Exercises for Week 5

The work handed in should be entirely your own. You can consult the textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 2.

Reading. Read Sections 2.3-2.4 of the textbook carefully.

1. Section 2.3 Exercise 11, 14.
2. Follow the steps and prove Theorem 2.14 on your own.

   We have known that, if V and W are vector spaces, and $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ are ordered bases for V and W respectively. Then, via the chosen bases, vectors can be expanded into a (column) of numbers:

   $$v \in V \Rightarrow v = \sum_{i=1}^{n} a_i v_i \Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n.$$

   Since these tuple of numbers depend on the choice of β, we denote the column vector by

   $$[v]^\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

   (The text book uses $[v]_\beta$ instead which is a bit awkward from what you'll show) We have also learnt in class that, any linear map $T : V \longrightarrow W$ is completely determined by its matrix $A = [T]_\beta^\gamma$ with respect to β and γ. Then we have

   $$[T(v)]^\gamma = [T]_\beta^\gamma \cdot [v]^\beta. \qquad (*)$$

   (1) Define the following linear map, for a fixed vector $v \in V$ by

   $$F_v : \mathbb{F} \longrightarrow V, \quad F_v(a) = av.$$

   Show that F_v is linear (state which axioms of vector spaces you use in the proof).

   (2) $\mathbb{F}$ as a 1-dimensional space over $\mathbb{F}$ has the standard basis $\alpha := \{1\}$. Compute the matrix $[F_v]_\alpha^\beta$.

   (3) Now use the compostion of linear maps giving rise to matrix multiplication to show that $(*)$ is true.

3. Section 2.4 Exercise 1, 2 (a) (d) (e), 3 (c), (d), 4, 6, 9, 12, 19, 22