Problem 5.2

$dX_t = (a_1 X_t + a_2)dt + g(X_t, t)dW_t$

in integral form this is

$X(t) - X(0) = \int_0^t (a_1 X_s + a_2)ds + \int_0^t g(X_s, s)dW_s$

Thus

$E(X(t) - X(0)) = E\int_0^t (a_1 X_s + a_2)ds + E(\int_0^t g(X_s, s)dW_s)$

By the martingale property and linearity of expectation and integration

$E(X(t)) - E(X(0)) = E(X(t) - X(0)) = \int_0^t (a_1 E(X_s) + a_2)ds + 0$

By substituting $m_t = E(X_t)$ we get

$m_t - m_0 = \int_0^t a_1 m_s + a_2 ds$

Which in differential form is the initial value problem

$\frac{dm_s}{ds} = a_1 m_s + a_2$
with initial condition $m_0 = E(X_0)$

I am terrible at solving ODE's, but Referencing Lessons 11A and 11B on integrating factors from Tenenbaum and Pollard's ODE's I get:

$$\frac{\partial m_t}{\partial t} - a_1 m_t = a_2$$

With integrating factor $e^{-a_1 t}$ we get

$$m_t = \frac{-a_2}{a_1}(1 - e^{a_1 t}) + m_0 e^{a_1 t}$$

an equivalent formulation of the solution.

Problem 6.4

Given the SDE $dX_t = \lambda(\mu - X_t)dt + \sigma dW_t$ we get coefficients $a_1 = -\lambda, a_2 = \lambda\mu$ matching exercise 5.2.

Let $Y_t = \phi(X_t, t) = X_t - m_t$. Thus as $\phi_t = dm_t, \phi_x = 1, \phi_{xx} = 0$ we get $dY_t = dX_t - dm_t$.

Thus as $dm_t = \lambda(\mu - m_t)dt$ from 5.2, it must be that

$dY_t = \lambda(\mu - X_t)dt + \sigma dW_t - \lambda(\mu - m_t)dt dY_t = -\lambda(X_t - m_t)dt + \sigma dW_t dY_t = -\lambda(Y_t)dt + \sigma dW_t$

This is known to have the solution

$$Y_t = \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dW_s$$

By transformation then,

$$X_t = \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dW_s + m_t X_t = \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dW_s + \mu(1 - e^{-\lambda t}) + m_0 e^{-\lambda t}$$

As $m_0 = x_0$ this result is equivalent to the result in the book.

Problem 6.5

Consider the SDE

$dS_t = dW_t$, i.e. $S_t = W_t$

Then for $Y_t = \phi(x, t) = \frac{1}{1-x}$ by the ito formula, we get that

$dY_t = 0dt + \phi_x(S_t, t)dS_t + \frac{1}{2}1^2\phi_{xx}(S_t, t)dt$

Thus as $\phi_x = (1 - x)^{-2}$ and $\phi_{xx} = 2(1 - x)^{-3}$ we get

$dY_t = (1 - S_t)^{-2}dS_t + \frac{2}{2}(1 - S_t)^{-3}dt = Y_t^2 dW_t + Y_t^3 dt$

Thus $Y_t = \frac{1}{1 - W_t}$ solves the differential equation above.

Problem 7.1

Expectation

Prove $E[\sum q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2 - 0.5\sum q(W_{t_i})\delta t] = 0$

Note that due to independence of $q(W_{t_i})$ and $(W_{t_{i+0.5}} - W_{t_i})$

$\sum E[q(W_{t_i})]E[(W_{t_{i+0.5}} - W_{t_i})^2] - 0.5\delta t E[\sum q(W_{t_i})]$

And note that

$$(W_{t_{i+0.5}} - W_{t_i}) \sim N(0, \frac{1}{\sqrt{2}}\delta t)$$

thus

$\sum E[q(W_{t_i})]0.5\delta t - 0.5\delta t E[\sum q(W_{t_i})] = 0$

Variance

Prove $V[\sum q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2 - 0.5\sum q(W_{t_i})\delta t]$ is of order $O(\delta t)$

Note the three major terms from factoring
$E[(\sum q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2 - 0.5\sum q(W_{t_i})\delta t)^2]$

$A = \sum_i q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2 \times \sum_j q(W_{t_j})(W_{t_{j+0.5}} - W_{t_j})^2 = 2\sum_{i<j}\sum_j q(W_{t_j})q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2(W_{t_{j+0.5}} - W_{t_j})^2 +$

$B = -2\sum_i q(W_{t_i})(W_{t_{i+0.5}} - W_{t_i})^2 \times 0.5\sum_j q(W_{t_j})\delta t$

$C = 0.25\sum_i q(W_{t_i})\sum_j q(W_{t_j})\delta t^2$

I realized I stopped this problem halfway through.

Problem 7.7

Given an Ito SDE

$$dX_t = rX_t dt + \gamma X_t dW_t$$

and corresponding Stratonovich SDE

$$d\hat{X}_t = r\hat{X}_t dt + \gamma\hat{X}_t \circ dW_t$$

we note that by equation 7.9, there is an ito SDE equivalent to the stratonovich SDE written as

$$d\bar{X}_t = \left(r - \frac{\gamma^2}{2}\right)\bar{X}_t dt + \gamma\bar{X}_t dW_t$$

Giving the key relationship that for any function $F(\cdot)$, it mus t be that $F(\hat{x}) = F(\bar{x})$

From the discussion surrounding 5.6, we know that

$$E(X_t) = E(X_0)e^{rt}$$

Thus from the discussion above

$$E(\hat{X}_t) = E(\bar{X}_t) = E(\hat{X}_0)e^{t(r - \frac{\gamma^2}{2})}$$

Similarly, as Exercise 6.3 gives us that

$$E(\log X_t) = \log X_0 + t(r - \frac{\gamma^2}{2})$$

Thus by subsitution

$$E(\log \hat{X}_t) = E(\log \bar{X}_t) = \log \hat{X}_0 + t(r - \frac{\gamma^2}{2}) - t\frac{\gamma^2}{2} E(\log \hat{X}_t) = E(\log \bar{X}_t) = \log \hat{X}_0 + tr$$

Problem PC 7.1 and Problem PC 7.2

See Submitted julia notebooks, html results, and pdf. They all say the same thing, but are different formats. I suggest reading the HTML.