Math 120, Practice final exam

1. [20 pts] Find a vector equation (of the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$) for the line of intersection of the planes $x + 2y - z = 4$ and $x - y + z = 1$.

2. [20 pts] Use the chain rule to find f_x and f_y at $(x, y) = (1, 0)$ for the following function:

$$f(u, v) = \ln(u - v), \quad u(x, y) = x\cos(y), \quad v(x, y) = \sin(xy^2)$$

3. [30 pts] Find the absolute maximum and absolute minimum of the function $f(x, y, z) = x - y^2 + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

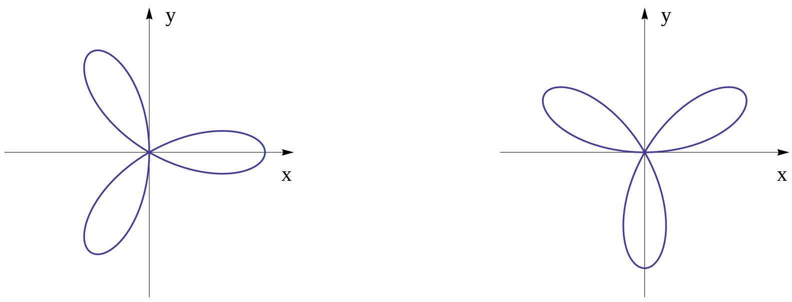
4. (a) [10 pts] If $f(x)$ and $g(y)$ are continuously differentiable functions on the intervals $[a, b]$ and $[c, d]$, respectively, and $R = [a, b] \times [c, d]$, show that

$$\int\int_R f'(x)g'(y)dA = (f(b) - f(a))(g(d) - g(c)).$$

   (b) [10 pts] Let D_1 be the square $[-1, 1] \times [-1, 1]$ and D_2 be the disk (in $\mathbb{R}^2$) of radius 1 centered at the origin. Is the following inequality true? Justify your answer.

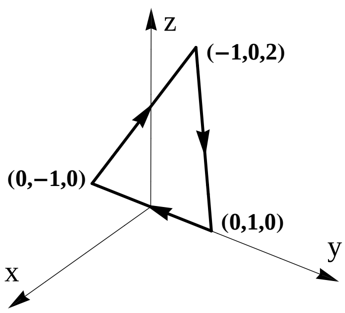
$$\iint_{D_1} (e^{x^2} + e^{y^2})dA \geq \iint_{D_2} (e^{x^2} + e^{y^2})dA.$$

5. (a) [5 pts] Which graph corresponds to the curve $r = \sin(3\theta)$? Give a brief reason to support your answer.



(b) [20 pts] Find the area enclosed by one leaf of $r = \sin(3\theta)$.

6. [30 pts] Let $\mathbf{F}(x, y, z) = \langle \sin(e^z) + y, -x + \tan(y), \cos(z^2) + x^2 \rangle$. Suppose C consists of three line segments, from $(0, -1, 0)$ to $(-1, 0, 2)$, then to $(0, 1, 0)$, then back to $(0, -1, 0)$ as shown the figure below. Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.



7. [30 pts] Let $\mathbf{F}(x, y) = \langle 2x(y+1) + y^3,\ \ 3y^2(x+1) + x^2 \rangle$. Let C be the curve parametrized by

$$\mathbf{r}(t) = \langle \pi^{\ln(t^2+1)} \sin(\pi t), t^{10} \cos(\pi t) \rangle$$

from $t = 0$ to $t = 1$. Evaluate the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

8. In each part (a) - (e), circle the correct statement. You do not need to justify your choices.

(a) [5 pts] Let $\mathbf{F}(x, y, z) = (0, 0, z^5)$, and let S be the unit sphere centered at $(0, 0, 0)$, oriented outwards. The integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is

i. positive

ii. negative

iii. zero

(b) [5 pts] Let $\mathbf{F}(x, y) = \langle y, -x \rangle$, and let C be the clockwise boundary of the rectangle with vertices $(0, 0)$, $(0, 1)$, $(2, 1)$, $(2, 0)$, as shown below. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is

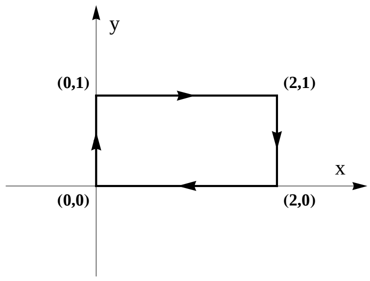
i. 4

ii. 2

iii. 0

iv. -2

v. -4



(c) [5 pts] Let $\mathbf{v}$ and $\mathbf{w}$ be vectors with three components. The product $(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} \times \mathbf{w})$ is

i. positive

ii. negative

iii. zero

iv. it can be any of the above, depending on the vectors

(d) [5 pts] The level curves of $f(x, y) = \dfrac{y}{x - y}$ are

i. parabolas

ii. hyperbolas

iii. lines

iv. planes

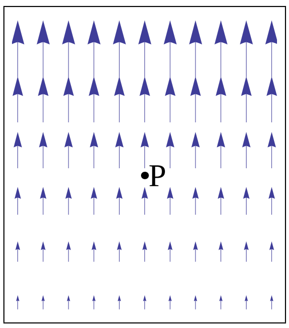
v. none of the above

(e) [5 pts] Shown to the right is a field $\mathbf{F}$ and a point P. The value of $\operatorname{div}(\mathbf{F})$ at P is

i. positive

ii. negative

iii. zero



9. Let C be the curve $z = 4 - x^2$, from $x = -2$ to $x = 2$. Let S be the surface of revolution, obtained by rotating C about the x-axis.

   (a) [15 pts] Find a parametric representation for S. Indicate the domain for your parameters.

   (b) [25 pts] Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for outward orientation of S, where $\mathbf{F}$ is the field

$$\mathbf{F} = \left\langle -\frac{z}{x^2 + y^2 + z^2},\ 0,\ \frac{x}{x^2 + y^2 + z^2} \right\rangle.$$

10. Let S be the cone parametrized by $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 2u \rangle$, $0 \le u \le 2$ and $0 \le v \le 2\pi$, with downward orientation.

   (a) [15 pts] Evaluate $\iint_S x^2\, dS$.

   (b) [15 pts] Let $\mathbf{F}(x, y, z) = \langle x + y, y^2, z \rangle$. Find $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

11. [30 pts] Let $\mathbf{F}(x, y, z) = \langle \cos(yz), yz + 1, z - \sin(x^2) \rangle$. Let V be the part of the unit ball given by the equations $x^2 + y^2 + z^2 \le 1$, $y \ge 0$, $z \le 0$. Let S be the surface of the solid, oriented outward. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.