Math 151b Midterm Exam

February 8, 2019

Notes. No discussions allowed for the exam. You have one day to finish the exam (Friday 5:00pm-Saturday 5:00pm). No late exams shall receive a grade.

You may use any result from Hatcher unless instructed to repeat the proof.

(i) Prove that, if $\{h^*\}$ is a reduced cohomology theory on CW complexes and $\delta : h^k \longrightarrow h^{k+1}$ is the coboundary map, then there is an isomorphism

$$h^*(X) \longrightarrow h^{*+k}(\Sigma^k X)$$

that is natural with respect to any continuous map of CW complexes.

(ii) Show that the complex

$$0 \longrightarrow \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3 \longrightarrow 0$$

where A is the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

is homotopic to the the complex

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{B} \mathbb{Z}^2 \longrightarrow 0$$

where B is the matrix

$$B = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

(iii) Inside $\mathbb{C}P^2$, consider the image of

$$\phi : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^2, \quad [x, y] \mapsto [x^2, xy, y^2].$$

Compute the homology class of $[\phi(\mathbb{C}P^1)]$ in $\mathrm{H}_2(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$.

(iv) The n-dimensional torus T^n can be identified with $\mathbb{R}^n/\mathbb{Z}^n$, so that any integral matrix $A \in \mathrm{M}(n, \mathbb{Z})$ induces an endomorphism of T^n. Show that the Lefschetz number of this endormophism of T^n is equal to $\det(I_n - A)$.

(v) Prove that, if a compact homology n-manifold X occurs as the boundary of an orientable acyclic homology $n+1$-manifold (i.e., a manifold with vanishing reduced homology), then X is a homology n-sphere. (Hint: Use Theorem 3.43, Poincare duality with boundary).