Math 113 Midterm

Don't forget to write down clearly your **Name**: and **ID number:**

1. True or False (10 points). Mark the box in front of a correct answer.

☐ The set $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, with the associative law $\times$, is a group with 1 as the unit.

☐ The modular numbers $\mathbb{Z}/(5)$ is a subgroup of $\mathbb{Z}$.

☐ The integers 5 and -6 are coprime.

☐ If $H_1,\ H_2$ are subgroups of G, then so is $H_1 \cup H_2$ (a subgroup of G).

☐ Any cyclic group is abelian.

☐ The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is an element of the group $GL(2, \mathbb{R})$.

☐ Under multiplication, the modular numbers without zero $\mathbb{Z}/(7)^* := \mathbb{Z}/(7)\backslash\{\overline{0}\}$ forms an abelian group.

☐ The element  $\in S_3$ has order 3.

☐ The map $\det : GL(2, \mathbb{R}) \longrightarrow \mathbb{R}^*$ is a group homomorphism.

☐ $\mathbb{Z}/(9)$ is a simple group.

2. Multiple choices (10 points). Mark the box in front of the correct answer.

(1) The partition of the set $T = \{1, 2, 3, 4, 5\} = \{1, 2, 3\} \sqcup \{4, 5\}$ defines an equivalence relation on T. Which of the following pairs of elements are **not** equivalent under this relation
☐ 1 and 1 ☐ 1 and 2 ☐ 1 and 3 ☐ 1 and 4

(2) Which of the following elements (inside their groups) has infinite order?
☐ $5 \in \mathbb{Z}$ ☐ $\overline{5} \in \mathbb{Z}/(100)$ ☐ $e^{\frac{2\pi i}{7}} \in \mathbb{C}^*$ ☐ $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL(2)$

(3) Consider a group of order 10^{100}. Which of the following number is **not** possibly the order of an element in this group?
☐ 3 ☐ 4 ☐ 5 ☐ 10

(4) Which of the following homomorphism is **not** an isomorphism?

☐ $\phi : (\mathbb{R}, +) \longrightarrow U(1) := \{z \in \mathbb{C}^* \big| |z| = 1\}, a \mapsto e^{2\pi i a}$
☐ $\phi : (\mathbb{R}, +) \longrightarrow (\mathbb{R}, +),\ \phi(a) = -a$
☐ $\exp : (\mathbb{R}, +) \longrightarrow (\mathbb{R}^{>0}, \times), a \mapsto e^a$
☐ $\phi : (\mathbb{R}, +) \longrightarrow \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \big| a \in \mathbb{R} \right\}, a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

(5) The subgroup $(42) + (30)$ of $\mathbb{Z}$ equals which of the following groups?

☐ (3) ☐ (6) ☐ (7) ☐ (12)

3. Subgroups and normal subgroups (7 points).

(a) Explain why $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$.

(b) Show that $H := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ is **not** a normal subgroup of $GL(2, \mathbb{R})$ by finding a $g \in GL(2, \mathbb{R})$ such that $gHg^{-1} \not\subset H$.

4. Homomorphisms from elements (8 points). Let G be a group, and $\mathrm{Hom}_{\mathrm{Group}}(\mathbb{Z}, G)$ be the set of all group homomorphisms from $\mathbb{Z}$ to G.

(a) Show that, fixing any element $x \in G$, the assignment

$$\phi_x : \mathbb{Z} \to G, a \mapsto x^a,$$

is a group homomorphism.

(b) On the other hand, given any homomorphism $\psi \in \mathrm{Hom}_{\mathrm{Group}}(\mathbb{Z}, G)$, define the element $y := \psi(1) \in G$. Show that, under the map of (a), we have

$$\phi_y = \psi \in \mathrm{Hom}_{\mathrm{Group}}(\mathbb{Z}, G),$$

that is, they agree as homomorphism of groups.