Exercises for Week 12

The work handed in should be entirely your own. You can consult Dummit and Foote, Artin and/or the class notes but nothing else. To receive full credit, justify your answer in a clear and logical way. Due Nov. 27.

Reading. Read Artin Sections 7.2, 6.12. Review section 5.1 for more background material on special orthogonal groups.

1. Let $Aut_{Gp}(G)$ be the set of automorphisms of G preserving the group structure.
   (a) Show that $Aut_{Gp}(G)$ is a group.
   (b) Show that conjugation gives rise to a group homomorphism $C : G \to Aut_{Gp}(G)$, $C(g) : G \to G, x \mapsto gxg^{-1}$. Determine the kernel of C.
   (c) The set of automorphisms in $Aut_{Gp}(G)$ that are in the image of C are called *inner automorphisms*. Show that inner automorphisms of a group G constitute a normal subgroup in $Aut_{Gp}(G)$.

2. Find all the conjugacy classes and class equations for the groups (a) C_n, (b) D_n, and (c) $H = \{\pm 1, \pm i, \pm j, \pm k\}$.

3. Determine the matrices that represent the following rotations of $\mathbb{R}^3$:
   (a) by angle θ about the y-axis.
   (b) by angle $\pi/4$ about the axis in the direction $(1, 1, 1)^t$.

4. Show that any element $A \in O(3, \mathbb{R}) \backslash SO(3, \mathbb{R})$ has -1 as an eigen-value. Mimic the proof of Euler's theorem to show that any such matrix can be conjugated, via a change of orthonormal basis, to one of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

5. Let G be the rotational symmetry group of the cube in $\mathbb{R}^3$, and let S be the set of four diagonal lines connecting opposite vertices. Determine the stabilizer of one of the diagonals.

6. In class we used the counting formula to find the order of the symmetry group of a tetrahedron. Use the same method to find the orders of the octahedron group and icosahedron group.