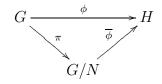
Exercises for Week 9

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Nov. 6.

Reading. Review Artin Sections 4.1-4.4 as we go along. Alternatively, consult any linear algebra book for the corresponding material. Read Artin Section 6.1-6.2.

1. The main difficulty we had with presentation is due to the following generalization of the first isomorphism theorem (FIT). Consider a surjective group homomorphism $\phi:G\to H$, and let $K:=\ker\phi$ be the kernel. Let N be a normal subgroup of G contained in K. Show that ϕ factors as group homomorphisms $\phi=\overline{\phi}\circ\pi$



(Hint: Define these maps explicitly as in the FIT). Show that $\overline{\phi}$ is surjective, and determine its kernel. In general, $\overline{\phi}$ is not an isomorphism. (Try to construct examples where it is not, and think about how this applies in the situation of group presentations!)

- 2. Let $\psi: V \to V$ be a linear operator on a vector space of dimension two. Suppose ψ is not a scalar multiple of the identity transformation. Show that there is a \mathbf{v} such that $\{\mathbf{v}, \psi(\mathbf{v})\}$ form a basis of V. Then compute the matrix of ψ with respect to this basis. (Try not to identify V with \mathbb{R}^2 !)
- 3. Prove that given any unit length vectors $\mathbf{u}, \mathbf{v} \in U$, there is an orthogonal transformation ϕ of the Euclidean space U that takes \mathbf{u} to \mathbf{v} .
- 4. Let $P:=\{p(x)\}$ be the space of polynomials over $\mathbb R$ of degree at most n. Consider $D=\frac{d}{dx}$ as a linear operator on P.
 - (a) Choose a basis of P to be $\{1, x, \dots, x^n\}$. Write down the matrix of D with respect to this basis.
 - (b) Choose a basis of P to be $\{1, x 1, \dots, (x 1)^n\}$. Write down the matrix of D with respect to this basis.
- 5. Consider the two dimensional Euclidean space \mathbb{R}^2 . Write down the matrices for the following operators.
 - (a) Reflection about the x-axis (y = 0).
 - (b) In general, the reflection about the ax + by = 0 line, where $(a, b) \in \mathbb{R}^2 \setminus \{0\}$.
- 6. Let $A \in O(2, \mathbb{R})$ be an orthogonal matrix. Determine when A has eigne-vectors in \mathbb{R}^2 , and find the eigenvalues. (Hint: Consider the cases when A has determinant ± 1 .)
- 7. Find matrix subgroups of $SO(2,\mathbb{R})$ that are isomorphic to (a) $\mathbb{Z}/(n)$ and (b) \mathbb{Z} .
- 8. Find a matrix subgroup of $O(n, \mathbb{R})$ that is isomorphic to S_n .