsto (y_{i+1} - x_{i+1}) f(\mathbf{x}, y_{i+1}) \in V_{n,i+1} .$$

In the diagrammatic description, we have

$$: \quad f(\mathbf{x}, y_i) \in V_{n,i} \mapsto x_1^{a_1} \cdots x_n^{a_n} y_i^b f(\mathbf{x}, y_i) \in V_{n,i}$$

$$: \quad f(\mathbf{x}, y_i) \in V_{n,i} \mapsto f(\mathbf{x}, y_{i-1}) \in V_{n,i-1}$$

$$: \quad f(\mathbf{x}, y_i) \in V_{n,i} \mapsto (y_{i+1} - x_{i+1}) f(\mathbf{x}, y_{i+1}) \in V_{n,i+1}.$$

Proof. This is a straightforward check. $\quad\square$

Next we will define elements which form a basis of $W(n, 1)$.

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ and $b \in \mathbb{Z}_{\geq 0}$. For $0 \leq i \leq j \leq n$, define

$$NE_{i,j}(\mathbf{a}, b) := \prod_{k=1}^n x_k^{a_k} y^b \psi_j \psi_{j-1} \cdots \psi_{i+1} e_i$$

For $0 \leq i < j \leq n$, define

$$SE_{i,j}(\mathbf{a}, b) := \prod_{k=1}^n x_k^{a_k} y^b \psi_{i+1} \psi_i \cdots \psi_j e_i$$

The proof of the next proposition is similar to the proof of [12, Proposition 2, Corollary 1]. See also [19, Proposition 4.9] and [20, Definition 2.7].

Proposition 3.4. We have the following facts about $W(n, 1)$ and its representation V_n.

1. The action of $W(n, 1)$ on V_n is faithful.
2. $W(n, 1)$ has a basis

$$\{NE_{i,j}(\mathbf{a}, b), SE_{i',j'}(\mathbf{a}', b')|0 \leq i \leq j \leq n, 0 \leq i' < j' \leq n, \mathbf{a}, \mathbf{a}' \in \mathbb{Z}_{\geq 0}^n, b, b' \in \mathbb{Z}_{\geq 0}\} .$$

4. Braid invariant

4.1. Bimodules

We recall the (W, W)-bimodules W_i for $i = 1, \dots, n-1$, and $W_{i,i+1}$ for $i = 1, \dots, n-2$ introduced in [12]. These bimodules were generalized in [7]. While all diagrams are supposed to be braid-like, we will occasionally, for visual reasons, have some pictures with cups and caps. These pictures could easily be converted to braid-like diagrams.

In order to define the bimodules W_i and $W_{i,i+1}$, we introduce the algebra $W((1^n)_{i,k}, 1)$ which is also a particular deformed Webster algebra and has been shown in [12, Section 3.4] to be isomorphic to a subalgebra of W.

Let $(1^n)_{i,k}$ be the sequence whose $n - k$ entries are 1 and i-th entry from left is k,

$$(1^n)_{i,k} = (1^{i-1}, k, 1^{n-i-k+1}),$$

and let $\text{Seq}((1^n)_{i,k}, 1)$ be the set of all sequences $\mathbf{i} = (i_1, ..., i_{n-k+2})$ composed of the sequence $(1^n)_{i,k}$ into which the symbol $\mathfrak{b}$ is inserted.

For instance, the set $\text{Seq}((1^4)_{2,2}, 1)$ is

$$\{(\mathfrak{b}, 1, 2, 1), (1, \mathfrak{b}, 2, 1), (1, 2, \mathfrak{b}, 1), (1, 2, 1, \mathfrak{b})\}.$$

Definition 4.1. $W((1^n)_{i,k}, 1)$ is the graded algebra over the ground field $\Bbbk$ generated by

- $e(\mathbf{i})$, where $\mathbf{i} \in \text{Seq}((1^n)_{i,k}, 1)$,
- x_j, where $1 \leq j < i$ or $i < j \leq n - k + 1$,
- y,
- ψ_j, where $1 \leq j \leq n - k + 1$,
- $E(d)$, where $1 \leq d \leq k$

satisfying the relations below.

$$\sum_{\mathbf{i} \in \text{Seq}((1^n)_{i,k}, 1)} e(\mathbf{i}) = 1 \tag{4.1}$$

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i}) \tag{4.2}$$

$$\psi_j e(\mathbf{i}) = e(\sigma_j(\mathbf{i}))\psi_j \tag{4.3}$$

$$\psi_j e(\mathbf{i}) = 0 \quad \text{if } (\mathbf{i}_j, \mathbf{i}_{j+1}) = (1,1), (1,k), (k,1) \tag{4.4}$$

$$\psi_j \psi_\ell = \psi_\ell \psi_j \quad \text{if } |j - \ell| > 1 \tag{4.5}$$

$$x_j, y \text{ and } E(d) \text{ are central} \tag{4.6}$$

$$\psi_j^2 e(\mathbf{i}) = \begin{cases} (y - x_j) e(\mathbf{i}) & \text{if } (\mathbf{i}_j, \mathbf{i}_{j+1}) = (1, \mathfrak{b}), (\mathfrak{b}, 1), \\ \sum_{a=0}^{k} (-1)^a E(a) y^{k-a} e(\mathbf{i}) & \text{if } (\mathbf{i}_j, \mathbf{i}_{j+1}) = (k, \mathfrak{b}), (\mathfrak{b}, k). \end{cases} \tag{4.7}$$

The degrees of the generators are

$$\deg(e(\mathbf{i})) = 0, \quad \deg(x_j) = 2, \quad \deg(y) = 2, \quad \deg(E(d)) = 2d, \quad \deg(\psi_j e(\mathbf{i})) = a$$
$$\text{if} \quad (\mathbf{i}_j, \mathbf{i}_{j+1}) = (\mathfrak{b}, a), (a, \mathfrak{b}),$$

where a is 1 or k.

We have a translation from $(1^n)_{i,k}$ to (1^n) which is obtained by replacing k of the sequence $(1^n)_{i,k}$ by $(\underbrace{1, \ldots, 1}_{k})$. This translation naturally induces the map ϕ : $\mathrm{Seq}((1^n)_{i,k}, 1) \to \mathrm{Seq}((1^n), 1)$.

We define the inclusion map Φ from $W((1^n)_{i,k}, 1)$ to $W((1^n), 1)$ by mapping idempotents $e(\mathbf{i})$ for $\mathbf{i} \in \mathrm{Seq}((1^n)_{i,k}, n)$ by

$$\Phi(e(\mathbf{i})) = e(\phi(\mathbf{i})),$$

mapping generators x_j by

$$\Phi(x_j) = \begin{cases} x_j & \text{if } 1 \le j < i, \\ x_{j+k-1} & \text{if } i < j \le n - k + 1, \end{cases}$$

mapping generators ψ_j by

$$\Phi(\psi_j) = \begin{cases} \psi_j(e_{j-1} + e_j) & j < i, \\ \psi_j \psi_{j+1} \cdots \psi_{j+k-1} e_{j+k-1} + \psi_{j+k-1} \psi_{j+k-2} \cdots \psi_j e_{j-1} & j = i, \\ \psi_{j+k-1}(e_{j+k-2} + e_{j+k-1}) & j > i \end{cases}$$

(where we recall the definition of e_i is given in (3.13)), and mapping generators $E(d)$ by

$$\Phi(E(d)) = \sum_{\substack{d_1 + d_2 + \cdots + d_k = d \\ d_1, \ldots, d_k \ge 0}} x_i^{d_1} x_{i+1}^{d_2} \cdots x_{i+k-1}^{d_k}.$$

This inclusion map gives rise to a left action of $W((1^n)_{i,2}, 1)$ on $\hat{e}_i W((1^n), 1)$ and a right action on $W((1^n), 1)\hat{e}_i$, where $\hat{e}_i$ is $\sum_{j \ne i} e_j$.

Definition 4.2. We define the bimodule W_i as the tensor product of two deformed Webster algebras $W((1^n), 1)$ over $W((1^n)_{i,2}, 1)$.

$$W_i := W((1^n), 1)\hat{e}_i \otimes_{W((1^n)_{i,2},1)} \hat{e}_i W((1^n), 1).$$

The following proposition follows directly from definitions.

Proposition 4.3. In the bimodule W_i, we have the following equalities.

$$e_k \otimes e_\ell = 0 \quad \text{if } k \neq \ell, \quad e_j \otimes e_j = e_j \otimes 1 = 1 \otimes e_j, \quad y \otimes 1 = 1 \otimes y, \tag{4.8}$$

$$x_\ell \otimes e_j = 1 \otimes x_\ell e_j \quad \text{if } \ell \neq i, i+1, \quad (x_i + x_{i+1}) \otimes e_j = 1 \otimes (x_i + x_{i+1})e_j,$$

$$x_i x_{i+1} \otimes e_j = 1 \otimes x_i x_{i+1} e_j, \tag{4.9}$$

$$\psi_\ell \otimes e_j = 1 \otimes \psi_\ell e_j \quad \text{if } \ell \neq i, i+1, \quad \psi_{i+1}\psi_i \otimes e_j = 1 \otimes \psi_{i+1}\psi_i e_j,$$

$$\psi_i \psi_{i+1} \otimes e_j = 1 \otimes \psi_i \psi_{i+1} e_j, \tag{4.10}$$

where $j \neq i$.

A graphical description of the bimodules W_i for $i = 1, \ldots, n-1$ was introduced in [12]. We consider collections of smooth arcs in the plane connecting n red points and 1 black point on one horizontal line with n red points and 1 black point on another horizontal line. The ith and $(i+1)$st red dots on one horizontal line must be connected to the ith and $(i+1)$st red dots on the other horizontal line by a diagram which has a thick red strand in the middle, which is given in (4.11). The arcs are colored in a manner consistent with their boundary points and are assumed to have no critical points (in other words no cups or caps). They are allowed to intersect, but no triple intersections are allowed. Arcs are allowed to carry dots. Two diagrams that are related by an isotopy that does not change the combinatorial types of the diagrams or the relative position of crossings are taken to be equal. The elements of the vector space W_i are formal linear combinations of these diagrams modulo the local relations for W along with the relations given in (4.12) and (4.13) which correspond to relations in Proposition 4.3.

The elements $e_j \otimes e_j$, where $0 \leq j < i$ or $i < j \leq n$, are represented by diagrams

$$e_j \otimes e_j = \quad \in W_i . \tag{4.11}$$

The second and third equations of (4.9) in terms of diagrams are

$$(4.12)$$

The second and third equations of (4.10) in terms of diagrams are

$$(4.13)$$

The bimodule W_i naturally inherits a p-DG structure from W as follows:

$$\partial_{W_i} := \partial_W \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_W .$$

One may twist the p-DG structure on W_i to obtain a new p-DG bimodule $W_i^{-e_1}$. As bimodules, $W_i = W_i^{-e_1}$ but the p-DG structure on the generator is twisted as follows:

$$\partial(1 \otimes e_j) = -(x_i + x_{i+1}) \otimes e_j \qquad (4.14)$$

In terms of a diagrammatic description, this twisted differential is

$$(4.15)$$

We use the notation $W_i^{-e_1}$ because the differential is twisted by the first elementary symmetric function in the dots corresponding to the bimodule generator connected to the thick red strand.

We will also need the bimodules $W_{i,i+1}$ for $i = 1, \ldots, n-2$.

Definition 4.4. We define the bimodule $W_{i,i+1}$ as the tensor product of two deformed Webster algebras $W((1^n), 1)$ over $W((1^n)_{i,3}, 1)$.

$$W_{i,i+1} := W((1^n), 1)\hat{e}_{i,i+1} \otimes_{W((1^n)_{i,3},1)} \hat{e}_{i,i+1} W((1^n), 1),$$

where $\hat{e}_{i,i+1} = \displaystyle\sum_{j \neq i, i+1} e_j$.

The following proposition follows directly from definitions.

Proposition 4.5. In the bimodule $W_{i,i+1}$, we have the following equalities.

$$e_k \otimes e_\ell = 0 \quad \text{if } k \neq \ell, \quad e_j \otimes e_j = e_j \otimes 1 = 1 \otimes e_j, \quad x_\ell \otimes e_j = 1 \otimes x_\ell e_j$$

$$\text{if } \ell \neq i, i+1, i+2, \tag{4.16}$$

$$\psi_\ell \otimes e_j = 1 \otimes \psi_\ell e_j \quad \text{if } \ell \neq i, i+1, i+2, \quad y \otimes 1 = 1 \otimes y, \tag{4.17}$$

$$\sum_{\substack{a+b+c=d \\ a,b,c \geq 0}} x_i^a x_{i+1}^b x_{i+2}^c \otimes e_j = 1 \otimes \sum_{\substack{a+b+c=d \\ a,b,c \geq 0}} x_i^a x_{i+1}^b x_{i+2}^c e_j \quad \text{where } d = 1, 2, 3, \tag{4.18}$$

$$\psi_{i+2}\psi_{i+1}\psi_i \otimes e_j = 1 \otimes \psi_{i+2}\psi_{i+1}\psi_i e_j, \quad \psi_i\psi_{i+1}\psi_{i+2} \otimes e_j = 1 \otimes \psi_i\psi_{i+1}\psi_{i+2}e_j, \tag{4.19}$$

where $j \neq i, i+1$.

For a diagrammatic description of $W_{i,i+1}$, we consider collections of smooth arcs in the plane connecting n red points and 1 black point on one horizontal line with n red points and 1 black point on another horizontal line. The ith, $(i+1)$st and $(i+2)$nd red dots on one horizontal line must be connected to the ith, $(i+1)$st and $(i+2)$nd red dots on the other horizontal line by a diagram which has a thick red strand in the middle, which is given in (4.20). The arcs are colored in a manner consistent with their boundary points and are assumed to have no critical points (in other words no cups or caps). They are allowed to intersect, but no triple intersections are allowed. Arcs are allowed to carry dots. Two diagrams that are related by an isotopy that does not change the combinatorial types of the diagrams or the relative position of crossings are taken to be equal. The elements of the vector space $W_{i,i+1}$ are formal linear combinations of these diagrams modulo the local relations for W along with the relations given in (4.21), (4.22), (4.23) and (4.24).
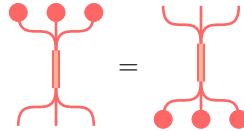
The elements $e_j \otimes e_j$, where $0 \leq j < i$ or $i+1 < j \leq n$, are represented by

$$e_j \otimes e_j = \qquad \in W_{i,i+1} . \tag{4.20}$$

The generating equations in (4.18) in terms of diagrams are
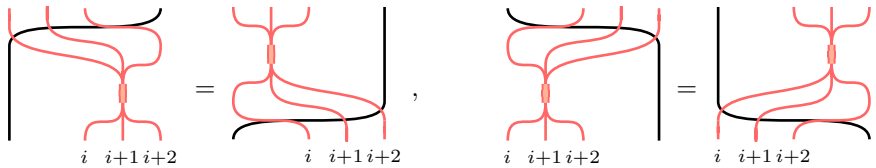
$$\tag{4.21}$$

$$\tag{4.22}$$

$$(4.23)$$

The equations in (4.19) in terms of diagrams are

$$(4.24)$$

The bimodule $W_{i,i+1}$ naturally inherits a p-DG structure from W where one sets the differential on the generator (4.20) to be zero.

Remark 4.6. We will sometimes denote the elements in the equalities of in (4.24) as a crossing between a black strand and a thick red strand. We take a similar convention for diagrams in (4.13).

4.2. Bases for bimodules

We construct bases of the bimodules just introduced.
Let

$$\aleph_1(\mathbf{a}, b, c) = \prod_{k=1}^{n} x_k^{a_k} y^b \psi_i \otimes \psi_i x_i^c e_i$$

$$\aleph_2(\mathbf{a}) = \prod_{k=1}^{n} x_k^{a_k} \psi_{i+1} \otimes \psi_{i+1} e_i$$

For $0 \leq j \leq n$ and $0 \leq \ell \leq n$ unless $j = \ell = i$, let

$$\aleph_3(\mathbf{a}, b, c, j, \ell) = \begin{cases} \prod_{k=1}^{n} x_k^{a_k} y^b \psi_\ell \psi_{\ell-1} \cdots \psi_{j+1} \otimes x_i^c e_j & \text{if } j < \ell \quad \text{and} \quad j \neq i \\ \prod_{k=1}^{n} x_k^{a_k} y^b \psi_{\ell+1} \psi_{\ell+2} \cdots \psi_j \otimes x_i^c e_j & \text{if } j > \ell \quad \text{and} \quad j \neq i \\ \prod_{k=1}^{n} x_k^{a_k} y^b \otimes x_i^c e_j & \text{if } j = \ell, j \neq i \\ \prod_{k=1}^{n} x_k^{a_k} y^b \otimes x_i^c \psi_\ell \psi_{\ell-1} \cdots \psi_{i+1} e_i & \text{if } j < \ell \quad \text{and} \quad j = i \\ \prod_{k=1}^{n} x_k^{a_k} y^b \otimes x_i^c \psi_{\ell+1} \psi_{\ell+2} \cdots \psi_i e_i & \text{if } j > \ell \quad \text{and} \quad j = i. \end{cases}$$

For $j < \ell$ and $j \neq i$, diagramatically $\aleph_3(\mathbf{a}, b, c, j, \ell)$ is given by
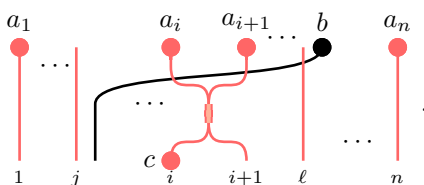
Lemma 4.7. For any $b_1, b_2, b_3 \geq 0$, the elements
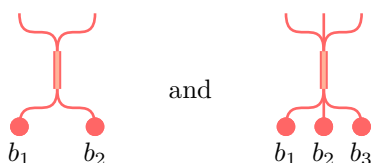
could be written as a linear combination of elements of the form
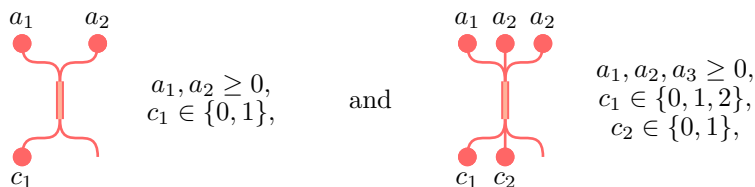
respectively.

Proof. This is a well-known result. See for example [2, Section 2.2]. □
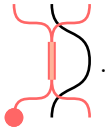
Proposition 4.8. The set

$$\{\aleph_1(\mathbf{a}, b, c), \aleph_2(\mathbf{a}), \aleph_3(\mathbf{a}, b, c, j, \ell) | a_1, \ldots, a_n, b \in \mathbb{Z}, c \in \{0, 1\}, 0 \leq j \leq n, 0 \leq \ell \leq n\}$$

is a spanning set of W_i.
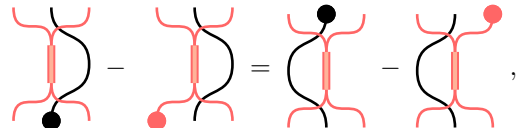
Proof. We will only highlight interesting features of the proof, which include cases where the black strand begins and ends in between the two red strands connected to the thick

red strand. First we will show that $\psi_{i+1} \otimes \psi_{i+1} x_i e_i$ is in the span of the proposed spanning set. The diagram for this element is
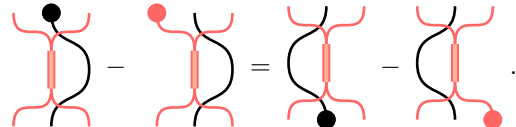
$$
\tag{4.25}
$$

Using relations (3.11), we have the following equations
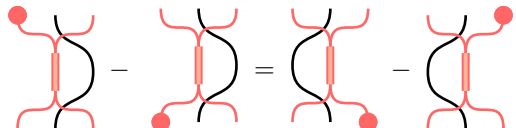
$$
\tag{4.26}
$$
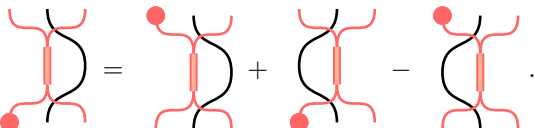
$$
\tag{4.27}
$$

Subtracting equation (4.27) from equation (4.26) yields

$$
\tag{4.28}
$$

Using symmetric function relations on (4.28) we get

$$
$$
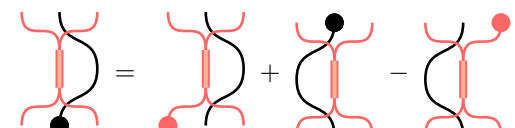
Thus a diagram with a portion as in (4.25) is in the span.

Consider a portion of a diagram of the form

$$
\tag{4.29}
$$

Then using relations (3.11), we have

$$
$$

which shows that a picture (4.29) is in the span of the proposed spanning set since we already showed the first term above is already in the proposed spanning set. □

In order to prove that the elements in Proposition 4.8 are actually a basis of W_i, we will construct a $(W(n,1), W(n,1))$-bimodule homomorphism $\phi_i \colon W_i \to \mathrm{Hom}_{\Bbbk}(V_n, V_n)$.

Consider the divided difference operator

$$D_i \colon \Bbbk[y_j][x_1, \ldots, x_n] \to \Bbbk[y_j][x_1, \ldots, x_n] \qquad f \mapsto \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

where f^{s_i} is the polynomial obtained from f by exchanging the variables x_i and x_{i+1} and keeping all other variables the same.

Proposition 4.9. There is a bimodule homomorphism $\phi_i \colon W_i \to \mathrm{Hom}_{\Bbbk}(V_n, V_n)$ inherited from the representation of $W(n,1)$ on V_n determined by

$$: \ f \in V_n \ \mapsto \ D_i(f) \in V_n \ .$$

Proof. It is a routine check using the fact that symmetric functions are invariant under the action of a divided difference operator. □

Proposition 4.10. The spanning set in Proposition 4.8 is a basis of W_i.

Proof. Clearly a collection of vectors in this bimodule could be linearly dependent only if the configuration of boundary points is the same. We will focus on the most interesting case. For simplicity, assume there is a dependence relation of the form

$$k^1_{a_1,a_2,b} \qquad + \quad k^2_{a_1,a_2,b} \qquad + \quad k^3_{a_1,a_2} \qquad = 0 \ . \qquad (4.30)$$

Applying the bimodule homomorphism ϕ_1 from above on the element 1 yields

$$k^2_{a_1,a_2,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}y_1^b + k^3_{a_1,a_2}x_1^{a_1}x_2^{a_2} = 0 \ .$$

This implies that $k^2_{a_1,a_2,b} = k^3_{a_1,a_2} = 0$ and thus $k^1_{a_1,a_2,b} = 0$. Therefore the elements in (4.30) are in fact linearly independent.

Checking linear independence for the other elements in the spanning set proceeds in a similar manner. □

Proposition 4.11. The following elements span $W_{i,i+1}$

$$\beth_1(\mathbf{a}, b, c_i, c_{i+1}) = \qquad\qquad b \geq 0, c_i = 0, 1, 2, c_{i+1} = 0, 1,$$

$$\beth_2(\mathbf{a}, c_i, c_{i+1}) = \qquad\qquad c_i = 0, c_{i+1} = 0, 1,$$

$$\beth_3(\mathbf{a}, b, c_i, c_{i+1}) = \qquad\qquad b \geq 0, c_i = 0, 1, 2, c_{i+1} = 0, 1,$$

$$\beth_4(\mathbf{a}, c_i, c_{i+1}) = \qquad\qquad c_i = 0, c_{i+1} = 0, 1,$$

$$\beth_5(\mathbf{a}, c_i, c_{i+1}) = \qquad\qquad (c_i, c_{i+1}) = (0, 0), (1, 0), (0, 1), (1, 1),$$

$$\beth_6(\mathbf{a}, b, c_i, c_{i+1}) = \qquad\qquad b \geq 0, c_i = 0, 1, 2, c_{i+1} = 0, 1,$$

$$\beth_7(\mathbf{a}, b, c_i, c_{i+1}) = \qquad\qquad b \geq 0, c_i = 0, 1, 2, c_{i+1} = 0, 1,$$

$$\beth_8(\mathbf{a}, c_i, c_{i+1}) = \text{[diagram]} \qquad (c_i, c_{i+1}) = (0,0), (1,0), (0,1), (2,0),$$

$$\beth_9(\mathbf{a}, b, c_i, c_{i+1}, j, \ell) = \text{[diagram]} \qquad \begin{array}{l} b \geq 0, c_i = 0, 1, 2, c_{i+1} = 0, 1, \\ 0 \leq j, \ell \leq n \\ \text{unless } \{j, \ell\} \subset \{i, i+1\}, \end{array}$$

where the diagram in $\beth_9(\mathbf{a}, b, c_i, c_{i+1}, j, \ell)$ represents any picture where the black strand begins after the jth red strand and ends after the ℓth red strand and has a minimal number of intersections with the red strands.

Proof. Once again, we highlight features of the proof that only involve cases where the black strand begins and ends in the segments connecting red boundary points connected to the thick red strands. First note that by Lemma 4.7, we could assume that the red dots appear in the following configuration
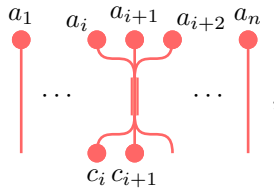
where all $a_i \in \mathbb{Z}_{\geq 0}$, $c_i \in \{0, 1, 2\}$, and $c_{i+1} \in \{0, 1\}$.

Next we will explain why we could assume that in (4.31) that there are no black dots.
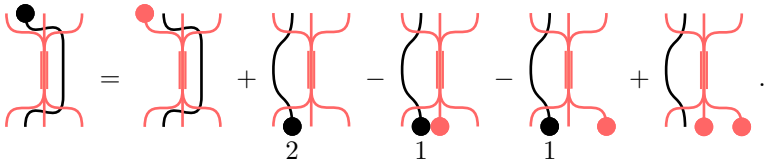
$$\text{[diagram]} \tag{4.31}$$

Using relations we have

$$\text{[diagram]} \tag{4.32}$$

so a black dot produces a linear combination of other elements in the proposed spanning set.

Next we will explain why we may assume that $c_i = 0$ in (4.31). As in (4.32) we have a relation

$$\tag{4.33}$$

Subtracting (4.33) from (4.32) yields
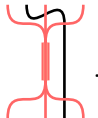
By Lemma 4.7, the third to last and last diagrams above are in the proposed span, and thus
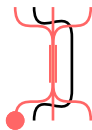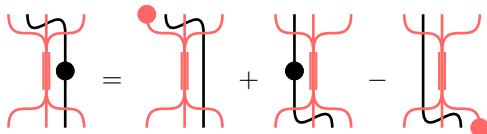
is in the span of other elements of the proposed spanning set.

Next we will show that we could assume that there are no black dots on diagrams of the form

$$\tag{4.34}$$

Using bimodule relations (3.11) and (4.24) we have

$$\tag{4.35}$$

Thus we may assume that there are no black dots on diagrams of the form (4.34).

Next we will show that elements of the form

$$(4.36)$$

are already in the span of the proposed spanning set. Using relations (3.11), the left side of (4.37) below could be rewritten as two double crossings of the black strand and the two bottom left red strands. Relations (4.24), then allow us to move the black strand up through the thick red strand. Applying relations (3.11) then implies (4.37).

$$(4.37)$$

Using (4.37) and (4.35) repeatedly, we get

Thus (4.36) is in the span of the proposed spanning set.

Similar techniques show that the other elements stated in the proposition complete a spanning set. $\square$

In order to prove that the elements in Proposition 4.11 are actually a basis of $W_{i,i+1}$, we use a $(W(n,1), W(n,1))$-bimodule homomorphism $\gamma_{i,i+1} \colon W_{i,i+1} \to \mathrm{Hom}_{\Bbbk}(V_n, V_n)$.

Proposition 4.12. There is a bimodule homomorphism $\gamma_{i,i+1} \colon W_{i,i+1} \to \mathrm{Hom}_{\Bbbk}(V_n, V_n)$ inherited from the representation of $W(n,1)$ on V_n determined by

$$\colon \; f \in V_n \; \mapsto D_i D_{i+1} D_i(f) \in V_n \; .$$

Proof. This is a routine calculation. $\square$

The next two results contain important facts used later on, but the proofs are just lengthy computations that the reader interested in a smooth flow of the text could safely ignore.

Proposition 4.13. The spanning set in Proposition 4.11 is a basis of $W_{i,i+1}$.

Proof. If $\epsilon_1' v_1 \epsilon_1, \epsilon_2' v_2 \epsilon_2 \in W_{i,i+1}$, where $\epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2'$ are idempotents, then there could only be a dependence relation if $\epsilon_1' = \epsilon_2'$ and $\epsilon_1 = \epsilon_2$. Using this fact, there are several cases to check. We supply the details for two non-trivial cases.

First consider a dependence relation of the form

$$(4.38)$$

Apply the homomorphism $\gamma_{i,i+1}$ to (4.38) and evaluate on 1 to get

$$k^1_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^{b} + k^7_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 .$$

This forces

$$k^1_{a_1,a_2,a_3,b} = k^7_{a_1,a_2,a_3} = 0 .$$

Then the dependence relation becomes

$$(4.39)$$

Now apply $\gamma_{i,i+1}$ to (4.39) and evaluate on x_1 to get

$$k^3_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b = 0 .$$

This implies

$$k^3_{a_1,a_2,a_3,b} = 0$$

and the dependence relations becomes

$$(4.40)$$

Now apply $\gamma_{i,i+1}$ to (4.40) and evaluate on x_2 to get

$$k^2_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b + k^8_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 .$$

This implies that $k^2_{a_1,a_2,a_3,b} = k^8_{a_1,a_2,a_3} = 0$ and the dependence relation becomes

$$(4.41)$$

Now apply $\gamma_{i,i+1}$ to (4.41) and evaluate on $x_2 x_3$ to get

$$k^5_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b = 0 .$$

Thus $k_5 = 0$ and the dependence relation becomes

$$(4.42)$$

Apply $\gamma_{i,i+1}$ to (4.42) and evaluate on $x_1 x_2$ to get

$$k^4_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b = 0 .$$

Thus $k^4_{a_1,a_2,a_3,b} = 0$ which implies $k^6_{a_1,a_2,a_3,b} = 0$. It is straightforward to show that each element in the spanning set gets sent to something non-zero under $\gamma_{i,i+1}$ by evaluating on an appropriate element. For instance,

$$(4.43)$$

Thus these elements of the spanning set are non-zero and linearly independent.

Next we will consider a dependence relation of the form

$$(4.44)$$

Applying the homomorphism $\gamma_{i,i+1}$ to (4.44) and evaluating on 1 yields the equation

$$k^1_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b + k^7_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 .$$

Thus $k_1 = k_7 = 0$ and the dependence relation becomes

$$\begin{aligned} k^2_{a_1,a_2,a_3,b} \;\; &+ k^3_{a_1,a_2,a_3,b} \;\; + k^4_{a_1,a_2,a_3,b} \\ &+ k^5_{a_1,a_2,a_3,b} \;\; + k^6_{a_1,a_2,a_3,b} \;\; + k^8_{a_1,a_2,a_3} \qquad (4.45) \\ &+ k^9_{a_1,a_2,a_3} \;\; + k^{10}_{a_1,a_2,a_3} \;\; = 0 \,. \end{aligned}$$

Now apply $\gamma_{i,i+1}$ to (4.45) and evaluate on x_1 to get

$$k^3_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b + k^8_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 \,.$$

Thus $k^3_{a_1,a_2,a_3,b} = k^8_{a_1,a_2,a_3} = 0$ and the dependence relation becomes

$$\begin{aligned} k^2_{a_1,a_2,a_3,b} \;\; &+ k^4_{a_1,a_2,a_3,b} \;\; + k^5_{a_1,a_2,a_3,b} \\ &+ k^6_{a_1,a_2,a_3,b} \;\; + k^9_{a_1,a_2,a_3} \;\; + k^{10}_{a_1,a_2,a_3} \;\; = 0 \,. \end{aligned}$$

$$(4.46)$$

Now apply $\gamma_{i,i+1}$ to (4.46) and evaluate on x_2 to get

$$k^2_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b - k^9_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 \,.$$

Thus $k^2_{a_1,a_2,a_3,b} = k^9_{a_1,a_2,a_3} = 0$ and the dependence relation becomes

$$\text{(4.47)}$$

Now apply $\gamma_{i,i+1}$ to (4.47) and evaluate on x_1^2 to get

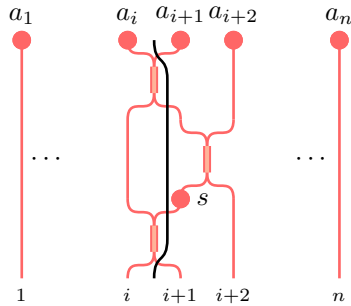$$k^5_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b + k^{10}_{a_1,a_2,a_3}x_1^{a_1}x_2^{a_2}x_3^{a_3} = 0 .$$

Thus $k^5_{a_1,a_2,a_3,b} = k^{10}_{a_1,a_2,a_3} = 0$ and the dependence relation becomes

$$\text{(4.48)}$$

Finally apply $\gamma_{i,i+1}$ to (4.48) and evaluate on x_1x_2 to get

$$k^4_{a_1,a_2,a_3,b}(y_1 - x_1)x_1^{a_1}x_2^{a_2}x_3^{a_3}y_1^b = 0 .$$

Thus $k^4_{a_1,a_2,a_3,b} = 0$ and consequently $k^6_{a_1,a_2,a_3,b} = 0$. Therefore these spanning elements are also linearly independent.

The other cases are checked in a similar fashion. □

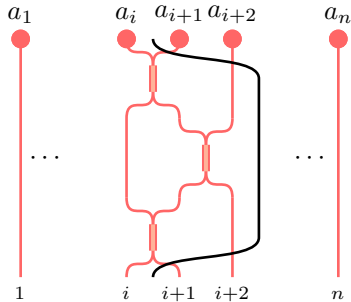**Proposition 4.14.** The following elements span the bimodule $W_i \otimes_W W_{i+1} \otimes_W W_i$.

$$\mathbf{J}_1(\mathbf{a}, b, r, s, t) = \qquad a_i \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}_{\geq 0}, r, s, t \in \{0,1\},$$

$$\beth_2(\mathbf{a}, s) \ = \ \cdots \qquad \cdots \qquad a_i \in \mathbb{Z}_{\geq 0}, s \in \{0, 1\},$$

$$\beth_3(\mathbf{a}) \ = \ \cdots \qquad \cdots \qquad a_i \in \mathbb{Z}_{\geq 0},$$

$$\beth_4(\mathbf{a}, b, r, s, t) \ = \ \cdots \qquad \cdots \qquad a_i \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}_{\geq 0}, r, s, t \in \{0, 1\},$$

$$\beth_5(\mathbf{a}, r, t) \ = \ \cdots \qquad \cdots \qquad a_i \in \mathbb{Z}_{\geq 0}, r, t \in \{0, 1\},$$

$$\beth_6(\mathbf{a}, b, r, s, t) \;=\;$$

$$a_i \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}_{\geq 0}, r, s, t \in \{0, 1\},$$

$$\beth_7(\mathbf{a}, r, t) \;=\;$$

$$a_i \in \mathbb{Z}_{\geq 0}, r, t \in \{0, 1\},$$

$$\beth_8(\mathbf{a}, t) \;=\;$$

$$a_i \in \mathbb{Z}_{\geq 0}, t \in \{0, 1\},$$

$$\beth_9(\mathbf{a}, b, r, s, t) \;=\;$$

$$a_i \in \mathbb{Z}_{\geq 0}, b \in \mathbb{Z}_{\geq 0}, r, s, t \in \{0, 1\},$$

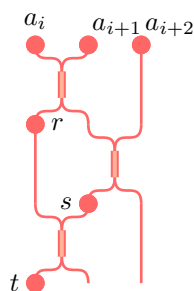$$\beth_{10}(\mathbf{a}, r, t) \;=\; \qquad\qquad\qquad\qquad\qquad a_i \in \mathbb{Z}_{\geq 0},\, r, t \in \{0, 1\},$$

$$\beth_{11}(\mathbf{a}, r) \;=\; \qquad\qquad\qquad\qquad\qquad a_i \in \mathbb{Z}_{\geq 0},\, r \in \{0, 1\},$$

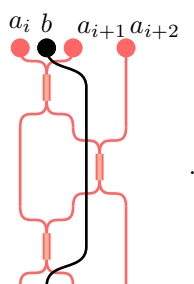$$\beth_{12}(\mathbf{a}, b, r, s, t, j, \ell) \;=\;$$

$$\begin{aligned}
& a_i \in \mathbb{Z}_{\geq 0},\, b \in \mathbb{Z}_{\geq 0},\\
& r, s, t \in \{0, 1\}\ 0 \leq j, \ell \leq n\\
& \text{unless } (j, \ell) = (i, i), (i, i+1),\\
& \qquad (i+1, i), (i+1, i+1),
\end{aligned}$$

where the diagram in $\beth_{12}(\mathbf{a}, b, r, s, t, j, \ell)$ represents any picture where the black strand begins after the jth red strand and ends after the ℓth red strand and has a minimal number of intersections with the red strands. Note that it could equally well be positioned in a northwest-southeast configuration.
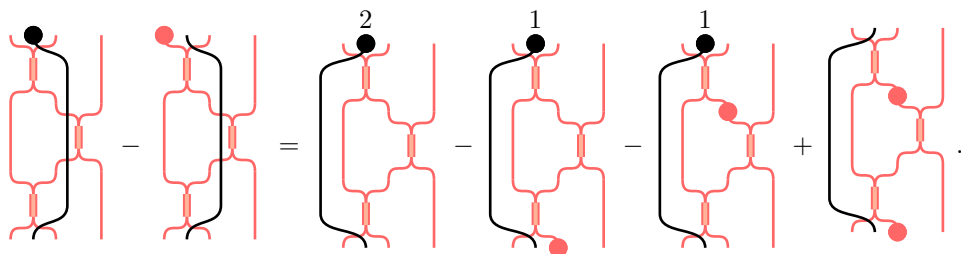
Proof. By Lemma 4.7, we may assume that any red dot configuration in the part of the picture connecting the i, $i+1$, and $i+2$ red boundary points is a linear combination of elements
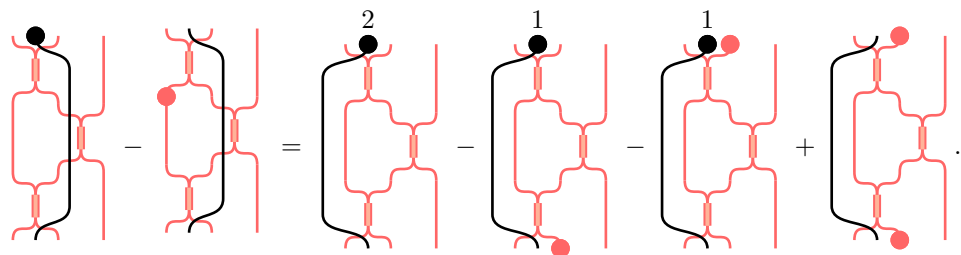
$$(4.49)$$

where $r, s, t \in \{0, 1\}$. Next we will show that we do not need any black dots on a diagram of the form
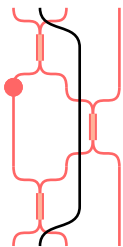

$$(4.50)$$

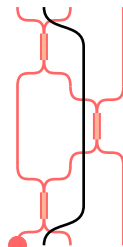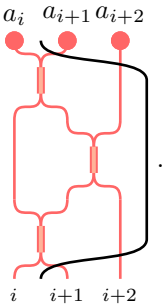This follows from the calculation

Next note that

Thus

is already in the span of the proposed spanning set. Similarly,
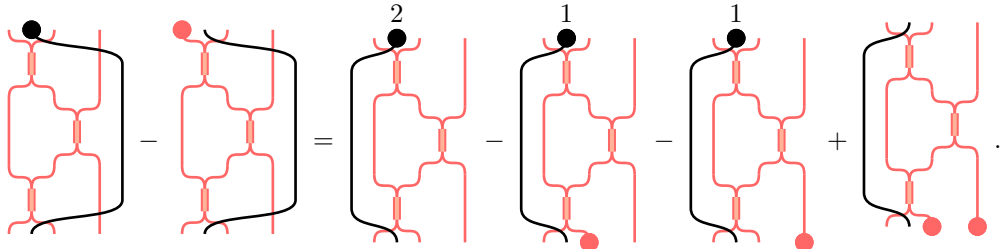

is also already in the span of the proposed spanning set.

Next we will show that we do not need black dots on a diagram of the form
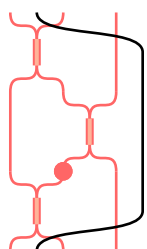
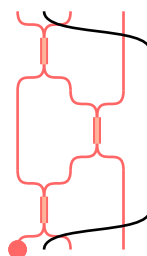

This follows from the calculation

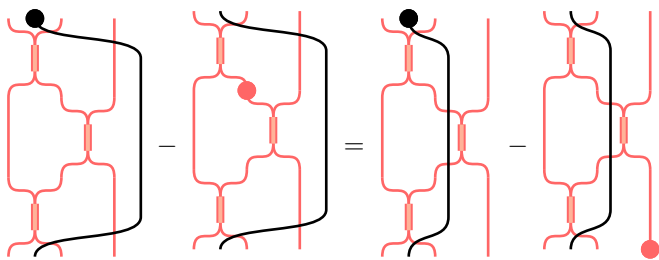Next note that

which shows that

is in the span of the proposed spanning set. The equality
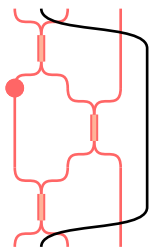

implies that

is in the span of the proposed spanning set. A longer computation using the equality
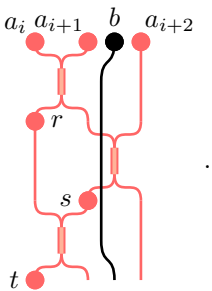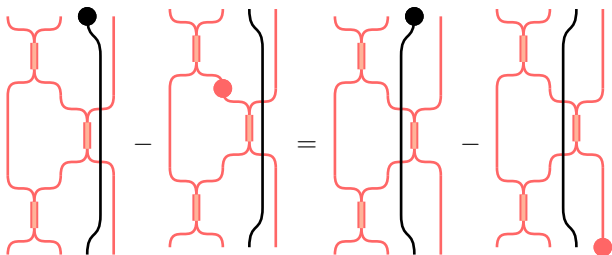
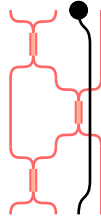implies that

is in the proposed span.

Now consider elements of the form

By Lemma 4.7, one may assume that $r, s, t \in \{0, 1\}$ and $a_i, a_{i+1}, a_{i+2}, b \in \mathbb{Z}_{\geq 0}$. Next note that the equation

implies that a black dot on a diagram

is already in the span of the proposed spanning set. The equation
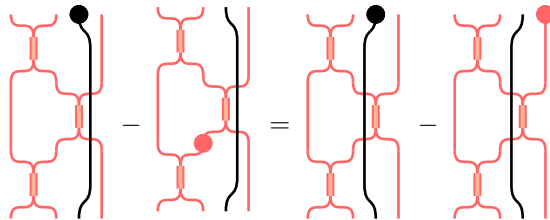
implies that

is already in the span of the proposed spanning set.

By Lemma 4.7, we may assume that dots on diagrams of the following form have the configuration
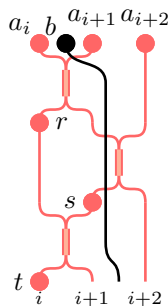
where $r, s, t \in \{0, 1\}$ and $b, a_i, a_{i+1}, a_{i+2} \in \mathbb{Z}_{\geq 0}$. Next note that

implies that
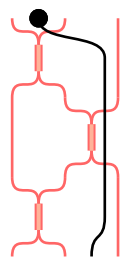
is in the span of the proposed spanning set. The equation
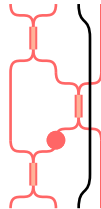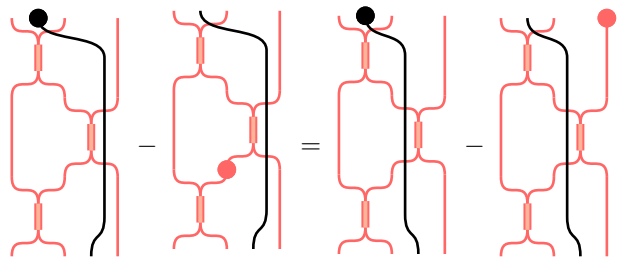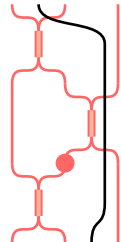

implies that

is in the span.

Next note that

Thus a diagram with a black dot on
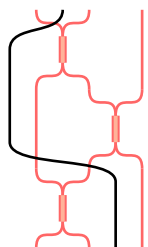
is already in the span of the proposed spanning set. The equation
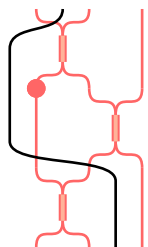
implies that

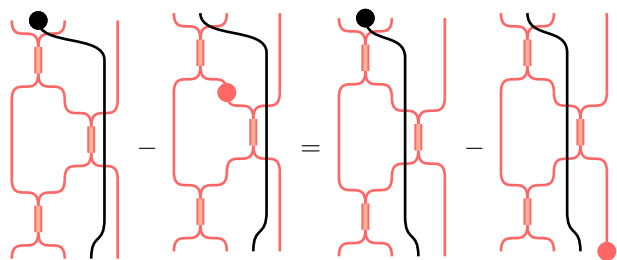

is in the span. The equation
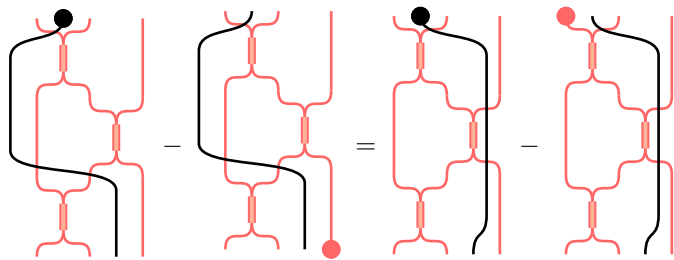
implies that
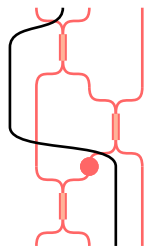
is in the span.

Similar manipulations show that the set of elements in the statement of the proposition, do in fact provide a spanning set for the bimodule. □

We will prove that the spanning set above is indeed a basis. This will utilize a bimodule homomorphism introduced in the next section.

4.3. Bimodule homomorphisms

There is a bimodule homomorphism

$$\epsilon_i \colon W_i \to W$$

determined by

                                                          (4.51)

It is clear that ϵ_i commutes with ∂.

There is a bimodule homomorphism

$$\iota_i \colon W \to W_i^{-e_1}\{-2\}$$

determined by

The fact that ι_i is a well-defined bimodule homomorphism is shown in [12, Proposition 17].

It is clear that $\iota_i \colon W \to W_i\{-2\}$ does not commute with ∂. However, we do have the following variation.

Proposition 4.15. The bimodule homomorphism $\iota_i \colon W \to W_i^{-e_1}\{-2\}$ commutes with the action of ∂.

Proof. We first check the proposition in the case that there is no black strand in between the ith and $(i+1)$st red strands. By definition

$$\iota_i \circ \partial \left(\Big|\Big| \quad \Big|\Big| \right) = 0 \ .$$

On the other hand,

where the four last terms come from the twisted p-DG structure on $W_i^{-e_1}$ (see (4.15)).

The second case we must consider is when the black strand lies between the ith and $(i+1)$st red strands. Once again

$$\iota_i \circ \partial \left(\;\middle|\;\middle|\;\middle|\; \right) = 0 \;.$$

Let

$$X \;=\; \text{(diagram)} - \text{(diagram)} \;\in\; W_i^{-e_1}\;.$$

We must show that $\partial(X) = 0$. By definition

$$\partial(X) \;=\; \text{(diagram)} + \text{(diagram)} - \text{(diagram)} - \text{(diagram)} - \text{(diagram)} - \text{(diagram)}$$
$$+ \text{(diagram)} + \text{(diagram)} \tag{4.52}$$

where the last four terms come from the twisting on $W_i^{-e_1}$. It follows that

$$\partial(X) \;=\; \text{(diagram)} - \text{(diagram)} - \text{(diagram)} + \text{(diagram)} \tag{4.53}$$

where the second, fifth, and sixth terms in (4.52) combine to become the second term in (4.53). Using the first set of relations of (3.9) and (4.13), we get that the above simplifies to

$$\partial(X) \;=\; \text{(diagram)} - \text{(diagram)} \;=\; 0. \quad \square$$

Proposition 4.16. There are bimodule homomorphisms $\alpha_{i,i+1} \colon W_{i,i+1} \to W_i \otimes_W W_{i+1} \otimes_W W_i$ and $\alpha_{i+1,i} \colon W_{i,i+1} \to W_{i+1} \otimes_W W_i \otimes_W W_{i+1}$ defined on the bimodule generator (4.20) by

$$\alpha_{i,i+1}: \qquad \longmapsto \qquad\qquad (4.54)$$

$$\alpha_{i+1,i}: \qquad \longmapsto \qquad\qquad . \qquad (4.55)$$

These homomorphisms commute with the action of ∂.

Proof. It is straightforward to check that these are bimodule maps. The differential ∂ annihilates both the generator (4.20) and its image under the homomorphisms. $\square$

There is a bimodule homomorphism $W_i \otimes_W W_{i+1} \otimes_W W_i \to W_i\{2\}$ defined as a composition of the maps ϵ and $\blacktriangle$ constructed in [12]. It is defined on generators in (4.56) and (4.57).

$$\longmapsto 0 \; , \qquad\qquad \longmapsto \qquad\qquad (4.56)$$

$$\longmapsto \qquad , \qquad\qquad \longmapsto \qquad\qquad (4.57)$$

The proof of the next result is similar to the proofs of Propositions 4.10 and 4.13, so we just supply the general strategy.

Proposition 4.17. The spanning set in Proposition 4.14 is a basis of $W_i \otimes_W W_{i+1} \otimes_W W_i$.

Proof. Let B^i_j be elements in the spanning set which are in the image of multiplication on the left by e_i and on the right by e_j.

In order to prove that the elements of B^i_{i+1} are linearly independent, one writes down a dependence relation and uses the bimodule homomorphisms ϕ_i from Proposition 4.9, and the bimodule homomorphism defined in (4.56) and (4.57) to conclude that all the coefficients in the dependence relation are zero. One proves in a similar way the linear independence of elements in B^i_i and B^{i+1}_i.

By applying the element

on top, the linear independence of elements in B^{i+1}_{i+1} follows from the linear independence of B^i_{i+1}.

Showing that all other sets B^i_j are linearly independent is routine.   □

4.4. Braid group action

As a consequence of the results of Section 4.3, there are complexes of $(W, W)\#H$-modules

$$\Sigma_i = \; W_i \xrightarrow{\;\epsilon_i\;} W \;, \qquad \Sigma'_i = \; W \xrightarrow{\;\iota_i\;} W^{-e_1}_i\{-2\} \;.$$

Lemma 4.18. There exists an isomorphism of $(W, W)\#H$-modules

$$W_i \otimes_W W_i \cong W_i \oplus W^{e_1}_i\{2\}$$

Proof. First note that using isotopies, black strands can be moved out of the way in these bimodules as in [12, Lemma 5]. Thus the proof reduces to the proof of [10, Lemma 4.3] or [17, Lemma 3.5].   □

Lemma 4.19. There is a short exact sequence of $(W, W)\#H$-modules which splits as (W, W)-bimodules

$$0 \to W_{i,i+1} \to W_i \otimes_W W_{i+1} \otimes_W W_i \to W^{e_1}_i \to 0.$$

Proof. The map $W_{i,i+1} \to W_i \otimes_W W_{i+1} \otimes_W W_i$ is just given by $\alpha_{i,i+1}$ defined in (4.54).

Recall there is a map $W_i \otimes_W W_{i+1} \otimes_W W_i \to W_i$ defined on generators in (4.56) and (4.57). This is clearly a surjection and a straightforward calculation shows that this surjection is a p-DG map.

Note that
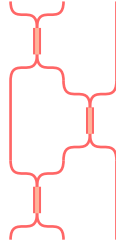
is in the kernel of $W_i \otimes_W W_{i+1} \otimes_W W_i \to W_i^{e_1}$. By a graded dimension count utilizing the bases of the bimodules from Propositions 4.10, 4.13, 4.17, we get the exactness of the sequence of the lemma.

Now we define a splitting map $W_i \otimes_W W_{i+1} \otimes_W W_i \to W_{i,i+1}$ by

 $\hspace{4cm}$ (4.58)

 $\hspace{4cm}$ (4.59)

Note that this splitting does not respect the p-DG structure.

Next we define a splitting map $W_i^{e_1} \to W_i \otimes_W W_{i+1} \otimes_W W_i$ by

 $\hspace{4cm}$ (4.60)

Note that this splitting map does not respect the p-DG structure either. All of these splitting maps are bimodule homomorphisms since they are compositions of bimodules maps described in detail in [12].

Thus there is a short exact sequence of $(W, W) \# H$-bimodules which splits as (W, W)-bimodules. $\quad \square$

The next crucial proposition is proved in a similar manner as in [10, Theorems 4.2, 4.4].

Proposition 4.20. The complexes of $(W, W) \# H$-modules satisfy the following relations in the relative homotopy category.

1. $\Sigma_i \circ \Sigma_i' \cong \mathrm{Id} \cong \Sigma_i' \circ \Sigma_i$,
2. $\Sigma_i \circ \Sigma_j \cong \Sigma_j \circ \Sigma_i$ for $|i - j| > 1$,
3. $\Sigma_i \circ \Sigma_j \circ \Sigma_i \cong \Sigma_j \circ \Sigma_i \circ \Sigma_j$ for $|i - j| = 1$.

We point out that, as in the usual homotopy category case, relative homotopy classes of p-DG bimodules over two p-DG algebras A and B give rise to functors from $\mathcal{C}^\partial(B, d_0)$ to $\mathcal{C}^\partial(A, d_0)$. Therefore, Proposition 4.20 can be regarded as an isomorphism of functors on the relative homotopy category of p-DG W-modules.

Proof. The first isomorphism follows as in [10, Theorem 4.2] or [17, Proposition 3.7] which use versions of Lemma 4.18.

The second item is clear.

The third isomorphism follows as in [10, Theorem 4.4] or [17, Proposition 3.9] which use versions of Lemma 4.19. $\quad \square$

Applying the p-extension functor $\mathcal{P}$, we obtain p-complexes of p-DG (W, W)-bimodules $T_i := \mathcal{P}(\Sigma_i)$ and $T_i' := \mathcal{P}(\Sigma_i')$ in the relative p-homotopy category $\mathcal{C}^\partial(W, \partial_0)$. Explicitly, these complexes look like:

$$T_i := \left(W_i \xrightarrow{=} \cdots \xrightarrow{=} W_i \xrightarrow{\epsilon_i} W \right) \tag{4.61}$$

$$T_i' := \left(W \xrightarrow{\iota_i} W_i^{-e_1}\{-2\} \xrightarrow{=} \cdots \xrightarrow{=} W_i^{-e_1}\{-2\} \right), \tag{4.62}$$

where the repeated terms appear $p - 1$ times. It should be pointed out that T_i and T_i' are just cones of the p-DG bimodule maps ϵ_i and ι_i in the relative homotopy category of p-DG (W, W)-bimodules. This is because $\mathcal{P}$ is exact (Proposition 2.4), and thus it sends the cones Σ_i and Σ_i' to cones in the relative p-homotopy category.

Theorem 4.21. The p-complexes of p-DG (W, W)-bimodules satisfy the following isomorphisms of functors on the relative p-homotopy category $\mathcal{C}^\partial(W, \partial_0)$.

1. $T_i \circ T_i' \cong \mathrm{Id} \cong T_i' \circ T_i$,
2. $T_i \circ T_j \cong T_j \circ T_i$ for $|i - j| > 1$,
3. $T_i \circ T_j \circ T_i \cong T_j \circ T_i \circ T_j$ for $|i - j| = 1$.

Proof. This follows from Proposition 4.20 by applying the functor $\mathcal{P}$ from Section 2. $\quad \square$

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