Exercises for Week 13

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Dec. 6 (Note the special date!).

Reading. Read Dummit and Foote Sections 7.1, 7.2, 10.1. Alternatively, read Artin Sections 11.1, 11.2, 14.1.

1. Let R and S be two rings.
   (a) Show that the product $R \times S$ is a ring under componentwise addition and multiplication.
   (b) Show that $R \times S$ is commutative iff both R and S are commutative.
   (c) Prove that, if $R = S$, $\{(r, r) | r \in R\} \subset R \times R$ is a subring.

2. The center of a ring R is $\{z \in R | z \cdot r = r \cdot z, \ \forall r \in R\}$. Prove that the center of a ring is a subring. Find the center for the $n \times n$-matrix ring $\mathrm{M}(n, \mathbb{R})$.

3. Use the definition of ring isomorphism to show that the group ring $F[\mathbb{Z}]$ is isomorphic to the ring of Laurent polynomials over F:
$$F[x, x^{-1}] := \{\sum_{i \in Z} a_i x^i | \text{all but finitely many } a_i \text{ is zero}\}.$$

4. Show that if M is an R-module, then $0 \cdot m = 0$, and $(-1) \cdot m = -m$ for for all $m \in M$.

5. Prove the statement we made in class: Modules over $\mathbb{Z}$ coincide with abelian groups.

6. Product modules. Consider a ring R and two given modules M, N over R. We define their *direct sum*
$$M \oplus N := \{(m, n) | m \in M, n \in N\},$$
   with the R-action $r \cdot (m, n) := (rm, rn), \forall r \in R, \ m \in M, \ n \in N$. Show that $M \oplus N$ is an R-module.

7. Let R be a ring, and M be an R-module. Let $R^* := \{x \in R | x \text{ has a left and right inverse.}\}$.
   (a) Show that $R^* \times M \to M$, $(r, m) \mapsto rm$ defines a group action.
   (b) When $R = FG$ is the group ring of a group G, and M is an F-vector space with a G action. Show that $G \subset (FG)^*$, and the action in (a), when restricted to G, agrees with the G-action on M.

8. Let F be a field, and X the following 2×2-matrix
$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
   Consider the $F[x]$ module $M := F^2$ by letting x act on M through the matrix X. Find a non-zero proper $F[x]$-submodule of M.