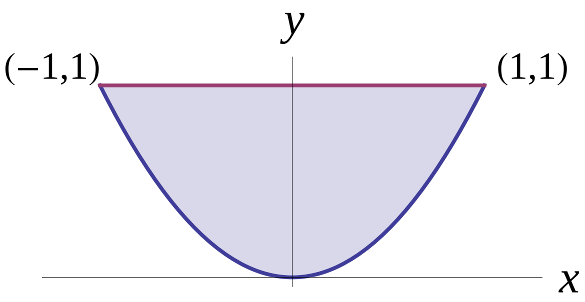
Math 120, Practice exam 2 solutions

1. First draw the region:



I. Check for critical points inside:

$$f_x = 3x^2 + y = 0$$
$$f_y = x + 3 = 0$$

The solution is $x = -3$, $y = -27$. This critical point is not in the region, so we do not include it on the list of candidates for our max/min.

II. Check the boundaries:

Top boundary is $y = 1$, for $-1 \le x \le 1$. There $f(x,1) = x^3+x+3$. The derivative $\frac{d}{dx}(x^3+x+3) = 3x^2+1$ is never zero, so there are no critical points on this boundary.

Bottom boundary is $y = x^2$, for $-1 \le x \le 1$. There $f(x,x^2) = x^3+x^3+3x^2 = 2x^3+3x^2$. The derivative $\frac{d}{dx}(2x^3+3x^2) = 6x^2+6x = 6x(x+1)$ is zero when $x = 0$ or $x = -1$. We get two critical points, $(0,0)$ and $(-1,1)$. Both of them are inside the region, with $-1 \le x \le 1$, so we include them on the list.

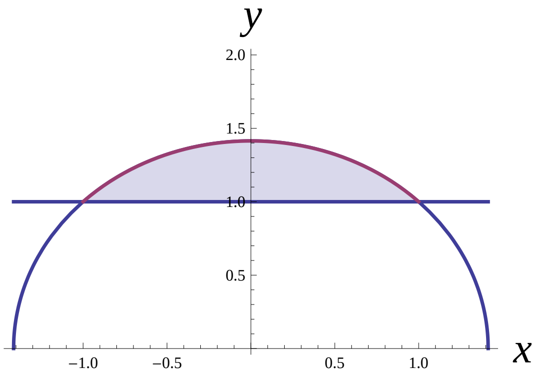
III. Include endpoints of the boundaries: $(-1,1)$ and $(1,1)$ (we already have the first one anyway, but not the second).

IV. List all the points, and compare values:

$$f(0,0) = 0$$
$$f(-1,1) = 1$$
$$f(1,1) = 5$$

The minimum value of the function on our region is 0, at $(0,0)$. The maximum value is 5, at $(1,1)$.

2. First draw the region:



The integral is difficult to do in cartesian coordinates, so we will convert to polar.

The curves intersect when $y = 1$ and $x^2 + y^2 = x^2 + 1 = 2$ so $x = \pm 1$ and $y = 1$. The angles corresponding to these endpoints are $\pi/4$ and $3\pi/4$, so those are the boundaries for θ.

Next we need boundaries for r.

The bottom boundary is $y = 1$. In polar coordinates, this says $r \sin\theta = 1$ or $r = 1/\sin\theta$.

The top boundary is $x^2 + y^2 = 2$ or $r^2 = 2$, so $r = \sqrt{2}$. We get the integral

$$\int_{\pi/4}^{3\pi/4} \int_{\frac{1}{\sin\theta}}^{\sqrt{2}} \frac{1}{r^3}\, r\, dr d\theta = \int_{\pi/4}^{3\pi/4} -\frac{1}{r}\bigg|_{\frac{1}{\sin\theta}}^{\sqrt{2}} d\theta = \int_{\pi/4}^{3\pi/4} \left(\sin\theta - \frac{1}{\sqrt{2}}\right) d\theta = \left(-\cos\theta - \frac{\theta}{\sqrt{2}}\right)\bigg|_{\pi/4}^{3\pi/4} = \sqrt{2} - \frac{\pi}{2\sqrt{2}}$$

3. The matching is 1C, 2E, 3A, 4D.

Explanation was not needed, but here is one possibility (there are many others):

Note that $\mathbf{F}_4$ has both coordinates positive, so all arrows point up and to the right. The only matching picture is D.

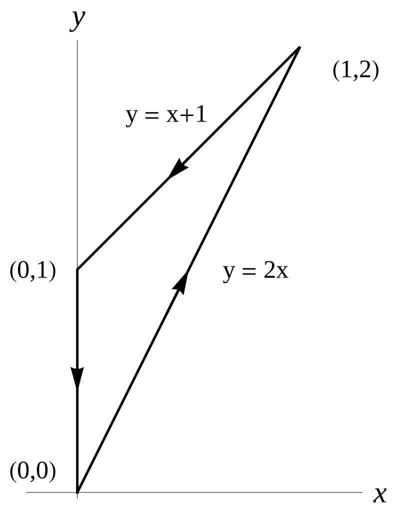
Next, notice that $\mathbf{F}_3$ has the second coordinate positive. The only remaining picture where arrows always point up is A.

For $\mathbf{F}_2$, the arrows will always be perpendicular to level curves of the function, which are lines with slope 1 ($\sin(x - y)$ is constant whenever $x - y$ is constant). The matching picture is E.

For $\mathbf{F}_1$, you can compute $\mathbf{F}_1 = (y, x)$. In first quadrant, the arrows will point up and to t he right (both x,y positive). In the third quadrant, they will point down and to the left (both x, y negative). The matching picture is C.

4. (a) $Q_x - P_y = 3x^2 \sin y - (4x + 4x^2 \sin y) = -4x$ so the field is not conservative.

   (b) C is a closed counterclockwise curve, the field $\mathbf{F}$ and all its derivatives are defined everywhere, so we can use Green's theorem.



The region is bounded by the lines between the given points, with equations $x = 0$, $y = 2x$ and $y = x + 1$. The integral becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) dA = \int_0^1 \int_{2x}^{x+1} (-4x) dy dx = -\frac{2}{3}$$

5. The curve is a counterclockwise circle with radius 2, which can be parametrized by $\mathbf{r} = \langle 2\cos t, 2\sin t \rangle$ for $0 \le t \le 2\pi$. This gives $\mathbf{r}' = \langle -2\sin t, 2\cos t \rangle$, and $ds = |\mathbf{r}'| dt = 2dt$.

$$\int_C x^2 ds = \int_0^{2\pi} 4\cos^2 t \, 2dt = \int_0^{2\pi} 8 \, \frac{1+\cos(2t)}{2} dt = \int_0^{2\pi} (4 + 4\cos(2t)) \, dt = 4t + 2\sin(2t) \Big|_0^{2\pi} = 8\pi$$

6. The field is defined everywhere, and $Q_x = P_y = \cos(x^2y^2) - 2x^2y^2 \sin(x^2y^2)$, so $\mathbf{F}$ is conservative.

   The potential function is difficult to find, so instead we take advantage of path independence. The given curve starts at $\mathbf{r}(-1) = \langle -1, 0 \rangle$ and ends at $\mathbf{r}(1) = \langle 1, 0 \rangle$. We can replace it with C', a straight line between these point, namely the x-axis.

   We can parametrize C' by $\mathbf{r}(t) = \langle t, 0 \rangle$ for $-1 \le t \le 1$. The field becomes $\mathbf{F} = \langle 0 + t^2, t\cos(0) + 0 \rangle = \langle t^2, t \rangle$. We also need $\mathbf{r}'(t) = \langle 1, 0 \rangle$. Putting it all together, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F} \cdot \mathbf{r}' dt = \int_{-1}^{1} t^2 dt = \frac{t^3}{3} \Big|_{-1}^{1} = \frac{2}{3}$$