Generalized Solution to Social Planner's Problem

2021-05-19

$$W(\theta_t) = \max_{x_t} F(\theta_t, x_t) + \beta W(\theta_{t+1})$$

x_t is a Vector of Choice Vars ($k \times 1$)

θ_t is a Vector of State Vars ($m \times 1$)

Envelope

$$\underset{m\times 1}{\frac{\partial W_t}{\partial \theta_t}} = \underset{m\times 1}{\frac{\partial F}{\partial \theta_t}} + \beta \underset{m\times m}{A_t} \underset{m\times 1}{\frac{\partial W_{t+1}}{\partial \theta_{t+1}}}$$

$$\frac{1}{\beta} A_t^{-1}\left(\frac{\partial W_t}{\partial \theta_t} - \frac{\partial F}{\partial \theta_t}\right) = \frac{\partial W_{t+1}}{\partial \theta_{t+1}}$$

Optimality

$$0 = \underset{k\times 1}{\frac{\partial F_t}{\partial x_t}} + \beta \underset{k\times m}{B_t} \underset{m\times 1}{\frac{\partial W_{t+1}}{\partial \theta_{t+1}}}$$

thus

$$0 = \frac{\partial F_t}{\partial x_t} + \cancel{\beta} B_t \frac{1}{\cancel{\beta}} A_t^{-1}\left(\frac{\partial W_t}{\partial \theta_t} - \frac{\partial F_t}{\partial \theta_t}\right)$$

$$-\frac{\partial F_t}{\partial x_t} + B_t A_t^{-1} \frac{\partial F_t}{\partial \theta_t} = B_t A_t^{-1} \frac{\partial W_t}{\partial \theta_t}$$

If $k = m$, then

$$\frac{\partial W_t}{\partial \theta_t} = -A_t B_t^{-1} \frac{\partial F_t}{\partial x_t} + \frac{\partial F_t}{\partial \theta_t}$$

But $k < m$ (usually $k+1 = m$ or $2k+1 = m$, in my problems)

So we need to construct extra rows.

- This can be done by taking the first $m-k$ rows of the optimality condition and iterating forward one period to $t+2$, then substituting in the envelope conditions twice. Add these conditions to the system, and you have an invertible system.

See Next Page.

Declare

Opt 1

$$0 = \frac{\partial F_t}{\partial x_t} + \beta B_t \frac{\partial W_{t+1}}{\partial \theta_{t+1}}$$

Opt 2

$$0 = \frac{\partial F_{t+1}}{\partial x_{t+1}} + \beta B_{t+1} \frac{\partial W_{t+2}}{\partial \theta_{t+2}}$$

Substitute

$$0 = \frac{\partial F_t}{\partial x_t} + B_t A_t^{-1}\left(\frac{\partial W_t}{\partial \theta_t} - \frac{\partial F_t}{\partial \theta_t}\right)$$

Substitute twice

$$0 = \frac{\partial F_{t+1}}{\partial x_{t+1}}(\theta_t\ x^*(\theta_t)) + \beta B_{t+1}(\theta_t)\left(\frac{A_{t+1}^{-1}}{\beta}\right)\left(\frac{\partial W_{t+1}}{\partial \theta_{t+1}} - \frac{\partial F_{t+1}}{\partial \theta_{t+1}}\right)$$

$$0 = \frac{\partial F}{\partial x}\left(\theta_{t+1}(\theta_t), x^*_{t+1}(\theta_t)\right) + B_{t+1}(\theta) A_{t+1}^{-1}(\theta_t)\left(-\frac{\partial F_{t+1}}{\partial \theta_{t+1}} + \frac{A_t^{-1}}{\beta}\left(\frac{\partial W_t}{\partial \theta_t} - \frac{\partial F_t}{\partial \theta_t}\right)\right)$$

$$0 = \frac{\partial F_{t+1}}{\partial x_{t+1}} + B_{t+1} A_{t+1}^{-1}\left(\frac{A_t^{-1}}{\beta}\frac{\partial W_t}{\partial \theta_t} - \frac{A_t^{-1}}{\beta}\frac{\partial F_t}{\partial \theta_t} - \frac{\partial F_{t+1}}{\partial \theta_{t+1}}\right)$$

$$0 = \frac{\partial F_{t+1}}{\partial x_{t+1}} - B_{t+1} A_{t+1}^{-1}\frac{\partial F_{t+1}}{\partial \theta_{t+1}} + \frac{B_{t+1} A_{t+1}^{-1} A_t^{-1}}{\beta}\frac{\partial F_t}{\partial \theta_t} + \frac{B_{t+1} A_{t+1}^{-1} A_t^{-1}}{\beta}\frac{\partial W_t}{\partial \theta_t}$$

Which is another Valid Set of Conditions.

Note that $\theta^*_{t+1}(\theta_t, x^*_t(\theta_t))$ etc, are function of the policy function

$$0 = \frac{\partial F^*}{\partial x}(\theta^*_{t+1}(\theta_t)) + B(\theta^*_{t+1}(\theta_t))\, A^{-1}(\theta^*_{t+1}(\theta_t))\left[\frac{\partial F^*}{\partial \theta}(\theta^*_{t+1}(\theta_t)) + \frac{A^{-1}(\theta_t)}{\beta}\left(\frac{\partial W}{\partial \theta}(\theta_t) + \frac{\partial F^*}{\partial \theta}(\theta_t)\right)\right]$$

i.e. they implicitly define the Policy function.

By appending the first M-K Rows of the Second to the first, we get a set of M expressions, which determine the M functions $\frac{\partial}{\partial \theta_t} W_t$. By Iterating the optimality Condition Back a single Period to t, we get an implicit definition of $X^*_t(\theta_t)$.