Exercises for Week 9

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Nov. 6.

Reading. Review Artin Sections 4.1-4.4 as we go along. Alternatively, consult any linear algebra book for the corresponding material. Read Artin Section 6.1-6.2.

1. The main difficulty we had with presentation is due to the following generalization of the first isomorphism theorem (FIT). Consider a surjective group homomorphism $\phi : G \to H$, and let $K := \ker \phi$ be the kernel. Let N be a normal subgroup of G contained in K. Show that ϕ factors as group homomorphisms $\phi = \overline{\phi} \circ \pi$

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & H \\ & G/N \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

   (Hint: Define these maps explicitly as in the FIT). Show that $\overline{\phi}$ is surjective, and determine its kernel. In general, $\overline{\phi}$ is not an isomorphism. (Try to construct examples where it is not, and think about how this applies in the situation of group presentations!)

2. Let $\psi : V \to V$ be a linear operator on a vector space of dimension two. Suppose ψ is not a scalar multiple of the identity transformation. Show that there is a $\mathbf{v}$ such that $\{\mathbf{v}, \psi(\mathbf{v})\}$ form a basis of V. Then compute the matrix of ψ with respect to this basis. (Try not to identify V with $\mathbb{R}^2$!)

3. Prove that given any unit length vectors $\mathbf{u}, \mathbf{v} \in U$, there is an orthogonal transformation ϕ of the Euclidean space U that takes $\mathbf{u}$ to $\mathbf{v}$.

4. Let $P := \{p(x)\}$ be the space of polynomials over $\mathbb{R}$ of degree at most n. Consider $D = \frac{d}{dx}$ as a linear operator on P.
   (a) Choose a basis of P to be $\{1, x, \ldots, x^n\}$. Write down the matrix of D with respect to this basis.
   (b) Choose a basis of P to be $\{1, x-1, \ldots, (x-1)^n\}$. Write down the matrix of D with respect to this basis.

5. Consider the two dimensional Euclidean space $\mathbb{R}^2$. Write down the matrices for the following operators.
   (a) Reflection about the x-axis ($y = 0$).
   (b) In general, the reflection about the $ax + by = 0$ line, where $(a, b) \in \mathbb{R}^2 \backslash \{0\}$.

6. Let $A \in O(2, \mathbb{R})$ be an orthogonal matrix. Determine when A has eigne-vectors in $\mathbb{R}^2$, and find the eigenvalues. (Hint: Consider the cases when A has determinant ± 1.)

7. Find matrix subgroups of $SO(2, \mathbb{R})$ that are isomorphic to (a) $\mathbb{Z}/(n)$ and (b) $\mathbb{Z}$.

8. Find a matrix subgroup of $O(n, \mathbb{R})$ that is isomorphic to S_n.