ZAMS Model Report

Yiwei Chai

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1 Introduction

Understanding the structure and evolution of stars is foundational to the field of astronomy. Many, if not most, topics of interest within the field rely on information obtained from observations of electromagnetic radiation produced by stars. However, stellar opacities prohibit direct observations of the internal structure of stars. Thus, it is important to develop stellar models in order to gain an understanding of internal stellar structures, and thus a star's consequent evolution.

In this project, I seek to create a rudimentary stellar model for a $1.33 M_\odot$ Zero Age Main Sequence (ZAMS) star. I assume a constant composition of fully ionised ideal gas, with H, He, and metallicity fractions of $X = 0.7$, $Y = 0.27$, and $Z = 0.03$ respectively. I also assume negligible rotation.

The internal structure and evolution of a star can be well described by the below four coupled ordinary differential equations relating its pressure (1), radius (2), luminosity (3), and temperature (4):

$$\frac{dP}{dm_r} = -\frac{Gm_r}{4\pi r^4} \tag{1}$$

$$\frac{dr}{dm_r} = -\frac{1}{4\pi r^2 \rho} \tag{2}$$

$$\frac{dl}{dm_r} = \epsilon \tag{3}$$

$$\frac{dT}{dm_r} = -\frac{Gm_r T}{4\pi r^4 \rho}\nabla \tag{4}$$

Modelling a ZAMS star therefore requires solving the boundary-value problem posed by these four coupled ODEs, for which the mass m_r enclosed within radius r is the dependent variable.

2 Methodology

The solution to the four coupled ODEs of stellar structure and evolution must be obtained numerically. Following *Teukolsky et al. 2011*, I have used the method of shooting to a fitting point. This method involves integrating from two sets of boundary values towards a chosen point between them, calculating the mismatch in the solutions at this chosen point, and then repeating the integration with updated boundary values until the two solutions converge.

For the stellar model, I set the fitting point at $m_r = 0.5M_*$, and the boundaries at two mass fraction points: one near the centre of the star ($m_r = 10^{-12}M_*$), and one near the surface ($m_r = 0.9999M_*$). In order to calculate values for the pressure, radius, luminosity, and temperature at these boundary points however, it is necessary to first calculate a number of other stellar parameters. A brief description of these various parameter calculations follows.

2.1 Calculation of Prerequisite Parameters

2.1.1 Opacity

I obtain the opacity κ by using `scipy.optimize.griddata` to interpolate linearly over a grid of opacity values corresponding to a set of temperature and density points. These points and values are taken from the OPAL table for a solar-type star with the same composition (i.e. $X = 0.7$, $Y = 0.27$, $Z = 0.03$). The parameter space is set as $3.75 < \log T < 7.5$ and $-9 < \log \rho < 3$.

2.1.2 Density

I obtain the density term ρ from the equation of pressure due to both gas and radiation:

$$\rho = \frac{(P - \frac{1}{3}aT^4)\mu}{N_A kT} \tag{5}$$

Here, P is the total pressure accounting for contributions from gas and radiation, and must be estimated. The problem with this method of obtaining the density is that, if the estimated P is less than $P_{rad} = \frac{1}{3}aT^4$, the calculated density becomes negative, which is an unphysical value. This issue and resulting complications for later parts of the model calculations are discussed in more detail in Results (Section 3).

2.1.3 Energy Generation Rate

In a $1.33M_\odot$ star, there are contributions from both the pp-chain and the CNO cycle to the energy generation rate ϵ_H. Thus, a calculation of ϵ_H must be two-fold. The contribution from the proton-proton chain can be described using:

$$\epsilon_{pp} = 2.57 \times 10^4 \psi f_{11} g_{11} \rho X_1^2 T_9^{2/3} e^{-3.381/T_9^1/3}, \tag{6}$$

$$g_{11} = (1 + 3.82T_9 + 1.51T_9^2 + 0.144T_9^3 - 0.0114T_9^4), \tag{7}$$

where $g_1 1$ is the Gaunt factor and psi 1 is used to correct for additional contributions to the energy generation if there is a non-negligible amount of ^{4}He. I make a slightly more detailed estimation of ψ based on Kippenhahn et al. 2012, in which ψ behaves as a step function dependent on temperature. Additionally, we assume a nondegenerate gas for which $E_D/kT << 1$, and take $f_{11} = e^{E_D/kT}$ to be a weak screening factor where,

$$\frac{E_D}{kT} = \frac{Z_1 Z_2 e^2}{r_D kT} = 5.92 \times 10^{-3} Z_1 Z_2 (\frac{\zeta \rho}{T_7^3})^{1/2} \tag{8}$$

Here, $\zeta \approx 1$.

The contribution from the CNO cycle can be described using:

$$\epsilon_{CNO} = 8.24 \times 10^{25} g_{14,1} X_{CNO} X_1 \rho T_9^{-2/3} e^{(-15.231 T_9^{-1/3} - (T_9/0.8)^2)}, \tag{9}$$

$$g_{14,1} = (1 - 2.00 T_9 + 3.41 T_9^2 - 2.43 T_9^3), \tag{10}$$

where X_{CNO} is the sum of the mass fractions of C, N, and O.

2.1.4 Nature of Energy Transport

It is important to understand the nature of energy transport ∇ within a star, as this will be necessary for temperature calculations. For $\nabla_{rad} \leq \nabla_{ad}$, pure diffusive radiative transfer is dominant, and $\nabla = \nabla_{rad}$. For $\nabla_{rad} > \nabla_{ad}$, adiabatic convection is dominant, and $\nabla = \nabla_{ad}$.

Assuming a fully ionised ideal gas ($\Gamma_2 = \frac{5}{3}$), the calculation of ∇_{ad} becomes relatively straightforward:

$$\nabla_{ad} = \frac{\partial \ln T}{\partial \ln P} = \frac{\Gamma_2 - 1}{\Gamma_2} = 0.4 \tag{11}$$

2.2 Calculation of Stellar Boundary Values

2.2.1 Stellar Centre

At the stellar centre, the four variables which must be calculated can be described by:

$$P = P_c - \frac{3G}{8\pi}(\frac{4\pi}{3}\rho_c)^{4/3} m^{2/3} \tag{12}$$

$$r = (\frac{3}{4\pi\rho_c})^{1/3} m^{1/3} \tag{13}$$

$$l = \epsilon_c m \tag{14}$$

$$T_{radc} = [T_c^4 - \frac{1}{2ac}(\frac{3}{4\pi})^{2/3} \kappa_c \epsilon_c \rho_c^{4/3} m^{2/3}]^{1/4} \tag{15}$$

$$T_{convc} = \exp[\ln T_c - (\frac{\pi}{6})^{1/3} G \frac{\nabla_{ad,c} \rho_c^{4/3}}{P_c} m^{2/3}] \tag{16}$$

Here, central pressure P_c and central temperature T_c are values that must be estimated. Additionally, the temperature value differs depending on whether energy transport at the core is radiative or convective.

2.2.2 Stellar Surface

At the stellar surface, I assume a known radius and luminosity. Thus, the variables to be calculated are pressure and temperature. I take the temperature to be equivalent to the effective temperature T_{eff}, which can be obtained from a rearrangement of the Stefan-Boltzmann law:

$$T_{eff} = \sqrt[4]{\frac{L_*}{4\pi R^2}} \tag{17}$$

At the stellar surface, the pressure is dominated by the opacity pressure $P_\tau = \frac{2}{3}\frac{g_s}{\kappa}(1+\frac{\kappa L_*}{4\pi c G M_*})$. Calculating this requires obtaining a value for the opacity, which in turn requires obtaining a value for the density. I use `scipy.optimize.minimize` to obtain a density at the point where the pressure due to opacity and the pressure due to radiation and ideal gas is close to equivalent. This then allows me to obtain a value for P_τ.

2.3 Initial Stellar Parameter Estimates

The final model takes an input of estimates for the central pressure, the total radius, the surface luminosity, and the central temperature, given the mass of the star being modelled.

I assume a constant density model will provide reasonable estimates for the central pressure and temperature. The equations describing these parameters for such a case are as follows:

$$P_c = \frac{3GM_*^2}{8\pi R_*^4} \tag{18}$$

$$T_c = 1.15 \times 10^7 \mu (\frac{M_*}{M_\odot})(\frac{R_*}{R_\odot})^{-1} \tag{19}$$

To obtain estimates for the total radius and surface luminosity, I use the following homology relations:

$$R \propto (\frac{M_*}{M_\odot})^{0.75} \tag{20}$$

$$L \propto (\frac{M_*}{M_\odot})^{3.5} \tag{21}$$

I assume these homology relations are reasonable as a $1.33M_\odot$ star is not massive, and thus its internal structure should not deviate too much from that of the sun (i.e. it has a radiative core and a convective envelope).

Table 1 provides the resulting estimated stellar parameter values input into the model.

3 Results and Discussion

After 27 iterations, my model successfully converges with fractional residuals on the order of 10^{-10}. The resulting solution for the stellar parameter values is fairly accurate, with the radius and luminosity being with 25% of the MESA value, and the temperature being within 5% of MESA. The full internal structure for a $1.33M_\odot$ star as calculated using my model

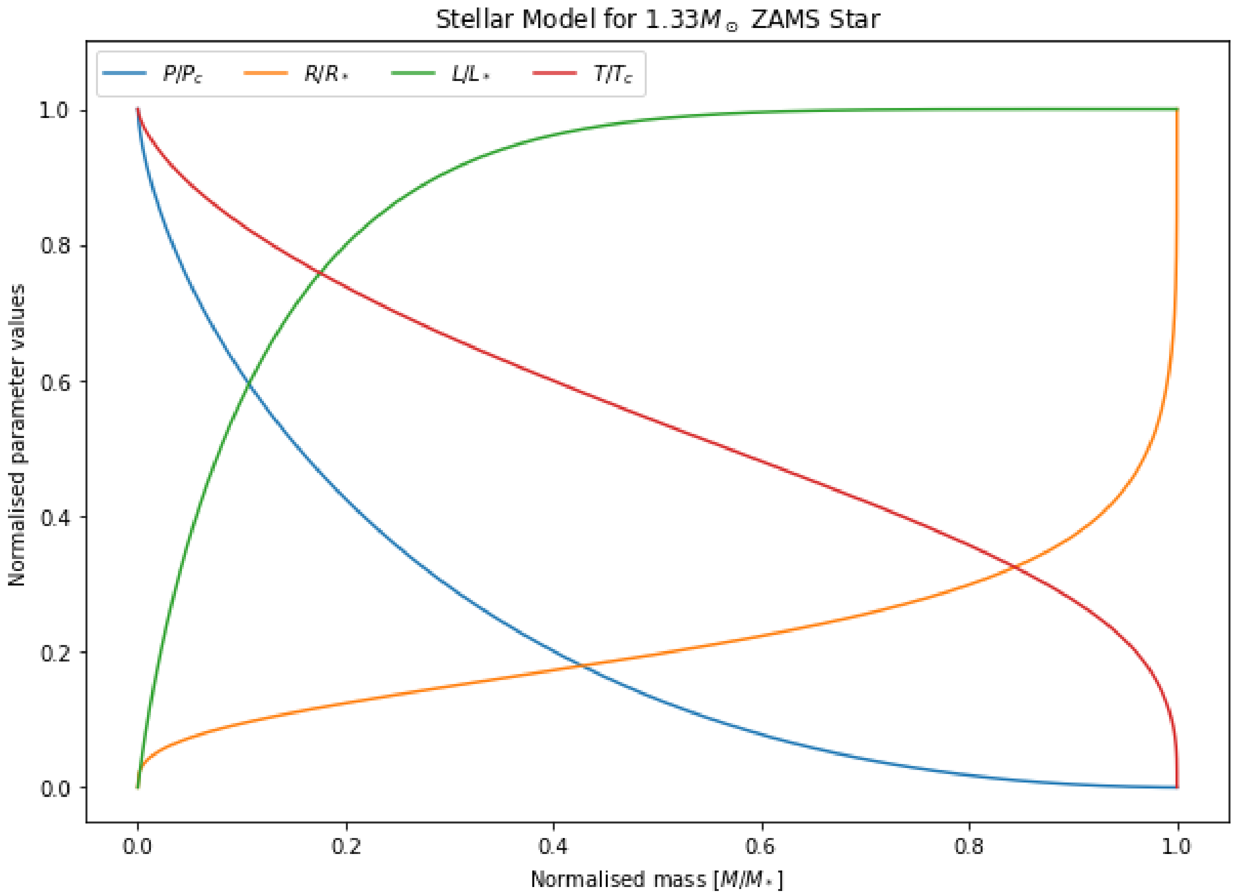

Figure 1: Stellar structure model for the pressure, radius, luminosity, and temperature of a $1.33M_\odot$ ZAMS star. Axes are normalised to the stellar parameter values output by the converged model.

is shown in Figure 1. A summary of the key parameter values, alongside the initial input estimates, is provided in Table 1.

	P [dyne/cm^2]	R [cm]	L [erg/s]	T [K]
Initial Estimate	2.62×10^{15}	6.96×10^{10}	3.84×10^{33}	7.08×10^{6}
Converged Model	1.72×10^{17}	1.05×10^{11}	7.59×10^{33}	1.61×10^{7}
MESA		9.13×10^{10}	9.69×10^{33}	1.69×10^{7}
% Error from MESA		14.4	21.2	4.24
Fractional Residuals	6.18×10^{-10}	-1.54×10^{-10}	-1.00×10^{-10}	-1.71×10^{-10}

Table 1: Comparison between stellar parameters. (Row 2) shows initial input estimates from constant density model and homology relations. (Row 3) shows output values from the converged model. (Row 4) shows parameters output by MESA for a star of $1.33M_\odot$ and $Z = 0.03$. (Row 5) gives the percentage error of my model values as compared to the MESA parameter values. Note that MESA does not output a value for pressure, so this term is calculated. (Row 6) gives the final fractional residuals at the fitting point.

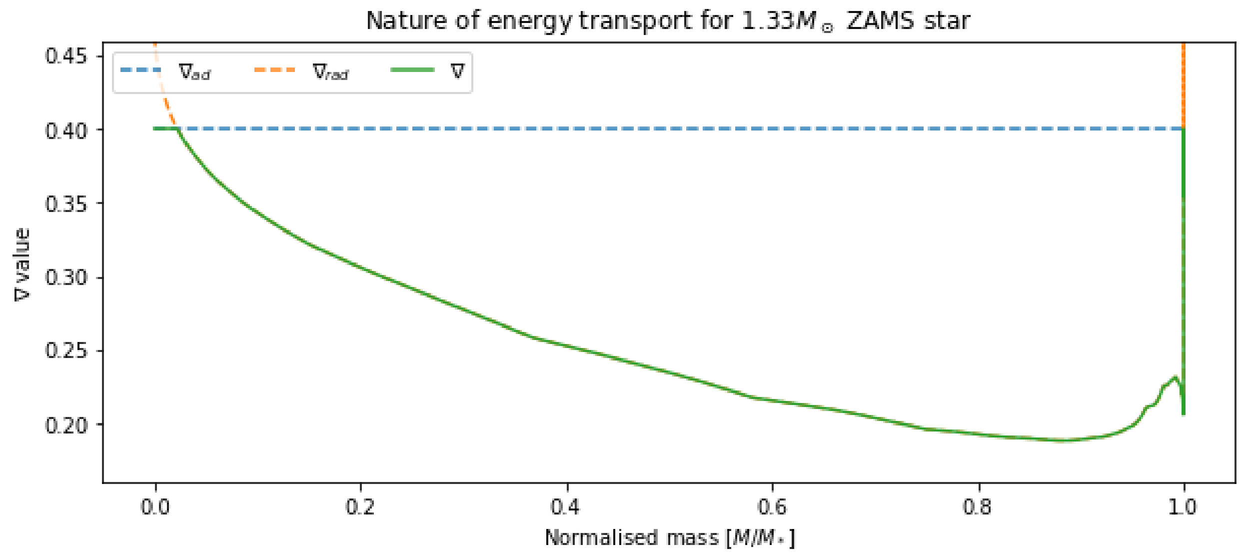

Figure 2: Plot of ∇ compared to ∇_{ad} and ∇_{rad} values as calculated using the stellar model. $\nabla > \nabla_{ad}$ at less than 0.1 M/M_* indicates a radiative core, and $\nabla < \nabla_{ad}$ beyond that indicates a convective envelope.

4 Conclusions

In this project, I seek to create a rudimentary model for the internal structure of a $1.33M_\odot$ star with a solar-like composition. I use the method of shooting to a fitting point to numerically solve the four coupled ODEs of stellar structure and evolution, and thus obtain the key stellar parameters of pressure, radius, luminosity and temperature. In creating this model, I neglect stellar rotation and assume that the star is composed of fully ionised ideal gas. My model converges in 27 iterations and outputs stellar parameter values within 25% of MESA values for a star with the same mass and composition. The less accurate values for radius and luminosity may be due to my assumption that a constant density model provides a reasonable initial pressure and temperature estimate.

All the code used to create this model, along with a table of all output stellar parameters calculated using the model, is publicly available in this Github repository: `https://github.com/youais/ZAMS_stellar_model`