Exercises for Week 5

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 9.

Reading. With Dummit and Foote, please read Sections 1.6, 3.1, 3.2. Alternatively, read Artin Sections 2.5, 2.6.

1. Let $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ be the map $f(a) = e^{2\pi i a}$. Prove that it's a group homomorphism and determine its kernel and image.

2. Let G be a group and $x \in G$ be an element of finite order n. Show that

$$\phi_x : \mathbb{Z}/(n) \longrightarrow \langle x \rangle, \quad \overline{a} \mapsto x^a$$

   is a well-defined group homomorphism, i.e., it's independent of representatives you choose for $\overline{a}$. Then prove that ϕ_x is an isomorphism.

3. For S_3, and $g = (12) =$ , write out explicitly what the conjugation-isomorphism-by-g does to all elements of S_3.

4. Prove that there is an isomorphism of groups

$$U(1) := \left\{ z \in \mathbb{C}^* \big| |z| = 1 \right\} \cong SO(2, \mathbb{R}) := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \big| \theta \in \mathbb{R} \right\},$$

   by explicitly constructing the isomorphisms.

5. Let $\phi \colon \mathbb{Z}/(12) \mapsto \mathbb{Z}/(3), \overline{a} \mapsto \overline{a}$. Prove that this is a well-defined group homomorphism, and find its kernel.

6. (a) Prove that if K is a normal subgroup of G, and H is an arbitrary subgroup of G, then $K \cap H \lhd H$.
   (b) If $\phi : G \to G'$ is a group homomorphism, and H is a subgroup of G. Use the previous part to determine that under what condition the restriction $\phi|_H : H \to G'$ is an injective homomorphism.

7. For S_3 and its normal subgroup A_3, write out the 2×2 multiplication table for the cosets (quotient group) S_3/A_3. Identify it with a more familiar group.

8. Prove that in any group G, and given elements $a, b \in G$, there is a conjugation automorphism of G taking ab into ba.