Exercises for Week 6

The work handed in should be entirely your own. You can consult the textbook and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Oct. 9.

Reading. Read Sections 2.5, 3.1, 3.2 of the textbook carefully.

(1) This exercise is intended to develop a useful notation to help you understand/memorize change of basis matrices.

Let $T : V \longrightarrow W$ be a linear map, and let $\beta = \{v_1, \ldots, v_n\}$, $\beta' = \{v'_1, \ldots, v'_n\}$ be ordered bases for V and $\gamma = \{w_1, \ldots, w_m\}$, $\gamma' = \{w'_1, \ldots, w'_m\}$ be ordered bases for W.

We know how to parametrize vectors in V (and similarly for W) by

$$v \in V \Longleftrightarrow v = \sum_{i=1}^{n} a_i v_i = \beta \cdot [v]^{\beta} = (v_1, \ldots, v_n) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Likewise, in terms of β', we have

$$v = \sum_{i=1}^{n} a_i v_i = \beta' \cdot [v]^{\beta'} = (v'_1, \ldots, v'_n) \cdot \begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix}.$$

(a) Show that the change of basis matrix $Q := [\mathrm{Id_V}]_{\beta}^{\beta'} = (q_{i,j})$ has the effect

$$\beta = \beta' \cdot Q \Longleftrightarrow (v_1, \ldots, v_n) = (v'_1, \ldots, v'_n) \cdot \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n,1} & q_{n,2} & \cdots & q_{n,n} \end{pmatrix}.$$

Thus $\beta' \cdot [v]^{\beta'} = v = \beta \cdot [v]^{\beta}$ implies that

$$\beta' \cdot [v]^{\beta'} = v = \beta \cdot Q[v]^{\beta} \Rightarrow [v]^{\beta'} = Q[v]^{\beta}.$$

(b) Define $T(\beta) := (T(v_1), T(v_2), \ldots, T(v_n))$, a row of vectors. Show that

$$T(\beta) = (w_1, \ldots, w_m) \cdot A.$$

where $A = [T]_{\beta}^{\gamma}$ is the matrix of T with respect to β and γ.

(c) Now, if $\beta = \beta' \cdot Q_1$ and $\gamma = \gamma' \cdot Q_2$ where $Q_1 = [\mathrm{Id}_V]_{\beta}^{\beta'}$ and $Q_2 = [\mathrm{Id}_W]_{\gamma}^{\gamma'}$ are the respective change of coordinate matrices, we then have

$$T(\beta) = T(\beta' \cdot Q_1) = T(\beta') \cdot Q_1$$

since T is linear. Combine this with part (b) and show that

$$A' = Q_2^{-1} A Q_1,$$

where $A' = [T]_{\beta'}^{\gamma'}$.

(2) 3.1 Exercises 1, 6.

(3) 3.2 Exercise 1, 3, 5 (b), (c), 7, 8.

(4) Prove that $\mathrm{rank}(A) = \mathrm{rank}(A^t)$ (Reduce both A and A^t into the simplest possible form via row and column operations. This is Corollary 2 of Section 2, but try not to look at the proof and directly prove it yourself.)