Exercises for Week 7

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 23.

Reading. With Dummit and Foote, please read Section 3.3. Alternatively, read Artin Sections 2.10, 2.11.

1. Prove that any index 2 subgroup H of G, i.e., $[G : H] = 2$, is normal. Use this to give another proof that $A_n \lhd S_n$.

2. Suppose $\phi : G \longrightarrow G'$ is a group homomorphism that does not collapse all of G onto the unit element of G' (usually we just say ϕ is non-trivial). Suppose G has order 20 and G' has order 35. What's the order of $\ker \phi$?

3. Consider the surjective group homomorphism

   $$\mathbb{Z}/(12) \longrightarrow \mathbb{Z}/(6), \overline{a} \mapsto \overline{a}.$$

   Work out the correspondence theorem in this particular situation by finding the subgroups of the these two groups that correspond to each other.

4. Show that $\mathbb{R}^* \cong \mathbb{R}^{>0} \times \mathbb{Z}/(2)$ by explicitly constructing an isomorphism and its inverse.

5. Let G be a group and $N \lhd G$ a normal subgroup.
   (a) Show that if G is cyclic, then so is G/N.
   (b) Show that if G is abelian, then so is G/N.

6. Show that there is an isomorphism of abelian groups

   $$\mathbb{Z}/(4) \times \mathbb{Z}/(3) \cong \mathbb{Z}/(12).$$

   (Hint: Find an element on the left hand side of order 12.)

7. Let G be a group and $G \times G$ its product with itself. Let $\Delta := \{(g, g) | g \in G\}$ be the *diagonal* subset.
   (a). Show that Δ is a subgroup of $G \times G$.
   (b). Show that if G is abelian, then Δ is a normal subgroup, and $G \times G/\Delta \cong G$.
   (c). Show by examining the case $G = S_3$ that, when G is not abelian, Δ is not a normal subgroup of $G \times G$.