Homework 1

January 11, 2019

(i) In ring theory, a module P over a ring A is called *projective* for any given surjective A-module map $\pi : M \twoheadrightarrow N$ and any map $\phi : P \longrightarrow N$, there is a map $\tilde{\phi} : P \longrightarrow M$ making the diagram commute

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\phi}"'] \arrow[d, "\phi"] \\ M \arrow[r, "\pi"] & N \end{tikzcd}, \qquad \pi \circ \tilde{\phi} = \phi.$$

Reprove Lemma 3.1 of the textbook with the notion of "free resolutions" replaced by "projective resolutions" (an exact sequence each of whose term is a projective module).

As you can see, since free modules are always projective, this generalized exercise contains Lemma 3.1 as a specical case.

(ii) Prove that, for a family of A-modules M_i, $i \in I$ and N, we have $\mathrm{Ext}^{\bullet}_A(\oplus_{i\in I} M_i, N) \cong \prod_{i\in I} \mathrm{Ext}^{\bullet}_A(M_i, N)$.

(iii) Prove that $\mathrm{Tor}^i_A(M, N) \cong \mathrm{Tor}^i_A(N, M)$ for any modules over a commutative ring A. You may find it useful to show first that $\mathrm{Tor}^i_A(M, N) = 0$, $i > 0$, if either M or N is a free A-module.

(ii) Exercises 3, 6, 8, 11 of Hatcher, Section 3.1.