

CATEGORIFYING HECKE ALGEBRAS AT PRIME ROOTS OF UNITY, PART I

BEN ELIAS AND YOU QI

ABSTRACT. We equip the type A diagrammatic Hecke category with a special derivation, so that after specialization to characteristic p it becomes a p -dg category. We prove that the defining relations of the Hecke algebra are satisfied in the p -dg Grothendieck group. We conjecture that the p -dg Grothendieck group is isomorphic to the Iwahori-Hecke algebra, equipping it with a basis which may differ from both the Kazhdan-Lusztig basis and the p -canonical basis. More precise conjectures will be found in the sequel.

Here are some other results contained in this paper. We provide an incomplete proof of the classification of all degree $+2$ derivations on the diagrammatic Hecke category, and a complete proof of the classification of those derivations for which the defining relations of the Hecke algebra are satisfied in the p -dg Grothendieck group. In particular, our special derivation is unique up to duality and equivalence. We prove that no such derivation exists in simply-laced types outside of finite and affine type A . We also examine a particular Bott-Samelson bimodule in type A_7 , which is indecomposable in characteristic 2 but decomposable in all other characteristics. We prove that this Bott-Samelson bimodule admits no nontrivial fantastic filtrations in any characteristic, which is the analogue in the p -dg setting of being indecomposable.

CONTENTS

1. Introduction	7692
2. The answer	7705
3. Preliminaries	7707
4. One color considerations	7708
5. Two distant colors	7717
6. Two adjacent colors	7721
7. Three colors	7737
8. Does the differential have divided powers?	7738
9. A surprising example with $n = 8$	7739
Acknowledgments	7740
References	7741

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1. INTRODUCTION

In order to understand what we do in this paper and why, it helps to know a bit about categorification at a root of unity, and its most popular techniques.

1.1. Categorification at a root of unity.

Definition 1.1. Let A be an algebra over a ground ring \mathbb{k} . A \mathbb{k} -linear map $d: A \rightarrow A$ is called an *even differential*¹ or a *derivation* if it satisfies the Leibniz rule

$$(1.1) \quad d(fg) = d(f)g + fd(g)$$

for any $f, g \in A$.

In his seminal paper [Kho16], Khovanov began the program of “categorification at a root of unity.” He defined a p -dg algebra to be a graded algebra A over a field \mathbb{k} of characteristic p , equipped with an even differential d which is homogeneous of degree 2, satisfying $d^p = 0$. A p -dg category is defined similarly. Khovanov defined the derived category of a p -dg category, and observed that its Grothendieck group is naturally a module over the p -th cyclotomic integers. More precisely, it is a module over \mathbb{O}_p , which is the extension of \mathbb{Z} by a variable ζ for which ζ^2 is a primitive p -th root of unity. To categorify something like the quantum group at a root of unity, one should hunt for an interesting p -dg category.

The work of Khovanov-Qi [KQ15] successfully applied this idea to the categorification of quantum groups. They took the quiver Hecke algebra or KLR algebra [KL11, Rou08] in simply-laced type, which was known to categorify the positive half of the quantum group at generic q , and equipped it with a degree 2 derivation d . After specialization to characteristic p , it becomes a p -dg category. They conjectured [KQ15, Conjecture 4.18] that this p -dg quiver Hecke algebra categorifies the positive half of the small quantum group at $q = \zeta$. They were able to prove this conjecture for \mathfrak{sl}_2 , and to prove in general that the defining relations of the small quantum group hold in the Grothendieck group [KQ15, Theorem 3.35, Theorem 4.14]. The conjecture has been proven by Andrew Stephens for small quantum \mathfrak{sl}_3 [Ste18].

Before continuing, let us note a common theme to the construction of p -dg algebras: the differential is independent of the prime p . Typically there is just one differential defined on an integral form of the algebra, which satisfies $d^p = 0$ only after specialization to characteristic p . Here is why.

Definition 1.2. An *even differential graded algebra*, or simply *edg-algebra*, (A, d) is a graded \mathbb{Z} -algebra A equipped with an even differential of degree 2. An edg-algebra is called a *gaea* (short for \mathbb{G}_a -equivariant algebra, or a metaphor for a “global” p -dg algebra) if the operator $d^{(k)} := \frac{d^k}{k!}$ can be defined over \mathbb{Z} .

A gaea specializes to a p -dg algebra after changing base from \mathbb{Z} to any field of characteristic p , because

$$d^p = p!d^{(p)} = 0.$$

¹A differential graded algebra has a degree one map called the differential which satisfies the super Leibniz rule. For an even degree map, there is no difference between the ordinary and the super Leibniz rule.

Example 1.3. Let $R = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring, graded such that $\deg(x_i) = 2$, and consider the operator

$$(1.2) \quad d = \sum_{i=1}^n x_i^2 \frac{\partial}{\partial x_i}.$$

This is the even differential determined by the Leibniz rule and the equation

$$(1.3) \quad d(x_i) = x_i^2$$

for all i . We call this the *standard differential* on the polynomial ring. The operator $d^{(k)}$ sends x_i^ℓ to $\binom{\ell+k-1}{k} x_i^{\ell+k}$, and is well-defined over \mathbb{Z} . One can prove that (after specialization to characteristic p) this p -dg algebra is quasi-isomorphic to the ground field \mathbb{k} , so that its p -dg Grothendieck group is just \mathbb{O}_p .

In this new terminology, Khovanov-Qi equipped the quiver Hecke algebra with the structure of a gaea. This illustrates the principle that, if you already have an additive categorification of an algebra at generic q , you should try to equip it with a differential (extending to a gaea) in order to categorify the specialization to any prime root of unity. As further illustration of this principle, we categorified the entire quantum group of \mathfrak{sl}_2 in [EQ16a, EQ16b], equipping Lauda's category \mathcal{U} and Khovanov-Lauda-Mackaay-Stosic's category $\dot{\mathcal{U}}$ (see [Lau10] and [KLMS12] respectively) with such a differential.

In this paper, we equip the diagrammatic Hecke category in finite and affine type A with the structure of a gaea, hoping to categorify the Iwahori-Hecke algebra at a root of unity. We also prove a negative result in any other simply-laced type.

Let us note that the differential d on the KLR algebra in type A_1 is, for all effective purposes, induced by the standard differential on the polynomial ring discussed in Example 1.3. What is more interesting is that there is a large family of possible differentials on the KLR algebra, but that only the standard differential (and its dual) gives rise to a p -dg algebra with the correct p -dg Grothendieck group! The same seems to be true of the diagrammatic Hecke category.

To reiterate, constructing a differential on the Hecke category is not the hard part, and it is only the first step. The real difficulty lies in computing the p -dg Grothendieck group. Let us explain some of the technology involved in understanding p -dg Grothendieck groups.

1.2. Fantastic filtrations. In an additive category, suppose we have a direct sum decomposition

$$X \cong \bigoplus_{j \in I} M_j$$

for some finite index set I . This produces a relation on the (split) Grothendieck group

$$(1.4) \quad [X] = \sum [M_j].$$

If (1.4) is a relation you want then you should prove it by constructing a direct sum decomposition. In practical terms, to prove that $X \cong \bigoplus M_j$, one must construct projection maps

$$p_j: X \rightarrow M_j$$

and inclusion maps

$$i_j: M_j \rightarrow X$$

which satisfy the basic axioms

$$\begin{aligned} (1.5a) \quad & p_j i_j = \mathrm{id}_{M_j}, \\ (1.5b) \quad & p_j i_k = 0 \quad \text{if } j \neq k, \\ (1.5c) \quad & \mathrm{id}_X = \sum_{j \in I} i_j p_j. \end{aligned}$$

Now consider the exact same scenario, but in a p -dg category. To make a long story short, (1.5) is not enough information to deduce that (1.4) holds in the p -dg Grothendieck group. The object X is not actually a direct sum of the objects M_j because the differential need not preserve the summands. Let us say this more precisely. The functor $\mathrm{Hom}(X, -)$ is (by definition) a representable left module over the category, where each piece $\mathrm{Hom}(X, Y)$ is equipped with a differential. Let $e_j = i_j p_j \in \mathrm{End}(X)$. Then the direct summand $\mathrm{Hom}(X, -)e_j \subset \mathrm{Hom}(X, -)$ need not be preserved by the differential on Hom spaces. It is not even desirable for $\mathrm{Hom}(X, -)e_j$ to be preserved by the differential, as this condition often fails in practice.

However, one might hope that the differential acts on the pieces of this decomposition in an “upper-triangular” fashion, which would equip X with a filtration whose subquotients might agree with the M_j . This idea is codified in [EQ16a, §5] in the notion of an *Fc-filtration*. This is shorthand for either a *fantastic filtration* or a *finite cell filtration*, your choice. A direct sum decomposition as in (1.5) is said to *lift to an Fc-filtration* if there exists a partial order on the index set I satisfying

$$(1.6) \quad p_j d(i_k) = 0 \quad \text{whenever } j \leq k.$$

If this can be done, then (1.4) still holds in the p -dg Grothendieck group.

The reader new to this theory should think of (1.6) as consisting of two separate statements. The first statement, $p_j d(i_k) = 0$ whenever $j < k$, implies that $\mathrm{Hom}(X, -)$ is filtered with subquotients $\mathrm{Hom}(X, -)e_j$. The second statement, $p_j d(i_k) = 0$ when $j = k$, implies that $\mathrm{Hom}(X, -)e_j$ and $\mathrm{Hom}(M_j, -)$ are isomorphic as p -dg modules (i.e. the natural isomorphism between these functors intertwines the differential). Since $\mathrm{Hom}(M_j, -)$ is representable, this implies that $\mathrm{Hom}(X, -)e_j$ is *cofibrant* and *compact*, which is needed for it to have a symbol in the Grothendieck group in the first place.

Remark 1.4. A simpler idea is that of a *dg-filtration* on an object X , which is a complete collection of orthogonal idempotents e_j satisfying $e_j d(e_k) = 0$ for $j < k$. This equips $\mathrm{Hom}(X, -)$ with a filtration by p -dg submodules which are additive summands. The additional data in an Fc-filtration goes one step further and proves that the subquotients $\mathrm{Hom}(X, -)e_j$ are isomorphic to some known representable modules $\mathrm{Hom}(M_j, -)$. For a dg-filtration, it is not obvious that the subquotients $\mathrm{Hom}(X, -)e_j$ will be cofibrant.

It is not really important which partial order on I gives rise to (1.6), only that some partial order should exist. Practically speaking, the method for determining if a direct sum decomposition lifts to an Fc-filtration is as follows. Construct an oriented graph $\Gamma_{I,d}$ with vertex set I , having an edge from k to j labeled by the degree $+2$ morphism $p_j d(i_k)$. Erase all edges with the zero label. If $\Gamma_{I,d}$ has no oriented cycles (and in particular, no loops) then it is possible to find a partial order on I satisfying (1.6), and consequently (1.4) holds.

Direct sum decompositions are somewhat fluid: there are usually many valid choices for the projection maps $p = \{p_j\}$ and inclusion maps $i = \{i_j\}$. Note that the graph under study depends on both the differential d and the choice of projection and inclusion maps, so we could denote it by $\Gamma_{I,d,p,i}$ to emphasize this point. What is rather amazing is that the Fc-filtration requirement is incredibly good at rigidifying the situation: while there may be large families of differentials d on a category, and large families of projection and inclusion maps, there is often a unique (up to symmetry) triple (d, p, i) such that $\Gamma_{I,d,p,i}$ has no cycles! If not unique, it is often severely restrictive.

Remark 1.5. Note that rescaling the inclusion and projection maps (e.g. multiplying p_j an invertible scalar κ , and multiplying i_j by κ^{-1} , for some j) will not change the graph, it will only rescale the edge labels. This is one symmetry we use freely below.

To illustrate this, let us return to the setting of categorified quantum groups. The quantum group has certain defining relations, such as $ef1_\lambda = fe1_\lambda + [\lambda]1_\lambda$, which are usually categorified in $\hat{\mathcal{U}}$ by direct sum decompositions (for which Lauda wrote down some inclusion and projection maps explicitly). Let us call these the *defining direct sum decompositions* in $\hat{\mathcal{U}}$. If a p -differential d on $\hat{\mathcal{U}}$ is to give rise to the correct Grothendieck group, then at the least it should be the case that $\Gamma_{I,d}$ has no cycles for all the defining direct sum decompositions (for some choice of projection and inclusion maps). Let us temporarily call such a differential *good*.

In [EQ16a, EQ16b] we first computed all the possible p -differentials on the categories \mathcal{U} and $\hat{\mathcal{U}}$. This produced a rather large family of differentials. Then in [EQ16a, Proposition 5.14] we computed which differentials in this family were good. That is, for each differential we computed the graphs $\Gamma_{I,d}$ for all the defining direct sum decompositions (and all possible inclusion and projection maps), and asked which graphs had no cycles. Happily, being good is a strong enough condition to pin down the differential and the inclusion and projection maps precisely! Ultimately, only two nonzero differentials d and \bar{d} were good, and these differentials are intertwined by the duality functor (which flips diagrams upside-down). Thus already one deduces that there are only two dual p -differentials d and \bar{d} which could possibly give rise to the correct Grothendieck group (caveat: see Remark 1.9), though one still needs to confirm that they do have the correct Grothendieck group.

In summary, that the defining direct sum decompositions lift to Fc-filtrations is typically a very restrictive property for a differential, and differentials which satisfy it are quite special and interesting. Not only that, but the projection and inclusion maps compatible with these differentials should be considered as particularly nice.

Remark 1.6. Perhaps most intriguingly, the special partial orders on the index sets I induced by the cycle-less graph $\Gamma_{I,d}$ are new and unfamiliar structures which were invisible before the introduction of the differential. To state a rough moral: categorification replaces structure coefficients (numbers) with multiplicity spaces (vector spaces), while p -dg categorification equips these multiplicity spaces with a filtration, whose shadow is now combinatorial (an oriented graph).

Remark 1.7. Essentially everything discussed in this section depended only on the differential as defined over \mathbb{Z} , but not on the choice of prime p or on the fact that $d^p = 0$. This is not obvious, as specializing to characteristic p could, in

theory, eliminate cycles in the graph $\Gamma_{I,d}$, but in practice it does not. In other words, fantastic filtrations should really be considered as a theory intrinsic to gaeas. At the current moment, the homological algebra of gaeas has not been developed to the same degree as p -dg algebras were in [Qi14], so whenever discussing the Grothendieck group we play it safe and talk only about p -dg algebras.

1.3. Computing the Grothendieck group. A useful tool towards computing the Grothendieck group has been the following result of Qi [Qi14], an analogue of the positive dg algebra case by [Sch11].

Proposition 1.8. *In the special case when A is a positively graded p -dg algebra,² the p -dg Grothendieck group is just the specialization at $q = \zeta$ of the original Grothendieck group.*

When A is not positively graded this result is often false, and great caution is required. Most interesting categorifications are not positively graded. In [EQ16a] we developed Fc-filtrations as part of a game which manipulates a p -dg category until, hopefully, we can apply Proposition 1.8. Let us elaborate on this method in the context of \mathcal{U} , which categorifies the quantum group $U_q(\mathfrak{sl}_2)$.

A common technique in the study of additive and abelian categories is to choose a projective generator and work instead with its endomorphism ring. Given a collection of self-dual indecomposable objects we can study the full subcategory \mathcal{P} in \mathcal{U} with those objects. Equivalently, letting P be the direct sum of these self-dual indecomposable objects, we can study the p -dg algebra $\text{End}(P)$. Now the size of morphism spaces between objects in \mathcal{U} is determined by a particular sesquilinear pairing on $U_q(\mathfrak{sl}_2)$, see [Lau10, §1.1]. The self-dual indecomposable objects in \mathcal{U} categorify Lusztig's canonical basis, and the pairing of canonical basis elements has only non-negative powers of q . Consequently, \mathcal{P} is a positively-graded category, i.e., $\text{End}(P)$ is a positively-graded algebra, from which one easily computes both the ordinary and the p -dg Grothendieck group of \mathcal{P} . The task is to relate the p -dg category \mathcal{U} with the p -dg category \mathcal{P} (the underlying additive categories are Morita equivalent, but the p -dg setting is more subtle).

In the additive setting, X is generated by \mathcal{P} if it can be expressed as a direct sum of the objects in \mathcal{P} . To show that a given set of objects is a generator, we need to find enough direct sum decompositions. In the p -dg world we need more; it is sufficient for X to be filtered by objects in \mathcal{P} via fantastic filtrations. That is, one other major implication of an Fc-filtration as above is that X will lie in the triangulated hull of $\{M_j\}$, inside the p -dg derived category.

Remark 1.9. At the moment we do not have results on the necessity of Fc-filtrations. That is, in theory X might be in the triangulated hull of $\{M_j\}$ even when there is no fantastic filtration, or (1.4) might hold, because no technology has been developed to provide an obstruction. However, we know of no examples where this happens.

There are other important direct sum decompositions, beyond the defining ones: for example, the idempotent decomposition known as the Stosic formula in [KLMS12, Theorem 5.6]. The remainder of the argument in [EQ16b] went as follows.

²A positively-graded p -dg algebra has its grading supported in non-negative degrees, with semisimple degree zero part, and the differential is trivial in degree zero.

- Find enough direct sum decompositions to decompose *any* object in the category \mathcal{U} as a direct sum of objects which are either indecomposable or contractible. (In this case, the defining decompositions and the Stosic formula were sufficient.)
- Prove that each of these direct sum decompositions lifts to an Fc-filtration, for the good differentials.
- Let P be the direct sum of all the non-contractible indecomposable objects. Deduce from the above that P generates the p -DG derived category of \mathcal{U} , and that $\text{End}(P)$ is positive, so that the p -dg Grothendieck group of \mathcal{U} agrees with the $q = \zeta$ specialization of the ordinary Grothendieck group of \mathcal{P} .

This method for computing the p -dg Grothendieck group has many obvious limitations. It worked for $\mathcal{U}(\mathfrak{sl}_2)$ because the category is relatively simple. We understand completely what all the indecomposable objects are, and we know enough explicit idempotent decompositions to take an arbitrary object and split it into indecomposables. In his PhD thesis, Andrew Stephens [Ste18] was able to extend this same method to categorify the positive half of quantum \mathfrak{sl}_3 , because the explicit idempotent decompositions were also constructed previously by Stosic [Sto11]. It seems hopeless to extend this method to \mathfrak{sl}_n in general: even the canonical basis of the positive half of the quantum group is unknown, so there is little hope of understanding the indecomposable objects explicitly. Some new techniques are clearly required to make progress beyond what is currently known.

1.4. The diagrammatic Hecke algebra and its differential. In this paper we initiate the program to categorify (Iwahori-)Hecke algebras at a root of unity, with preliminary positive results, useful negative results, and extremely intriguing conjectures.

The Hecke algebra in type A is a deformation of the group algebra of the symmetric group. It is categorified by the monoidal category of Soergel bimodules, which are certain bimodules over the polynomial ring from Example 1.3. The appropriate integral form of this categorification is the diagrammatic Hecke category, as introduced by Elias-Khovanov in type A [EK10]. Just as the symmetric group is generated by its simple reflections, the adjacent transpositions $s_i = (i, i+1)$, the diagrammatic Hecke category is generated by certain objects $B_i = B_{s_i}$. Tensor products of these objects are commonly called *Bott-Samelson* bimodules or objects. The diagrammatic Hecke category encodes morphisms between Bott-Samelson bimodules as planar diagrams. A basis for these morphism spaces was constructed in [EW16], called the *double leaves basis*.

Following the motif from the parallel world of quantum groups, if we want to categorify the Hecke algebra at a prime root of unity, we should equip the diagrammatic Hecke category with the structure of a gaea. In this paper, we only examine simply-laced type. As in the outline of [EQ16b], we first compute all possible differentials on the category, obtaining a large family of even differentials. Then, for each of the defining idempotent decompositions, we compute the corresponding graph and determine for which differentials the graph has no cycles. We temporarily call such a differential *good*. Once again, this constraint is enough to pin down the differential precisely: only two dual differentials (d and \bar{d}) are good, and could

possibly induce the correct p -dg Grothendieck group. We explicitly check all possible choices of projection and inclusion maps for each of the defining idempotent decompositions.

Remark 1.10. Some of the defining idempotent decompositions require dg-filtrations rather than Fc-filtrations, because the summands in question are not Bott-Samelson bimodules and thus not pre-existing objects in the diagrammatic Hecke category. In fact, they require a slight generalization which mixes the concepts of a dg-filtration and an Fc-filtration, see §6.4.

Remark 1.11. A differential on the Hecke category induces a differential on its polynomial ring, which for the Elias-Khovanov version of the diagrammatic category is assumed to be $\mathbb{k}[x_1, \dots, x_n]$ with its standard action of S_n . This polynomial ring has a standard differential, where $d(x_i) = x_i^2$. In our classification we do not assume that this is the differential on the polynomial ring. Instead, we prove that when the differential is good, the p -dg polynomial ring is forced to be isomorphic to the standard one. Moreover, we prove that Hecke category, if constructed instead using the $(n-1)$ -dimensional reflection representation of S_n , does not admit a good differential!

Remark 1.12. There are a number of other compatibilities one might desire from a differential. The first is that it is compatible with the categorical Schur-Weyl duality of Mackaay-Stošić-Vaz [MSV13], which gives a functor from the Hecke category of S_n to the categorification of quantum \mathfrak{gl}_n . The second is that the singular Hecke 2-category, which contains the Hecke category as an endomorphism category, should have a p -differential which restricts to our chosen differential. The third is that the thick calculus of the first author [Eli16a] should have a p -differential which restricts to our chosen differential. All these are satisfied by the good differential; we briefly discuss some of these compatibilities in this paper.

In practice, the classification of differentials is a three-step process. By the Leibniz rule, any differential on an algebra is determined by its action on the generators. A generator is sent to some morphism of degree 2 higher, living in a finite-dimensional morphism space, so we can specify this morphism by a number of parameters (its coefficients in the double leaves basis). Now we impose three constraints on these parameters. The first is that the relations are preserved by the differential; this ensures that the differential is well-defined. The second is that the divided power $d^{(k)}$ is well-defined over \mathbb{Z} ; this need only be checked on the generators, since

$$(1.7) \quad d^{(k)}(fg) = \sum_{i+j=k} d^{(i)}(f)d^{(j)}(g)$$

by the Leibniz rule. The third is that the defining idempotent decompositions should have no cycles. Only the first two constraints need to be checked when classifying general differentials rather than good differentials.

As noted above, our main result is a classification of all good differentials on the Hecke category (up to object-fixing isomorphism): there are only two, d and \bar{d} . We also go most of the way towards classifying the general differential, although we do not quite finish the job. The general differential has many more non-zero parameters than the good differential. For example, a general differential applied to the 4-valent vertex gives a sum of three diagrams with particular coefficients,

while a good differential applied to the 4-valent vertex is zero. Consequently, it is much easier to check that the good differential satisfies all the relations of the diagrammatic Hecke category than it is to check the general case. We are able to check every relation in the general case except the most complicated one, the so-called Zamolodchikov relation associated to parabolic subgroups of type A_3 . Perhaps it is only laziness which prevents us from finishing this calculation, although it is a surprisingly thorny one.³

The Hecke category was generalized to all Coxeter groups by Elias-Williamson in [EW16]. The computations done here also have implications for possible differentials in other types and for other realizations (see [EW16, §3.1]). In particular, we prove that there is no nonzero good differential for any realization in any simply-laced type except for finite and affine type A (our results are sufficient to classify the good differentials in affine type A as well). Work in progress of the first author and Lars Thorge Jensen is exploring differentials in some non-simply-laced types, using a realization which is central extension of the root realization.

Let us reiterate that the good differential has already been discovered in some sense, in the algebraic context of Soergel bimodules. Good differentials on the nilHecke algebra have been studied in the work of Beliakova-Cooper [BC18] and Kitchloo [Kit13]. In a different direction Khovanov and Rozansky in [KR16] construct an action of the positive half of the Witt algebra \mathcal{W}^+ on all Bott-Samelson bimodules and on certain complexes thereof. This led to an action of \mathcal{W}^+ on triply graded knot homology (just as predecessors Beliakova-Cooper and Kitchloo defined an action of the Steenrod algebra on the characteristic p versions).

The \mathcal{W}^+ action equips Bott-Samelson bimodules with a differential. This differential on the objects (the Bott-Samelson bimodules) can be used in standard fashion to construct a differential on morphism spaces between objects. A generator of the \mathcal{W}^+ action gives rise to our good differential. Again, the existence of this differential is no surprise at this point. Our paper has a different set of goals: to prove the key homological properties of d (i.e. it is good), to prove the uniqueness of d (the lack of other good differentials), and to examine other types, en route to computing the p -dg Grothendieck group of the diagrammatic Hecke category.

1.5. The Grothendieck group? Having classified the good differentials, the next step is to compute the p -dg Grothendieck group of the Elias-Khovanov category for these differentials. Now it is clear that the methods of [EQ16b] will no longer suffice, for several reasons which we now discuss.

Indecomposable objects B_w in the Hecke category (up to isomorphism and grading shift) are in bijection with elements w in the symmetric group S_n , and they appear as direct summands inside $B_{i_1} \otimes \cdots \otimes B_{i_d}$ whenever $s_{i_1} \cdots s_{i_d}$ is a reduced expression for w . Aside from these facts, the indecomposable objects B_w are extremely mysterious. Unlike the case of quantum \mathfrak{sl}_2 , the size and structure of the indecomposable object B_w depends on the characteristic of the base field! We write ${}^p B_w$ to indicate the dependence of this object on the characteristic. The smallest example where the size of ${}^p B_w$ depends on p occurs when $p = 2$ for 28 elements $w \in S_8$, but examples become ever more frequent as n grows larger.

³The interested reader is welcome to finish this calculation for us and write an appendix! That said, we are not sure why anyone should care about the complete family of differentials anyway; the good differentials seem at the moment like the only interesting ones.

In characteristic zero the indecomposable objects 0B_w categorify the Kazhdan-Lusztig basis, by results of Soergel [Soe90]. The size of morphism spaces is determined by a sesquilinear form on the Hecke algebra, and Kazhdan-Lusztig basis elements pair positively, so the endomorphism ring of $\bigoplus {}^0B_w$ is positively graded. When p is large relative to n , pB_w will “agree” with 0B_w for all w , and continue to categorify the Kazhdan-Lusztig basis. Note that, as proven by Williamson [Wil17], the prime p must grow at least exponentially with n for this statement to hold! When p is small relative to n , the endomorphism ring of $\bigoplus {}^pB_w$ need not be positively graded. This is the first nail in the coffin.

To decompose an arbitrary object in the Hecke category into indecomposables, we would need to be able to find the idempotent projecting to B_w inside a reduced expression, and would need to be able to decompose $B_w \otimes B_s$ for each $w \in S_n$ and each simple reflection s . Essentially nothing is known about these idempotents, and it is an incredibly difficult open problem to study them! Without an explicit idempotent decomposition, there are currently no tools which could prove that some decomposition lifts to an Fc-filtration or a dg-filtration. This is the second nail in the coffin.

For all these reasons, we are forced to abandon the previous methods in p -dg theory and search for something new, which will be the focus of the sequel to this paper.

Aside from the method of [EQ16b], there are only a few other tools in the literature which can be used to compute the p -dg Grothendieck group. Recently, the second author and Sussan [QS18] have developed the notion of a p -dg cellular algebra and a p -dg quasi-hereditary algebra, and proven that their p -dg Grothendieck groups are $q = \zeta$ specializations of the ordinary Grothendieck group. However, their concepts only apply to cellular algebras over the base field \mathbb{k} , rather than cellular algebras over other rings. In the lingo, their technology works for quasi-hereditary algebras but not for affine quasi-hereditary algebras in the sense of Kleschchev [Kle15]. The double leaves basis is a cellular basis, but with base ring R (the polynomial ring in n variables), which itself has a nontrivial differential. One might hope that the techniques of Qi-Sussan can be adapted to the more general setting, but this is not an easy adaptation. Still, the cellular structure on the Hecke category is one of the most powerful weapons in the arsenal, and we expect that any successful approach will use it.

Despite the lack of tools, we still believe the end result.

Conjecture 1.13. *The diagrammatic Hecke category associated to S_n , when equipped with a good differential (either d or \bar{d}) and specialized to characteristic p , has p -dg Grothendieck group isomorphic to the Hecke algebra at the appropriate root of unity.*

1.6. Computations for small n . The methods of [EQ16b] do suffice for small values of n , where we can compute all the idempotent decompositions by hand, and where the size of the indecomposable objects does not depend on the characteristic. For each $w \in S_n$ and simple reflection s with $ws > w$, the direct summands of $B_w \otimes B_s$ all⁴ have the form B_z for various $z \in S_n$. We let $I_{w,s} \subset S_n$ denote

⁴We are working in characteristic zero, or assuming that the object B_w and its direct sum decomposition agree with the characteristic zero setting in the Grothendieck group. This holds when $n < 8$ in any characteristic.

the set of such z (with multiplicity). If the graph $\Gamma_{I_{w,s},d}$ has no cycles, then the decomposition

$$(1.8) \quad B_w B_s \cong \bigoplus_{z \in I_{w,s}} B_z$$

is fantastically filtered.

Theorem 1.14. *If $n \leq 4$, $w \in S_n$ and $s \in S$, then $\Gamma_{I_{w,s},d}$ has no cycles. As a consequence, any Bott-Samelson splits into indecomposable objects via a fantastic filtration, and Conjecture 1.13 holds for $n \leq 4$.*

The proof of Theorem 1.14 is by direct, straightforward, and exhaustive computation, and we have chosen not to write it up. Preliminary calculations have verified that Theorem 1.14 continues to hold true for interesting examples with $n = 5$ as well.

Already one observes some mysterious phenomena. The graph $\Gamma_{I_{w,s},d}$, when it has no cycles, induces a special partial order \prec on $I_{w,s}$. In order to prove that $\Gamma_{I_{w,s},d}$ has no cycles in general, it would help to know in advance what this partial order will be.

For example, let $n = 3$ and let $s = (12)$ and $t = (23)$ inside S_3 . In the additive setting we know that

$$B_{st} \otimes B_s \cong B_{sts} \oplus B_s,$$

so that d induces a partial order on the set $\{s, sts\}$. This partial order happens to be $s \prec sts$. Meanwhile,

$$B_{ts} \otimes B_t \cong B_{sts} \oplus B_t,$$

with partial order⁵ $sts \prec t$. Already it is clear that \prec is not determined by the Bruhat order. One might still pray that it is governed by some kind of lexicographic order or convex order on roots, but one would be disappointed.

Letting $u = (34)$ inside S_4 , we can compute various other partial orders, including

$$(1.9a) \quad us \prec usts \quad \text{from } I_{ust,s},$$

$$(1.9b) \quad usts \prec stsuts \prec sutu \quad \text{from } I_{sutsu,t},$$

$$(1.9c) \quad sutu \prec su \quad \text{from } I_{sut,u}.$$

There are no cycles in any of these individual sets, but there is a cycle if the sets $I_{w,s}$ are all placed together! In other words, there is no partial order on S_n which restricts to the partial order on each $I_{w,s}$. This makes the partial orders on $I_{w,s}$ especially mysterious.

Remark 1.15. This cycle

$$(1.10) \quad us \rightarrow usts \rightarrow stsuts \rightarrow sutu \rightarrow us$$

is not the only one. One can replace $usts$ with $stsu$, or $sutu$ with $utus$, to obtain another cycle. In addition, there is a cycle

$$(1.11) \quad tsu \rightarrow sts \rightarrow tstut \rightarrow tut \rightarrow tsu,$$

and one can replace $tstut$ with either $tutst$ or with t .

⁵In both cases, \bar{d} induces the opposite partial order.

1.7. Computations for $n = 8$. In finite characteristic for larger values of n , as already noted, the decomposition of tensor products into indecomposable objects is different from in characteristic zero. Some summands which split in characteristic zero will instead get “glued together.” This does not change the Grothendieck group itself! The Grothendieck group of the diagrammatic Hecke category is the Hecke algebra in any characteristic. What it does change is the basis of the Grothendieck group given by the symbols of indecomposable objects. In characteristic p , one defines pB_w as the top summand of the Bott-Samelson object associated to a reduced expression of w , i.e. the unique direct summand of this Bott-Samelson which is not a direct summand of any shorter Bott-Samelson.⁶ These pB_w enumerate the isomorphism classes of indecomposable objects up to grading shift. The basis $[{}^0B_w] = b_w$ is the usual Kazhdan-Lusztig basis, also called the *0-canonical basis*, while $[{}^pB_w] = {}^pb_w$ is now called the *p -canonical basis*.

Let us give a concrete example, in the symmetric group S_8 . Let X denote the Bott-Samelson bimodule associated to the sequence of simple reflections

$$(s_3, s_2, s_1, s_5, s_4, s_3, s_2, s_6, s_5, s_4, s_3, s_7, s_6, s_5),$$

which is a reduced expression for an element $w \in S_8$. Let $y = s_2s_3s_2s_5s_6s_5$, which is the longest element of a parabolic subgroup. In characteristic zero (and any odd finite characteristic), there is a direct sum decomposition⁷

$$(1.12) \quad X \cong B_w \oplus B_y.$$

However, in characteristic 2, X is indecomposable. What is happening is that the integral form of the Hecke category contains morphisms $p: X \rightarrow B_y$ and $i: B_y \rightarrow X$ such that $p \circ i = 2 \operatorname{id}_{B_y}$. These morphisms p and i span their respective Hom spaces over \mathbb{Z} . Now change base to a field. If 2 is invertible then $\frac{i p}{2}$ is an idempotent projecting to B_y . If $2 = 0$ one cannot construct any splitting of i or p ; consequently, $X = {}^2B_w$ is indecomposable, and on the Grothendieck group, ${}^2b_w = b_w + b_y$.

In similar fashion, it need not be the case that $\Gamma_{I_{w,s},d}$ has no cycles in order for Conjecture 1.13 to hold! Let us call an object p -dg *indecomposable* if it does not have any idempotents fitting into a dg-filtration. A refinement of Conjecture 1.13 might say that a Bott-Samelson associated to a reduced expression (for w) has an Fc-filtration with a unique p -dg indecomposable summand dB_w not appearing inside any Fc-filtration of a shorter Bott-Samelson. In theory, these objects dB_w might descend to a basis of the p -dg Grothendieck group, the *d -canonical basis*. The p -dg Grothendieck group might still be the Hecke algebra, even if the d -canonical basis is an unexpected one. For example, what if the graph for the decomposition (1.12) has cycles? Then X has no idempotents fitting into a dg-filtration, and $X = {}^dB_w$ is p -dg indecomposable. Morally speaking, this is no worse than having X be indecomposable in characteristic 2.

We did not bring up this specific example for no reason.

Theorem 1.16. *The decomposition (1.12) is not Fc-filtered in any characteristic. If i and p are the morphisms discussed above, then $d(i) \neq 0$, $d(p) \neq 0$, and $d(p)i \neq 0$ (so the graph has both a cycle and a loop).*

⁶One can prove that it is independent (up to isomorphism) of the choice of reduced expression.

⁷Thanks to Lars Thorge Jensen for his computer calculations and his help finding this and other accessible examples. This decomposition was verified by his programs.

The proof is by direct and nasty computation, and we do not write it up. In §9 we record the conclusion of our efforts for posterity, writing down p , $d(p)$, i , $d(i)$, and $d(p)i$.

Theorem 1.16 is very surprising, because it implies that (if Conjecture 1.13 is true) the d -canonical basis of the Hecke algebra is different from both the 0-canonical basis and the p -canonical basis! For S_8 in characteristic $p > 2$, the p -canonical basis agrees with the 0-canonical basis, but the d -canonical basis contains ${}^d b_w = b_w + b_y$.

Remark 1.17. Having just emphasized that it is not important that every direct sum decomposition is an Fc-filtration, we wish to re-emphasize that the defining direct sum decompositions (like $B_s B_s \cong B_s(-1) \oplus B_s(1)$) must be Fc-filtrations, or the ring structure on the p -dg Grothendieck group would be incorrect.

1.8. A preview of part II. As noted above, new techniques are required to compute the p -dg Grothendieck group of the Hecke category, and to understand Fc-filtrations in the absence of explicit decompositions. In the paper [EQ21], we introduce some new techniques (some of them conjectural) which we hope will fit the bill. Here is a quick preview.

First, we introduce what we call the *counterdifferential*, a new structure which exists in both the Hecke and the quantum group settings. This is a derivation z of degree -2 (satisfying the Leibniz rule), giving a new gaea structure. Since most of the generating morphisms live in the minimal nonzero degree of their respective Hom spaces, z will kill these generators. So z is determined by what it does to the polynomial ring $R = \mathbb{k}[x_1, \dots, x_n]$, where it sends $x_k \mapsto 1$ for each k . In the sequel we will prove the following basic structural results.

- Letting h denote the degree operator (which acts on morphisms of degree k by multiplication by the scalar k), the triple (d, h, z) acts as an \mathfrak{sl}_2 -triple, making all Hom spaces into \mathfrak{sl}_2 -representations.
- The triple (\bar{d}, h, z) is also an \mathfrak{sl}_2 -triple.
- Each Hom space is a free R -module with its double leaves basis, as noted above. The R -span of any double leaf is preserved by z . Moreover, there is a partial order on the set of double leaves such that d sends each double leaf to the R -span of double leaves which are weakly lower in the partial order. Thus the double leaves basis equips all Hom spaces with a very particular kind of filtration.

One should think of the \mathfrak{sl}_2 -representations which appear in the Hom spaces as roughly being filtered by coVerma modules. After all, the polynomial ring in one variable $\mathbb{k}[x]$ is precisely the coVerma module $\nabla(0)$ (the polynomial ring in multiple variables is more complicated). Hom spaces are large infinite-dimensional modules, but may also have small finite-dimensional submodules. For example, the \mathbb{k} -span of the identity element 1 inside R is a finite-dimensional \mathfrak{sl}_2 -subrepresentation. We propose that it is no accident that the (one-dimensional) finite part of R agrees precisely with the span inside R of the units!

To give another example, whenever $s \in S$ is a simple reflection,

$$\mathrm{Hom}(B_s B_s B_s B_s, B_s)$$

is a free R -module of graded rank $q(q + q^{-1})^4$. Note that $B_s B_s B_s B_s$ splits into 8 shifted copies of B_s , with graded rank $(q + q^{-1})^3$, so one should expect to find 8 projection maps inside $\mathrm{Hom}(B_s B_s B_s B_s, B_s)$. Amazingly, for any fantastic filtration

picking out these 8 summands, the span of the 8 projection maps will be an 8-dimensional \mathfrak{sl}_2 -subrepresentation of $\mathrm{Hom}(B_s B_s B_s B_s, B_s)$, and this is the maximal finite-dimensional part of that infinite-dimensional representation.

More generally, we conjecture the following algorithm to find all the projection maps in a fantastic filtration on a Bott-Samelson bimodule X . Consider the \mathfrak{sl}_2 -representation $V = \bigoplus_{w \in W} \mathrm{Hom}(X, B_w)$ of all morphisms to indecomposable objects (there are ways of modeling this representation without needing to understand the indecomposable objects). Find the maximal finite-dimensional subrepresentation F_1 ; this should be spanned by projection maps. Then, take the quotient of V by $J \cdot F_1$, where J is the Jacobson radical of the category. Call this quotient V_1 . Now repeat, finding the maximal finite-dimensional subrepresentation of V_1 , and so forth. Not only do we conjecture that sufficiently many projection maps can be found inside these finite-dimensional subquotients, and that this will produce a fantastic filtration on X , but also that the existence of a fantastic filtration on X is equivalent to the effectiveness of this procedure. Further details will await in the sequel to [EQ21].

Remark 1.18. One should think of this conjecture as a new kind of Hodge theory. The relative hard Lefschetz theorem (see [EW21, Theorem 1.2] for the theorem in this context) implies (in characteristic zero) that multiplicity spaces (appropriately defined) of an indecomposable summand in a Bott-Samelson bimodule satisfy the hard Lefschetz property with respect to the appropriate Lefschetz operator, meaning that they are finite-dimensional \mathfrak{sl}_2 -representations. Meanwhile, we are stating that the entire Hom space has the structure of an \mathfrak{sl}_2 -representation (with a very different raising operator), whose finite-dimensional \mathfrak{sl}_2 -subrepresentation is related to the multiplicity space. Note that d is not a Lefschetz operator; it is more like an “infinitesimal Lefschetz operator.” Moreover, we also conjecture some positivity properties for d , analogous to the relative Hodge-Riemann bilinear relations.

Remark 1.19. The \mathfrak{sl}_N link homologies are defined using categories of \mathfrak{sl}_N -foams. The diagrammatic Hecke category for S_n admits a monoidal functor to \mathfrak{sl}_N -foams for each N , see e.g. [Vaz10, MV10]. The diagrammatic Hecke category for S_n also has a monoidal ideal for each $2 \leq N$. This ideal is zero when $N > n$, and otherwise it is generated by the indecomposable object B_{w_N} , where w_N is the longest element of S_N after its typical embedding into S_n . The functor to \mathfrak{sl}_N -foams should annihilate this monoidal ideal, though we are unsure where to find this statement, cf. [EMTW20, Chapter 22]. When $N = 2$, the computations in this paper imply that the monoidal ideal is preserved by the action of d and z , so that the quotient still admits an action of \mathfrak{sl}_2 by derivations. We expect the same statement to hold for any N .

Note furthermore that many link homologies (e.g. Khovanov homology) use a further monoidal quotient, where one also kills positive degree symmetric polynomials to obtain finite dimensional morphism spaces. This ideal is preserved by d , but not by z in general. However, the ideal is preserved by z in certain finite characteristics. In characteristic 2, the operator z descends to the one used by Shumakovitch in [Shu14, Section 3], while in characteristic $p > 2$, it agrees with the p -nilpotent operator discovered by Wang [Wan21]. See also [QRSW22] for connections with the foam approach.

2. THE ANSWER

Let us state the end result of our computations. The rest of the paper will comprise the proof of these results.

We will not review the diagrammatic Hecke category here. See [EW16] for details. We fix a Coxeter system (W, S) with a realization, having polynomial ring R with an action of W . The degrees in R are doubled, so that the simple roots $\{\alpha_s\}_{s \in S}$ have degree 2. To avoid potential confusion, we refer to the simple roots and other homogeneous polynomials of the same degree as *linear polynomials* (rather than degree 2 polynomials). Since R is the endomorphism ring of the monoidal identity, a differential on the diagrammatic Hecke category induces a differential on R . One should not confuse the differential d with the divided difference operators ∂_s associated to each $s \in S$, see (3.1). In the pictures below, blue represents s , red represents some t with $m_{st} = 3$, and green represents some u with $m_{su} = 2$.

Theorem 2.1. *Let d be an even differential of degree $+2$ on the diagrammatic Hecke category in simply laced type. Then there exist linear polynomials $g_s, \bar{g}_s \in R$ for each $s \in S$, such that the differential is defined on the generating diagrams by the following formulas.*

$$(2.1a) \quad d \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = g_s \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array},$$

$$(2.1b) \quad d \left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \right) = \bar{g}_s \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array},$$

$$(2.1c) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}^{-g_s},$$

$$(2.1d) \quad d \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}^{-\bar{g}_s},$$

$$(2.1e) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}^{-g_u} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}^{g_s} + \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}^{g_u - g_s},$$

$$(2.1f) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = A \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + B \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + C \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + D \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} + f.$$

In this final formula, we have

$$(2.2a) \quad A = -\partial_s(g_t),$$

$$(2.2b) \quad B = -\partial_t(\bar{g}_s),$$

$$(2.2c) \quad C = \partial_t(\bar{g}_t) - \partial_t(g_s) - \partial_s(g_t),$$

$$(2.2d) \quad D = \partial_s(g_s) - \partial_s(\bar{g}_t) - \partial_t(\bar{g}_s),$$

$$(2.2e) \quad f = g_s - g_t - (\partial_s(g_s) + \partial_t(g_s))\alpha_s + (\partial_t(g_t) + \partial_s(g_t))\alpha_t.$$

These formulas, together with a differential on R , determine the differential on the category. Let

$$(2.3) \quad z_s = g_s + \bar{g}_s \in R.$$

Then the differential on R satisfies the following properties:

$$(2.4a) \quad d(w(f)) = w(d(f)) \quad \text{for all } w \in W, f \in R,$$

$$(2.4b) \quad d(\alpha_s) = \alpha_s z_s,$$

$$(2.4c) \quad z_s \in R^s,$$

$$(2.4d) \quad z_s \in R^u \quad \text{when } m_{su} = 2,$$

$$(2.4e) \quad s(\alpha_t)\partial_s(z_t) = \partial_s(\alpha_t)(z_s - z_t) \quad \text{when } m_{st} > 2.$$

To state the converse, let us call the data of an even differential on R and a collection of linear polynomials g_s, \bar{g}_s satisfying (2.4) by the name of a potential differential. Then a potential differential induces a differential on the diagrammatic Hecke category via the formulas (2.1) if and only if the A_3 Zamolodchikov relation is sent to zero.

In other words, we did not check the A_3 Zamolodchikov relation, and are unsure whether it is sent to zero by any potential differential, or if there are additional requirements to be met. We suspect there are no additional requirements.

Definition 2.2. Let us call a differential d on the Hecke category *good* if the defining idempotent decompositions can be lifted to fantastic filtrations.

We will make this more precise later. In type A the defining decompositions lift the relations

$$(2.5a) \quad b_s b_s = v b_s + v^{-1} b_s,$$

$$(2.5b) \quad b_s b_u = b_u b_s \quad \text{if } m_{su} = 2,$$

$$(2.5c) \quad b_s b_t b_s - b_s = b_t b_s b_t - b_t \quad \text{if } m_{st} = 3$$

in the Hecke algebra. Note that the zero differential is good.

Theorem 2.3. A differential is good (in simply laced type) if and only if it satisfies the following properties.

(1) These equations hold.

$$(2.6a) \quad d(g_s) = g_s^2,$$

$$(2.6b) \quad \bar{g}_s = s(g_s),$$

$$(2.6c) \quad u(g_s) = g_s \quad \text{whenever } m_{su} = 2,$$

$$(2.6d) \quad st(g_s) = g_t \quad \text{whenever } m_{st} = 3.$$

This last equation implies that if $g_s = 0$ for some s , then $g_t = \bar{g}_t = 0$ for all t in the same connected component of the Coxeter graph.

(2) If $g_s \neq 0$ then $g_s \notin R^s$.

(3) If $g_s \neq 0$ and $m_{st} = 3$ then exactly one of these two possibilities holds. We encode which one holds using an orientation on the corresponding edge in the Coxeter graph.

(a) g_t is fixed by s , and $g_s = t(g_t)$ is fixed by sts . We orient the edge from t to s .

(b) g_s is fixed by t , and $g_t = s(g_s)$ is fixed by sts . We orient the edge from s to t .

(4) The orientation is consistent in that, for any parabolic subgroup of type A_3 , the middle vertex is neither a source nor a sink.

Finally, a potential differential which satisfies the properties listed in this theorem will send the A_3 Zamolodchikov relation to zero, so it does induce a differential on the diagrammatic Hecke category.

Corollary 2.4. *The only connected simply-laced Coxeter groups which admit a consistent orientation have type A_n , type \tilde{A}_n , or type A_∞ , and they each have precisely two consistent orientations.*

Proof. There is no way to consistently orient the D_4 Coxeter graph, which is contained inside any connected simply-laced Coxeter group aside from those listed above. \square

When the differential is good, the formula for the differential simplifies. The scalar $\kappa = \partial_s(g_s)$ is independent of the choice of s in a connected component of the Coxeter graph. We have

$$(2.7a) \quad d \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = 0,$$

$$(2.7b) \quad d \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = 0 \quad \text{if } t \rightarrow s,$$

$$(2.7c) \quad d \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = \kappa \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) - \kappa \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) \quad \text{if } s \rightarrow t,$$

and can deduce that

$$(2.7d) \quad d \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = -\kappa \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}, \quad d \left(\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \right) = \kappa \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array}.$$

Finally, we prove that there are only two good differentials up to equivalence.

Theorem 2.5. *Up to an automorphism of the Hecke category in type A_n , if there is a good derivation then we can assume that R contains the polynomial ring $\mathbb{k}[x_1, \dots, x_n]$ with its usual S_n action and differential $d(x_i) = x_i^2$, and we can assume that either $g_{(i,i+1)} = x_i$ for all i with $\kappa = 1$, or that $g_{(i,i+1)} = x_{i+1}$ for all i with $\kappa = -1$. An exception is when \mathbb{k} has characteristic 2, in which case it is possible that one may need to impose the relation $\sum x_i = 0$.*

3. PRELIMINARIES

We start by fixing a realization of a Coxeter system (W, S) . For more on realizations, see [EW16, Section 3.1]. Later we will assume that the Coxeter system is simply-laced, but not at first.

Recall that $\partial_s: R \rightarrow R$ denotes the Demazure operator of a simple reflection $s \in S$, which is defined by the formula

$$(3.1) \quad \partial_s(f)\alpha_s = f - s(f),$$

and satisfies the twisted Leibniz rule

$$(3.2) \quad \partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g).$$

The Cartan matrix encodes the values of $\partial_s(\alpha_t)$ for various simple reflections.

We always assume that Demazure surjectivity holds, see [EW16, Assumption 3.7]. The implication is that for each $s \in S$ there exists some linear polynomial ϖ_s such that

$$(3.3) \quad \partial_s(\varpi_s) = 1.$$

For example, if $m_{st} = 3$ we could set $\varpi_s = -\alpha_t$. We do not assume that ϖ_s is a fundamental weight (i.e. that $\partial_t(\varpi_s) = 0$ for $t \neq s$).

We also assume (cf. [EWS16, Definition 1.7]) that

- (\star) the linear terms in R^s and the linear terms in R^t together span the linear terms in R , whenever $s \neq t$.

This rules out the degenerate possibility that the fixed hyperplanes of s and t coincide. Both Demazure surjectivity and (\star) are requirements for the Hecke category to behave optimally (though they are sometimes replaced by related assumptions, like reflection faithfulness).

Let us try to define a differential on the Hecke category, by defining it as generally as possible, and then determining what constraints are imposed by the fact that it must preserve the relations of the category. We will simultaneously determine what additional constraints are imposed if we want the differential to be good. We do not assume that our differential is invariant under the symmetries of the Hecke category (vertical and horizontal flips, rotation by 180 degrees, Dynkin diagram automorphisms).⁸

The reader interested in type A can fix $n \geq 2$ and work with the standard realization associated to S_n . The base ring R has the form

$$(3.4) \quad R = \mathbb{k}[x_1, \dots, x_n]$$


with its usual action of S_n , and $\deg x_i = 2$. There is a *standard differential* on the polynomial ring R , namely

$$(3.5) \quad d(x_i) = x_i^2$$

for all $1 \leq i \leq n$. We will not assume that this is the induced differential on R , although this will eventually be a consequence of our computations.

Below, s will be an arbitrary simple reflection and will be drawn using the color blue, u will be distant from s and drawn as green, and t will be either be another arbitrary simple reflection, or will be adjacent to s ($m_{st} = 3$) and drawn using the color red.

4. ONE COLOR CONSIDERATIONS

4.1. Dots and polynomials. One of the generating morphisms in the diagrammatic Hecke category is the s -colored enddot , a morphism $B_s \rightarrow \mathbb{1}$ of degree $+1$. The enddot generates $\text{Hom}(B_s, \mathbb{1})$ as a free rank 1 module over R . Thus for any differential we have

$$(4.1) \quad d(\text{enddot}_s) = g_s \text{enddot}_s$$

for some linear polynomial $g_s \in R$.

⁸It will turn out that any differential must be invariant under horizontal flip, but not typically under the other symmetries.

Consider the s -colored startdot $\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}$, a morphism $\mathbb{1} \rightarrow B_s$ of degree $+1$. For similar reasons we have

$$(4.2) \quad d\left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}\right) = \bar{g}_s \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}$$

for some linear polynomial $\bar{g}_s \in R$.

Remark 4.1. We will eventually see that the differential on the entire category is determined by the values of g_s and g'_s .

Now we compute that

$$(4.3) \quad d\left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}\right) = (g_s + \bar{g}_s) \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}.$$

Since the barbell is equal to multiplication by α_s

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} = \alpha_s,$$

we deduce that

$$(4.4) \quad d(\alpha_s) = \alpha_s(g_s + \bar{g}_s).$$

It turns out that the linear polynomial $g_s + \bar{g}_s$ is more intrinsic than either g_s or \bar{g}_s . Henceforth we write

$$(4.5) \quad z_s = g_s + \bar{g}_s,$$

so that

$$(4.6) \quad d(\alpha_s) = \alpha_s z_s$$

and

$$(4.7) \quad d\left(\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}\right) = (z_s - g_s) \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}.$$

If the differential on R is known, then z_s is determined by (4.6). Conversely, the differential on R must be such that $d(\alpha_s)$ is a multiple of α_s , which is not true of the most general differential.

Remark 4.2. When $s = (i, i + 1)$ we have $\alpha_s = x_i - x_{i+1}$. The standard differential satisfies

$$(4.8) \quad d(x_i - x_{i+1}) = x_i^2 - x_{i+1}^2 = (x_i - x_{i+1})(x_i + x_{i+1})$$

and thus $z_s = x_i + x_{i+1}$. Later on we will deduce that for a good differential one has $g_s = x_i$ and $\bar{g}_s = x_{i+1}$, or vice versa (this will be the difference between d and \bar{d}). Again, we will not assume anything about the differential on R or z_s or g_s just yet, but we recommend that the reader verify the formulas below with these specializations, as a motivational sanity check.

Now we examine the polynomial forcing relations. We have

$$(4.9) \quad \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} \partial_s(f) + \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} s(f).$$

Applying the differential to both sides we have

$$(4.10) \quad \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} d(f) = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} z_s \partial_s(f) + \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} d(\partial_s(f)) + \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} d(s(f)).$$

We resolve the LHS by applying (4.9) again. In the result, the equality of coefficients of $\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array}$ is

$$(4.11) \quad \partial_s(d(f)) = z_s \partial_s(f) + d(\partial_s(f)),$$

and the equality of coefficients of \prod is

$$(4.12) \quad s(d(f)) = d(s(f)).$$

By (4.12) we deduce that the differential d commutes with the action of the symmetric group S_n . A consequence is that d preserves the invariant subring R^s , for all simple reflections s .

In fact, (4.11) already follows from the fact that d commutes with the symmetric group action. By definition of the Demazure operator we know that

$$\alpha_s \partial_s(f) = f - s(f),$$

which appeared above as (3.1). Taking the differential of both sides of (3.1) and using (4.12) we get that

$$\alpha_s z_s \partial_s(f) + \alpha_s d(\partial_s(f)) = d(f) - s(d(f)) = \alpha_s \partial_s(d(f)).$$

Dividing both sides by α_s (a non-zero-divisor in R) we recover (4.11).

In the special case when $f = \alpha_s$, (4.11) reduces⁹ to

$$\partial_s(\alpha_s z_s) = 2z_s.$$

However, we already knew from the twisted Leibniz rule that

$$\partial_s(\alpha_s z_s) = 2z_s - \alpha_s \partial_s(z_s)$$

from which we deduce that

$$(4.13) \quad \partial_s(z_s) = 0, \quad z_s \in R^s.$$

We could have deduced that $z_s \in R^s$ in a different way. Since $\alpha_s^2 \in R^s$, and R^s is preserved by d , we see that

$$d(\alpha_s^2) = 2\alpha_s^2 z_s \in R^s,$$

which implies that $z_s \in R^s$. We have brought up this alternative method because we can use it for other purposes as well. For example, suppose that $m_{su} = 2$, so that s and u are *distant* simple reflections. Since $\alpha_s \in R^u$, we must have $d(\alpha_s) \in R^u$, from which we deduce that

$$(4.14) \quad z_s \in R^u \quad \text{when } m_{su} = 2.$$

Now consider the case when $f = \alpha_t$ for some $s \neq t$, so that $\partial_s(f)$ is some scalar. Then (4.11) reduces to

$$(4.15) \quad \partial_s(\alpha_t z_t) = \partial_s(\alpha_t) z_s.$$

Applying the twisted Leibniz rule, we get

$$(4.16) \quad s(\alpha_t) \partial_s(z_t) = \partial_s(\alpha_t)(z_s - z_t).$$

Both sides are zero when $m_{st} = 2$. When $m_{st} > 2$, so that $\partial_s(\alpha_t)$ is nonzero, we deduce that $z_s - z_t$ is proportional to the root $s(\alpha_t)$.

In the special case when $m_{st} = 3$ we find that

$$(4.17) \quad (\alpha_s + \alpha_t) \partial_s(z_t) = z_t - z_s.$$

Applying ∂_t to both sides of the equation, we get

$$(4.18) \quad \partial_s(z_t) = -\partial_t(z_s).$$

⁹We will often use without mention the fact that d kills any scalar multiple of the identity, and hence any polynomial of degree zero. This is a consequence of the Leibniz rule for differentials.

Remark 4.3. The first major consequence of (4.16) is that one need not have an incredibly large realization in order to find a p -differential d . If the Coxeter graph is connected, then the subspace spanned by $\{\alpha_s, z_s\}_{s \in S}$ is at most one dimension higher than the subspace spanned by the roots. One should think that there is a realization spanned by the simple roots and a single new element z , and that each z_s can be written as a linear combination of the roots and z . Let R' denote the subring of R generated by the roots and z . Then, so long as $d(z) \in R'$, our computations will imply that the Hecke category, when defined over R' , will be preserved by the differential. In the standard setup for S_n , we might let $z = x_1$, for example.

Remark 4.4. We have imposed many conditions on the elements z_s and on the differential, making the situation seem quite overdetermined, and the reader may already be convinced that in type A the standard differential on $R = \mathbb{k}[x_1, \dots, x_n]$ is the only one which could satisfy them all. This intuition is entirely false. An arbitrary S_n -invariant differential on R has the form

$$(4.19) \quad d(x_i) = Ax_i^2 + Bx_i \sum_{j \neq i} x_j + C \sum_{j \neq i} x_j^2 + D \sum_{j < k, j \neq i \neq k} x_j x_k,$$

for some scalars A, B, C, D . For any such differential, one can compute that $d(x_i - x_j) = z_{ij}(x_i - x_j)$ where

$$(4.20) \quad z_{ij} = (A - C)(x_i + x_j) + (B - D) \sum_{k \neq i, j} x_k.$$

One can verify (4.16) in this general setting. So, in fact, the constraints on z_s which we have deduced above all are consequences of the fact that d is S_n -invariant.

The extra condition that $d^p = 0$ in characteristic p will restrict the situation considerably, but it is not clear precisely by how much.

4.2. Trivalent vertices. We now compute the differential applied to the trivalent vertices Y and X , which have degree -1 .

It is not difficult to prove that every degree $+1$ map $B_s \rightarrow B_s \otimes B_s$ is a linear combination of taking Y and adding a linear polynomial to one of the three regions. (This is because any element of the double leaves basis can be obtained from Y by breaking some lines.) Consequently, let us assume that

$$(4.21) \quad d \left(\text{Y} \right) = \begin{array}{c} \text{Y} \\ f_1 \end{array} + \begin{array}{c} \text{Y} \\ f_2 \end{array} + \begin{array}{c} \text{Y} \\ f_3 \end{array}.$$

Remark 4.5. This description of a degree $+1$ morphism is not unique, in that different values of f_1, f_2, f_3 can give rise to the same morphism. If we wanted to make it unique, we could assert that f_1 and f_2 are each scalar multiples of ϖ_s . To prove this, set

$$(4.22) \quad g_1 = f_1 - \partial_s(f_1)\varpi_s.$$

Then $\partial_s(g_1) = 0$ so $g_1 \in R^s$. So, sliding g_1 across the s -wall, we can replace f_1 with $\partial_s(f_1)\varpi_s$, and replace f_3 with $f_3 + g_1$. We can do the same with f_2 .

Consider the counit relation

$$(4.23) \quad \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}.$$

Applying the differential to both sides, we get

$$(4.24) \quad 0 = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} g_s.$$

If $f_1 \notin R^s$ then this equation can not possibly hold. If $f_1 \in R^s$ we could rewrite (4.21) such that $f_1 = 0$ by setting the new f_3 equal to the old $f_1 + f_3$. So we can assume that $f_1 = 0$. But the horizontal reflection (i.e. flip across a vertical axis) of (4.23) also holds, from which we deduce that (without loss of generality) $f_2 = 0$ as well. Finally, we see that $f_3 = -g_s$. Thus

$$(4.25) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}^{-g_s}.$$

An almost identical computation with the unit relation will give that

$$(4.26) \quad d \left(\begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array}^{-\bar{g}_s}.$$

We have now pinned down the differential on the trivalent vertices, given the choice of g_s and \bar{g}_s . Note that the differential of the cup and cap are not trivial. Using the fact that a cup or cap is the composition of a dot and a trivalent vertex, and forcing the polynomials outside of the cup or cap, we get that

$$(4.27) \quad d \left(\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} \right) = \partial_s(g_s) \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} g_s - s(\bar{g}_s),$$

$$(4.28) \quad d \left(\begin{array}{c} \text{---} \\ \cap \\ \text{---} \end{array} \right) = -\partial_s(g_s) \begin{array}{c} \text{---} \\ \cap \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \cap \\ \text{---} \end{array} \bar{g}_s - s(g_s).$$

In (4.27) we have used that $\partial_s(-\bar{g}_s) = \partial_s(g_s)$.

We should now check that the remaining one-color relations are satisfied for any such differential. Consider the needle relation

$$(4.29) \quad \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} = 0.$$

Applying the differential to the LHS we get

$$(4.30) \quad d \left(\begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array}^{-z_s} = (-z_s) \cdot \begin{array}{c} \text{---} \\ \bigcirc \\ \text{---} \end{array} = 0$$

since $z_s \in R^s$. This agrees with the differential applied to the RHS, as desired.

The remaining one-color relations (namely: associativity, coassociativity, and Frobenius associativity) are all easily checked.

4.3. The quadratic relation and its defining idempotent decomposition.

The quadratic relation $b_s^2 = [2]b_s$ in the Hecke algebra is categorified by a direct sum decomposition

$$(4.31) \quad B_s B_s \cong B_s(1) \oplus B_s(-1)$$

in the Hecke category. Here (1) represents the grading shift. To specify the convention: within the shifted bimodule $R(1)$, the identity element lives in degree -1 .

To show this direct sum decomposition (4.31) we should choose projections p_{+1}, p_{-1} and inclusions i_{+1}, i_{-1} from the two factors $B_s(+1)$ and $B_s(-1)$ respectively. Let $I = \{+1, -1\}$ be the indexing set for this decomposition.

The projection map p_{-1} must be a scalar multiple of $\text{---}\overline{\text{---}}\text{---}$, for degree reasons. Similarly, the inclusion i_{+1} must be a scalar multiple of $\text{---}\overline{\text{---}}\text{---}$. These scalars must be invertible in order for (1.5) to hold, and rescaling the projection and inclusion maps will not change the graph $\Gamma_{I,d}$, see Remark 1.5. Thus we can assume that $p_{-1} = \text{---}\overline{\text{---}}\text{---}$ and $i_{+1} = \text{---}\overline{\text{---}}\text{---}$ precisely.

The projection map p_{+1} must have the form

$$(4.32) \quad p_{+1} = A \text{---}\overline{\text{---}}\text{---} + B \text{---}\overline{\text{---}}\text{---} + \text{---}\overline{\text{---}}\text{---} f$$

for some linear polynomial f and some scalars A and B , since this is a general morphism of degree $+1$ written in the double leaves basis. Similarly, we have

$$(4.33) \quad i_{-1} = A' \text{---}\overline{\text{---}}\text{---} + B' \text{---}\overline{\text{---}}\text{---} + \text{---}\overline{\text{---}}\text{---} f'.$$

In order for (1.5) to hold, we need

$$(4.34a) \quad A' = 1 = A,$$

$$(4.34b) \quad B' + B = -2,$$

$$(4.34c) \quad f' + f = \alpha_s.$$

For this computation we have used the formula

$$(4.35) \quad \text{---}\overline{\text{---}}\text{---} = \text{---}\overline{\text{---}}\text{---} + \text{---}\overline{\text{---}}\text{---} - 2 \text{---}\overline{\text{---}}\text{---} + \text{---}\overline{\text{---}}\text{---} \alpha_s.$$

Recall that the graph $\Gamma_{I,d}$ has edges labeled by $p_j d(i_k)$ for each $j, k \in I$, but with zero-labeled edges removed. We now determine, for any possible choice of projections and inclusions, when the graph the graph $\Gamma_{I,d}$ has no loops and cycles. Let us set $\kappa_s = \partial_s(g_s)$, which will appear because we use

$$(4.36) \quad \text{---}\overline{\text{---}}\text{---} = \kappa_s \text{---}\overline{\text{---}}\text{---}$$

frequently in this computation. In this section we shorten κ_s to κ . Note also that $\partial_s(\bar{g}_s) = -\kappa$, since $g_s + \bar{g}_s = z_s$ is killed by ∂_s .

We have

$$(4.37a) \quad p_{+1} d(i_{+1}) = \text{---}\overline{\text{---}}\text{---} - B \kappa \text{---}\overline{\text{---}}\text{---} - \text{---}\overline{\text{---}}\text{---} \kappa f = - \text{---}\overline{\text{---}}\text{---} (1+B) \kappa - \text{---}\overline{\text{---}}\text{---} [s(g_s) + \kappa f],$$

$$(4.37b) \quad p_{-1} d(i_{-1}) = \text{---}\overline{\text{---}}\text{---} - B' \kappa \text{---}\overline{\text{---}}\text{---} - \text{---}\overline{\text{---}}\text{---} \kappa f' = \text{---}\overline{\text{---}}\text{---} (1+B')(-\kappa) + \text{---}\overline{\text{---}}\text{---} [s(\bar{g}_s) - \kappa f'].$$

In particular, the graph $\Gamma_{I,d}$ will have a loop at vertex $+1$ unless

$$(4.38a) \quad (1+B)\kappa = 0,$$

$$(4.38b) \quad s(g_s) + \kappa f = 0,$$

and a loop at vertex -1 unless

$$(4.38c) \quad (1+B')\kappa = 0,$$

$$(4.38d) \quad s(\bar{g}_s) - \kappa f' = 0.$$

Note that (4.38c) and (4.38a) are equivalent via (4.34b). In order for (4.38) to hold, we have two options.

Suppose $\kappa = 0$ so that $g_s \in R^s$. Then (4.38b) and (4.38d) hold if and only if $g_s = \bar{g}_s = 0$. Thus also $z_s = 0$. Then the differential is zero on every s -colored diagram: dots, trivalent vertices, and barbells (i.e. α_s).

Now suppose $\kappa \neq 0$, so that $g_s \neq s(g_s) \neq 0$. Then $B = B' = -1$ in order for (4.38a) to hold. Moreover, combining (4.38b) and (4.38d) and (4.34c) we get

$$(4.39) \quad \alpha_s = f + f' = \frac{s(\bar{g}_s - g_s)}{\partial_s(g_s)}$$

or equivalently (since $s(\alpha_s) = -\alpha_s$)

$$(4.40) \quad \partial_s(g_s)\alpha_s = g_s - \bar{g}_s.$$

However, by definition of the Demazure operator we have

$$(4.41) \quad \partial_s(g_s)\alpha_s = g_s - s(g_s)$$

and we conclude that

$$(4.42) \quad \bar{g}_s = s(g_s).$$

In addition to this interesting constraint on the differential, the idempotent decomposition (for which the graph has no cycles) is uniquely determined, since

$$(4.43) \quad f = \frac{-\bar{g}_s}{\kappa}, \quad f' = \frac{g_s}{\kappa}.$$

In fact, we encourage the reader to doublecheck the following intriguing equality:

$$(4.44) \quad \kappa p_{+1} = d(p_{-1}), \quad -\kappa i_{-1} = d(i_{+1}).$$

We discuss why it is intriguing in the next section.

We have just ensured that $\Gamma_{I,d}$ has no loops, but we also need to check that it has no cycles, which means checking that either $p_{-1}d(i_{+1}) = 0$ or $p_{+1}d(i_{-1}) = 0$.

We have

$$(4.45) \quad p_{-1}d(i_{+1}) = -\kappa \begin{array}{c} \cdots \\ \vdots \\ \text{---} \bullet \text{---} \\ \vdots \\ \cdots \end{array}.$$

Consequently, this edge is zero in the $\kappa = 0$ case, and nonzero otherwise. If $\kappa \neq 0$, then the remaining edge $p_{+1}d(i_{-1})$ had better be zero, or there will be a cycle.

Now for the nastiest computation. We only give the answer; the $*$ s represent two different polynomials.

$$(4.46a) \quad p_{+1}d(i_{-1}) = \begin{array}{c} \cdots \\ \vdots \\ \text{---} \bullet \text{---} \\ \vdots \\ \cdots \end{array} * + \begin{array}{c} \cdots \\ \vdots \\ \text{---} \bullet \text{---} \\ \vdots \\ \cdots \end{array} *.$$

The coefficient of $\begin{array}{c} \cdots \\ \vdots \\ \text{---} \bullet \text{---} \\ \vdots \\ \cdots \end{array}$ is

$$(4.46b) \quad -\kappa(B'f + Bf' + BB'\alpha_s)$$

and the coefficient of $\text{---}\text{---}\text{---}$ is

$$(4.46c) \quad d(f') - \kappa f f' - f' z_s.$$

When $\kappa \neq 0$, the requirement that $B = B' = -1$ from (4.38a), combined with (4.34c), will already imply that (4.46b) is zero. Thus when $\kappa \neq 0$ and there are no loops in $\Gamma_{I,d}$, then there are no cycles in $\Gamma_{I,d}$ if and only if

$$d(f') = \kappa f f' + f' z_s.$$

Plugging in the known values of f and f' we get

$$\frac{d(g_s)}{\kappa} = \frac{g_s(z_s - \bar{g}_s)}{\kappa} = \frac{g_s^2}{\kappa}$$

or in other words

$$(4.47) \quad d(g_s) = g_s^2.$$

Here we have found a very interesting constraint! Of course, the other case when $g_s = 0$ also satisfies this constraint.

In conclusion, we have proven the following result.

Proposition 4.6. *There is a fantastically filtered idempotent decomposition $B_s B_s \cong B_s(+1) \oplus B_s(-1)$ if and only if $d(g_s) = g_s^2$, $\bar{g}_s = s(g_s)$, and one of the following possibilities holds:*

- $g_s = 0$, or
- $g_s \notin R^s$.

In both cases, the differential of the cap and cup has a simplified formula.

$$(4.48) \quad d\left(\text{---}\text{---}\text{---}\right) = -\kappa_s \text{---}\text{---}\text{---}, \quad d\left(\text{---}\text{---}\text{---}\right) = \kappa_s \text{---}\text{---}\text{---},$$

where $\kappa_s = \partial_s(g_s)$.

4.4. An alternate approach. The reader should confirm that the condition that $d(g_s) = g_s^2$ is equivalent to the condition that $d(d(i_{+1})) = 0$. We want to briefly elaborate on the relationship between $d^2(i_{+1}) = 0$ and (4.44); namely, we will prove the existence of an Fc-filtration using only two spare assumptions, with no further computation!

So let us begin again in our attempt to construct an Fc-filtration, noting once more that i_{+1} and p_{-1} are forced to be trivalent vertices (after rescaling) because they live in one-dimensional Hom spaces. Our two assumptions are:

$$(4.49a) \quad d(d(i_{+1})) = 0,$$

$$(4.49b) \quad p_{-1}d(i_{+1}) = \lambda \text{id}_{B_s},$$

where λ is some invertible scalar. We claim that setting $i_{-1} := \lambda^{-1}d(i_{+1})$ and $p_{+1} := -\lambda^{-1}d(p_{-1})$ will yield an Fc-filtration.

First we check that this gives an idempotent decomposition, and our computation is just an exercise in the Leibniz rule. It is clear that $p_{-1}i_{+1} = 0$ for degree reasons, and $p_{-1}i_{-1} = \text{id}_{B_s}$ is given by (4.49b). Now

$$(4.50) \quad p_{+1}i_{+1} = -\lambda^{-1}d(p_{-1})i_{+1} = -\lambda^{-1}(d(p_{-1}i_{+1}) - p_{-1}d(i_{+1})) = -\lambda^{-1}(-\lambda \text{id}_{B_s}) = \text{id}_{B_s}.$$

Here we used the Leibniz rule together with the fact that $p_{-1}i_{+1} = 0$. Continuing,

$$(4.51) \quad p_{+1}i_{-1} = -\lambda^{-1}d(p_{-1})i_{-1} = -\lambda^{-1}(d(p_{-1}i_{-1}) - p_{-1}d(i_{-1})) = 0.$$

Here we used the Leibniz rule together with the facts that $d(i_{-1}) = 0$ from (4.49a), and $d(p_{-1}i_{-1}) = 0$ since $p_{-1}i_{-1}$ is the identity.

Now we check that $\Gamma_{I,d}$ has no cycles. But $d(i_{-1}) = 0$ so there are no oriented edges leaving the vertex -1 . There will be an edge from $+1$ to -1 , since $p_{-1}d(i_{+1}) = \lambda \text{id}_{B_s}$. But there will be no loop at $+1$, since $p_{+1}d(i_{+1}) = \lambda p_{+1}i_{-1} = 0$. In other words, after knowing (4.49a) and (4.49b), the existence of an Fc-filtration follows from general principles!

Note that (4.49b) is analogous to the definiteness of the Lefschetz form in Hodge theory. Given morphisms $p_1, p_2 \in \text{Hom}(B_s B_s, B_s)$ in degree $-k$, one can consider

$$(4.52) \quad p_1 \circ d^k(\overline{p_2}),$$

a degree zero endomorphism of B_s . Here $\overline{p_2}$ is the vertical flip of p_2 , living in $\text{Hom}(B_s, B_s B_s)$. By picking out the coefficient of the identity map, we get a bilinear pairing

$$(4.53) \quad p_1 \circ d^k(\overline{p_2}) = \langle p_1, p_2 \rangle \cdot \text{id}_{B_s}.$$

Then (4.49b) implies that the form in degree -1 is (positive or negative) definite. This is not surprising for a form on a one-dimensional space, but we use this as an example. In many similar decompositions, the corresponding forms appear to have a signature which matches the Hodge-Riemann bilinear relations.

This section was a preview of the conjectures in the sequel to [EQ21], where we push this idea as far as we can take it.

4.5. Implications in type A. Suppose we are in the setting of Proposition 4.6, and that $R = \mathbb{k}[x_1, \dots, x_n]$.

Let $s = (i, i+1)$. For the standard differential, $d(\alpha_s) = \alpha_s z_s$, where $z_s = x_i + x_{i+1}$. Since $z_s \neq 0$, we must have $g_s \neq 0$, so $g_s \notin R^s$. The only nonzero linear polynomials g which satisfy $d(g) = g^2$ are $g = x_i$ for some i . The only g for which $d(g) = g^2$ and $g + s(g) = z_s$ are $g = x_i$ and $g = x_{i+1}$.

Let us briefly consider non-standard differentials on R , and ask whether we can find g_s, z_s , etcetera as above.

Example 4.7. Consider the case when $n = 2$ and the differential satisfies

$$(4.54) \quad d(x_1) = x_2^2 - 2x_1x_2, \quad d(x_2) = x_1^2 - 2x_1x_2.$$

Then

$$(4.55) \quad d(x_1 - x_2) = (x_1 - x_2)(-x_1 - x_2)$$

so that $z_s = -(x_1 + x_2)$. Meanwhile, if ρ is a primitive third root of unity and $g = \rho x_1 + \rho^2 x_2$, the reader can verify that $d(g) = g^2$ and $g + s(g) = -(x_1 + x_2)$.

Example 4.8. We have computed and found the most general differential on $\mathbb{k}[x_1, x_2]$ which works. It is S_2 -invariant and satisfies

$$(4.56) \quad d(x_1) = Ax_1^2 - 2Cx_1x_2 + Cx_2^2.$$

The case $C = 0$ is the standard differential. Then $g = a_1x_1 + a_2x_2$ satisfies $d(g) = g^2$ and $g + s(g) = z_s$, when a_1 and a_2 are distinct roots of the quadratic equation

$$(4.57) \quad y^2 + y(C - A) + C(C - A) = 0.$$

4.6. Relation to singular Soergel calculus. A diagrammatic calculus for singular Soergel bimodules (also called the *(diagrammatic) Hecke 2-category*) in type A is long-standing work in progress of Elias-Williamson. Diagrammatic calculus for dihedral groups is due to Elias [Eli16b], where one can also find a review of the background. We will not provide further review here. Let us examine what kind of differentials could exist on the Hecke 2-category, and how they would restrict to the ordinary Hecke category. We will be brief and only provide summary results.

The one-color generators of the Hecke 2-category are oriented cups and caps, as below.



They also span their morphism spaces up to the action of R (in the white region). Consequently, any differential on the Hecke 2-category must send these diagrams to a multiple of themselves by a linear polynomial; for the four diagrams pictured we call these linear polynomials $g_s, \bar{f}_s, f_s, \bar{g}_s$ respectively. Checking that the isotopy/biadjunction relations are preserved by the differential will immediately imply that $f_s = -g_s, \bar{f}_s = -\bar{g}_s$. This then implies (4.1), (4.2), (4.25) and (4.26). Checking the remaining relations again gives the same constraints on g_s .


Remark 4.9. This is a more restrictive and easier way to find a formula for (4.25). In the computation of §4.2, the fact that $f_1 = f_2 = 0$ is a consequence of the differential being restricted from the Hecke 2-category, rather than being a consequence of the unit relation.


In similar fashion, the fact that d preserves R^s is forced upon any differential on the Hecke 2-category, rather than being a consequence. After all, R^s is the endomorphism ring of the identity 1-morphism of the parabolic subset s .

The defining idempotent decomposition is the decomposition of R as an (R^s, R^s) -bimodule into two shifted copies of R^s . That this descends to an Fc-filtration gives exactly the same two possibilities as in Proposition 4.6, although the computation is quite different.

5. TWO DISTANT COLORS

5.1. 4-Valent vertices. Let s and u be distant. Let us recall the assumption (\star) from §3, which states that all linear polynomials are in the span of R^s and R^u . One implication (with Demazure surjectivity) is the existence of a linear polynomial ϖ_s for which $\partial_s(\varpi_s) = 1$ and $\partial_u(\varpi_s) = 0$, and similarly for a linear polynomial ϖ_u .

We now compute the differential applied to the generating 4-valent vertex , also known as the *crossing*, which has degree 0.

Once again, by examination of the double leaves basis, one can deduce that any degree +2 map $B_s \otimes B_u \rightarrow B_u \otimes B_s$ is a linear combination of taking  and placing a linear polynomial in one of the four regions. Here is a useful consequence of (\star) which we wish to record.

Lemma 5.1. *Let f be an arbitrary linear polynomial in R . Then*

$$(5.1) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} = 0.$$

The diagrams are 4-valent vertices with two green lines and two blue lines. The regions are labeled with $-f$ or f . The first diagram has $-f$ in the top-left region. The second has f in the top-right region. The third has $-f$ in the bottom-left region. The fourth has f in the bottom-right region.

Proof. By (\star) we can write $f = g + h$ where $g \in R^s$ and $h \in R^u$. Take the copies of $+f$ on the right and left, decompose them as $g + h$, and slide g across the s -strand and h across the u -strand. Now we have $-f + g + h = 0$ in both the top and bottom, so the result is zero. \square

Remark 5.2. In fact, an analogous lemma applies to any $2m_{st}$ -valent vertex: the alternating sum of placing f in each region is zero. The proof is the same.

Now consider an arbitrary degree $+2$ morphism, which is a linear combination of $\begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array}$ with various linear polynomials. Using Lemma 5.1 we can remove the polynomial from the leftmost region. Using the same simplification as in Remark 4.5, we can assume that the polynomial on top is a multiple of ϖ_s and the polynomial on bottom a multiple of ϖ_u , by modifying the polynomial on the right. Thus the degree $+2$ morphism $d(\begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array})$ has the following form.

$$(5.2) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} b\varpi_s \\ \diagup \diagdown \\ \text{---} \end{array} + \begin{array}{c} \diagup \diagdown \\ a\varpi_u \\ \text{---} \end{array} + \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} f.$$

One could also slide these polynomials to the left, giving the equivalent statement

$$(5.3) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \diagup \diagdown \\ b\varpi_s + a\varpi_u \\ \text{---} \end{array} + \begin{array}{c} \diagup \diagdown \\ \text{---} \end{array} f.$$

Let us check the relation which slides a dot through a crossing. There are actually four such relations, depending on where one puts the dot. First we place the dot in the upper left.

$$(5.4) \quad \begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} = \begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array}.$$

Taking the differential of both sides, we get

$$(5.5) \quad \begin{array}{c} g_u + b\varpi_s \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \diagdown \\ \bullet \end{array} f + a\varpi_u = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} g_u.$$

In order to break the blue strand with zero coefficient, we need $\partial_s(g_u + b\varpi_s) = 0$, or in other words

$$(5.6) \quad b = -\partial_s(g_u).$$

Then, given that $g_u + b\varpi_s \in R^s$, the equality of both sides is equivalent to

$$f + a\varpi_u + b\varpi_s + g_u = g_u,$$

or in other words

$$f + a\varpi_u + b\varpi_s = 0.$$

Similarly, checking the relation with a dot on the upper right gives

$$(5.7) \quad a = \partial_u(g_s).$$

Thus we deduce that

$$(5.8) \quad f = \partial_s(g_u)\varpi_s - \partial_u(g_s)\varpi_u.$$

This pins down the differential of the crossing exactly, given the known differential of the dots. Note that if $g_s \in R^u$ and $g_u \in R^s$ then $d(\begin{array}{c} \text{---} \\ \diagup \diagdown \\ \text{---} \end{array}) = 0$, and otherwise the differential is nonzero.

Let us simplify the answer. Adding and subtracting g_u to the region on top, the polynomial in that region is

$$(5.9) \quad -g_u + (g_u - \partial_s(g_u)\varpi_s).$$

Since $(g_u - \partial_s(g_u)\varpi_s)$ is s -invariant, it slides through to the rightmost region, leaving $-g_u$ behind. Similarly, we can add and subtract g_s to the region on bottom, and slide $(\partial_u(g_s)\varpi_u - g_s)$ through the u -wall to the rightmost region. What remains is

$$(5.10) \quad d \left(\begin{array}{c} \text{diagram: blue and green strands crossing} \end{array} \right) = \begin{array}{c} \text{diagram: blue strand top, green bottom, label } -g_u \end{array} + \begin{array}{c} \text{diagram: green strand top, blue bottom, label } g_s \end{array} + \begin{array}{c} \text{diagram: blue and green strands crossing, label } g_u - g_s \end{array}.$$

Using Lemma 5.1 we can shuffle around these polynomials to obtain other nice descriptions of $d(\text{diagram: blue and green strands crossing})$ as well. For example,

$$(5.11) \quad d \left(\begin{array}{c} \text{diagram: blue and green strands crossing} \end{array} \right) = \begin{array}{c} \text{diagram: blue strand top, green bottom, label } -g_u \end{array} + \begin{array}{c} \text{diagram: green strand top, blue bottom, label } g_s + g_u \end{array} + \begin{array}{c} \text{diagram: blue and green strands crossing, label } -g_s \end{array}.$$

Let us check the cyclicity of the 4-valent vertex. We have

$$(5.12) \quad d \left(\begin{array}{c} \text{diagram: blue cap and cup with green strand} \end{array} \right) = \begin{array}{c} \text{diagram: blue strand top, green bottom, label } -g_s \end{array} + \begin{array}{c} \text{diagram: green strand top, blue bottom, label } g_s - g_u \end{array} + \begin{array}{c} \text{diagram: blue and green strands crossing, label } * \end{array} + \begin{array}{c} \text{diagram: blue and green strands crossing, label } g_s + \bar{g}_s \end{array},$$

where $* = g_u - g_s - \bar{g}_s$. We used (5.10) to get four of these terms (which, instead of appearing on the top, right, and bottom as in (5.10), now appear on the right, bottom, and left respectively because of the twisting). The remaining four terms came from taking the differential of the blue cap and cup. Now notice that $g_s + \bar{g}_s = z_s \in R^u$, so it can be slid through the green strand. This cancels four of the terms, yielding

$$(5.13) \quad d \left(\begin{array}{c} \text{diagram: blue cap and cup with green strand} \end{array} \right) = \begin{array}{c} \text{diagram: blue strand top, green bottom, label } -g_s \end{array} + \begin{array}{c} \text{diagram: green strand top, blue bottom, label } g_s - g_u \end{array} + \begin{array}{c} \text{diagram: blue and green strands crossing, label } g_u \end{array}.$$

But this agrees with the formula (5.10) after swapping the colors s and u , as desired.

We claim that checking the relations below is completely straightforward, and we leave it as an exercise to the reader.

- Sliding a trivalent vertex through a crossing.
- Crossings are inverse isomorphisms: $\begin{array}{c} \text{diagram: blue and green strands crossing} \end{array} \circ \begin{array}{c} \text{diagram: blue and green strands crossing} \end{array} = \begin{array}{c} \text{diagram: two parallel blue strands} \end{array}.$
- The $A_1 \times A_1 \times A_1$ Zamolodchikov relation.

5.2. The commuting relation and its idempotent decomposition. One of the defining relations of the Hecke algebra is the commuting relation $b_s b_u = b_u b_s$, when $m_{su} = 2$. This is lifted by an isomorphism

$$(5.14) \quad B_s B_u \cong B_u B_s$$

in the Hecke category, and the inverse isomorphisms p and i are given by the 4-valent vertices. In fact, up to rescaling, these are the only choices of isomorphisms.

One should think that $B_s B_u$ has a decomposition with a single term, so that I has size 1. We still need to check that $\Gamma_{I,d}$ has no cycles, or in other words, that there is no loop at the single vertex. In other words, we need to check that $pd(i) = 0$ in order for the isomorphism to lift to a fantastic filtration, and for $[B_s][B_u] = [B_u][B_s]$ in the p -dg Grothendieck group.

Proposition 5.3. *The isomorphism $B_s B_u \cong B_u B_s$ is an isomorphism of p -DG objects if and only if*

$$(5.15) \quad \text{X} \circ d \left(\text{X} \right) = 0,$$

if and only if $g_s \in R^u$ and $g_u \in R^s$, if and only if $d(\text{X}) = 0$.

Proof. The first equivalence is definitional, and the remainder are very straightforward computations. \square

5.3. Implications in type A. For the standard differential, as deduced in §4.5, when $s = s_i = (i, i+1)$ then either $g_{s_i} = x_i$ or $g_{s_i} = x_{i+1}$. In either case, $g_s \in R^u$ whenever u is distant from s , as desired.

5.4. Relation to thick calculus. There is very little additional perspective added from considering thick calculus (or singular calculus) for two distant colors. For completeness, we include a brief discussion.

In the thick calculus, there is an object $B_{s,u}$ corresponding to the parabolic subset $\{s, u\}$, which we draw with an olive strand. There are splitting and merging maps which give inverse isomorphisms between $B_{s,u}$ and the tensor product $B_s B_u$, as well as the tensor product $B_u B_s$. Here is the drawing of the isomorphism $B_{s,u} \rightarrow B_u B_s$.



The differential applied to this map will place a linear polynomial in each of the regions. Using (\star) , there is no need to put a polynomial in the region on top, so the polynomials must go on the left and right.

$$(5.16) \quad d(\text{Y}) = f_1 \text{Y} + \text{Y} f_2.$$

Moreover, by moving polynomials in R^{su} from left to right, we can assume that $f_1 = b' \varpi_s + a' \varpi_u$ for some scalars a', b' .

The inverse isomorphism comes from flipping the diagram upside down, and applying the differential to the relation stating that they are inverse isomorphisms implies

$$(5.17) \quad d(\text{Y}) = -f_1 \text{Y} + \text{Y} f_2.$$

For these inverse isomorphisms to give p -dg isomorphisms, it is immediate to compute that $f_1 = f_2 = 0$.

The isomorphism $B_{s,u} \rightarrow B_s B_u$ has a similar picture, with polynomials h_1 and h_2 instead. Composing the morphisms $B_s B_u \rightarrow B_{s,u} \rightarrow B_u B_s$ is supposed to give the 4-valent vertex, and the relations in the thick calculus are derived from this. So, without any appreciable difference from the computation in the rest of this section, we deduce that

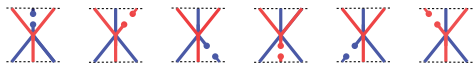
$$(5.18) \quad h_1 - f_1 = f_2 - h_2 = \partial_s(g_u) \varpi_s - \partial_u(g_s) \varpi_u.$$

Thus, unlike the ordinary diagrammatic Hecke category, a differential on the thick calculus is *not* uniquely determined by what happens to the dots. One can choose f_1 and f_2 freely, and then h_1 and h_2 are determined by (5.18). However, if the differential is to be good (i.e. it satisfies Proposition 5.3) then $f_1 = f_2 = h_1 = h_2 = 0$, and this additional flexibility disappears.

6. TWO ADJACENT COLORS

6.1. Preliminaries. Let s and t be adjacent, so that $m_{st} = 3$. Our next task is to compute the differential applied to the generating 6-valent vertex, which has degree 0. So let us examine for a time the space of degree +2 morphisms $B_s B_t B_s \rightarrow B_t B_s B_t$.

Inside this space we have six *broken 6-valent vertices* which we call the 12 o'clock break, the 2 o'clock break, etcetera.

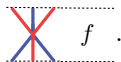


In fact, these six morphisms only span a four-dimensional subspace. There is a relation

$$(6.1) \quad \text{12 o'clock break} + \text{6 o'clock break} = \text{2 o'clock break} + \text{8 o'clock break},$$

together with its rotations around the clock. Using this, one can prove that 12, 2, 4, and 6 o'clock form a basis for this four-dimensional subspace.

Now consider the larger subspace spanned by the broken 6-valent vertices, and by morphisms obtained from the 6-valent vertex by adding a linear polynomial in some region. Forcing a polynomial from one region to another is possible by (4.9) at the cost of breaking some strands. Consequently, this subspace is spanned by the four broken 6-valent vertices above, and by the morphisms of the form



Meanwhile, there are five double leaves of degree +2, so the entire space of degree +2 morphisms is spanned by the previous subspace and one more morphism, which we can take to be



Thus we can assume that

$$(6.2) \quad d \left(\text{6-valent vertex} \right) = A \cdot \text{12 o'clock break} + B \cdot \text{2 o'clock break} + C \cdot \text{4 o'clock break} + D \cdot \text{6 o'clock break} + E \cdot \text{double leaf} + \text{6-valent vertex} \cdot f$$

for some scalars A, B, C, D, E and some linear polynomial f .

At this point, our computations are linear combinations of many very similar diagrams, and it helps to introduce some new and extremely abusive notation. First, instead of drawing a broken strand, we will merely draw a strand marked with a coefficient. Thus if A is a scalar then

$$(6.3) \quad \text{strand with coefficient } A := A \cdot \text{strand}.$$

Second, with the understanding that we are only interested in morphisms of a particular degree, we **superimpose** diagrams rather than adding them together! So, if we knew we were discussing morphisms $B_s \rightarrow B_s$ of degree 2, then

$$\begin{array}{c} \text{---} \\ f_1 \quad A \quad f_2 \\ \text{---} \end{array}$$

is actually shorthand for the sum

$$\begin{array}{c} \text{---} \\ f_1 \quad \text{---} \\ \text{---} \end{array} + A \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ f_2 \end{array}.$$

This notation is horribly abusive (oh, if only our mothers could see us now!) but being able to draw a large linear combination succinctly has benefits both for the page count and for the readability and understandability of this paper. For sanity, we will always use the olive color when we use this particular abuse of notation.

So, instead of (6.2) we can write the very compact

$$(6.4) \quad d\left(\begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array} f + E \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}.$$

To give some more examples, here is a rewriting of (6.1), after multiplication by A .

$$(6.5) \quad \begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array}.$$

Here is a useful equation involving the trivalent vertex.

$$(6.6) \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} - A\alpha_s.$$

Finally, before we begin the computation proper, let us set up notation for some important scalars. Let

$$(6.7) \quad \kappa_{ss} = \partial_s(g_s), \quad \kappa_{st} = \partial_s(g_t), \quad \kappa_{ts} = \partial_t(g_s), \quad \kappa_{tt} = \partial_t(g_t),$$

$$(6.8) \quad \bar{\kappa}_{ss} = \partial_s(\bar{g}_s), \quad \bar{\kappa}_{st} = \partial_s(\bar{g}_t), \quad \bar{\kappa}_{ts} = \partial_t(\bar{g}_s), \quad \bar{\kappa}_{tt} = \partial_t(\bar{g}_t).$$

Here are some things we know about these scalars. Since $g_s + \bar{g}_s \in R^s$ and similarly for t , we have

$$(6.9a) \quad \kappa_{ss} + \bar{\kappa}_{ss} = 0, \quad \kappa_{tt} + \bar{\kappa}_{tt} = 0.$$

Using (4.18) we have

$$(6.9b) \quad \kappa_{st} + \bar{\kappa}_{st} + \kappa_{ts} + \bar{\kappa}_{ts} = 0.$$

6.2. Pinning down the differential. We will pin down the coefficients A, B, C, D, E, f by checking the “death by pitchfork” relations, which say that putting a dot on one input to a 6-valent vertex, and merging its two neighbors with a trivalent vertex, will yield the zero morphism. In this first example, we put the dot at 12 o'clock.

$$(6.10) \quad \begin{array}{c} \text{Diagram: A 6-valent vertex with a red dot at the top input. Two blue strands enter from the bottom, and two red strands exit from the top.} \end{array} = 0.$$

Applying the differential to both sides we get

$$(6.11) \quad 0 = \begin{array}{c} \text{Diagram: A 6-valent vertex with a red dot at the top input. The left side is labeled with } A, C, B, D, f \text{ and a polynomial } *. \end{array} - E \cdot \begin{array}{c} \text{Diagram: A red strand with a dot at the top, and a blue arc below it with a dot at the bottom.} \end{array}.$$

The polynomial $*$ is $g_s - \bar{g}_t$. Now, the contributions of the terms with B, D , and f are all zero, since they have a subdiagram with (6.10) inside. The A term just creates a copy of $A\alpha_s$ which is added to $*$. Then $*$ can be forced to the right using (4.9), and only the term which breaks the strand will survive. Consequently, the result is

$$(6.12) \quad 0 = (C + \partial_t(g_s - \bar{g}_t + A\alpha_s)) \cdot \begin{array}{c} \text{Diagram: A 6-valent vertex with a red dot at the top input. The left side is labeled with } A, C, B, D, f \text{ and a polynomial } *. \end{array} - E \cdot \begin{array}{c} \text{Diagram: A red strand with a dot at the top, and a blue arc below it with a dot at the bottom.} \end{array}.$$

These two diagrams are linearly independent, from which we conclude that

$$(6.13) \quad E = 0,$$

$$(6.14) \quad C - A + \kappa_{ts} - \bar{\kappa}_{tt} = 0.$$

Similarly, we can put the pitchfork on bottom, putting the dot on 6 o'clock. An entirely similar argument shows that ($E = 0$ again, and)

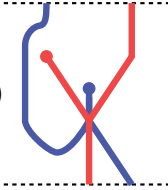
$$(6.15) \quad D - B + \bar{\kappa}_{st} - \kappa_{ss} = 0.$$

In general, the two-color Hecke category has an automorphism which flips a diagram upside-down and swaps the colors s and t . This automorphism preserves the 6-valent vertex. The effect of this symmetry will be to swap g_s with \bar{g}_t , \bar{g}_s with g_t , A with B , C with D , κ_{st} with $\bar{\kappa}_{ts}$, etcetera, while fixing E and f . Thus this symmetry interchanges (6.14) with (6.15).

Now we put the dot on 10 o'clock, and take the differential.

$$(6.16) \quad 0 = \begin{array}{c} \text{Diagram: A 6-valent vertex with a red dot at the top input. The left side is labeled with } A, C, B, D, f \text{ and a polynomial } *. \end{array}.$$

This time the polynomial $*$ is $g_t - \bar{g}_s - g_s$, with contributions coming from the red dot, the blue trivalent vertex, and the blue cup. Since $g_s + \bar{g}_s = z_s \in R^s$, it can be slid out of this region. Thus the only surviving terms are

$$(6.17) \quad 0 = (A + \kappa_{st}) \cdot$$


From this (and the corresponding computation for 8 o'clock, given by symmetry) we deduce that

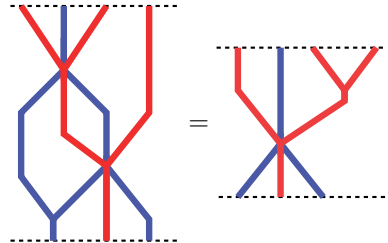
$$(6.18) \quad A = -\kappa_{st}, \quad B = -\bar{\kappa}_{ts}.$$

Then (6.14) and (6.15) give us

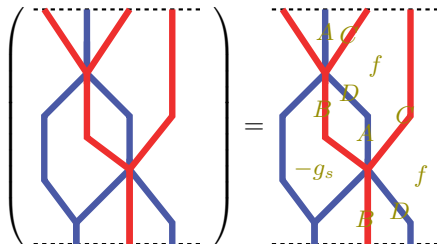
$$(6.19) \quad C = \bar{\kappa}_{tt} - \kappa_{ts} - \kappa_{st}, \quad D = \kappa_{ss} - \bar{\kappa}_{st} - \bar{\kappa}_{ts}.$$

Thus we have solved for A, B, C, D, E . If we resolved the 2 o'clock and 4 o'clock relations we would get equations for $\partial_s(f)$ and $\partial_t(f)$, but it is easier to check other relations to determine f precisely.

6.3. Checking the relations. The relation called *2-colored associativity* is the equality

$$(6.20) \quad$$


We apply the differential to the left hand side.

$$(6.21) \quad d \left(\text{Diagram} \right) = \text{Diagram with labels } A, B, C, D, E, f, -g_s.$$


Now we simplify this morphism. The central red strand, if broken, yields the zero morphism; consequently one of the B terms does not contribute, and the $-g_s$ term can be forced across this red line to become $-t(g_s)$. This polynomial can be forced further across the blue line, breaking this line with a coefficient of $-\partial_s(t(g_s))$, which is added to the previous coefficient for breaking this line, namely $A + D$. Note that

$$t(g_s) = g_s - \kappa_{ts}\alpha_t$$

so that

$$(6.22) \quad \partial_s(t(g_s)) = \kappa_{ss} + \kappa_{ts}.$$

In particular, the line is broken with coefficient

$$(6.23) \quad A + D - \partial_s(t(g_s)) = -\kappa_{st} + \kappa_{ss} - \bar{\kappa}_{st} - \bar{\kappa}_{ts} - \kappa_{ss} - \kappa_{ts} = 0,$$

where we used (6.9) to deduce that the coefficient was zero. Thus what remains at the end is

$$(6.24) \quad d \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \begin{array}{c} \text{Diagram 2} \end{array},$$

where the polynomial $*$ is $f - st(g_s)$.

Meanwhile, on the right hand side of 2-colored associativity, we have

$$(6.25) \quad d \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \begin{array}{c} \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \end{array}.$$

Here $*$ represents $-g_t - C\alpha_t$, and the second equality followed from (6.6).

Applying the differential to both sides of (6.20), we have a linear combination of terms which look like either side of (6.20) except with one strand broken, and the coefficients on each strand match up perfectly. We also have f in the rightmost region, matching perfectly, and a polynomial in the upper right region, which does not obviously match. So all that is required is for these polynomials to agree, namely,

$$(6.26) \quad f = -g_t - C\alpha_t + st(g_s).$$

This solves for the polynomial f .

To simplify further, note that

$$st(g_s) = g_s - \kappa_{ss}\alpha_s - \kappa_{ts}(\alpha_s + \alpha_t)$$

so that (using (6.19)) we have

$$(6.27) \quad f = g_s - g_t - (\kappa_{ss} + \kappa_{ts})\alpha_s - (\bar{\kappa}_{tt} - \kappa_{st})\alpha_t.$$

There is another version of 2-colored associativity (which is not a rotation of this one), which can be checked by flipping upside-down and swapping colors, a symmetry we have previously discussed. This yields the equality

$$(6.28) \quad f = \bar{g}_t - \bar{g}_s - (\bar{\kappa}_{tt} + \bar{\kappa}_{st})\alpha_t - (\kappa_{ss} - \bar{\kappa}_{ts})\alpha_s.$$

We must confirm that these two equations for f are consistent. Taking the difference, we have

$$g_s + \bar{g}_s - g_t - \bar{g}_t - (\kappa_{ts} + \bar{\kappa}_{ts})\alpha_s + (\kappa_{st} + \bar{\kappa}_{st})\alpha_t$$

which is equal to

$$z_s - z_t - \partial_t(z_s)\alpha_s + \partial_s(z_t)\alpha_t.$$

Using (4.18) and (4.17), this is zero as desired.

The next relation we should check is when a dot is placed on the 6-valent vertex. We can check what happens if a blue dot is placed, and determine what happens for a red dot by symmetry. Confirming that the differential preserves this relation is extremely tedious but straightforward, and uses only the tricks already used above, so we leave it to the reader.

Finally, we should check the cyclicity relation. We postpone the 3-color relations until the next chapter.

First we must discuss the differential of the other 6-valent vertex, from $B_t B_s B_t \rightarrow B_s B_t B_s$. In the next chapter, the 6-valent vertex $B_s B_t B_s \rightarrow B_t B_s B_t$ will be denoted ϕ , and the 6-valent vertex $B_t B_s B_t \rightarrow B_s B_t B_s$ will be denoted ψ .

We can deduce many things about ψ by applying the symmetry of the two-color Hecke category which swaps s and t ; this will swap g_s with g_t , \bar{g}_s with \bar{g}_t , etcetera. Thus we have

$$(6.29) \quad d \left(\begin{array}{c} \text{Diagram: A 6-valent vertex with three blue strands and three red strands crossing in a specific pattern.} \end{array} \right) = \begin{array}{c} \text{Diagram: A 6-valent vertex with three blue strands and three red strands, labeled with A', B', C', D', and f'.} \end{array},$$

where

$$(6.30a) \quad A' = -\kappa_{ts}, \quad B' = -\bar{\kappa}_{st},$$

$$(6.30b) \quad C' = \bar{\kappa}_{ss} - \kappa_{ts} - \kappa_{st}, \quad D' = \kappa_{tt} - \bar{\kappa}_{st} - \bar{\kappa}_{ts},$$

$$(6.30c) \quad f' = g_t - g_s - (\kappa_{tt} + \kappa_{st})\alpha_t - (\bar{\kappa}_{ss} - \kappa_{ts})\alpha_s.$$

Using (6.9) one can see that

$$(6.31) \quad D' = -C, \quad C' = -D.$$

Now applying the formula (6.4), and taking the differential of cups and caps as well, we get

$$(6.32) \quad d \left(\begin{array}{c} \text{Diagram: A 6-valent vertex with three blue strands and three red strands, including a cup and a cap.} \end{array} \right) = \begin{array}{c} \text{Diagram: A 6-valent vertex with three blue strands and three red strands, labeled with A, B, C, D, f-g_s, g_t, -g_t, and g_s-bar.} \end{array}.$$

Our goal will be to simplify this linear combination until it agrees with (6.29), which we do by forcing all the polynomials to the rightmost region, one strand at a time.

Recall that $A = -\kappa_{st} = -\partial_s(g_t)$. If we try to force g_t from the leftmost region across its neighboring blue strand (10 o'clock), we get $s(g_t)$ on the other side, plus a broken strand with scalar $+\partial_s(g_t)$. Thus the 10 o'clock break has overall coefficient $A + \partial_s(g_t) = 0$. Similarly, if we force $-\bar{g}_t$ across its neighboring blue strand (6 o'clock), we get $-s(\bar{g}_t)$ on the other side, and break the strand with coefficient

$-\partial_s(\bar{g}_t) = -\bar{\kappa}_{st} = B'$. Combining these two manipulations we have

$$(6.33) \quad d \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \begin{array}{c} \text{Diagram 2} \end{array}.$$

Diagram 1: A braid with two strands, one red and one blue. The red strand starts at the bottom left, goes up, loops around the blue strand, and goes back down. The blue strand starts at the bottom right, goes up, loops around the red strand, and goes back down. The diagram is enclosed in large parentheses with a 'd' to the left.

Diagram 2: A braid with two strands, one red and one blue. The red strand starts at the bottom left, goes up, and crosses over the blue strand. The blue strand starts at the bottom right, goes up, and crosses under the red strand. The diagram is enclosed in large parentheses with a 'd' to the left. Labels include $s(g_t)$, $f - g_s$, C' , D , \bar{g}_s , B' , and $B' - s(g_t)$.

In similar fashion, we continue to force more polynomials to the right. Forcing $s(g_t)$ across its red neighbor will produce $ts(g_t)$ on the other side, and break the strand with coefficient $\partial_t(s(g_t)) = \kappa_{tt} + \kappa_{st}$. Adding this to C , the overall coefficient on the 12 o'clock break will be

$$C + \kappa_{tt} + \kappa_{st} = \bar{\kappa}_{tt} + \kappa_{tt} - \kappa_{ts} = -\kappa_{ts} = A'.$$

To get the coefficient of the 2 o'clock break, we need to force $f - g_s + ts(g_t)$ across its neighboring blue strand. In the same way that we deduced (6.27) from (6.26), one can also deduce that

$$(6.34) \quad f = g_s + C'\alpha_s - ts(g_t).$$

Hence $f - g_s + ts(g_t) = C'\alpha_s$, and it breaks the 2 o'clock strand with coefficient $\partial_s(C'\alpha_s) = 2C'$. Adding this to the existing coefficient $D = -C'$, we get an overall coefficient of C' .

The ultimate coefficient of the 4 o'clock break will be

$$B - \partial_t(s(\bar{g}_t)) = -\bar{\kappa}_{ts} - \bar{\kappa}_{tt} - \bar{\kappa}_{st} = D'.$$

Putting this together, we get

$$(6.35) \quad d \left(\begin{array}{c} \text{Diagram 1} \end{array} \right) = \begin{array}{c} \text{Diagram 2} \end{array},$$

Diagram 1: Same as in (6.33).

Diagram 2: A braid with two strands, one red and one blue. The red strand starts at the bottom left, goes up, and crosses over the blue strand. The blue strand starts at the bottom right, goes up, and crosses under the red strand. The diagram is enclosed in large parentheses with a 'd' to the left. Labels include A' , C' , B' , and D' .

where

$$(6.36) \quad * = \bar{g}_s + s(f - g_s + ts(g_t)) - ts(\bar{g}_t) = \bar{g}_s - C'\alpha_s - ts(\bar{g}_t).$$

It remains to show that $* = f'$. We leave this to the reader, having done enough similar computations.

6.4. The braid relation and its defining idempotent decomposition: Abstractions. The braid relation on the Hecke algebra, reinterpreted in the Kazhdan-Lusztig presentation, is

$$(6.37) \quad b_s b_t b_s - b_s = b_t b_s b_t - b_t,$$

as both sides are actually descriptions of the Kazhdan-Lusztig basis element b_{sts} . The categorification of this statement is the direct sum decompositions

$$(6.38) \quad B_s B_t B_s \cong M \oplus B_s, \quad B_t B_s B_t \cong M' \oplus B_t,$$

together with the isomorphism

$$(6.39) \quad M \cong M'.$$

However, M and M' are not objects in the diagrammatic Hecke category, but are only objects in the Karoubi envelope, being the image of certain idempotents.

In §1 we discussed the practical way to prove a direct sum decomposition, see (1.5). Analogously, one wishes to prove (6.38) and (6.39) practically, but using only morphisms between Bott-Samelson bimodules. For sake of brevity let us write $X = B_s B_t B_s$ and $Y = B_t B_s B_t$. One should provide morphisms

$$\begin{aligned} i_s: B_s \rightarrow X, \quad i_t: B_t \rightarrow Y, \quad p_s: X \rightarrow B_s, \quad p_t: Y \rightarrow B_t, \\ \phi: X \rightarrow Y, \quad \psi: Y \rightarrow X, \end{aligned}$$

satisfying

$$(6.40a) \quad p_s i_s = \text{id}_s, \quad p_t i_t = \text{id}_t,$$

$$(6.40b) \quad \phi i_s = 0, \quad \psi i_t = 0, \quad p_s \psi = 0, \quad p_t \phi = 0,$$

$$(6.40c) \quad \text{id}_X = \psi \phi + i_s p_s, \quad \text{id}_Y = \phi \psi + i_t p_t.$$

One should think that the maps ϕ and ψ pass through the common summand $M \cong M'$, so their composition is the idempotent projecting to this summand. In particular, M is to be identified with the object $(X, \psi \phi)$ in the Karoubi envelope, and M' with $(Y, \phi \psi)$. The map ϕ induces an isomorphism

$$\bar{\phi}: (X, \psi \phi) \rightarrow (Y, \phi \psi)$$

and similarly ψ induces the inverse isomorphism $\bar{\psi}$. Now we ask what extra conditions produce the appropriate relations on the Grothendieck group in the p -dg setting.

Remark 6.1. Everything we say in this section, including the main result (Proposition 6.2), is easily adaptable to the general situation where one has

$$X \cong M \oplus P, \quad Y \cong M' \oplus Q, \quad M \cong M'$$

in some p -dg category, where P and Q are genuine objects, while M and M' are only objects in the Karoubi envelope.

The identity of X is decomposed as a sum of two orthogonal idempotents $e_1 = i_s p_s$ and $e_2 = \psi \phi$. In order for the decomposition $X \cong M \oplus B_s$ to be a dg-filtration, we need¹⁰ either $d(e_1)e_2 = 0$ or $d(e_2)e_1 = 0$. Now

$$(6.41) \quad d(e_1)e_2 = d(i_s p_s)\psi \phi = i_s d(p_s)\psi \phi$$

since $p_s \psi = 0$. Clearly $d(e_1)e_2 = 0$ if $d(p_s)\psi \phi = 0$. Conversely, by postcomposing with p_s , if $d(e_1)e_2 = 0$ then $d(p_s)\psi \phi = 0$. Using a similar argument, precomposing with either ψ or ϕ , we see that

$$(6.42) \quad d(e_1)e_2 = 0 \iff d(p_s)\psi = 0.$$

We will use this pre- and post-composition trick several times below.

Similarly,

$$(6.43) \quad d(e_2)e_1 = 0 \iff d(\phi)i_s = 0.$$

¹⁰As a reminder, this condition will imply that either $\text{Hom}(X, -)e_1$ or $\text{Hom}(X, -)e_2$ is preserved by the differential, as a left p -dg module over the category, and that the other one is the quotient of $\text{Hom}(X, -)$ by the first. It also implies the analogous condition for right p -dg modules.

This seems entirely analogous, but the proof is slightly trickier. Clearly

$$(6.44) \quad d(e_2)e_1 = \psi d(\phi)i_s p_s$$

so

$$(6.45) \quad d(e_2)e_1 = 0 \iff \psi d(\phi)i_s = 0 \iff \phi \psi d(\phi)i_s = 0.$$

But in fact $\phi\psi$ can be replaced with id_{tst} here, since the difference is $i_t p_t d(\phi)i_s$, and

$$(6.46) \quad p_t d(\phi)i_s = d(p_t \phi)i_s - d(p_t)\phi i_s = 0.$$

In addition to checking that the decomposition on X is a dg-filtration, we need to confirm that the image of $i_s p_s$ has the appropriate p -dg structure, which amounts to checking that

$$(6.47) \quad d(p_s)i_s = 0$$

as a degree 2 endomorphism of B_s .

Analogous conditions need to hold for the decomposition of Y .

Finally, we need to confirm that $M = (X, \psi\phi)$ and $M' = (Y, \phi\psi)$ are isomorphic as p -dg modules, with their induced differentials. Recall that, if (X, e) and (Y, f) are two objects in the Karoubi envelope, then the induced differential \bar{d} on $\text{Hom}((X, e), (Y, f)) = f \text{Hom}(X, Y)e$ is

$$(6.48) \quad \bar{d}(f\xi e) = f d(f\xi e)e,$$

where $\xi \in \text{Hom}(X, Y)$ and d is the usual differential on $\text{Hom}(X, Y)$. If $\bar{\phi}$ and $\bar{\psi}$ are to induce inverse isomorphisms of p -dg modules then we need

$$(6.49) \quad \bar{d}(\bar{\phi})\bar{\psi} = 0$$

and (equivalently)

$$(6.50) \quad \bar{d}(\bar{\psi})\bar{\phi} = 0.$$

The first equation unravels to

$$\phi \psi d(\phi) \psi \phi = 0$$

which is equivalent by pre- and post-composition to

$$(6.51) \quad \psi d(\phi) \psi = 0.$$

The second equation is equivalent to

$$(6.52) \quad \phi d(\psi) \phi = 0.$$

To convince the reader that (6.51) and (6.52) are equivalent, let us pre- and post-compose (6.51) with ϕ . The result is

$$\phi \psi d(\phi) \psi \phi = \phi \psi d(\phi \psi) \phi - \phi \psi \phi d(\psi) \phi = \phi \psi d(\phi \psi) \phi - \phi d(\psi) \phi.$$

So $\psi d(\phi) \psi = 0$ will imply that $\phi d(\psi) \phi = 0$ so long as $\phi \psi d(\phi \psi) \phi = 0$. But since $d(\text{id}_{tst}) = 0$, we know that $d(\phi \psi) = -d(i_t p_t)$. Then

$$\psi d(\phi \psi) \phi = -\psi d(i_t) p_t \phi - \psi i_t d(p_t) \phi = 0 + 0 = 0,$$

as desired.

Together, all these conditions imply that X is fantastically filtered by M and B_s in the Karoubi envelope, that Y is fantastically filtered by M' and B_t , and that $M \cong M'$ as p -dg objects. In conclusion, we have proven the following result.

Proposition 6.2. *Given maps $i_s, p_s, i_t, p_t, \phi, \psi$ as in (6.40), then consider the following graph.*

$$(6.53) \quad d(p_s)i_s \hookrightarrow B_s \xrightleftharpoons[d(p_s)\psi]{d(\phi)i_s} M' \xrightleftharpoons[\phi d(\psi)\phi]{\psi d(\phi)\psi} M \xrightleftharpoons[d(\psi)i_t]{d(p_t)\phi} B_t \hookleftarrow d(p_t)i_t.$$

Erase the edges with zero labels (noting that if one edge between M and M' is zero, then so is the other). If the resulting graph has no cycles or loops, then the idempotent decompositions (6.38) and (6.39) lift to fantastic filtrations, the objects M and M' are cofibrant, and the braid relation (6.37) holds in the p -dg Grothendieck group.

Remark 6.3. The existence of a fantastic filtration is useful not only because it implies a relation in the Grothendieck group, but also because it implies, e.g., that $B_s B_t B_s$ is in the triangulated hull of B_s and M , and hence in the triangulated hull of $\{B_s, B_t, B_t B_s B_t\}$. Since all the other objects are cofibrant, the 2/3 rule in triangulated categories implies that M is cofibrant.

Remark 6.4. One should think of the pair of edges between M and M' as being like a loop. There is a partial idempotent completion where we could add a new object Z corresponding to either M or M' , and rewrite the idempotent decompositions as

$$X \cong B_s \oplus Z, \quad Y \cong B_t \oplus Z.$$

Then we could use the ordinary theory of Fc-filtrations to analyze these decompositions. We would get two graphs corresponding to the right and left halves of the graph in Proposition 6.2, and in each graph there would be a loop at Z which corresponds to the edges between M and M' .

6.5. The braid relation and its defining idempotent decomposition: Computations. Now let us compute the graph of Proposition 6.2. We should note that all maps i_s, p_s, ϕ , etcetera are uniquely determined up to scalar, living in one-dimensional Hom spaces, so there is only one graph to compute. The maps ϕ and ψ are 6-valent vertices, and the remaining maps are pitchforks.

First we check the loop at B_s . We have

$$(6.54) \quad d(p_s)i_s = \text{Diagram} ,$$


where $*$ is the polynomial $\alpha_t(g_t - \bar{g}_s)$. Note that $p_s i_s$ is the same picture with a red barbell α_t inside. Applying d to p_s produces the extra factor of $g_t - \bar{g}_s$.

For (6.54) to be zero we need $*$ to be s -invariant, which requires that $g_t - \bar{g}_s$ is proportional to $s(\alpha_t) = \alpha_s + \alpha_t$. This is a new condition! The scalar of proportionality can be determined by applying Demazure operators. Since $\partial_s(s(\alpha_t)) = \partial_t(s(\alpha_t)) = 1$, we see that

$$(6.55) \quad g_t - \bar{g}_s = (\kappa_{st} - \bar{\kappa}_{ss})(\alpha_s + \alpha_t) = (\kappa_{tt} - \bar{\kappa}_{ts})(\alpha_s + \alpha_t).$$

Checking the loop at B_t , we get the analogous equation

$$(6.56) \quad g_s - \bar{g}_t = (\kappa_{ts} - \bar{\kappa}_{tt})(\alpha_s + \alpha_t) = (\kappa_{ss} - \bar{\kappa}_{st})(\alpha_s + \alpha_t).$$

Now we check the pair of edges between M and M' . We have

$$(6.57) \quad \psi d(\phi) \psi = \begin{array}{c} \text{Diagram with two diamond shapes, one red and one blue, connected by a vertical line. The top diamond is red and labeled A, C, f. The bottom diamond is blue and labeled B, D. The vertical line is blue. The top and bottom edges are dashed lines. The left and right edges are solid lines. The top and bottom edges are labeled A, C, f and B, D respectively. The vertical line is labeled f. } \end{array}.$$

The A term and the B term vanish. We encourage the reader to confirm that the C term, the D term, and the f term are linearly independent!¹¹ Thus

$$(6.58) \quad \psi d(\phi) \psi = 0 \iff C = D = f = 0.$$

Using (6.31), this will imply that $C' = D' = 0$ as well (and also $f' = 0$).

Note that $f = -g_t - C\alpha_t + st(g_s)$ by (6.26), so if $f = C = 0$ then

$$(6.59) \quad g_t = st(g_s).$$

If $g_s = 0$ then $g_t = 0$ and vice versa. Recall that Proposition 4.6 left us with two cases for each color. Either $g_s = \bar{g}_s = 0$ or $g_s \neq \bar{g}_s = s(g_s) \neq 0$, and similarly for t . We can now observe that we are in the same case for both s and t : either $g_s = g_t = 0$, or $g_s \neq 0 \neq g_t$. In the case where $g_s = g_t = 0$ one deduces that the differential is also zero on both 6-valent vertices, so it is zero on any diagram with colors in $\{s, t\}$. This is a valid solution, but one which makes the computations trivial.

Let us restrict our attention to the other case. Thus we assume that

$$(6.60) \quad \bar{g}_s = s(g_s), \quad \bar{g}_t = t(g_t), \quad g_t = st(g_s).$$

We leave it as an exercise to verify that (6.60) implies that $C = D = f = f' = 0$ as well as (6.55) and (6.56). Thus (6.60) is the only assumption we need. The reader attempting this exercise will be helped by the formula

$$(6.61) \quad st\partial_s(h) = \partial_t(st(h))$$

for any polynomial h .

Recall also the scalar $\kappa_s = \partial_s(g_s)$ which we used in a previous chapter. Then $g_t = st(g_s)$ and (6.61) imply that

$$(6.62) \quad \kappa_s = \partial_s(g_s) = \partial_t(g_t) = \kappa_t.$$

We write

$$(6.63) \quad \kappa := \kappa_s = \kappa_t.$$

¹¹After all, M is supposed to be isomorphic to the indecomposable Soergel bimodule B_{sts} . By the Soergel hom formula $\text{End}^2(B_{sts})$ is spanned by: linear polynomials times the identity (the f term), a degree-2 map factoring through B_{st} (the D term), and a degree 2 map factoring through B_{ts} (the C term).

Remark 6.5. If $g_t = st(g_s)$ then $\bar{g}_t = tst(g_s)$. So it makes sense that $g_s - \bar{g}_t = g_s - tst(g_s)$ is proportional to $\alpha_s + \alpha_t$, since tst is a reflection and $\alpha_s + \alpha_t$ is its root.

Now we analyze the two remaining pairs of edges, under the requirements that the loops vanished above. We have

$$(6.64) \quad d(p_t)\phi = \text{Diagram},$$

where $*$ is the polynomial $g_s - \bar{g}_t = g_s - tst(g_s)$. This polynomial can be forced out, and only the term which breaks the pitchfork will survive, so

$$(6.65) \quad d(p_t)\phi = 0 \iff \partial_t(g_s - \bar{g}_t) = 0.$$

However, we have already seen that $g_s - \bar{g}_t$ is proportional to $\alpha_s + \alpha_t$, which is not t -invariant. Thus $\partial_t(g_s - \bar{g}_t) = 0$ if and only if $g_s - \bar{g}_t = 0$! Consequently,

$$(6.66) \quad d(p_t)\phi = 0 \iff g_s = \bar{g}_t = sts(g_s).$$

Meanwhile,

$$(6.67) \quad d(\psi)i_t = \text{Diagram}.$$

Only the B' term will contribute, and it will contribute $\partial_t(B'\alpha_s) = -B'$. Recall that $B' = -\partial_s(\bar{g}_t)$. Using (6.61), we can also observe that

$$(6.68) \quad -\partial_s(\bar{g}_t) = \partial_t(g_s).$$

Then

$$(6.69) \quad d(\psi)i_t = 0 \iff B' = 0 \iff \partial_s(\bar{g}_t) = 0 \iff \partial_t(g_s) = 0.$$

Only one of these two conditions (6.66) and (6.69) need hold, so either $g_s = \bar{g}_t = sts(g_s)$ or $B' = 0 = \partial_s(\bar{g}_t) = \partial_t(g_s)$, but not necessarily both. That is, g_s is either fixed by the reflection t or by the reflection sts . If both hold, then $g_s = 0$.

Similarly, either $d(p_s)\psi = 0$ or $d(\phi)i_s = 0$, from which we deduce that either $g_t = \bar{g}_s = sts(g_t)$ or $B = 0 = \partial_t(\bar{g}_s) = \partial_s(g_t)$. These two conditions are equivalent to the conditions above, but in the reverse order: if g_s is fixed by sts then $g_t = st(g_s)$ is fixed by s , and if g_s is fixed by t then g_t is fixed by sts .

Let us consider one possibility, where $B = 0$, so that g_t is fixed by s and g_s is fixed by sts . In this case both $A = B = 0$, so that the differential kills ϕ ! We leave the reader to determine that $B' = -A' = \partial_t(g_s) \neq 0$. Moreover,

$$(6.70) \quad \partial_t(g_s) = \partial_t(tst(g_s)) = -\partial_t(st(g_s)) = -\partial_t(g_t) = -\kappa.$$

Thus, $A' = \kappa$ and $B' = -\kappa$ and our differential satisfies

$$(6.71) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0, \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}.$$

Analogously, the other possibility, where $B' = 0$, also satisfies $A' = 0$ and $B = -A = \partial_s(g_t)$, and

$$(6.72) \quad \partial_s(g_t) = \partial_s(st(g_s)) = \partial_s(s(g_s)) = -\kappa,$$

so that $A = \kappa$ and $B = -\kappa$ and

$$(6.73) \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = 0, \quad d \left(\begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array}.$$

These are our two options.

Let us summarize.

Proposition 6.6. *The graph from Proposition 6.2 has no loops or cycles if and only if one of the following three possibilities holds.*

- (1) $g_s = g_t = 0$, and the differential is zero on every diagram with colors in $\{s, t\}$.
- (2) Equation (6.60) holds, and g_t is fixed by s , and $g_s = t(g_t)$ is fixed by sts . Then the differential on 6-valent vertices is given by (6.71), where $\kappa = \partial_s(g_s) = \partial_t(g_t)$.
- (3) Equation (6.60) holds, and g_s is fixed by t , and $g_t = s(g_s)$ is fixed by sts . Then the differential on 6-valent vertices is given by (6.73), where $\kappa = \partial_s(g_s) = \partial_t(g_t)$.

In other words, any good differential must satisfy one of these three possibilities.

Example 6.7. For the standard differential on $\mathbb{k}[x_1, \dots, x_n]$, we deduced in §4.5 that either $g_i = x_i$ or $g_i = x_{i+1}$. Ere now it seemed we could make this choice for each i independently. However, Proposition 6.6 forces the choices of g_i and the choice of g_{i+1} to be related to each other, since $g_{i+1} = s_i s_{i+1} g_i$. If $g_i = x_i$ for some i then $g_{i+1} = x_{i+1}$, and consequently $g_j = x_j$ for all j . If $g_i = x_{i+1}$ then $g_{i+1} = x_{i+2}$, and consequently $g_j = x_{j+1}$ for all j .

6.6. Implications in simply laced types. Let (W, S) be an irreducible Coxeter group with $m_{st} \in \{2, 3\}$ for all $s, t \in S$.

If $g_s = 0$ for any $s \in S$, then by Proposition 6.6 we must have $g_t = 0$ for all t in the same connected component of the Coxeter graph as s . This is the boring case.

Otherwise, Proposition 6.6 gives one a dichotomy for each pair $s, t \in S$ with $m_{st} = 3$. We will encode this with an orientation on the edges in the Coxeter graph: the edge points from s to t if g_s is fixed by t .

Now suppose that $s, t, u \in S$ generate a copy of A_3 inside W , with $m_{su} = 2$. Suppose that the $s - t$ edge is oriented from s to t . Then g_s is fixed by t . We also know that g_s is fixed by u , thanks to Proposition 5.3. Now, $g_t = st(g_s) = s(g_s)$, so that

$$u(g_t) = us(g_s) = su(g_s) = s(g_s) = g_t.$$

Consequently g_t is fixed by u , and the $t - u$ edge is oriented from t to u . Similarly, we leave the reader to deduce that when the $s - t$ edge is oriented from t to s , the $t - u$ edge must be oriented from u to t . In particular, this implies that every copy of A_3 must be consistently oriented: t is never a source or a sink. This fact is sadly not true in general because of Proposition 6.8.

Proposition 6.8. *There is no good differential in any simply laced type outside of finite or affine type A .*

Proof. Outside of finite and affine type A , the Coxeter graph must have a subgraph of type D_4 . There is no way to orient D_4 such that every copy of A_3 inside is consistent. \square

6.7. Implications in type A . We continue to let g_i denote g_{s_i} when $s_i = (i, i+1)$, for $1 \leq i \leq n-1$. As just noted in §6.6, any nonzero good differential must induce a consistent orientation on the Dynkin diagram.

Let us begin by quickly discussing the case where the realization is the standard one, with polynomial ring $\mathbb{k}[x_1, \dots, x_n]$, and the differential is assumed to be the standard one, with $d(x_i) = x_i^2$. We have proven that there are only two nonzero good differentials possible which extend the standard differential on the polynomial ring, the one where $g_i = x_i$ for all i , and the one where $g_i = x_{i+1}$ for all i . Technically we still need to check the three-color relations, but this is done in the next chapter. We refer to the differential where $g_i = x_i$ for all i as having the *standard orientation*, and the other differential as having the *reverse orientation*. In Manin-Schechtman theory, braid relations $s_i s_{i+1} s_i \rightarrow s_{i+1} s_i s_{i+1}$ can be given the lexicographic orientation, and the set of all reduced expressions for a given element becomes a semioriented graph (the commuting braid relations $s_i s_j = s_j s_i$ have no orientation). This graph has a unique source and a unique sink, up to commuting braid relations. See [Eli16a] for more details. For the standardly oriented good differential, d will kill any 6-valent vertex corresponding to a reverse-oriented braid relation. Vice versa, for the reverse oriented good differential, 6-valent vertices corresponding to standardly oriented braid moves are killed. Let us refer to the p -dg category associated to the standard orientation as the *standard p -dg diagrammatic Hecke category*.

Now we work with an arbitrary realization and its polynomial ring R , but under the assumption that its diagrammatic Hecke category possesses a good differential. For ease of discussion we will assume the Dynkin diagram gets the standard orientation; the other case can be handled similarly, or by duality. We will prove that our p -dg category is isomorphic to the standard p -dg diagrammatic Hecke category, perhaps after extension of the realization.

We know that $g_1 \in R^{s^2}$. We also know that $g_1 \in R^{s^j}$ for all $j \geq 3$, by Proposition 5.3. Thus g_1 is invariant in all simple reflections but the first, so it is invariant in the parabolic subgroup $S_1 \times S_{n-1}$. In other words, g_1 has the same stabilizer in the symmetric group as does x_1 in the standard polynomial ring. Similarly, g_i has the same stabilizer as x_i for all $1 \leq i \leq n-1$, which one can check in the same way, or can confirm since $g_i = s_{i-1} s_{i-2} \cdots s_2 s_1(g_1)$. Let us denote $s_{n-1}(g_{n-1})$ by g_n ; it has the same stabilizer as x_n .

Let $\kappa = \partial_1(g_1)$, which is also equal to $\partial_i(g_i)$ for all i by (6.61). Note that, by the usual definition of Demazure operators, for all $1 \leq i \leq n-1$ one has

$$(6.74) \quad g_i - g_{i+1} = g_i - s_i(g_i) = \kappa \alpha_i.$$

In particular, the span of the g_i contains all the simple roots, and has dimension at least $n - 1$. It is also preserved by the differential, since $d(g_i) = g_i^2$. Thus we may as well restrict our realization and our attention to the span of $\{g_i\}_{i=1}^n$.

Proposition 6.9. *There is a ring homomorphism from $\mathbb{k}[x_1, \dots, x_n]$ to the subring of R generated by g_1, \dots, g_n , sending $x_i \mapsto g_i$. This homomorphism is S_n -equivariant and intertwines the standard differential with the differential on R . In fact, it is an isomorphism so long as the characteristic of \mathbb{k} is not 2. If the characteristic is 2, it is either an isomorphism or is the quotient map by the ideal generated by $e_1 = x_1 + \dots + x_n$.*

Proof. The first two sentences are straightforward, so it remains to show that the homomorphism is an isomorphism. Let I be the kernel of the map. If g_1, \dots, g_n is linearly independent, then they must be algebraically independent (since they are linear terms inside a polynomial ring), and $I = 0$.

Suppose to the contrary that there were a linear dependence $\sum a_i g_i = 0$ for some scalars a_i . Let $y = \sum_{i \geq 3} a_i g_i$, so that $a_1 g_1 + a_2 g_2 + y = 0$. Since y is fixed by s_1 , so must be $a_1 g_1 + a_2 g_2$. Since $g_2 = s_1(g_1) \neq g_1$, this implies that $a_1 = a_2$. By similar arguments, $a_2 = a_3$, etcetera, and all the scalars a_i are equal. So the only possible linear dependence relation is $a(g_1 + \dots + g_n)$ for some nonzero scalar a . Since the linear polynomials form a free \mathbb{k} -module in any realization, a relation of the form $a(g_1 + \dots + g_n) = 0$ for $a \neq 0$ implies that $g_1 + \dots + g_n = 0$.

Thus if I is nonzero then I contains the symmetric polynomial $e_1 = x_1 + \dots + x_n$. Since there is at most one linear relation, at least $n - 1$ of the elements g_i are algebraically independent. If I is any bigger than the ideal generated by e_1 then this contradicts the algebraic independence of $\{g_1, \dots, g_{n-1}\}$.

Now I is preserved by the differential, and $d(e_1) = \sum x_i^2 = e_1^2 - 2e_2$. Thus $2e_2 \in I$. Unless the characteristic of \mathbb{k} is 2, this gives the desired contradiction. \square

Remark 6.10. Similarly, $d(e_k) = e_1 e_k - (k + 1)e_{k+1}$ for all k . Ignoring the case of small primes, any ideal preserved by I which contains e_1 will also contain e_k for all $1 \leq k \leq n$. Thus $\mathbb{k}[x_1, \dots, x_n]/I$ is finite-dimensional, as a quotient of the coinvariant ring. But its image inside R is infinite dimensional, containing at least a polynomial ring with $n - 1$ generators. This is a contradiction.

The conclusion is that our realization with good differential need not be the standard one with the standard differential, but it (or the relevant part of it, the subring generated by the g_i) is equivariantly isomorphic to the standard realization with the standard differential.

Example 6.11. There is an S_3 -invariant isomorphism $\mathbb{k}[x_1, x_2, x_3] \rightarrow \mathbb{k}[y_1, y_2, y_3]$ sending $x_1 \mapsto y_2 + y_3$, $x_2 \mapsto y_1 + y_3$, and $x_3 \mapsto y_1 + y_2$. This intertwines the differential, when

$$d(y_i) = y_i^2 + y_i y_j + y_i y_k - y_j y_k$$

for all $\{i, j, k\} = \{1, 2, 3\}$. So, even when the realization is the standard one, we cannot and should not rule out the possibility that R is equipped with a non-standard differential, because it might be a non-standard differential isomorphic to a standard differential.

The above proposition showed that the underlying polynomial rings of the standard realization and our good realization are isomorphic (outside of the possible

characteristic 2 exception), but we still need to show that the associated Hecke categories are isomorphic categories (via an isomorphism which intertwines the differential). This is slightly subtle, because the isomorphism $\mathbb{k}[x_1, \dots, x_n] \rightarrow R$ sending $x_i \mapsto g_i$ need not send simple roots to simple roots! Instead,

$$(6.75) \quad \alpha_i = x_i - x_{i+1} \mapsto g_i - g_{i+1} = \kappa \alpha_i,$$

and roots are rescaled by the scalar κ .

Proposition 6.12. *Assume that the Hecke category is equipped with a good differential d . The homomorphism $\mathbb{k}[x_1, \dots, x_n] \rightarrow R$ sending $x_i \mapsto g_i$ lifts to a functor between the standard p -dg Hecke category and the version equipped with d . This functor rescales each s -colored enddot by κ , and each s -colored startdot by 1. Thus $\alpha_s \mapsto \kappa \alpha_s$, as noted above. The merging trivalent vertex is rescaled by 1, and the splitting trivalent vertex is rescaled by κ^{-1} . The 4- and 6-valent vertices are rescaled by 1. This functor is an isomorphism, except in the case of characteristic 2, when it is either an isomorphism or the kernel is generated by the polynomial e_1 .*

Proof. Let us assume that the map $\mathbb{k}[x_1, \dots, x_n] \rightarrow R$ is an isomorphism (as it must be outside of characteristic 2). In [EH] we classify all autoequivalences of the Hecke category which fix the objects B_s for each s . Any such automorphism is determined uniquely by how it rescales the start and end dots, and how it affects the remainder of the polynomial ring (beyond the part spanned by roots), and these choices can be made arbitrarily. So, the functor defined above is an autoequivalence. Checking that it intertwines the differential is straightforward. Modifying these results to account for the possible e_1 kernel in characteristic 2 is straightforward. \square

6.8. Relations to thick calculus. The thick calculus has a new object $B_{s,t}$ (drawn as purple) and several new morphisms: a trivalent vertex



$B_{s,t}B_s \rightarrow B_{s,t}$ together with variants thereof; splitters $B_{s,t} \rightarrow B_sB_tB_s$ and $B_{s,t} \rightarrow B_tB_sB_t$, and mergers $B_sB_tB_s \rightarrow B_{s,t}$ and $B_tB_sB_t \rightarrow B_{s,t}$, pictured below.



The composition of a splitter and a merger gives the 6-valent vertex $B_sB_tB_s \rightarrow B_tB_sB_t$. The composition of a merger with a splitter gives the identity map of $B_{s,t}$.

Once again there is not very much new to say about differentials on the thick calculus. Here are the highlights.

- Because all the direct summands of $B_sB_tB_s$ are actually objects in the thick diagrammatic category, one need not worry about the abstractions of §6.4, but can work directly with fantastic filtrations.
- The differential on the trivalent vertex is forced to be the same as for an ordinary trivalent vertex (e.g. for the map $B_{s,t}B_s \rightarrow B_{s,t}$, put $-\bar{g}_s$ in the middle on the bottom).
- One can compute the general differential on the splitters and mergers. The computation is just as nasty and thorny as the one above for the general differential of the 6-valent vertices. Just as in §5.4 there is one extra degree of freedom: the polynomial next to the splitter and the polynomial next to

the merger can be arbitrary so long as they add up to the polynomial f from (6.4). Also as in §5.4, this additional freedom disappears when restricting to good differentials.

For posterity, here is the good differential on the thick calculus, when the orientation on the Dynkin diagram is $s \rightarrow t$.

$$\begin{aligned}
 (6.76a) \quad & d \left(\begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching right} \end{array} \right) = \begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching right} \\ \text{Label } -\bar{g}_s \end{array}, & d \left(\begin{array}{c} \text{Thin blue line branching left} \\ \text{Thick vertical line} \end{array} \right) = \begin{array}{c} \text{Thin blue line branching left} \\ \text{Thick vertical line} \\ \text{Label } -\bar{g}_s \end{array}, \\
 (6.76b) \quad & d \left(\begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching left} \end{array} \right) = \begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching left} \\ \text{Label } -g_s \end{array}, & d \left(\begin{array}{c} \text{Thin blue line branching right} \\ \text{Thick vertical line} \end{array} \right) = \begin{array}{c} \text{Thin blue line branching right} \\ \text{Thick vertical line} \\ \text{Label } -g_s \end{array}, \\
 (6.76c) \quad & d \left(\begin{array}{c} \text{Thin blue line branching left} \\ \text{Thin red line branching right} \\ \text{Thick vertical line} \end{array} \right) = 0, & d \left(\begin{array}{c} \text{Thick vertical line} \\ \text{Thin red line branching left} \\ \text{Thin blue line branching right} \end{array} \right) = 0, \\
 (6.76d) \quad & d \left(\begin{array}{c} \text{Thin red line branching left} \\ \text{Thin blue line branching right} \\ \text{Thick vertical line} \end{array} \right) = \kappa \begin{array}{c} \text{Thin red line branching left} \\ \text{Thin blue line branching right} \\ \text{Thick vertical line} \\ \text{Label } \kappa \end{array}, & d \left(\begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching left} \\ \text{Thin red line branching right} \end{array} \right) = -\kappa \begin{array}{c} \text{Thick vertical line} \\ \text{Thin blue line branching left} \\ \text{Thin red line branching right} \\ \text{Label } -\kappa \end{array}.
 \end{aligned}$$

7. THREE COLORS

It remains to check the Zamolodchikov relations associated to finite rank 3 Coxeter subgroups. Since we are in simply laced type, there are only three possible rank 3 subgroups: $A_1 \times A_1 \times A_1$, $A_1 \times A_2$, and A_3 .

When the differential is good, the differential kills all 4-valent vertices, and kills half of the 6-valent vertices. There is a version of each Zamolodchikov relation where the differential kills both sides, and thus the relation is checked trivially! This is because the Zamolodchikov relations are equalities between two oriented paths in the reduced expression graph. For the reverse-oriented good differential, all oriented paths are sent to zero by the differential. Rotating the relations by 180 degrees, we obtain an equivalent relation which is an equality between reverse-oriented paths in the reduced expression graph, and these are sent to zero by the standardly oriented good differential.

Remark 7.1. It is a good exercise for the reader learning diagrammatics to compute directly that the standardly oriented good differential preserves the oriented version of the Zamolodchikov relation (the version that it does not just send to zero).

We have checked the $A_1 \times A_1 \times A_1$ and $A_1 \times A_2$ relations for an arbitrary differential, and they hold; the techniques required for this check have all been discussed above. We tried to check the A_3 relation for an arbitrary differential, but it was a surprisingly thorny computation, and we gave up.

8. DOES THE DIFFERENTIAL HAVE DIVIDED POWERS?

Yes, at least for the good differentials. We did not bother to check the general differential.

Consider the graded ring $T = \mathbb{Z}[x]$ with differential $d(x) = x^2$. Let M be a free T -module of rank 1, generated by the element m , and equip with an edg-structure (i.e. a degree +2 derivation) where $d_M(m) = ax \cdot m$ for some $a \in \mathbb{Z}$. We call this edg-module M_a . Then it is easy to compute that

$$(8.1) \quad d_M^{(k)} x^\ell \cdot m = \binom{\ell + k + a - 1}{k} x^{\ell+k} \cdot m$$

for all $\ell \geq 0$. Thus the divided powers are defined over \mathbb{Z} .

Remark 8.1. More generally, if $T = \mathbb{k}[x]$ for some field of characteristic p , and M_a is defined as above for some $a \in \mathbb{k}$, then $d^p = 0$ on M_a if and only if a lives in the prime field $\mathbb{F}_p \subset \mathbb{k}$.

Now consider the diagrammatic Hecke category over \mathbb{Z} , and let θ be a generator. We ask whether $d^{(k)}(\theta)$ lives inside the \mathbb{Z} -form as well. Obviously this holds if $d(\theta) = 0$, so we can consider only those differentials which do not kill θ .

Suppose that θ is an s -colored enddot. Then $d(\theta) = g_s \theta$, and $d(g_s) = g_s^2$. Thus we can consider the sub-edg-algebra $T = \mathbb{Z}[g_s] \subset R$. The subset $T \cdot \theta$ is closed under the differential, and is isomorphic as an edg module over T to M_1 . In particular, $d^{(k)}(\theta)$ is well-defined over \mathbb{Z} . The same argument works for the s -colored startdot.

The analogous argument also works for the s -colored trivalent vertices, where T acts by putting a polynomial in the appropriate region. This time $T \cdot \theta$ is isomorphic to M_{-1} instead.

Any good differential kills every 4-valent vertex, and one of the two 6-valent vertices. We need only check what happens to the other 6-valent vertex. We do the case when the orientation is $s \rightarrow t$.

Let α , β , and γ denote the following three diagrams.

$$\alpha = \begin{array}{c} \text{---} \cdot \text{---} \\ \diagup \quad \diagdown \\ \text{---} \cdot \text{---} \end{array}, \quad \beta = \begin{array}{c} \text{---} \cdot \text{---} \\ \diagup \quad \diagdown \\ \text{---} \cdot \text{---} \end{array}, \quad \gamma = \begin{array}{c} \text{---} \cdot \text{---} \\ \diagup \quad \diagdown \\ \text{---} \cdot \text{---} \end{array}.$$

Lemma 8.2. *The differential acts by the following formulae.*

$$(8.2a) \quad d(\alpha) = 2g_s \alpha - \kappa \gamma,$$

$$(8.2b) \quad d(\beta) = 2\bar{g}_t \beta + \kappa \gamma,$$

$$(8.2c) \quad d(\gamma) = 2(g_s + \bar{g}_t) \gamma.$$

Here, multiplication by a polynomial means putting that polynomial in the leftmost region (or rightmost region, it happens to be equal).

Proof. The proof is straightforward. Let us derive the first equality, as the others are similar. Applying the differential to α we get a sum of three terms:

- The differential applied to the broken strand on top (the pair of dots). This places z_s in the top region.
- Breaking the top strand again, with a plus sign and a factor of κ . This contributes $\kappa \alpha_s$ to the top region.
- Breaking the bottom strand, with a minus sign and a factor of κ . This contributes $-\kappa \gamma$.

Then one observes that $z_s + \kappa\alpha_s = (g_s + \bar{g}_s) + (g_s - \bar{g}_s) = 2g_s$. Since g_s is t -fixed, it slides across the t -colored strand to the leftmost or rightmost region. \square

Lemma 8.3. *When the orientation is $s \rightarrow t$, the differential applied iteratively to the 6-valent vertex*

$$\phi: B_s B_t B_s \rightarrow B_t B_s B_t$$

is equal to

$$(8.3) \quad d^k(\phi) = k! \kappa (g_s^{k-1} \alpha - \bar{g}_t^{k-1} \beta - \kappa \left(\sum_{a+b=k-2} g_s^a \bar{g}_t^b \right) \gamma)$$

for any $k \geq 1$. Consequently, $d^{(k)}$ is well-defined integrally.

Proof. We have $d(\phi) = \kappa(\alpha - \beta)$, and

$$(8.4) \quad d^2(\phi) = \kappa(2g_s \alpha - 2\kappa\gamma - 2\bar{g}_t \beta).$$

This proves the cases $k = 1, 2$. The inductive step is a simple exercise in the Leibniz rule. A helpful observation is that whenever $d(x) = x^2$ and $d(y) = y^2$ we have

$$(8.5) \quad d\left(\sum_{a+b=m} x^a y^b\right) = m x^{m+1} + (m-1)xy \sum_{a'+b'=m-1} x^{a'} y^{b'} + m y^{m+1}.$$

\square

Putting it all together, we have proven the following result.

Theorem 8.4. *For any good differential d on the diagrammatic Hecke category in simply-laced type, $d^{(k)}$ is defined integrally.*

9. A SURPRISING EXAMPLE WITH $n = 8$

Let us work in the diagrammatic Hecke category for S_8 associated to the standard realization with polynomial ring $R = \mathbb{Z}[x_1, \dots, x_8]$. Equip it with the standard differential, where $g_{s_i} = x_i$ and $\bar{g}_{s_i} = x_{i+1}$ and $\kappa = 1$. We also extend this differential to the thicker calculus which includes objects B_{sts} for $m_{st} = 3$, using (6.76).

Let X denote the Bott-Samelson bimodule associated with the sequence

$$\underline{w} = (s_3, s_2, s_1, s_5, s_4, s_3, s_2, s_6, s_5, s_4, s_3, s_7, s_6, s_5).$$

Let $y = s_3 s_2 s_3 s_5 s_6 s_5$. To describe the indecomposable object B_y we can most easily use thick calculus, since $B_y \cong B_{s_3 s_2 s_3} B_{s_5 s_6 s_5}$.

Here is a degree 0 map $p: X \rightarrow B_y$ and $i: B_y \rightarrow X$ which span their respective degree 0 Hom spaces.

$$(9.1) \quad p = \text{diagram} \quad i = \text{diagram}.$$

To summarize these maps in words: the four (grayscale) strands with colors in $\{1, 4, 7\}$ get dotted off; the five (red and blue) strands with colors in $\{2, 3\}$ get merged into $B_{s_3 s_2 s_3}$ (colored purple) with two thick trivalent vertices; the five (aqua and green) strands with colors in $\{5, 6\}$ get merged into $B_{s_5 s_6 s_5}$ (colored teal) with two thick trivalent vertices. It is a very fun exercise to compute that

$$(9.2) \quad pi = 2 \text{id}_{B_y}.$$

A computation yields

$$(9.3) \quad -d(p) = \text{diagram},$$

where again we use our abusive olive-colored sum notation and broken line notation from §6.1. To use words again, $-d(p)$ is the sum of three terms with the same underlying diagram as p : one which breaks the second strand colored s_3 with coefficient $+1$, one which breaks the last strand colored s_6 with coefficient $+1$, and one which places the polynomial $x_6 - x_1$ in the center. Another computation yields

$$(9.4) \quad +d(i) = \text{diagram}.$$

One can actually derive the second computation from the first, using various symmetries: rotation by 180 degrees, and the Dynkin diagram automorphism.

It is not immediately obvious that $d(p) \neq 0$, since it is not expressed as a linear combination of light leaves with polynomials on the right, but it is not too difficult to justify. For example, the polynomial x_1 is fixed by s_i for $i \geq 1$ and slides through the entire diagram; none of the other terms can possibly contribute a polynomial involving x_1 . Similarly, $d(i) \neq 0$ as can be seen from the x_8 term.

Finally, a computation yields

$$(9.5) \quad d(p)i = \text{diagram}.$$

Note that the split-merge appearing on the rightmost part of this picture is equal to the identity of $B_{s_5 s_6 s_5}$, but if the s_5 -colored strand is broken (with coefficient -1), it yields a nonzero degree $+2$ endomorphism of $B_{s_5 s_6 s_5}$. Similarly with the leftmost part of the picture and the identity of $B_{s_3 s_2 s_3}$.

As a consequence of this computation, B_y is not a summand of X in any dg-filtration, for any prime! This example shows that the conjectural d -canonical basis does not agree with the p -canonical basis. See §1.7 for further discussion.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403

Email address: belias@uoregon.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22901

Email address: yq2dw@virginia.edu