Linear Algebra and Matrix Theory Midterm Exam

March 8, 2016

Don't forget to write down clearly your

Name: ____________________ **Net ID:** ____________________

Instructions.

- The exam book contains 6 basic problems, worth 120 points. Out of the last five problems, please choose **four** to answer and receive a grading. **Please circle the problems you choose to be graded on the front page marking table below**.
- The total time for the exam is 75 minutes.
- No books, notes or calculators are allowed.
- Read the following story before opening the exam book!

  A mathematician, a physicist, and an engineer were traveling through Scotland when they saw a black sheep through the window of the train.

  "Aha," says the engineer, "I see that Scottish sheep are black."

  "Hmm," says the physicist, "You mean that some Scottish sheep are black."

  "No," says the mathematician, "All we know is that there is at least one sheep in Scotland, and that at least one side of that one sheep is black!"

  So, please be precise with your answers just as the mathematician in the story!
- Good luck with the exam!

CIRCLE THE PROBLEMS TO BE GRADED BELOW

Problem Number	Points
1	
2	
3	
4	
5	
6	
Total Points	

1. True or False (20 points). Mark "T" (True) in front of a correct statement and "F" (False) in front of a wrong one. No justification needed. V stands for a vector space over some field $\mathbb{F}$.

(1) ____ Any vector v in a vector space V has a unique $u \in V$ such that $u + v = 0_V$.

(2) ____ If U_1, U_2 are two subspaces of a vector space V, so is $U_1 \cup U_2$ a subspace of V.

(3) ____ Let $\mathbb{F}$ be a field. $\dim(\mathrm{M}_{3\times 5}(\mathbb{F})) = 8$.

(4) ____ If S is a generating set for a vector space V, then any subset of S also generates V.

(5) ____ The set of complex numbers $\mathbb{C}$ has dimension 2 over real numbers $\mathbb{R}$.

(6) ____ The linear map $\int_0^x dt : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R}), f \mapsto \int_0^x f(t)dt$ is onto.

(7) ____ The map $T : \mathrm{M}_{2\times 2}(\mathbb{R}) \longrightarrow \mathrm{M}_{2\times 2}(\mathbb{R}), T(A) := A^t$ is one-to-one.

(8) ____ Any linear map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ has a 1-dimensional kernel.

(9) ____ A 3×5 matrix may have rank 5.

(10) ____ Adding an elementary square matrix E to another square matrix A does not change the rank: $\mathrm{rank}(E + A) = \mathrm{rank}(A)$.

2. Basis (20 points). Let $\{u_1, u_2, u_3\}$ be a basis for a vector space V.

(a) Is the set of vectors $\{u_1 - u_2, u_2 - u_3, u_3 - u_1\}$ also a basis for V? Justify your answer (10 points).

(b) Is the set of vectors $\{u_1, u_1 - u_2, u_1 - u_2 - u_3\}$ also a basis for V? Justify your answer (10 points).

3. Ordered basis and matrix (20 points) Let β be the standard basis of $\mathbb{R}^2$:

(a) Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the reflection linear transformation of $\mathbb{R}^2$ about the y axis. Please find the matrix for S with respect to the standard basis (6 points).

(b) Let $R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map of performing a counterclockwise rotation by 60°. Please determine the matrix of R with respect to the standard basis (8 points).

(c) Compute the matrices of the composition linear transformation $R \circ S$ and $S \circ R$ with respect to the standard basis β of $\mathbb{R}^2$ (6 points).

4. Polynomials (20 points). Show that, if a polynomial $f(x)$ of degree less than or equal to 2015 satisfies $f(i) = 2016$, for all integers i in between 1 and 2016 (both ends included!), then the polynomial must be the constant polynomial $f(x) = 2016$.

5. Isomorphisms (20 points). Let $A \in \mathrm{M}_n(\mathbb{F})$ be an invertible $n \times n$ matrix.
(a) Show that A^t is an invertible matrix as well (8 points).

(b) Show that the following map

$$f : \mathrm{M}_n(\mathbb{F}) \longrightarrow \mathrm{M}_n(\mathbb{F}), \quad f(B) := A^t BA$$

is linear and an isomorphism (12 points).

6. Matrix inversion (20 points). Let a, b, c be any element of some field $\mathbb{F}$. (a) Consider the set of matrics

$$B := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{F} \right\}$$

(a) Show that, for any $A_1, A_2 \in B$, their matrix product is still in B (10 points).

(b)Are the matrices in B always invertible? If so, find the inverse of

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

If not, justify your answer (10 points).